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I study random-matching economies where at money coexists with real assets, and no restrictions are imposed on payment arrangements. I emphasize informational asymmetries about asset fundamentals to explain the partial illiquidity of real assets and the usefulness of at money. The liquidity of the real asset, as measured by its transaction velocity, is shown to depend on the discrepancy of its dividend across states as well as policy. I analyze how monetary policy a¤ects payment arrangements, asset prices, and welfare.

Key words: Fiat money, payments, private information, liquidity, asset prices.

JEL code: D82, D83, E40, E50

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1 Introduction

I study random-matching economies where fiat money coexists with real assets and no restrictions are imposed on payment arrangements. I investigate the relationship between assets' intrinsic characteristics and liquidity, and I analyze how monetary policy affects payment arrangements, asset prices, and welfare.

In accordance with the finance literature, I emphasize private information frictions to explain why some assets are more costly or more difficult to trade than others. A key assumption of the theory is that agents have a privileged knowledge of the fundamental value of the assets they hold in their portfolios. For instance, an individual is better informed about the future dividend of a stock he owns, the value of a project he has undertaken, or his ability to repay his debt.

Following the recent monetary literature pioneered by Kiyotaki and Wright (1989), I consider an environment where some trades occur within bilateral meetings, and terms of trade are the outcome of a bargaining game. Payment arrangements are therefore explicit and endogenous. A temporal double-coincidence-of-wants problem makes the use of a medium of exchange necessary. There are two types of assets in the economy: a fiat object with no intrinsic value–fiat currency–and a one-period-lived asset that promises some output in the future. Both assets can be used as media of exchange.

As in Lagos and Wright (2005), there are periodic rounds of centralized and decentralized trades, assets are perfectly divisible, and individuals' portfolios are unrestricted. The presence of incomplete information about the dividend of the real asset makes the determination of the terms of trade in bilateral meetings a harder problem. I consider a simple bargaining game where buyers make take-it-or-leave-it offers, and I sharpen the predictions of this game by appealing to the Intuitive Criterion (Cho and Kreps, 1987).[1]

I study two versions of the model corresponding to different structures for the asset market. In the first version, real assets are not homogeneous (they generate dividends of different sizes) and they can only be traded in bilateral meetings. The asset is viewed as private equity or bilateral credit. In the second version, the real asset, interpreted as a publicly traded stock, is homogenous but it is subject to aggregate dividend shocks. It can be exchanged, and hence priced, both in centralized and decentralized markets.

For both versions of the model, fiat money can be valued even though there are no restrictions on the use of assets as media of exchange. The presence of informational asymmetries enlarges the set of parameter

[1] In Lagos and Wright (2005) the terms of trade in bilateral matches are determined by the Nash solution. Since standard axiomatic bargaining solutions are not applicable to games with incomplete information, I adopt a strategic bargaining game–the ultimatum game–which has been widely used in the search money literature.

values under which fiat money has a positive price. Fiat money is useful because in some states buyers spend only a fraction of their asset holdings even though their consumption is inefficiently low (in a sense to be made precise later), i.e., the real asset is partially illiquid.

The model is used to analyze the relationship between asset liquidity–as measured by transaction velocity–and fundamentals. The asset becomes less liquid as the dispersion of the dividends across states increases. Moreover, if the real asset is valueless in some states then it becomes fully illiquid. So, the model can rationalize the exclusive use of fiat currency as a medium of exchange.

The model has implications for asset prices. Provided that the stock of real assets is not too large and inflation is not too low, the price of the real asset can also rise above its expected dividend and exhibit a liquidity premium. The imperfect substitutability between fiat money and the real asset–buyers only spend their real assets when their cash holdings are exhausted–manifests itself by a rate-of-return differential. The illiquidity premium paid to the real asset tends to increase as the asset becomes riskier and more abundant.

Finally, the model delivers insights for the linkages between monetary policy and asset prices. Monetary policy affects an asset's return when the quantity of the real asset is not too large and inflation is in some intermediate range. An increase in inflation induces a reallocation of individuals' portfolios towards real assets that are not as liquid as currency but have a higher rate of return. Consequently, the model predicts a negative relationship between inflation and assets' expected returns. The optimal monetary policy is such that the real asset in the high-dividend state is illiquid, i.e., its transaction velocity and liquidity premium are zero.

1.1 Related literature

This paper aims at providing foundations for some of the trading restrictions found in recent search monetary models with multiple assets. Aruoba and Wright (2003), Aruoba, Waller and Wright (2007), Berentsen, Menzio and Wright (2007) and Telyukova and Wright (2007) introduce capital, loans or stocks into the Lagos-Wright model. Following Freeman (1985), they assume that all claims on future output can be counterfeited at no cost, and hence they cannot be used as means of payment.[2] Lagos (2006) introduces "Lucas trees" but restricts their use as means of payment in a fraction of the trades (because of legal or

[2] Aruoba and Wright (2003) and Aruoba, Waller and Wright (2007) also refer to the lack of portability of capital goods to justify the assumption that capital cannot be used as means of payment in decentralized markets. They assume that agents have their capital physically fixed in place at production sites. Telyukova and Wright (2007, Section 4) lay down an extension of their model with "Lucas trees," in which agents pay a fixed cost if they use their real assets as means of payment.

institutional reasons).[3] Lester, Postlewaite and Wright (2007) endogenize this fraction of trades by assuming that agents have to invest in a costly technology to verify the authenticity of the assets held by other agents.

Kiyotaki and Moore (2005) introduce a similar trading restriction in a dynamic general equilibrium model with two assets (capital and land). They assume that investing agents can sell at most a fraction $\theta \in (0, 1)$ of their capital. In their footnote 6, they provide some explanations for why capital may not be perfectly liquid: "*there may be different qualities of capital, and buyers may be less informed than sellers so that there is adverse selection in the second-hand market*." This is precisely the avenue I follow in this paper.[4]

In contrast to the literature above, the extent to which capital is used as means of payment is endogenous and it depends on policy and the characteristics of the asset. The theory can justify the complete illiquidity of capital, as in Aruoba, Waller and Wright (2007), if capital is valueless in some states. Relative to Kiyotaki and Moore (2005) and Lagos (2006), my model links the illiquidity of capital to the properties of its dividend process and to policy.

The idea of explaining asset liquidity by a private information problem is omnipresent in both the finance and the monetary literature. Asymmetries of informations are used to explain transaction costs in financial markets (e.g., Kyle, 1985; Glosten and Milgrom, 1985), security design (e.g., DeMarzo and Duffie, 1999), and capital structure choices (e.g., Myers and Majluf, 1984). The monetary literature has resorted to private information problems to explain the role of money when goods are of unknown quality (e.g., Williamson and Wright, 1994; Banarjee and Maskin, 1996) or when individuals have private information about their ability to repay their debt (e.g., Jafarey and Rupert, 2001).[5] Closer to my model, Velde, Weber and Wright (1999) explain Gresham's law with an adverse selection problem in a search environment with a fixed supply of indivisible coins of different qualities.[6]

In accordance with Wallace's (1996) dictum, I make no restriction on the use of assets as means of payment. In the same vein, Aiyagari, Wallace and Wright (1996), Wallace (1996, 2000) and Cone (2005) emphasize asset divisibility, or lack of divisibility, to explain the coexistence of money and interest-bearing assets and the liquidity structure of asset yields. Zhu and Wallace (2007) make bonds illiquid by constructing

[3] See Shi (2004) for a similar assumption in a search model with fiat money and nominal bonds.

[4] Similarly, Zhu (2006, Section 4) discusses how one could introduce capital into his OLG model with search, and he argues that to maintain the transactions role of money, "one could assume some private information about the quality of capital, similar to the private information problem on the quality of goods in Williamson and Wright (1994)."

[5] Berentsen and Rocheteau (2004) introduce a moral hazard similar to Williamson and Wright (1994) into a model with divisible money. The "counterfeit" consumption good is perishable, it has no value, and only a pooling mechanism is considered.

[6] Li (1995) constructs a related model, in which there is quality uncertainty about commodity monies.

a pairwise-efficient trading mechanism that yields a surplus to the buyer that is equal to the one he would obtain if he would trade with money only (the "cash-in-advance" twist) and had all the bargaining power. In contrast to Aiyagari, Wallace and Wright (1996), Shi (2004) and Zhu and Wallace (2007), this paper is not an attempt to explain the coexistence of fiat money and risk-free government bonds. (One could substitute currency by risk-free bonds, like in Lagos (2006).)

Finally, Lagos and Rocheteau (2006) and Geromichalos, Licari and Suarez-Lledo (2007) study a complete information version of the model in this paper. Money is useful provided that the capital stock in the economy is small, and if money and capital coexist they have the same rate of return. In contrast, in my model the presence of money is always useful irrespective of the size of the capital stock, and if money and capital coexist then capital dominates money in its rate of return.

2 Environment

Time is discrete, starts at $t = 0$, and continues forever. Each period has two subperiods, a morning (AM) followed by an afternoon (PM), where different activities take place. There is a continuum of agents divided into two types, called *buyers* and *sellers*, who differ in terms of when they produce and consume. The labels *buyers* and *sellers* indicate agents' roles in the PM market. There are two consumption goods, one produced in the AM and the other in the PM. Consumption goods are perishable.

Agents live for three subperiods. Buyers and sellers from generation t are born at the beginning of period t, and they die at the end of the AM in period $t+1$. (See Figure 1.) Let $\mathcal{B}_t$ denote the set of buyers from generation t, $\mathcal{S}_t$ the set of sellers from generation t, and $\mathcal{J}_t = \mathcal{B}_t \cup \mathcal{S}_t$.[7] The measures of buyers and sellers are normalized to 1.

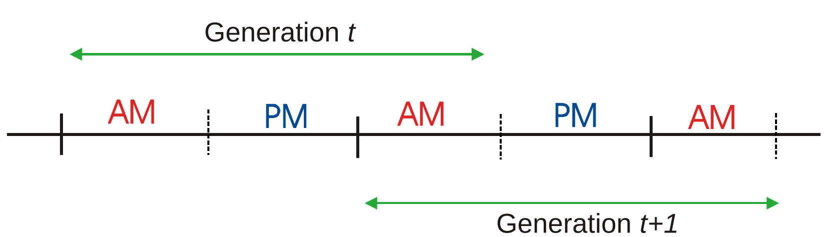

Figure 1: Overlapping generations structure

[7] This overlapping-generations structure facilitates the presentation of the model. For a related environment, see Zhu (2006) and Zhu and Wallace (2007). The assumption of alternating market structures is borrowed from Lagos and Wright (2005).

Buyers produce in the first AM of their lives while sellers produce in the PM. This heterogeneity will generate a temporal double-coincidence problem.[8] The utility of a buyer born at date t is

$$U_t^b = -\ell_t + u(q_t) + \beta x_{t+1}, \tag{1}$$

where x_t is the AM consumption of period t, ℓ_t is the AM disutility of work, q_t is the PM consumption, and $\beta \in (0,1)$ is a discount factor. The utility function $u(q)$ is twice continuously differentiable, $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$, and $u''(q) < 0$. The production technology in the AM is linear with labor as the only input, $y_t = \ell_t$. Buyers' endowment of labor is unlimited when young.

The utility of a seller born at date t is

$$U_t^s = -c(q_t) + \beta x_{t+1}, \tag{2}$$

where q_t is the PM production. The cost function $c(q)$ is twice continuously differentiable, $c(0) = c'(0) = 0$, $c'(q) > 0$, $c''(q) \geq 0$ and $c(q) = u(q)$ for some $q > 0$. Let q^* denote the solution to $u'(q^*) = c'(q^*)$.

At the beginning of his life, each buyer is endowed with $A > 0$ units of a one-period-lived real asset. The asset is perfectly divisible, uncounterfeitable, and perfectly durable over its lifetime. Each unit of the asset held by buyer $j \in \mathcal{B}_t$ yields $\kappa_{j,t+1}$ units of AM-output delivered in $t+1$, and it fully depreciates subsequently. The real dividend can take two values, $\kappa_{j,t+1} \in \{\kappa_\ell, \kappa_h\}$, where $0 < \kappa_\ell < \kappa_h$. Let $\mathcal{B}_t^h \equiv \{j \in \mathcal{B}_t : \kappa_{j,t+1} = \kappa_h\}$ denote the subset of buyers from generation t endowed with high-dividend assets, and $\mathcal{B}_t^\ell \equiv \mathcal{B}_t \backslash \mathcal{B}_t^h$ the subset of buyers endowed with low-dividend assets.

I will consider two versions of the model, which differ in terms of the description of the dividend shock κ. In the first version, the $\kappa_{j,t}$'s are the realizations of i.i.d. random variables. In this case, the measures of $\mathcal{B}_t^h$ and $\mathcal{B}_t^\ell$ are constant over time and equal to $\pi_h \in (0,1)$ and $\pi_\ell = 1 - \pi_h$, respectively. In the second version, $\kappa_{j,t}$ is independent of j and it is interpreted as an aggregate shock. Then, $\mathcal{B}_t^h = \mathcal{B}_t$ with probability $\pi_h \in (0,1)$ and $\mathcal{B}_t^h = \varnothing$ with complement probability $\pi_\ell = 1 - \pi_h$. Denote $\bar{\kappa} = \pi_h \kappa_h + \pi_\ell \kappa_\ell$. For both versions of the model, buyers who enter the PM have some private information about the quality of their real asset holdings, while sellers are uninformed.

Fiat money is durable, perfectly divisible, and it can be held in any nonnegative amount. The quantity of money per buyer in the PM of period t is denoted M_t. It grows at a constant gross rate, $\gamma \equiv M_{t+1}/M_t$,

[8] The description of the temporal double coincidence problem comes from Rocheteau and Wright (2005). The assumption that sellers cannot produce when young is relaxed in Appendix F without affecting the results.

where $\gamma > \beta$. New money is injected, or withdrawn if $\gamma < 1$, by lump-sum transfers $T_t = (\gamma - 1)M_{t-1}$, or taxes if $\gamma < 1$, to the young buyers.[9]

In the AM, there is a competitive market where agents can trade goods and fiat money. I will make different assumptions about whether the real asset can be traded or not in the AM. No other assets (such as bonds) are available in this market.

In the PM, each seller is matched bilaterally with a buyer drawn at random from $\mathcal{B}_t$.[10] All trades in the PM are *quid pro quo*, and matched agents can transfer any nonnegative quantity of PM-output and any quantity of their asset holdings. Agents can only trade the physical asset and not claims on future output.[11] In order to guarantee that there is an essential role for a medium of exchange, there is no public record of individuals' trading histories.[12]

Terms of trade in the PM are determined according to a simple bargaining game: The buyer makes an offer that the seller accepts or rejects. If the offer is accepted then the trade is implemented. At the end of the PM, agent pairs split apart.

3 Social optimum

Consider the problem of a social planner who chooses an allocation in order to maximize the sum of utilities of all agents in the economy. The planner assigns the Pareto-weights β^t to all agents from generation t, i.e., it values equally the consumption of one unit of AM-good and the disutility cost to produce one such unit by any agent alive in period t.[13]

Let $\mathcal{M}_t \subset \mathcal{B}_t \times \mathcal{S}_t$ denote the set of bilateral matches composed of one buyer and one seller in the PM of period t. The expression for social welfare is then

$$\mathcal{W} = \sum_{t \geq 1} \beta^t \int_{j \in \mathcal{J}_{t-1}} x_t(j) dj - \sum_{t \geq 0} \beta^t \int_{j \in \mathcal{B}_t} \ell_t(j) dj + \sum_{t \geq 0} \beta^t \int_{(j,j') \in \mathcal{M}_t} \{u[q_t(j)] - c[q_t(j')]\} \, d(j,j'). \quad (3)$$

[9] If $\gamma < 1$ the government can force all young buyers to pay taxes in the AM. However, it has no enforcement power in the PM, and it does not observe agents' trading histories. In a related model, Andolfatto (2007) considers the case where the government has limited coercion power–it cannot confiscate output and cannot force agents to work–and the payment of lump-sum taxes is voluntary: agents can avoid paying taxes by not accumulating money balances. He shows that if agents are sufficiently impatient, then the Friedman rule is not incentive-feasible, i.e., there is an induced lower bound on deflation.

[10] It would be easy to introduce search frictions so that the measure of bilateral matches in the PM is less than one, as is standard in search monetary literature.

[11] See footnote 22 for an interpretation of the real asset as bilateral credit.

[12] If trading histories were publicly observable, then some good allocations could be implemented through the threat of trigger strategies. See Kocherlakota (1998) for a detailed presentation of this argument.

[13] The choice of the welfare metric can be justified by the (observational) equivalence between the infinitely-lived-agent model of Lagos and Wright (2005) and the OLG model with search and risk-neutral old of Zhu (2006). Waller (2007) shows that the same optimal PM allocations would be derived under alternative weights for the welfare function.

The first integral on the right-hand side of (3) corresponds to the AM-consumption of all old agents from $t = 1$ onwards. The second term is the AM disutility of production of the young buyers from $t = 0$ onwards. The last term is buyers' consumption net of sellers' disutility of production in bilateral matches formed in the PM subperiods.

The planner observes the realizations of the dividend shocks $\{\kappa_{j,t}\}$ at the beginning of period t. It is subject to the following feasibility constraints:

$$\int_{j\in\mathcal{J}_{t-1}} x_t(j)dj \leq \int_{j\in\mathcal{B}_t} \ell_t(j)dj + A\int_{j\in\mathcal{B}_{t-1}} \kappa_{j,t}dj, \quad \forall t \geq 1 \tag{4}$$

$$q_t(j) \leq q_t(j'), \qquad \forall (j,j') \in \mathcal{M}_t, \quad \forall t \geq 0. \tag{5}$$

Feasibility constraint (4) requires agents' AM-consumption in period t to be at most equal to the aggregate production in that period, including the output generated by the stock of assets, A. Feasibility condition (5) indicates that the buyer's consumption in a bilateral match is no greater than the seller's production in that match.

The planner's problem can be rewritten as a sequence of static problems, i.e.,

$$\max_{x_t,\ell_t,q_t} \int_{j\in\mathcal{J}_{t-1}} x_t(j)dj - \int_{j\in\mathcal{B}_t} \ell_t(j)dj + \int_{(j,j')\in\mathcal{M}_t} \{u[q_t(j)] - c[q_t(j')]\}\, d(j,j') \tag{6}$$

subject to (4) and (5). The planner is indifferent on how to allocate the AM-goods between agents. The optimal consumption and production in bilateral matches satisfy $q_t(j) = q_t(j') = q^*$ for all $(j,j') \in \mathcal{M}_t$.

4 Payments under private information

In this section, I consider an economy without fiat money, where only real assets can be used as media of exchange. There is no market in the AM: all trades occur in bilateral meetings in the PM. This version of the model is consistent with the dividend shock being idiosyncratic or aggregate.

This section has two purposes. One is to investigate how private information affects the capacity of an asset to serve as a means of payment, thereby providing a benchmark to compare with the monetary economies studied later. The second purpose is to analyze in detail the bargaining game under incomplete information in a simple environment.

The bargaining game between a buyer and a seller in the PM has the structure of a signaling game.[14] A

[14] See Appendix B for a more detailed presentation of signaling games. If one rescales the buyer's payoff as $u(q)/\kappa - d$ and the seller's payoff as $-c(q)/\kappa + d$, then the bargaining game has the basic take-it-or-leave-it set-up defined in Kreps and Sobel (1994, p. 855).

strategy for the buyer specifies an offer $(q, d) \in \mathbb{R}_+ \times [0, A]$, where q is the output produced by the seller and d is the transfer of asset by the buyer, as a function of the buyer's type (i.e., the future dividend of his asset holdings). A strategy for the seller is an acceptance rule that specifies the set $\mathcal{A} \subseteq \mathbb{R}_+ \times [0, A]$ of acceptable offers.

The buyer's payoff is $[u(q) - \beta\kappa d]\, \mathbb{I}_{\mathcal{A}}(q, d) + \beta\kappa A$, where $\mathbb{I}_{\mathcal{A}}(q, d)$ is an indicator function that is equal to one if $(q, d) \in \mathcal{A}$. If an offer is accepted, then the buyer enjoys his utility of consumption in the PM, $u(q)$, net of the utility he forgoes by transferring d units of his asset to the seller, $-\beta\kappa d$. The seller's payoff is $-c(q) + \beta\kappa d$. The seller uses the information conveyed by (q, d) to update his prior belief about the quality of the asset held by the buyer. Let $\lambda(q, p) \in [0, 1]$ represent the updated belief of a seller that the buyer holds a high-dividend asset ($\kappa = \kappa_h$).

An equilibrium of the bargaining game is a profile of strategies for the buyer and the seller, and a belief system λ. The equilibrium concept is sequential equilibrium. The buyer chooses an offer that maximizes his surplus, taking as given the acceptance rule of the seller. The seller chooses optimally to reject or accept offers given his posterior belief. If (q, p) corresponds to an equilibrium offer, then $\lambda(q, p)$ is derived from the seller's prior belief according to Bayes's rule. If (q, p) is an out-of-equilibrium offer, then the seller's belief is arbitrary (to some extent discussed later).

For a given belief system, the set of acceptable offers for a seller is

$$\mathcal{A}(\lambda) = \{(q, d) \in \mathbb{R}_+ \times [0, A] : -c(q) + \beta \{\lambda(q, d)\kappa_h + [1 - \lambda(q, d)]\, \kappa_\ell\}\, d \geq 0\}\,. \tag{7}$$

For an offer to be acceptable, the seller's disutility of production in the PM, $-c(q)$, must be compensated by his expected discounted utility in the next AM, $\beta\mathbb{E}_\lambda[\kappa]\, d$, where the expectation is with respect to the random dividend of the asset. I assume that a seller agrees to any offer that makes him indifferent between accepting or rejecting a trade.[15] The problem of a buyer holding an asset of quality κ is then

$$\max_{q, d \leq A} [u(q) - \beta\kappa d]\, \mathbb{I}_{\mathcal{A}}(q, d). \tag{8}$$

Sellers' beliefs following out-of-equilibrium offers are largely arbitrary. The equilibrium concept is refined by using the Intuitive Criterion proposed by Cho and Kreps (1987).[16] Denote U_h^b the surplus of an $h-$type

[15] A similar tie-breaking assumption is used in Rubinstein (1985, Assumption B-3).

[16] The Intuitive Criterion is a refinement supported by much of the signalling literature. An equilibrium that fails the Intuitive Criterion gives an outcome that is not strategically stable in the sense of Kohlberg and Mertens (1986). Also, Inderst (2002) provides a strategic foundation of the selection procedure underlying the Intuitive Criterion in a search environment with

buyer and U^b_ℓ the surplus of an $\ell-$type buyer in a proposed equilibrium of the bargaining game. This proposed equilibrium fails the Intuitive Criterion if there is an unsent offer $(\tilde{q}, \tilde{d})$ such that the following is true:

$$u(\tilde{q}) - \beta\kappa_h\tilde{d} > U^b_h \tag{9}$$

$$u(\tilde{q}) - \beta\kappa_\ell\tilde{d} < U^b_\ell \tag{10}$$

$$-c(\tilde{q}) + \beta\kappa_h\tilde{d} \geq 0. \tag{11}$$

According to (9), the unsent offer $(\tilde{q}, \tilde{d})$ would make an $h-$type buyer strictly better off if it were accepted. According to (10), the unsent offer $(\tilde{q}, \tilde{d})$ would make an $\ell-$type buyer strictly worse off. According to (11), the offer is acceptable provided that the seller believes it comes from an $h-$type.[17]

I next turn to the definition of an equilibrium. Time is not introduced explicitly in the definition since there is no state variable linking the different generations. Moreover, the seller's acceptance rule is not included; it appears as a constraint in the buyer's problem.

Definition 1 *An equilibrium is a list of strategies for buyers and a belief system for sellers,* $\langle [q(j), d(j)]_{j\in\mathcal{B}^\ell\cup\mathcal{B}^h}, \lambda\rangle$*, such that: (i)* $[q(j), d(j)]$ *is solution to (8) with* $\kappa = \kappa_\ell$ *for all* $j \in \mathcal{B}^\ell$ *and* $\kappa = \kappa_h$ *for all* $j \in \mathcal{B}^h$*; (ii)* $\lambda : \mathbb{R}_+ \times [0, A] \rightarrow [0, 1]$ *satisfies Bayes' rule whenever possible and the Intuitive Criterion.*

Buyers of the same type are allowed to use different (pure) strategies. All sellers are assumed to use the same belief system λ, and hence the same acceptance rule. An equilibrium offer (q, d) is defined as pooling if it is in the support of the distribution of offers made by both $h-$type and $\ell-$type buyers, i.e., $\lambda(q, d) \in (0, 1)$.

Lemma 1 *In equilibrium, there is no pooling offer.*

The left panel of Figure 2 illustrates the argument in the proof of Lemma 1. Consider an equilibrium with a pooling offer $(\bar{q}, \bar{d})$. The surpluses of the two types of buyers at the proposed equilibrium are denoted $U^b_\ell \equiv u(\bar{q}) - \beta\kappa_\ell\bar{d}$ and $U^b_h \equiv u(\bar{q}) - \beta\kappa_h\bar{d}$. The indifference curves U^b_ℓ and U^b_h in Figure 2 represent the

endogenous participation. See Riley (2001) for a survey of the applications of the Intuitive Criterion (and other refinements) in various contexts. It has been used in monetary theory by Nosal and Wallace (2007) and Dutu, Nosal and Rocheteau (2006); in the corporate finance literature by Noe (1989) and DeMarzo and Duffie (1999); in bargaining theory by Rubinstein (1985, Assumption B-1); and recently in the literature on global games by Angeletos, Hellwig and Pavan (2006). For sake of completeness, the model is also analyzed under the alternative refinement from Mailath, Okuno-Fujiwara and Postlewaite (1993) in Appendix C.

[17] The inequality in (11) is weak as a result of the tie-breaking rule (7) according to which sellers accept offers that make them indifferent between accepting and rejecting.

set of offers (q, d) that generate the equilibrium surpluses. They exhibit a single-crossing property, which is key to obtain a separating equilibrium.[18] The participation constraint of a seller who believes he is facing an $h-$type buyer is represented by the frontier $U_h^s \equiv \{(q,d) : -c(q) + \beta\kappa_h d = 0\}$. The offer $(\bar{q}, \bar{d})$ is located above U_h^s since it is accepted when $\lambda < 1$. The shaded area indicates the set of offers that raise the utility of an $h-$type buyer (offers to the right of U_h^b) but reduce the utility of an $\ell-$type buyer (offers to the left of U_ℓ^b) and that are acceptable by sellers, provided that $\lambda = 1$ (offers above U_h^s). These offers satisfy (9)-(11) so that the proposed equilibrium with a pooling offer $(\bar{q}, \bar{d})$ violates the Intuitive Criterion. In order to separate himself, an $h-$type buyer reduces his PM consumption as well as his transfer of his asset to the seller. Provided that the reduction in q is sufficiently large relative to the reduction in d, an $\ell-$type buyer would never choose such an offer because his asset is less valuable than the one of an $h-$type buyer.

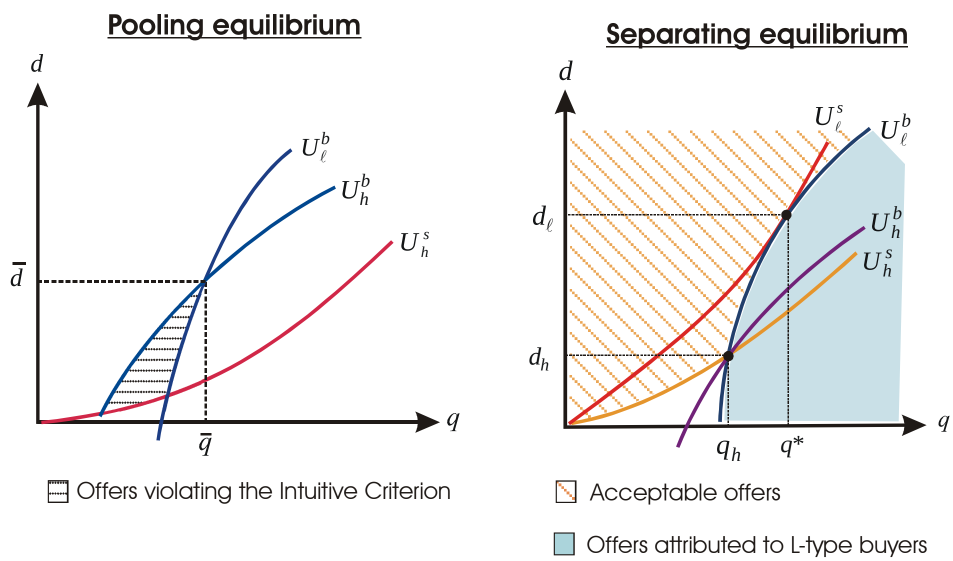

Figure 2: Pooling vs separating equilibria

I now characterize the equilibrium offers. The only way an $\ell-$type buyer can achieve a higher payoff than the one he would get in a game with complete information is by making an offer that a seller would attribute to an $h-$type buyer with positive probability, i.e., $\lambda(q, d) > 0$, which has been ruled out by Lemma 1. Hence, since the complete information payoff can always been achieved, i.e., $\lambda(q, d) \geq 0$, the offer of an

[18] For a definition of the *single-crossing property*, see Kreps and Sobel (1994, p. 855). The slopes of the indifference curves are $\frac{dd}{dq}\Big|_{U_\ell^b} = \frac{u'(q)}{\beta\kappa_\ell} < \frac{dd}{dq}\Big|_{U_h^b} = \frac{u'(q)}{\beta\kappa_h}$. Hence, U_ℓ^b intersects U_h^b by below.

$\ell-$type buyer solves

$$U(\kappa_\ell) = \max_{q,d \le A} [u(q) - \beta\kappa_\ell d] \quad \text{s.t.} \quad -c(q) + \beta\kappa_\ell d \ge 0. \tag{12}$$

I characterize next the offer (q_h, d_h) made by an $h-$type buyer. From the Intuitive Criterion, an $h-$type buyer can always increase his payoff as long as by so doing he does not give incentives to an $\ell-$type buyer to imitate him. Hence, (q_h, d_h) solves:

$$U(\kappa_h) = \max_{q,d \le A} [u(q) - \beta\kappa_h d] \quad \text{s.t.} \quad -c(q) + \beta\kappa_h d \ge 0 \tag{13}$$

$$\text{s.t.} \quad u(q) - \beta\kappa_\ell d \le U(\kappa_\ell). \tag{14}$$

From (13)-(14) the buyer maximizes his expected surplus subject to the participation constraint of the seller, where the seller has the correct belief that he faces an $h-$type buyer, and subject to the incentive-compatibility condition according to which an $\ell-$type buyer cannot be made better-off by offering (q_h, d_h).[19]

Proposition 1 *There exists a unique equilibrium (up to the belief system λ), and it is such that $\ell-$type buyers trade*

$$q_\ell = \min\left[q^*, c^{-1}(\beta\kappa_\ell A)\right] \tag{15}$$

$$d_\ell = \min\left[\frac{c(q^*)}{\beta\kappa_\ell}, A\right], \tag{16}$$

while $h-$type buyers trade (q_h, d_h) that satisfies

$$c(q_h) = \frac{\kappa_h}{\kappa_\ell}[u(q_h) - U(\kappa_\ell)] \tag{17}$$

$$d_h = \frac{u(q_h) - U(\kappa_\ell)}{\beta\kappa_\ell}. \tag{18}$$

Furthermore, $d_h < d_\ell$ and $q_h < q_\ell \le q^*$.

If the quantity of asset is large enough, then the trade in $\ell-$type matches is efficient, $q = q^*$. In contrast, if the value of the asset is less than the disutility incurred by the seller to produce q^*, then the $\ell-$type buyer cannot ask for the efficient quantity of output. In both cases, the buyer appropriates the whole surplus of the match.

[19] Suppose there is a separating equilibrium where the expected payoff of the $\ell-$type is $U(\kappa_\ell)$ and the expected payoff of the $h-$type is $\hat{U}(\kappa_h) \in [0, U(\kappa_h))$. Replace $U(\kappa_\ell)$ in (14) by $U(\kappa_\ell) - \varepsilon$ with $\varepsilon > 0$, and denote $U^\varepsilon(\kappa_h)$ the associated payoff for the $h-$type buyer. The set of acceptable and feasible offers is compact. From the Theorem of the Maximum, $U^\varepsilon(\kappa_h)$ is continuous in ε and $\lim_{\varepsilon\to 0} U^\varepsilon(\kappa_h) = U(\kappa_h)$. Hence, there is an $\varepsilon > 0$ such that $U^\varepsilon(\kappa_h) > \hat{U}(\kappa_h)$. The associated offer satisfies (9)-(11) so that the proposed equilibrium violates the Intuitive Criterion.

Equation (17) determines a unique $q_h < q_\ell$. Given q_h, d_h is determined by (18). The most noticeable feature of this solution is that $q_h < q_\ell$, which implies $d_h < A$ and $q_h < q^*$. Buyers holding high-dividend assets only trade a fraction of their assets in the PM market even though their consumption is inefficiently low. This illiquidity—the fact that they spend strictly less than they would in a complete information environment—is a consequence of the need for buyers in the high state to separate themselves from buyers in the low state.

Buyers' offers are illustrated in the right panel of Figure 2 (in the case where the constraint $d_\ell \leq A$ does not bind). The offer of the $\ell-$type buyer is at the tangency point between the iso-surplus curve of the seller, $U_\ell^s \equiv -c(q) + \beta\kappa_\ell d = 0$, and the iso-surplus curve of the buyer, U_ℓ^b. In order to satisfy the seller's participation constraint and (14), type$-h$ buyers make offers to the left of U_ℓ^b and above U_h^s. The utility-maximizing offer is at the intersection of the two curves.

A belief system consistent with the offers in Proposition 1 is such that sellers attribute all offers that violate (14) to $\ell-$type buyers, and all other out-of-equilibrium offers to $h-$type buyers. (See the right panel of Figure 2.) So, larger trades that involve the transfer of a large quantity of an asset suffer from less favorable terms of trade.

I now turn to the normative properties of the equilibrium. If $\kappa_\ell A \geq c(q^*)/\beta$, then the value of the low-dividend asset is large enough to trade the first-best quantity, q^*. Under complete information the economy achieves its first-best. In contrast, if the quality of the asset is private information, then the equilibrium allocation is inefficient. The $\ell-$type buyers consume q^*, but $h-$type buyers consume $q_h < q^*$. If $\kappa_\ell A < c(q^*)/\beta$, then the quantities traded in the PM are inefficiently low in all matches, i.e., $q_h < q_\ell < q^*$.[20]

5 Fiat money and payment arrangements

In this section fiat money is introduced as a competing means of payment. I ask whether fiat money can acquire some positive value in exchange, and whether it helps mitigate the inefficiencies associated with the adverse selection problem in the PM and the partial illiquidity of the real asset. I study how the rate of return of fiat currency affects asset liquidity and payment arrangements. Finally, the model will

[20] One could also ask whether there exists an incentive-feasible trading mechanism that implements the first-best allocation in the absence of fiat money. Consider a direct mechanism that maps the buyer's type κ into an offer (q, d). Suppose $q_h = q_\ell = q^*$. Then, incentive-compatibility requires $d_h = d_\ell = d$. So the outcome is pooling, in contrast to the outcome of our bargaining game. The trade (q^*, d) satisfies the seller's individual rationality constraint if $-c(q^*) + \beta\bar{\kappa}d \geq 0$. Similarly, buyers are willing to participate if $u(q^*) - \beta\kappa_h d \geq 0$. Thus, the first-best is incentive-feasible provided that $A \geq c(q^*)/\beta\bar{\kappa}$ and $\kappa_h/\bar{\kappa} \leq u(q^*)/c(q^*)$, i.e., there is no shortage of the asset and the discrepancy between the dividends in the different states is not too large.

provide microfoundations, and closed-form expressions, for some of the trading restrictions found in the recent monetary literature.

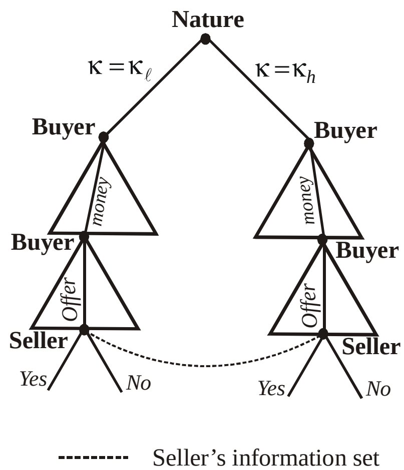

Figure 3: Game tree

I depart from the previous section by opening a competitive market in the AM where agents can trade fiat money for goods. The dividend shocks, $\kappa_{j,t}$, are independent and identically distributed across buyers, i.e., the measures of the subsets $\mathcal{B}_t^h$ and $\mathcal{B}_t^\ell$ are time-invariant.[21] The real asset, which is not homogenous, is only traded in bilateral meetings in the PM. (In the next section, the asset be traded in the AM market in order to investigate implications for asset prices.) One can think of the asset as private equity or bilateral credit (IOUs).[22]

The sequence of events is summarized by the game tree in Figure 3. First, for every buyer, Nature chooses the dividend size $\kappa \in \{\kappa_\ell, \kappa_h\}$. A buyer learns the future dividend of his asset holdings before trading in the AM market. Second, the buyer chooses his real balances as a function of his type. Third, he enters a bilateral match in the PM, and he makes an offer to an uninformed seller, who accepts or rejects it. From the seller's standpoint, the buyer j he is matched with has been chosen at random from the pool of all

[21] Sellers cannot pool the risk associated with the random quality of the asset they receive in the PM. For instance, they learn the quality of the asset as soon as they leave the PM, before they can trade with each other.

[22] Many assets, such as corporate bonds, private equity, derivatives and swaps, are traded in bilateral meetings, in over-the-counter markets. See Duffie, Garleanu and Pedersen (2005) for a formalization of such markets using a search model. A version of the model with IOUs would adopt a similar interpretation as in Jafarey and Rupert (2001). Buyers receive an endowment A in the last period of their lives with probability $\kappa \in (0,1)$ and nothing with the complement probability $1-\kappa$. Hence, buyers default with probability $1-\kappa$, and the expected value of a claim of d units of future output is κd. For such credit arrangements to be feasible, one needs to assume that buyers are able to commit (but not sellers).

buyers, i.e., $\Pr\left(j \in \mathcal{B}_t^h\right) = \pi_h$ and $\Pr\left(j \in \mathcal{B}_t^\ell\right) = \pi_\ell$. The buyer's portfolio is assumed to be non-observable by the seller in the match, and hence it cannot be used to condition the seller's acceptance decision (i.e., all histories with the same offer are part of the same information set).[23]

Denote p_t the price of the AM-good in period t. I focus on steady-state equilibria where aggregate real balances, M_t/p_t, are constant over time. Then, $p_{t+1}/p_t = \gamma$.

A buyer makes two decisions consecutively: his real balances in the AM and the offer to make in the PM. Hence, the strategy of a buyer is defined as a list (z, q, d, τ) function of his type κ, where z is the choice of real balances (expressed in terms of the AM good of the current period), q is the buyer's consumption in the PM, d the transfer of the real asset, and τ the transfer of real balances. The optimal strategy maximizes the buyer's utility (excluding the lump-sum transfer, T) subject to the seller's acceptance rule. It solves

$$\max_{z,q,d\le A,\tau\le z} \left[-z + u(q) + \beta\kappa(A-d) + \frac{\beta}{\gamma}(z-\tau)\right] \tag{19}$$

$$\text{s.t.} \quad -c(q) + \beta\left\{\lambda(q,d,\tau)\kappa_h + \left[1-\lambda(q,d,\tau)\right]\kappa_\ell\right\}d + \frac{\beta}{\gamma}\tau \ge 0. \tag{20}$$

The unspent real balances of the buyer generate a flow of utility, $\beta(z-\tau)/\gamma$, because consumption takes place in the next AM and real balances depreciate at rate γ.[24] Since buyers' real balances are not observable, the seller's updated belief, λ, only depends on the offer made by the buyer, (q, d, τ).

Lemma 2 *Any buyer's strategy, (z, q, d, τ), such that $z > \tau$, is strictly dominated.*

Since it is costly to hold money—the gross inflation rate is larger than the discount factor—and since real balances have no signaling function, it is a dominant strategy for a buyer to bring the exact amount he plans to spend in a bilateral match.

From Lemma 2, buyers' strategies can be restricted to triples (q, d, ω), where $\omega \equiv \beta z/\gamma = \beta\tau/\gamma$ indicates both the real balances of the buyer (discounted and expressed in terms of the next period's AM good) and the real money transfer in the PM to the seller. The buyer's problem, (19)-(20), can then be reduced to:

$$\max_{q,d\le A,\omega} \left\{-(1+i)\omega + u(q) - \beta\kappa d\right\} \quad \text{s.t.} \quad -c(q) + \left\{\lambda(q,d,\omega)\kappa_h + \left[1-\lambda(q,d,\omega)\right]\kappa_\ell\right\}d + \omega \ge 0, \tag{21}$$

where $\lambda(q, d, \omega)$ is the seller's posterior belief (with a slight abuse of notation), and $i \equiv (\gamma - \beta)/\beta > 0$ is the cost of holding real balances.

[23] This assumption simplifies the presentation by reducing the extent to which the buyer can signal his type. In Section 6 I consider a model with a different information structure where buyers' portfolios are common knowledge in the match.

[24] Suppose the buyer hands over m_t units of money to the seller. These m_t units of money buy m_t/p_{t+1} units of AM goods in period $t+1$ or, equivalently, $(m_t/p_t)(p_t/p_{t+1}) = (m_t/p_t)/\gamma$.

Definition 2 *An equilibrium is a list of buyers' strategies and a belief system for sellers,* $\langle [q(j), d(j), \omega(j)]_{j \in \mathcal{B}}, \lambda \rangle$, *such that: (i)* $[q(j), d(j), \omega(j)]$ *is solution to (21) with* $\kappa = \kappa_\ell$ *for all* $j \in \mathcal{B}^\ell$ *and* $\kappa = \kappa_h$ *for all* $j \in \mathcal{B}^h$*; (ii)* $\lambda : \mathbb{R}_+ \times [0, A] \times \mathbb{R}_+ \to [0, 1]$ *satisfies Bayes's rule whenever possible and the Intuitive Criterion.*

Using a similar argument to the one in Lemma 1, the next proposition establishes that the equilibrium is separating.

Lemma 3 *In any equilibrium, there is no pooling offer.*

From Lemma 3, an $\ell-$type buyer cannot do better than his complete-information payoff, which solves

$$U(\kappa_\ell) = \max_{q, d \le A, \omega \ge 0} \{-(1+i)\omega + u(q) - \beta\kappa_\ell d\} \quad \text{s.t.} \quad -c(q) + \beta\kappa_\ell d + \omega \ge 0. \tag{22}$$

If $c(q^*) \le \beta\kappa_\ell A$, then $q_\ell = q^*$, $d_\ell = c(q^*)/\beta\kappa_\ell$, and $\omega = 0$. If $c(q^*) > \beta\kappa_\ell A$, then $d_\ell = A$, $\omega_\ell = [c(q_\ell) - \beta\kappa_\ell A]^+$ (where $[x]^+ \equiv \max(x, 0)$) and q_ℓ solves

$$\frac{u'(q_\ell)}{c'(q_\ell)} \le 1 + i, \tag{23}$$

with a strict equality if $\omega_\ell > 0$. So $\ell-$type buyers accumulate real balances if the value of their real asset is not large enough to purchase q^* and if i is sufficiently small.

As in the previous section, the Intuitive Criterion selects the equilibrium that is Pareto efficient (from the standpoint of buyers' interim payoffs) in the class of separating equilibria.[25] Hence, the $h-$type buyer makes an offer that maximizes his payoff subject to the seller's acceptance rule and the condition that the offer must not be imitated by $\ell-$type buyers, i.e., (q_h, d_h, ω_h) solves

$$\max_{q, d \le A, \omega} \{-(1+i)\omega + u(q) - \beta\kappa_h d\} \tag{24}$$

$$\text{s.t.} \quad \omega - c(q) + \beta\kappa_h d \ge 0 \tag{25}$$

$$-(1+i)\omega + u(q) - \beta\kappa_\ell d \le U(\kappa_\ell). \tag{26}$$

Lemma 4 *There is a unique solution,* (q_h, d_h, ω_h)*, to (24)-(26) and it solves:*

$$\omega_h = \frac{\kappa_h \left\{ \left[u(q_h) - \frac{\kappa_\ell}{\kappa_h} c(q_h) \right] - U(\kappa_\ell) \right\}}{(1+i)\kappa_h - \kappa_\ell} \tag{27}$$

$$d_h = \frac{U(\kappa_\ell) - [u(q_h) - (1+i)c(q_h)]}{[(1+i)\kappa_h - \kappa_\ell]\beta} \tag{28}$$

[25] See footnote 19 for a formal argument.

and

$$u'(q_h) - (1+i)\,c'(q_h) \leq 0 \quad \text{“}=\text{”} \quad \textit{if } \omega_h > 0. \tag{29}$$

Assuming $\omega_h > 0$, the problem of an $h-$type buyer can be solved recursively. First, (29) determines q_h. Given q_h, (27) and (28) determine ω_h and d_h.[26]

The next proposition determines the conditions for fiat money to be valued in equilibrium.

Proposition 2 ***(Existence of monetary equilibrium)***

There exists $i_1 \geq 0$ and $i_2 > i_1$, such that the following is true.

1. *If $i < i_1$, then there is a unique monetary equilibrium, and it is such that all buyers accumulate real balances. Furthermore, $i_1 > 0$ iff $\kappa_\ell A < c(q^*)/\beta$.*

2. *If $i \in [i_1, i_2)$, then there is a unique monetary equilibrium, and it is such that only $h-$type buyers accumulate real balances.*

3. *If $i \geq i_2$, then there is no equilibrium where fiat money is valued.*

Proposition 2 is illustrated in Figure 4. The condition for the existence of a monetary equilibrium in part 1 of Proposition 2 is identical to the one in the complete-information economy. Indeed, with complete information fiat money is valued if and only if $\omega_\ell > 0$ (since $\omega_h < \omega_\ell$) or, equivalently,

$$i < i_1 \equiv \frac{u'(q_\ell)}{c'(q_\ell)} - 1, \tag{30}$$

where $q_\ell = \min\left[q^*, c^{-1}(\beta\kappa_\ell A)\right]$. The condition (30) requires A to be small enough.

Part 2 of Proposition 2 is new. If $i > i_1$, then there is no monetary equilibrium with complete information. In contrast, if buyers have some private information, then a monetary equilibrium exists, provided that i is not greater than $i_2 \equiv \frac{u'(\hat{q})}{c'(\hat{q})} - 1$, where $\hat{q} < q^*$ is the solution to (17). The threshold i_2 is bounded away from zero, for any level of the stock of assets, A. In particular, if $A \geq c(q^*)/\beta\kappa_\ell$, then $U(\kappa_\ell) = u(q^*) - c(q^*)$, so that both $\hat{q} < q^*$ and $i_2 > 0$ are independent of A. So the private information problem enlarges the set of parameter values under which fiat money is valued.

[26] A belief system that is consistent with (22) and (24)-(26) is $\lambda(q_h, d_h, \omega_h) = 1$, $\lambda(q_\ell, d_\ell, \omega_\ell) = 0$ and, for out-of-equilibrium offers,

$$\begin{aligned} \lambda(q,d,\omega) &= 0 \quad \forall (q,d,\omega) \text{ s.t. } -(1+i)\omega + u(q) - \beta\kappa_\ell d > U(\kappa_\ell) \\ \lambda(q,d,\omega) &= 1 \quad \text{otherwise.} \end{aligned}$$

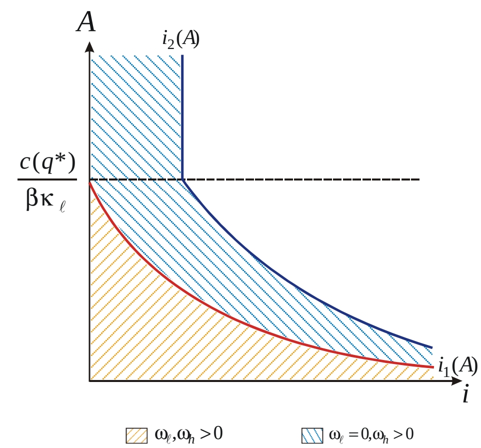

Figure 4: Existence of a monetary equilibrium

A distinctive feature of search-theoretic monetary models is their ability to endogenize payment arrangements in decentralized trades (e.g., Kiyotaki and Wright, 1989). The next proposition describes the payment arrangements in the PM and asset liquidity as a function of fundamentals. Liquidity is measured by the fraction of the stock of the asset that is used as means of payment in the PM. Such transaction velocities are denoted by $\mathcal{V}_h \equiv d_h/A$ and $\mathcal{V}_\ell \equiv d_\ell/A$.

Proposition 3 ***(Payments, liquidity, and fundamentals)***

1. *In all monetary equilibria,* $\omega_h > \omega_\ell \geq 0$, $d_h < d_\ell \leq A$, $q_h < q^*$ *and* $q_\ell \in [q_h, q^*]$.
2. $d\mathcal{V}_h/d\kappa_h < 0$ *and* $d\mathcal{V}_h/d\kappa_\ell > 0$.
3. $d\mathcal{V}_\ell/d\kappa_h = 0$; $d\mathcal{V}_\ell/d\kappa_\ell < 0$ *if* $c(q^*) < \beta\kappa_\ell A$ *and* $\mathcal{V}_\ell = 1$ *otherwise.*
4. *As* $\kappa_\ell \to 0$, $\mathcal{V}_h \to 0$, *and* $\omega_h, \omega_\ell \to \omega$ *where* ω *solves* $u'\left[c^{-1}(\omega)\right]/c'\left[c^{-1}(\omega)\right] = 1 + i$.

According to part 1 of Proposition 3, the high-dividend asset is partially illiquid in the sense that buyers only spend a fraction of their assets, $d_h < A$, even though their PM consumption is inefficiently low, $q_h < q^*$. As a consequence of this illiquidity, $h-$type buyers accumulate more real balances than $\ell-$type buyers. By holding onto a fraction of his real asset, the buyer is able to signal its quality to the seller; he uses the liquid asset to finance the rest of his consumption.[27] Notice that this payment pattern is significantly different from the one that would prevail in the complete-information economy: $h-$type buyers would accumulate fewer real balances and consume (weakly) more than $\ell-$type buyers.

According to part 2, the velocity of the high-dividend asset increases with the size of the low-state dividend, κ_ℓ, and it decreases with κ_h. To understand this result, notice from (26) that $\ell-$type buyers enjoy an informational rent equal to $\beta(\kappa_h - \kappa_\ell)d_h$. As κ_ℓ gets closer to κ_h, this informational rent shrinks, and the incentive-compatibility constraint of the $\ell-$type buyer is relaxed, which improves the liquidity of the asset in the high-dividend state.[28] Conversely, as $\kappa_h - \kappa_\ell$ increases, the informational asymmetries become more severe, which makes the incentive-compatibiliy condition more binding. According to part 3, the velocity of the low-dividend asset decreases with κ_ℓ but it is unaffected by κ_h.

In the case where the dividend in the low state approaches 0 (part 4 of Proposition 3), the adverse selection problem is so severe that the real asset ceases to be traded. Fiat money becomes the only means of payment.[29] This result rationalizes cash-in-advance-like constraints.

In the case where $i < i_1$, one can get the following closed-form expression for the velocity of the high-dividend asset,

$$\mathcal{V}_h = \frac{i\kappa_\ell}{(1+i)\kappa_h - \kappa_\ell}. \tag{31}$$

This expression makes a connection between this model and the approaches of Kiyotaki and Moore (2005) and Lagos (2006). In Kiyotaki and Moore (2005), agents can only sell a fraction $\theta \in (0,1)$ of their illiquid asset (capital) to raise funds; in Lagos (2006), agents can use their illiquid asset ("Lucas' trees") in a fraction of θ of the matches. In both cases, the parameter θ is exogenous.[30] In my model, assuming $i < i_1$, buyers spend all their capital in a fraction π_ℓ of the matches, and they spend a fraction $\frac{i\kappa_\ell}{(1+i)\kappa_h - \kappa_\ell}$ of their capital in the remaining π_h matches. The illiquidity of capital is endogenous: it depends on the intrinsic characteristics

[27] This result is reminiscent to some of the findings of the liquidity-based model of security design from DeMarzo and Duffie (1999). They consider the problem faced by a firm that needs to raise funds by issuing a security backed by real assets. The issuer has private information regarding the distribution of cash flows of the underlying assets. Using the Intuitive Criterion, they show that a signaling equilibrium exists in which the seller receives a high price for the security by retaining some fraction of the issue.

[28] This result is related to the findings in Banerjee and Maskin (1996), according to which the good that serves as the medium of exchange is the one for which the discrepancy between qualities is smallest.

[29] Strictly speaking, the $\ell-$type buyers use the real asset in payments ($d_\ell = A$) but because $\kappa_\ell \to 0$ the amount of output they buy with it approaches 0. This result is related to the threat of counterfeiting in Nosal and Wallace (2007).

[30] If I assume the same trading restriction as in Kiyotaki-Moore in a version of the model with homogenous assets ($\kappa_h = \kappa_\ell = \kappa$) then $q = q^*$ iff $\theta A \geq c(q^*)/\beta\kappa$ in which case fiat money is not valued and the velocity of capital is $\mathcal{V} = c(q^*)/\beta\kappa A$. If $\theta A < c(q^*)/\beta\kappa$ then $\mathcal{V} = \theta$. If I assume the same restriction as in Lagos then the following is true. If $A \geq c(q^*)/\beta\kappa$ then $q = q^*$ in a fraction θ of the trades and $\mathcal{V} = \theta c(q^*)/\beta\kappa A$. If $A < c(q^*)/\beta\kappa$ then $\mathcal{V} = \theta$.

of the asset (κ_ℓ and κ_h) as well as monetary policy (i).

In the case where $i \in (i_1, i_2)$ then, from (28), asset velocity satisfies

$$\mathcal{V}_h = \frac{u(q_\ell) - c(q_\ell) - \max_q [u(q) - (1+i)c(q)]}{[(1+i)\kappa_h - \kappa_\ell]\beta A}, \tag{32}$$

where $q_\ell = \min\left[c^{-1}(\beta\kappa_\ell A), q^*\right]$. The fraction of the real asset that is used as means of payment is still a function of the dividend process and inflation, but it is no longer independent of the stock of the asset: it decreases with A provided that A is sufficiently large.

The next Proposition investigates the effects of monetary policy on payment arrangements and liquidity.

Proposition 4 ***(Monetary policy and liquidity)***

1. *If $i < i_2$ then $d\omega_h/di < 0$, $d\mathcal{V}_h/di > 0$ and $d\mathcal{V}_\ell/di = 0$.*

2. *In addition, if $i < i_1$ then $d\omega_\ell/di < 0$.*

3. *As $i \to 0$, $\mathcal{V}_h \to 0$ and $q \to q^*$ in all trades.*

Inflation lowers the rate of return of fiat money, and hence it induces buyers to reduce their real balances. While the liquidity of the low-dividend asset is independent of monetary policy, inflation raises the velocity of the high-dividend asset. Since $\omega_h > \omega_\ell$ an increase in i makes it less attractive for an $\ell-$type buyer to imitate an $h-$type buyer. For instance, in the case where $i < i_1$, the incentive-compatibility condition (26) at equality yields $\beta d_h(\kappa_h - \kappa_\ell) = i(\omega_h - \omega_\ell)$. An increase in i relaxes the incentive-compatibility constraint allowing the $h-$type buyer to transfer a larger quantity of his real asset in the PM.

As the cost of holding money is driven to 0, the equilibrium allocation approaches the first best.[31] The optimal monetary policy is such that the high-dividend asset is illiquid, i.e., $h-$type buyers trade with money only. Moreover, if $\beta\kappa_\ell A > c(q^*)$ then $\ell-$type buyers do not accumulate real balances. So, buyers specialize in different means of payment according to their types.[32]

[31] Recall that even if the Friedman rule is optimal, it might not be incentive-feasible if the government has limited coercion power. See footnote 9.

[32] While fiat money and the real asset coexist as means of payment in equilibrium for all $i > 0$, there is an equilibrium at $i = 0$ where $d_\ell = 0$ and $\omega_\ell = c(q^*)$ so that only money is used. To see this, notice from (22) that at $i = 0$ the choices of ω and d are perfect substitutes for $\ell-$type buyers: they only care about the total expected resources they give up, $\beta\kappa_\ell d + \omega$. Hence, the equilibrium allocation is only upper-hemi continuous at $i = 0$.

6 Asset pricing and liquidity

This section investigates the implications of the model for asset prices. I analyze the relationship between assets intrinsic characteristics, liquidity, and returns. The model also provides a channel through which monetary policy affects asset prices.

The model is amended as follows. The real asset is described as a short-lived homogenous "Lucas tree" subject to an aggregate dividend shock. All capital goods yield a high dividend (i.e., $\kappa_{j,t} = \kappa_h$ for all j) with probability π_h, and a low dividend (i.e., $\kappa_{j,t} = \kappa_\ell$ for all j) with complement probability π_ℓ. Both fiat money and capital are traded in a competitive market in the AM.[33] This extension allows the real asset to be priced. In order to prevent the asset price from revealing buyers' private information, it is assumed that buyers learn the future dividend of the real asset when they enter the PM, after they chose their portfolios. Finally, to simplify the analysis of the bargaining game, a buyer's portfolio is common knowledge in a match in the PM.[34]

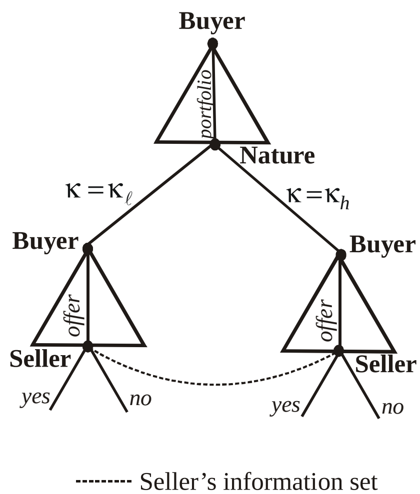

Figure 5: Game tree

The sequence of events is summarized by the game tree of Figure 5. First, newborn buyers make a portfolio choice in the AM market. Second, they receive a private and fully informative signal about the

[33]As indicated in Section 2, money and capital are the only assets that are traded in the competitive market in the AM. It is shown in the Appendix A6 that even if sellers could produce when young they would have no strict incentives to accumulate capital or hold real balances.

[34]If buyers' porfolios were private information then one would have to specify the seller's belief regarding the portfolio of the buyer in the match, which would open the possibility of multiple equilibria.

future dividend of the real asset. Then, they enter the PM and get matched with sellers. An implication of this timing is that the buyer's portfolio does not convey any information about κ. Upon entering the bargaining game, and irrespective of the buyer's portfolio he observes, the seller assigns probability π_h to the event $\kappa = \kappa_h$ and probability π_ℓ to the event $\kappa = \kappa_\ell$. Once the buyer has made his offer (q, d, τ), the seller updates his initial belief. Let $\lambda(q, d, \tau; \omega, a)$ denote the seller's belief that $\kappa = \kappa_h$ conditional on the offer (q, d, τ) being made. The seller's posterior belief, λ, is also a function of the buyer's portfolio (which is known to the seller). In the following, this dependence will be left implicit.

The strategy of a buyer is composed of a portfolio choice (ω, a) in the AM and an offer (q, d, τ) in the PM contingent on the history (ω, a, κ). The buyer's offer solves:

$$[q(\omega, a, \kappa), d(\omega, a, \kappa), \tau(\omega, a, \kappa)] = \arg\max_{q,d,\tau} \left[u(q) - \beta\kappa d - \frac{\beta}{\gamma}\tau \right] \tag{33}$$

$$\text{s.t.} \quad - c(q) + \lambda(q, d, \tau)\beta\kappa_h d + [1 - \lambda(q, d, \tau)]\, \beta\kappa_\ell d + \frac{\beta}{\gamma}\tau \geq 0 \tag{34}$$

$$\frac{\beta}{\gamma}\tau \leq \omega, \quad d \leq a. \tag{35}$$

Let define the buyer's surplus in the PM as $S^j(\omega, a) \equiv u(q) - \beta\kappa_j d - \beta\tau/\gamma$ for $j \in \{\ell, h\}$ where (q, d, τ) is a solution to (33)-(35) when the buyer's state is (ω, a, κ_j).

In the AM, buyers choose their portfolios in order to maximize their expected surplus in the PM net of the cost of holding real balances and capital, i.e.,

$$\max_{\omega,a} \left\{ -i\omega - (\phi - \beta\bar{\kappa})\, a + \pi_h S^h(\omega, a) + \pi_\ell S^\ell(\omega, a) \right\}, \tag{36}$$

where $i = (\gamma - \beta)/\beta$ is the cost of holding real balances, and $\phi - \beta\bar{\kappa}$ is the cost of investing in capital, the difference between its price and its expected discounted dividend. Since sellers cannot produce in the AM, only buyers hold some capital and market-clearing implies

$$\int_{j\in\mathcal{B}} a(j)dj = A. \tag{37}$$

Definition 3 *An equilibrium is a list of portfolios, buyers' strategies in the PM, the price of capital, and a belief system for sellers,* $\langle [\omega(j), a(j)]_{j\in\mathcal{B}}, [q(\cdot; j), d(\cdot; j), \tau(\cdot; j)]_{j\in\mathcal{B}}, \phi, \lambda \rangle$ *such that: (i)* $[\omega(j), a(j)]$ *is solution to (36) for all* $j \in \mathcal{B}$*; (ii)* $[q(\omega, a, \kappa; j), d(\omega, a, \kappa; j), \tau(\omega, a, \kappa; j)]$ *is solution to (33)-(35) for all* $j \in \mathcal{B}$ *and for all* (ω, a, κ)*; (iii)* λ *satisfies Bayes' rule whenever possible and the Intuitive Criterion. (iv)* ϕ *solves (37).*

The next lemmas characterize the equilibrium offers in the PM. The Intuitive Criterion is applied in every subgame following a portfolio choice (ω, a) by a buyer.

Lemma 5 *Consider a buyer in the PM with ω units of real balances and a units of capital. If $\kappa = \kappa_\ell$ then the equilibrium terms of trade $(q_\ell, d_\ell, \tau_\ell)$ are solution to*

$$(q_\ell, d_\ell, \tau_\ell) = \arg\max_{q,\tau,d} \left[u(q) - \beta\kappa_\ell d - \frac{\beta}{\gamma}\tau \right] \tag{38}$$

$$s.t. \quad -c(q) + \beta\kappa_\ell d + \frac{\beta}{\gamma}\tau \geq 0 \tag{39}$$

$$\frac{\beta}{\gamma}\tau \leq \omega, \quad d \leq a. \tag{40}$$

If $\kappa = \kappa_h$ then the equilibrium terms of trade (q_h, d_h, τ_h) solve

$$(q_h, d_h, \tau_h) = \arg\max_{q,\tau,d} \left[u(q) - \beta\kappa_h d - \frac{\beta}{\gamma}\tau \right] \tag{41}$$

$$s.t. \quad -c(q) + \beta\kappa_h d + \frac{\beta}{\gamma}\tau \geq 0 \tag{42}$$

$$u(q) - \beta\kappa_\ell d - \frac{\beta}{\gamma}\tau \leq S^\ell(\omega, a) \tag{43}$$

$$\frac{\beta}{\gamma}\tau \leq \omega, \quad d \leq a. \tag{44}$$

The equilibrium of the bargaining game is separating. In the low-dividend state, buyers make their complete information offer, i.e., $q_\ell = \min\left[q^*, c^{-1}(\beta\kappa_\ell a + \omega)\right]$ and $\beta\kappa_\ell d_\ell + \frac{\beta}{\gamma}\tau_\ell = c(q_\ell)$. The buyers' surplus in this case is $S^\ell(\omega, a) = \hat{\mathcal{S}}\left(\omega + \beta\kappa_\ell a\right)$, which only depends on their total wealth. In the high-dividend state, buyers choose the separating offer that maximizes their surplus. Pooling offers are ruled-out by a reasoning analogous to the one in the previous sections: if there were a pooling offer then buyers could deviate in the high-dividend state and signal the true state of the world by demanding less output and offering less capital.

Lemma 6 *For any $(\omega, a) \in \mathbb{R}_+^2$, there is a unique solution (q_h, d_h, τ_h) to (41)-(44) and it is such that (42) and (43) hold at equality.*

1. *If $\omega \geq c(q^*)$ then*

$$q_h = q^* \tag{45}$$

$$\tau_h = \frac{\gamma}{\beta}c(q^*) \tag{46}$$

$$d_h = 0. \tag{47}$$

2. If $\omega < c(q^*)$ then $\tau_h = \gamma\omega/\beta$ and $(q_h, d_h) \in [0, q_\ell] \times [0, a]$ is solution to:

$$\beta\kappa_h d_h = c(q_h) - \omega \quad (48)$$

$$u(q_h) - c(q_h) + \left(1 - \frac{\kappa_\ell}{\kappa_h}\right)[c(q_h) - \omega] = u(q_\ell) - c(q_\ell), \quad (49)$$

where $q_\ell = \min\left[q^*, c^{-1}(\omega + \beta\kappa_\ell a)\right]$. Moreover, if $a > 0$ then $q_h < q_\ell$ and $d_h < a$.

Lemma 6 offers a *pecking order* theory of payment choices: agents with a consumption opportunity finance it with cash first, and they use their risky assets as a last resort.[35] They choose not to spend all their capital goods, even when q_h is inefficiently low, in order to signal the high future dividend of the real asset.

From (48) and (49), the fraction $\theta_h \equiv d_h/a$ of his capital that a buyer spends in the PM is a function of his portfolio, (ω, a), as well as the characteristics of the dividend process, (κ_ℓ, κ_h). For instance, θ_h decreases with ω and κ_h, but it increases with κ_ℓ. The fact that θ_h is affected by real balances offers a channel through which monetary policy affects the liquidity of the real asset. At the margin, the fraction of capital that is used as means of payment is

$$\frac{dd_h}{da} = \frac{\kappa_\ell c'(q_h)\left[u'(q_\ell) - c'(q_\ell)\right]}{c'(q_\ell)\left[\kappa_h u'(q_h) - \kappa_\ell c'(q_h)\right]} \in (0, 1). \quad (50)$$

If $a > c(q^*)/\beta\kappa_\ell$ then $q_\ell = q^*$ and $dd_h/da = 0$. A marginal unit of capital has no direct liquidity value in the high-dividend state; it influences the terms of trade only indirectly, through the surplus of the buyer in the low-dividend state, by relaxing the incentive-compatibility constraint. But if $a > c(q^*)/\beta\kappa_\ell$ then the liquidity needs in the low-dividend state are satiated, and hence an additional unit of capital does not affect the terms of trade in the high-dividend state.

Let S^χ_ω and S^χ_a denote the partial derivatives of the surplus function $S^\chi(\omega, a)$ for $\chi \in \{\ell, h\}$. These quantities represent the liquidity values of fiat money and capital in the state χ. It is shown in the Appendix A (proof of Lemma 7) that

$$S^\ell_\omega = \frac{S^\ell_a}{\beta\kappa_\ell} = \frac{u'(q_\ell)}{c'(q_\ell)} - 1. \quad (51)$$

A marginal unit of asset (expressed in terms of its discounted value in the next AM market) allows the buyer to purchase $1/c'(q_\ell)$ units of PM output, which is valued according to the marginal surplus of the match,

[35] The term "pecking order" was coined by Myers (1984, p.581). It describes the predictions of models of capital structure choices under private information. According to the pecking order theory, firms with an investment opportunity prefer internal finance (nondistributed dividends). If external finance is required then they issue the safest security first, and they use equity as a last resort.

$u'(q_\ell) - c'(q_\ell)$. In the high-dividend state,

$$S^h_\omega = \Delta(q_h)\left[\frac{u'(q_\ell)}{c'(q_\ell)} - \frac{\kappa_\ell}{\kappa_h}\right] \quad (52)$$

$$S^h_a = \Delta(q_h)\beta\kappa_\ell\left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right], \quad (53)$$

where $\Delta(q) = [u'(q) - c'(q)] \,/\, \left[u'(q) - \frac{\kappa_\ell}{\kappa_h}c'(q)\right]$.

Consider a buyer who accumulates an additional unit of capital. How does this marginal unit impact on his surplus in the PM in the high-dividend state? Provided that $q_\ell < q^*$, an additional unit of capital raises the surplus of the buyer in the low-dividend state by S^ℓ_a, and hence it relaxes the incentive-compatibility constraint (43). Suppose the buyer increases his consumption by dq_h and delivers dd_h units of capital in exchange. In order to make the trade acceptable by sellers, $dd_h = c'(q_h)dq_h/\beta\kappa_h$. A buyer in the low-dividend state who imitates this offer can extract an informational rent equal to $\beta(\kappa_h - \kappa_\ell)dd_h$, i.e., his surplus increases by $[u'(q_h) - \frac{\kappa_\ell}{\kappa_h}c'(q_h)]dq_h$. Therefore, the buyer in the high-dividend state can only afford to consume dq_h solution to $[u'(q_h) - \frac{\kappa_\ell}{\kappa_h}c'(q_h)]dq_h = S^\ell_a$, and his utility increases by $S^h_a = [u'(q_h) - c'(q_h)]\,dq_h$, which gives (53).

It can be seen from (51) and (53) that $S^h_a < S^\ell_a$ (unless $q_\ell = q^*$) so that the capital stock has a lower liquidity value in the high-dividend state. Moreover, from (52)-(53),

$$\frac{\pi_\ell S^\ell_a + \pi_h S^h_a}{\beta\bar{\kappa}} = \frac{\kappa_\ell}{\bar{\kappa}}\left[\pi_\ell S^\ell_\omega + \pi_h S^h_\omega - \pi_h\Delta(q_h)\left(1 - \frac{\kappa_\ell}{\kappa_h}\right)\right] < \pi_\ell S^\ell_\omega + \pi_h S^h_\omega.$$

So, the expected liquidity value of capital, expressed as a fraction of its fundamental price, is less than the liquidity value of fiat currency. This observation will be useful in the following to explain the rate of return differential between assets.

Given the solution to the bargaining problem in the PM, I proceed backward and solve the buyer's portfolio problem in the AM.

Lemma 7 *If $\phi > \beta\bar{\kappa}$ then there is a unique solution to (36) and it satisfies*

$$-i + \pi_h S^h_\omega(\omega, a) + \pi_\ell S^\ell_\omega(\omega, a) \leq 0 \quad \text{“} = \text{”} \quad \textit{if } \omega > 0. \quad (54)$$

$$-\phi + \beta\bar{\kappa} + \pi_h S^h_a(\omega, a) + \pi_\ell S^\ell_a(\omega, a) \leq 0 \quad \text{“} = \text{”} \quad \textit{if } a > 0. \quad (55)$$

If $\phi = \beta\bar{\kappa}$ then ω is uniquely determined by (54) and $a \in [\frac{c(q^*)-\omega}{\beta\kappa_\ell}, \infty)$. If $\phi < \beta\bar{\kappa}$ then there is no solution to (36).

If the price of capital is greater than its fundamental value, i.e., $\phi - \beta\bar{\kappa} > 0$, then the composition of the buyer's optimal portfolio is unique. This result is a consequence of Lemma 6 according to which fiat money is a preferred means of payment, i.e., the two assets are not perfect substitutes. If the price of capital coincides with its fundamental value, $\phi = \beta\bar{\kappa}$, then buyers hold enough wealth to buy the first-best quantity of output when $\kappa = \kappa_\ell$ and the buyer's choice of capital is indeterminate. In contrast, the choice of real balances is always unique.

The next proposition proves existence of the equilibrium and it characterizes the allocations.

Proposition 5 ***(Equilibrium allocations and prices)***

An equilibrium exists and it is such that the price of capital, $\phi \in [\beta\bar{\kappa}, \beta\bar{\kappa} + i\beta\kappa_\ell]$, and the allocation $(\omega, q_\ell, q_h, d_h, \tau_h)$ are uniquely determined. For all $A > 0$, there is a $i_0(A) > 0$ such that the equilibrium is monetary if and only if $i < i_0(A)$.

From (51), (53) and (55), the equilibrium price of capital satisfies

$$\phi = \beta\bar{\kappa} + \beta\kappa_\ell \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right]\left[1 - \pi_h \left(1 - \frac{\kappa_\ell}{\kappa_h}\right) \frac{c'(q_h)}{u'(q_h) - \frac{\kappa_\ell}{\kappa_h} c'(q_h)}\right]. \tag{56}$$

The second term on the right-hand side of (56) is the liquidity component of the asset price. It is positive if and only if $q_\ell < q^*$ and $\kappa_\ell > 0$. If $q_\ell = q^*$ then buyers have enough wealth to buy q^* in the low-dividend state so that a marginal unit of capital is not useful as a means of payment. If $\kappa_\ell \to 0$ then capital has no value in the low-dividend state, and hence it does not provide liquidity in the PM. From (54), the liquidity value of fiat money is equal to i which, from (51)-(53), satisfies

$$i = \pi_h \left[\frac{u'(q_h) - c'(q_h)}{u'(q_h) - \frac{\kappa_\ell}{\kappa_h} c'(q_h)}\right]\left[\frac{u'(q_\ell)}{c'(q_\ell)} - \frac{\kappa_\ell}{\kappa_h}\right] + \pi_\ell \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right]. \tag{57}$$

According to the right-hand side of (57) fiat money provides some liquidity services whenever $q_\ell < q^*$ or $q_h < q^*$.

A monetary equilibrium exists for all A provided that the cost of holding real balances, i, is sufficiently low. This result contrasts with the complete-information economy where the equilibrium is monetary only if the capital stock is not large enough to allow buyers to trade q^* when $\kappa = \kappa_\ell$ (See Appendix E). Money is useful, even for large values of A, because it overcomes the illiquidity of capital in the high-dividend state, i.e., it relaxes the incentive-compatibility constraint faced by buyers. Consequently, the set of parameter values under which $\omega > 0$ is larger in the economy with private information (see Figure 6).

The next proposition describes the effects of monetary policy on the liquidity and expected return of the real asset. The liquidity of capital is measured by its transaction velocity, and its liquidity premium is defined by $\mathcal{L} = (\phi - \beta\bar{\kappa})/\phi$.[36] The expected return of capital is $R_a = \bar{\kappa}/\phi$.

Proposition 6 ***(Monetary policy, liquidity, and returns.)***

1. *If $i < i_0(A)$ then $d\mathcal{V}_h/di > 0$.*
2. *For all $i > 0$, there is $\bar{A}(i) \in [0, c(q^*)/\beta\kappa_\ell]$ such that:*
   (a) *For all $A \geq \bar{A}(i)$, $\mathcal{L} = 0$ and $R_a = \beta^{-1}$.*
   (b) *For all $A < \bar{A}(i)$, $\mathcal{L} > 0$ and $R_a < \beta^{-1}$. Moreover, if $i < i_0(A)$ then $d\mathcal{L}/di > 0$ and $dR_a/di < 0$.*
3. *As $i \to 0$, $\mathcal{V}_h \to 0$, $\mathcal{L} \to 0$ and $R_a \to \beta^{-1}$.*

The price of capital can depart from its fundamental value and exhibit a liquidity premium. This liquidity component emerges if capital is relatively scarce, i.e., $A < c(q^*)/\beta\kappa_\ell$, and inflation is sufficiently large, $i > \bar{A}^{-1}(A)$. On the contrary, if inflation is too low then the liquidity needs in the low-dividend state are exhausted.[37] An obvious requirement for monetary policy to be effective is that fiat money is valued, which necessitates that inflation is not too large, $i < i_0(A)$.

An increase in the inflation rate raises the price of capital, and its liquidity premium, through a substitution effect that induces buyers to hold fewer real balances but more capital. Since capital is in fixed supply, its price goes up and the fraction of capital that is used as means of payment in the high-dividend state ($\mathcal{V}_h$) increases. As a corollary of these findings, the model predicts a negative relationship between inflation and expected asset returns.[38]

If capital is sufficiently abundant to allow buyers to consume q^* in the low-dividend state then the price of capital is equal to its fundamental value–which is independent of monetary policy–and its expected rate of return is equal to the gross discount rate.

[36] Since the payment arrangement in the low-dividend state is indeterminate when $\phi = \beta\bar{\kappa}$, I focus on the velocity in the high-dividend state. The velocity of the asset in the low-yield state is equal to 1 if $A < \bar{A}(i)$. In the case where $A > \bar{A}(i)$, one could adopt the convention that buyers use their money first, i.e., $\mathcal{V}_\ell = [c(q^*) - \omega]/\beta\kappa_\ell A$ and $d\mathcal{V}_\ell/di > 0$ provided that $i < i_0(A)$.

[37] The expression for $\bar{A}(i)$ is provided in the proof of Proposition 6. If $i < i_0(A)$, i.e., fiat money is valued, it can be shown that $\bar{A}(i)$ is strictly increasing.

[38] The negative relationship between equity returns and inflation has been extensively documented. See Marshall (1992) for references. Theoretical models of this relationship are provided by Danthine and Donaldson (1986) and Marshall (1992). Both models assume the liquidity services of fiat money through a money-in-the-utility-function assumption or a shopping time technology.

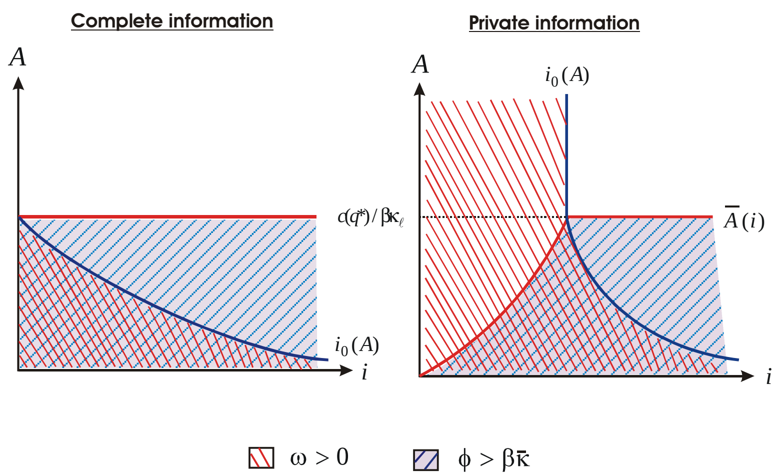

Figure 6: Types of equilibria: Liquidity premium and value of money

The optimal policy drives the cost of holding money to 0, and it exhausts the liquidity of the real asset. As i tends to zero then q_ℓ and q_h approach q^*.[39] In the high-dividend state, buyers trade with money only ($d_h \to 0$) while in the low-dividend state buyers are indifferent between using money or capital as means of payment. The price of capital converges to its fundamental value ($\phi \to \beta\bar{\kappa}$).

Next, I look at the implications of the model for the rates of return of fiat money ($R_m = \gamma^{-1}$) and capital ($R_a = \bar{\kappa}/\phi$).

Proposition 7 ***(Rate of return dominance)***

In any monetary equilibrium, $R_a > R_m$.

The expected rate of return of capital is always greater than the rate of return of fiat money (provided that it is valued). So, the model generates a rate-of-return differential between the two assets without resorting to restrictions on payment arrangements. This rate-of-return differential is not an obvious consequence of the difference of risks associated with each asset. Indeed, because of linear preferences with respect to AM consumption, the riskiness of capital would not affect its rate of return if it were not used as a means of payment in the PM. For instance, if A is sufficiently abundant then capital has no liquidity value at the

[39] Since the equilibrium correspondence is only upper-hemi continuous at $i = 0$, I focus on the equilibria that are obtained by taking the limit as i approaches 0. Moreover, it is worth recalling that the Friedman rule might not be feasible if one assumes limited coercion power by the government. See footnote 9. Also, the Friedman rule may not longer be optimal if agents have strictly concave preferences and face idiosyncratic trading shocks. See Zhu (2006) and Waller (2007).

margin and $R_a = \beta^{-1}$, independently of the dividend process. Risk matters here because, in the presence of private information, it affects the liquidity value of capital relative to the one of fiat money.

As showed in the Appendix E, such rate-of-return dominance pattern can also emerge from an economy with complete information. The private information problem, however, reduces the liquidity premium that accrues to the real asset, and it increases the rate of return differential between fiat money and risky capital. In particular, the liquidity premium of capital, $(\phi - \beta\bar{\kappa})/\phi$, is bounded above by $\kappa_\ell i/\bar{\kappa}$, which tends to zero as the dividend in the low state becomes small. Moreover, provided that the capital stock is sufficiently large ($A > \bar{A}(i)$), the rate of return of capital is maximum and equal to the gross discount rate, $R_a = \beta^{-1}$. In this case, an additional unit of capital has no liquidity value in the PM.[40]

The rate of return differential between risk-free fiat money and risky capital depends on the relative liquidity of both assets, which in turn depends on their intrinsic characteristics, such as their rate of return and risk. I end this section by illustrating this point through a simple numerical example. I adopt the following specifications: $u(q) = 2\sqrt{q}$, $c(q) = q$, $\beta = 0.95$, $\kappa_\ell = 1 - \sigma$, $\kappa_h = 1 + \sigma$, $\pi_h = \pi_\ell = 0.5$ and $A = 1$. The mean of the dividend is equal to 1 while its variance is σ^2. I consider the effects of a change in σ on the velocity of the asset and its liquidity premium.

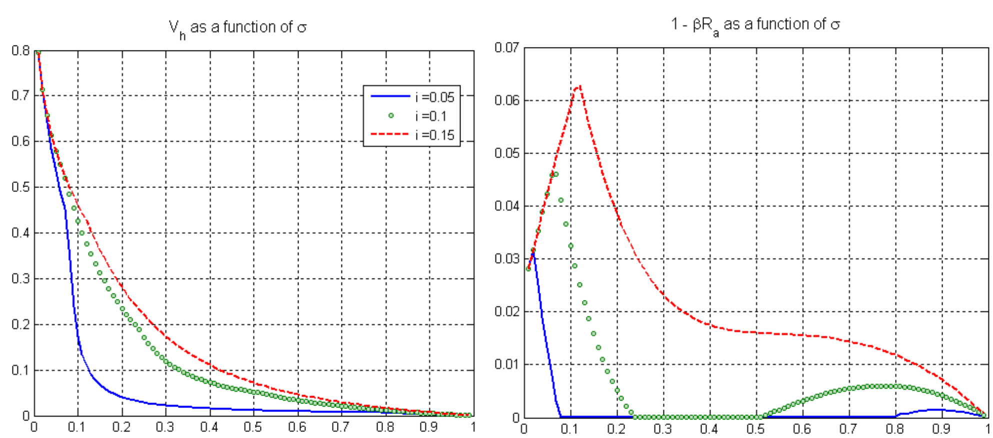

Figure 7: Asset liquidity

[40]These results can have interesting empirical implications for asset pricing puzzles (provided that one reinterprets currency as risk-free bonds). Indeed, Lagos (2006) showed that a standard search model of exchange can generate an equity premium as large as in the data (for plausible degrees of risk aversion) provided that equity is partially illiquid. While the illiquidity arises from legal or institutional restrictions in Lagos (2006), it is directly related to the dividend process here.

The left panel of Figure 7 represents the velocity of the asset in the high state, $\mathcal{V}_h = d_h/A$. (Recall that in the low state the payment can be indeterminate.) As σ increases, the fraction of the asset that is used as means of payment in the high-dividend state decreases. As σ approaches one, the real asset becomes fully illiquid.

The right panel of Figure 7 plots the liquidity premium defined as $\frac{\phi-\beta\bar{\kappa}}{\phi} = 1 - \beta R_a$. Recall that the rate of return of fiat money is constant and equal to γ^{-1}. Hence, as the liquidity premium decreases the rate-of-return differential increases. The relationship between the liquidity premium and risk is nonmonotonic. An increase in σ makes the asset more illiquid in the high-dividend state so that the liquidity premium should fall. But the decrease in κ_ℓ makes liquidity more valuable in the low-dividend state. As σ is sufficiently large, the liquidity premium decreases with risk, and it tends to 0 as σ approaches one. So, the rate-of-return differential is maximum provided that the real asset is sufficiently risky.

7 Conclusion

I have formalized economies where fiat money coexists and competes with a one-period lived real asset as means of payment. I complied with the Wallace (1996) dictum by placing no restrictions on the use of assets as media of exchange. The usefulness of fiat money in the model arises from a private information problem about the fundamental value of real assets. Some agents are informed about the future dividend of capital goods while others are uninformed. These informational asymmetries make real assets partially illiquid thereby providing microfoundations for some of the trading restrictions, or liquidity constraints, found in the recent monetary literature. I have investigated the relationship between asset liquidity and fundamentals, the implications for asset pricing, and the links between monetary policy, liquidity and asset returns.

In order to obtain a unique outcome for the bargaining game, the notion of sequential equilibrium has been refined using the Intuitive Criterion of Cho and Kreps (1987). While the Intuitive Criterion is widely accepted in the literature on signaling games (see footnote 16), alternative approaches can offer a complementary view of the game. In Appendix C it is shown that the equilibrium selected by the Intuitive Criterion is undefeated (in the sense of Mailath, Okuno-Fujiwara and Postlewaite, 1993) if the probability of the high-dividend state is sufficiently small or if inflation is not too large ($i < \kappa_h/\bar{\kappa} - 1$). This result provides some confidence that the results are not overly sensitive to the equilibrium refinement.

In terms of extensions, one could investigate the effect of liquidity on capital formation by letting the real asset be produced in the AM (as in Lagos and Rocheteau (2006)). The dividend shocks can be made persistent to study liquidity and asset prices over the cycle. For some questions (e.g., endogenous information acquisition) it might also be desirable to endow sellers with some market power (e.g., through competitive price posting). Finally, it would certainly be worthwhile to calibrate a version of the model in order to see how well it does to explain some asset pricing puzzles, as in Lagos (2006), or to quantify the effects of monetary policy on capital accumulation and output, as in Aruoba, Waller and Wright (2007).

References

[1] Aiyagari, S. Rao, Wallace, Neil, and Randall Wright (1996). "Coexistence of money and interest-bearing securities," Journal of Monetary Economics 37, 397–419.

[2] Andolfatto, David (2007). "Incentives and the limits to deflationary policy," Simon Fraser University Working Paper.

[3] Angeletos, George-Marios, Hellwig, Christian, and Alessandro Pavan (2006). "Signaling in a global game: Coordination and policy traps," Journal of Political Economy 114, 452-484.

[4] Aruoba, S. Borağan and Randall Wright (2003). "Search, money and capital: A neoclassical dichotomy," Journal of Money, Credit and Banking 35, 1086-1105.

[5] Aruoba, S. Borağan, Waller, Christopher, and Randall Wright (2007). "Money and capital: A quantitative analysis," University of Pennsylvania Working Paper.

[6] Banarjee, Abhijit and Eric Maskin (1996). "A Walrasian theory of money and barter," Quarterly Journal of Economics 111, 955-1005.

[7] Bansal, Ravi and John Coleman (1996). "A monetary explanation of the equity premium, term premium, and risk-free rate puzzles," Journal of Political Economy 104, 1135-1171.

[8] Berentsen, Aleksander and Guillaume Rocheteau (2004). "Money and information," Review of Economic Studies 71, 915-944.

[9] Berentsen, Aleksander, Menzio, Guido, and Randall Wright (2007). "Inflation and unemployment in the long-run," University of Pennsylvania Working Paper.

[10] Cho, In-Koo and David Kreps (1987). "Signaling games and stable equilibria," Quarterly Journal of Economics 102, 179–221.

[11] Cone, Thomas (2005). "Anticipated inflation and rates of return in a search-theoretic monetary model," Journal of Monetary Economics 52, 567-574.

[12] Danthine, Jean-Pierre and John Donaldson (1986). "Inflation and asset prices in an exchange economy," Econometrica 54, 585-605.

[13] DeMarzo, Peter and Darrell Duffie (1999). "A liquidity-based model of security design," Econometrica 67, 65-99.

[14] Duffie, Darrell, Nicolae Gârleanu, and Lasse Heje Pedersen (2005). "Over-the-counter markets," Econometrica 73, 1815–47.

[15] Dutu, Richard, Nosal, Ed, and Guillaume Rocheteau (2006). "Signaling in a model of currency circulation under private information," Working Paper of the Federal Reserve Bank of Cleveland.

[16] Freeman, Scott (1985). "Transactions costs and the optimal quantity of money," Journal of Political Economy 93, 146-157.

[17] Geromichalos, Athanasios, Licari, Juan Manuel, and Jose Suarez-Lledo (2007). "Monetary policy and asset prices," Review of Economic Dynamics 10, 761-779.

[18] Glosten, Lawrence and Paul Milgrom (1985). "Bid, ask and transaction prices in a specialist market with heterogeneously informed traders," Journal of Financial Economics 14, 71-100.

[19] Hicks, John (1935). "A suggestion for simplifying the theory of money," Economica 2, 1-19

[20] Inderst, Roman (2002). "Contractual signaling in a market environment," Games and Economic Behavior 40, 77-98.

[21] Jafarey, Saqib and Peter Rupert (2001). "Limited commitment, money, and credit," Journal of Economic Theory 99, 22-58.

[22] Kiyotaki, Nobuhiro and John Moore (2005). "Liquidity and asset prices," International Economic Review 46, 317-349.

[23] Kocherlakota, Narayana (1998): "Money is Memory," Journal of Economic Theory, 81, 232-251.

[24] Kohlberg, Elon and Jean-Francois Mertens (1986). "On the strategic stability of equilibria," Econometrica 54, 1003-1037.

[25] Kreps, David and Joel Sobel (1994). "Signaling," In Handbook of Game Theory with Economic Applications, chapter 25, pp. 849-867, Edited by Robert Aumann and Sergiu Hart.

[26] Kyle, Albert (1985). "Continuous auctions and insider trading," Econometrica 53, 1315-1335.

[27] Lagos, Ricardo (2006). "Asset prices and liquidity in an exchange economy," New-York University Working Paper.

[28] Lagos, Ricardo and Guillaume Rocheteau (2006). "Money and capital as competing media of exchange," Journal of Economic Theory (Forthcoming).

[29] Lagos, Ricardo and Randall Wright (2005). "A unified framework for monetary theory and policy analysis," Journal of Political Economy 113, 463–484.

[30] Lester, Benjamin, Postlewaite, Andrew, and Randall Wright (2007). "Information and Liquidity," University of Pennsylvania Working Paper.

[31] Li, Yiting (1995). "Commodity money under private information," Journal of Monetary Economics 36, 573-592.

[32] Mailath, George, Okuno-Fujiwara, Masahiro, and Andrew Postlewaite (1993). "Belief-based refinements in signaling games," Journal of Economic Theory 60, 241-276.

[33] Marshall, David (1992). "Inflation and asset returns in a monetary economy," Journal of Finance 47, 1315-1342.

[34] Myers, Stewart (1984). "The capital structure puzzle," Journal of Finance 39, 575-592.

[35] Myers, Stewart and Nicholas Majluf (1984). "Corporate financing and investment decisions when firms have information that investors do not have," Journal of Financial Economics 13, 187-221.

[36] Noe, Thomas (1989). "Capital structure and signaling game equilibria," Review of Financial Studies 1, 331-355.

[37] Nosal, Ed and Neil Wallace (2007). "A model of (the threat of) counterfeiting," Journal of Monetary Economics 54, 229-246.

[38] Riley, John (2001). "Silver signals: Twenty-five years of screening and signaling," Journal of Economic Literature 39, 432-478.

[39] Rocheteau, Guillaume and Randall Wright (2005). "Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium," Econometrica 73, 175-202.

[40] Rubinstein, Ariel (1985). "A bargaining model with incomplete information about time preference," Econometrica 53, 1151-1172.

[41] Shi, Shouyong (2004). "Liquidity, interest rates and output," University of Toronto Working Paper.

[42] Telyukova, Irina and Randall Wright (2007). "A model of money and credit with an application to the credit card debt puzzle," Review of Economic Studies (Forthcoming).

[43] Velde, François, Weber, Warren, and Randall Wright (1999). "A model of commodity money, with applications to Gresham's law and the debasement puzzle," Review of Economic Dynamics 2, 291–323.

[44] Wallace, Neil (1996). "A dictum for monetary theory," Federal Reserve Bank of Minneapolis Quarterly Review 22, 20-26.

[45] Wallace, Neil (2000). "A model of the liquidity structure based on asset indivisibility," Journal of Monetary Economics 45, 55-68.

[46] Waller, Christopher (2007). "Dynamic taxation, private information and money," Working Paper of the University of Notre Dame.

[47] Williamson, Stephen and Randall Wright (1994). "Barter and monetary exchange under private information," American Economic Review 84, 104–123.

[48] Zhu, Tao (2006). "An overlapping-generations model with search," Journal of Economic Theory (Forthcoming).

[49] Zhu, Tao and Neil Wallace (2007). "Pairwise trade and coexistence of money and higher-return assets," Journal of Economic Theory 133, 524-535.

A. Proofs of lemmas and propositions

Proof of Lemma 1. Suppose there is an equilibrium offer such that $\lambda(\bar{q},\bar{d}) \in (0,1)$. Hence, $U_h^b = u(\bar{q}) - \beta\kappa_h\bar{d}$ and $U_\ell^b = u(\bar{q}) - \beta\kappa_\ell\bar{d}$. The payoff of an $\ell-$type buyer is bounded below by his complete information payoff, i.e.,

$$U_\ell^b \geq \max_{q,d\leq A} [u(q) - \beta\kappa_\ell d] \quad \text{ s.t. } \quad -c(q) + \beta\kappa_\ell d \geq 0.$$

Since $\kappa_\ell > 0$, $U_\ell^b > 0$ and hence $\bar{q} > 0$. Furthermore, $(\bar{q},\bar{d})$ is accepted by sellers if

$$-c(\bar{q}) + \beta\left\{\lambda(\bar{q},\bar{d})\kappa_h + \left[1-\lambda(\bar{q},\bar{d})\right]\kappa_\ell\right\}\bar{d} \geq 0.$$

To show that the proposed equilibrium violates the Intuitive Criterion, consider an out-of-equilibrium offer $(\tilde{q},\tilde{d})$ such that $\tilde{d} = \bar{d} - \varepsilon$, where $\varepsilon \in \left(0, \bar{d} - c(\bar{q})/\beta\kappa_h\right)$, and $\tilde{q} < \bar{q}$ satisfies (9)-(10) or, equivalently,

$$u(\bar{q}) - \beta\kappa_h\varepsilon < u(\tilde{q}) < u(\bar{q}) - \beta\kappa_\ell\varepsilon. \tag{58}$$

Since $\lambda(\bar{q},\bar{d}) < 1$, $c(\bar{q}) < \beta\kappa_h\bar{d}$ and $\left(0, \bar{d} - c(\bar{q})/\beta\kappa_h\right)$ is not empty. Moreover, $U_h^b \geq 0$ implies $u(\bar{q}) - \beta\kappa_h\varepsilon \geq \beta\kappa_h\left(\bar{d}-\varepsilon\right)$. For any $\varepsilon \in \left(0, \bar{d} - c(\bar{q})/\beta\kappa_h\right)$, $\bar{d} - \varepsilon > 0$ and there is a $\tilde{q} \geq 0$ that satisfies (58).

From (58), $(\tilde{q},\tilde{d})$ satisfies (9)-(10). Moreover, $\varepsilon < \bar{d} - c(\bar{q})/\beta\kappa_h$ implies $c(\bar{q}) < \beta\kappa_h\left(\bar{d}-\varepsilon\right)$. From (58), $u(\bar{q}) - u(\tilde{q}) > 0$ and therefore $c(\bar{q}) - c(\tilde{q}) > 0$. So (11) is satisfied as well. ■

Proof of Proposition 1. The allocation in $\ell-$type matches, (15) and (16), is derived directly from (12). The rest of the proof focuses on the allocation in $h-$type matches. It proceeds in three parts. First, I establish that both the seller's participation constraint and the incentive-compatibility condition (14) are binding. Second, it is shown that the solution to (13)-(14) is unique and it is such that $d_h < d_\ell$ and $q_h < q_\ell$. Third, I specify a belief system λ consistent with these offers.

(i) The set of admissible values for (q,d) being compact (closed and bounded) and the buyer's objective function being continuous, a solution to (13)-(14) exists. It is straightforward to check that this solution cannot be such that neither the seller's participation constraint nor (14) bind. Suppose first that the seller's participation constraint binds while (14) is slack. Then, $q_h = \min\left[c^{-1}(\beta\kappa_h A), q^*\right] \geq q_\ell = \min\left[c^{-1}(\beta\kappa_\ell A), q^*\right]$ (from (15)) and $d_h = c(q_h)/\beta\kappa_h$. Thus,

$$u(q_h) - \beta\kappa_\ell d_h = u(q_h) - \frac{\kappa_\ell}{\kappa_h}c(q_h) > u(q_h) - c(q_h).$$

Since $q_h \geq q_\ell$ then $u(q_h) - \beta\kappa_\ell d_h > u(q_\ell) - c(q_\ell) = U(\kappa_\ell)$ and (14) is violated. A contradiction.

Suppose next that (14) binds while the seller's participation constraint is slack. Substitute $u(q_h)$ by its expression given by (14) into the $h-$buyer's objective function to get

$$U(\kappa_h) = \max_{d \leq A} (\kappa_\ell - \kappa_h)\beta d + U(\kappa_\ell)$$

which yields $d_h = 0$ and $U(\kappa_h) = u(q_h) = U(\kappa_\ell)$. So the seller's participation constraint, $-c(q_h) + \beta\kappa_h d_h = -c(q_h) < 0$, is violated. A contradiction.

Consequently, the solution to (13)-(14) is such that both the seller's participation constraint and (14) bind. Substitute $d_h = c(q_h)/\beta\kappa_h$ into (14) to get (17)-(18).

(ii) Equation (17) can be rewritten as:

$$c(q_h) = \frac{\kappa_h}{\kappa_h - \kappa_\ell}\left\{U(\kappa_\ell) - [u(q_h) - c(q_h)]\right\}. \tag{59}$$

The term between brackets on the right-hand side of (59) is strictly decreasing from $U(\kappa_\ell)$ to 0 as q_h varies from 0 to $q_\ell \leq q^*$, while the left-hand side of (59) is strictly increasing from 0 to $c(q_\ell)$ as q_h varies from 0 to q_ℓ. Hence, there is a unique $q_h \in (0, q_\ell)$ that solves (59) and it is the unique solution to (13)-(14). To see this, from (17),

$$u(q_h) - c(q_h) = U(\kappa_\ell) - c(q_h)\left(1 - \frac{\kappa_\ell}{\kappa_h}\right).$$

Hence, the solution to (17) corresponding to the lowest value for q_h is the one that maximizes the $h-$type buyer's payoff, $u(q_h) - c(q_h)$.

Finally, from the fact that $h-$buyers prefer weakly (q_h, d_h) to (q_ℓ, d_ℓ) and $\ell-$buyers prefer weakly (q_ℓ, d_ℓ) to (q_h, d_h),

$$\beta\kappa_\ell(d_\ell - d_h) \leq u(q_\ell) - u(q_h) \leq \beta\kappa_h(d_\ell - d_h). \tag{60}$$

Since $q_\ell > q_h$ then $d_h < d_\ell$.

(iii) A belief system consistent with the offers (q_ℓ, d_ℓ) and (q_h, d_h) is as follows. Bayes' rule requires $\lambda(q_\ell, d_\ell) = 0$ and $\lambda(q_h, d_h) = 1$. For all other (out-of-equilibrium) offers,

$$\begin{aligned} \lambda(q,d) &= 0 \quad \text{if} \quad u(q) - \beta\kappa_\ell d > U(\kappa_\ell) \\ \lambda(q,d) &= 1 \quad \text{otherwise.} \end{aligned}$$

One can verify that (q_ℓ, d_ℓ) and (q_h, d_h) are solutions to (8) given $\lambda(q,d)$. Any offer such that $u(q) - \beta\kappa_\ell d > U(\kappa_\ell)$ is assigned to an $\ell-$type buyer and it is such that $-c(q) + \beta\kappa_\ell d < 0$ (by definition of $U(\kappa_\ell)$). Hence it is rejected by sellers. Consequently, $U(\kappa_\ell)$ is the highest payoff attainable by an $\ell-$type buyer. Similarly, the solution to (13)-(14) is also the solution to (8) since any offer that violates (14) is rejected by sellers and any offer that satisfies (14), except (q_ℓ, d_ℓ), is attributed to an $h-$type buyer. ■

Proof of Lemma 2. A buyer's strategy is defined as his choice of real balances, z, and the subsequent offer in the PM, (q, d, τ). Consider two strategies $\text{s}= (z, q, d, \tau)$ such that $\tau < z$ and $\text{s}' = (\tau, q, d, \tau)$. The two strategies prescribe the same offer in the PM but differ in terms of the choice of real balances. The seller's strategy is an acceptance set $\mathcal{A}$ such that (q, d, τ) is accepted if it is an element of $\mathcal{A}$. The strategy s' strictly dominates s if it generates a strictly higher payoff irrespective of the seller's acceptance set, $\mathcal{A}$. Let $\mathbb{I}_{\mathcal{A}}$ denote the indicator function that is equal to one if its argument is in $\mathcal{A}$. The buyer's expected utility if he chooses s' is

$$-\frac{\tau}{\gamma}(\gamma-\beta)+\left\{u\left(q\right)-\beta\kappa d-\frac{\beta}{\gamma}\tau\right\}\mathbb{I}_{\mathcal{A}}(q,d,\tau)>-\frac{z}{\gamma}(\gamma-\beta)+\left\{u\left(q\right)-\beta\kappa d-\frac{\beta}{\gamma}\tau\right\}\mathbb{I}_{\mathcal{A}}(q,d,\tau),$$

for any $\mathcal{A}$ (since $\tau < z$). Hence, s is strictly dominated. ■

Proof of Lemma 3. A proposed equilibrium fails the Intuitive Criterion if there is an unsent offer $(\tilde{q}, \tilde{d}, \tilde{\omega})$ that satisfies:

$$-(1+i)\tilde{\omega}+u\left(\tilde{q}\right)-\beta\kappa_h\tilde{d} \;>\; U_h^b \tag{61}$$

$$-(1+i)\tilde{\omega}+u\left(\tilde{q}\right)-\beta\kappa_\ell\tilde{d} \;<\; U_\ell^b \tag{62}$$

$$\tilde{\omega}-c(\tilde{q})+\beta\kappa_h\tilde{d} \;\geq\; 0 \tag{63}$$

where U_h^b and U_ℓ^b are the buyers' equilibrium payoffs defined as their expected surplus in the PM net of the cost of holding real balances (but excluding the lump-sum transfer T). Suppose there is an equilibrium offer such that $\lambda(\bar{q}, \bar{d}, \bar{\omega}) \in (0, 1)$ with $\bar{d} > 0$. Hence, buyers' equilibrium payoffs are

$$U_h^b \;\equiv\; -(1+i)\bar{\omega}+u\left(\bar{q}\right)-\beta\kappa_h\bar{d}$$

$$U_\ell^b \;\equiv\; -(1+i)\bar{\omega}+u\left(\bar{q}\right)-\beta\kappa_\ell\bar{d}.$$

This offer satisfies the seller's participation constraint, i.e.,

$$-c(\bar{q})+\left\{\lambda(\bar{q},\bar{d},\bar{\omega})\kappa_h+\left[1-\lambda(\bar{q},\bar{d},\bar{\omega})\right]\kappa_\ell\right\}\beta\bar{d}+\bar{\omega}\geq 0. \tag{64}$$

In order to prove that the proposed equilibrium violates the Intuitive Criterion, consider an out-of-equilibrium offer $(\tilde{q}, \tilde{d}, \tilde{\omega})$ such that $\tilde{\omega} = \bar{\omega}$, $\tilde{d} = \bar{d} - \varepsilon$ where $\varepsilon \in \left(0, \bar{d} + [\bar{\omega} - c(\bar{q})] / \beta\kappa_h\right) \cap (0, \bar{d}]$, and it satisfies (61)-(62) or, equivalently,

$$u(\bar{q}) - \beta\varepsilon\kappa_h < u(\tilde{q}) < u(\bar{q}) - \beta\varepsilon\kappa_\ell. \tag{65}$$

Since $\lambda(\bar{q}, \bar{d}, \bar{\omega}) < 1$, (64) implies $c(\bar{q}) < \beta\kappa_h\bar{d} + \bar{\omega}$ and $\left(0, \bar{d} + [\bar{\omega} - c(\bar{q})] / \beta\kappa_h\right)$ is non-empty. The requirement $U_h^b \geq 0$ implies $u(\bar{q}) - \beta\kappa_h\bar{d} \geq 0$ and hence $u(\bar{q}) - \beta\varepsilon\kappa_h \geq \beta\kappa_h\left(\bar{d} - \varepsilon\right) \geq 0$. So for any $\varepsilon \in \left(0, \bar{d} + [\bar{\omega} - c(\bar{q})] / \beta\kappa_h\right) \cap (0, \bar{d}]$ there is a $\tilde{q} \geq 0$ satisfying (65). From (65), $(\tilde{q}, \tilde{d}, \tilde{\omega})$ satisfies (61)-(62). Moreover, $c(\bar{q}) < \beta\kappa_h\left(\bar{d} - \varepsilon\right) + \bar{\omega}$. From (65), $u(\bar{q}) - u(\tilde{q}) > 0$ and hence $c(\bar{q}) > c(\tilde{q})$. So, (63) is also satisfied.

Finally, to show that there is no pooling offer with $d = 0$ it is enough to notice that

$$\max_{q,\omega\geq 0} \{-(1+i)\omega + u(q)\} \quad \text{s.t.} \quad -c(q) + \omega \geq 0$$

is less than $U(\kappa_\ell)$, the complete-information payoff of the $\ell-$type buyer defined in (22). ■

Proof of Lemma 4. The proof proceeds in two parts. First, it establishes that the constraints (25) and (26) are binding. Second, it proves that the solution to (24)-(26) exists and is unique.

(i) The constraints (25) and (26) are binding.

It is straightforward to show that the solution to (24)-(26) cannot be such that neither (25) nor (26) bind. Assume that the seller's participation constraint binds while (26) is slack. Then,

$$(q_h, d_h, \omega_h) = \max_{q,d\leq A,\omega\geq 0} \{-(1+i)\omega + u(q) - \beta\kappa_h d\} \quad \text{s.t.} \quad -c(q) + \beta\kappa_h d + \omega = 0.$$

If $c(q^*) \leq \beta\kappa_h A$ then $q_h = q^*$ and $d_h = c(q^*)/\beta\kappa_h$. Otherwise, $d_h = A$, $c(q_h) = \omega_h + \beta\kappa_h A$ and $\frac{u'(q_h)}{c'(q_h)} \leq 1+i$ with an equality if $\omega_h > 0$. From the comparison with (22), it can be checked that $q_h \geq q_\ell$ and $\omega_h \leq \omega_\ell$ with at least one strict inequality. Hence,

$$-i\omega_h + u(q_h) - c(q_h) > U(\kappa_\ell) = -i\omega_\ell + u(q_\ell) - c(q_\ell).$$

Since $c(q_h) = \beta\kappa_h d_h + \omega_h$ the previous inequality gives

$$-(1+i)\omega_h + u(q_h) - \beta\kappa_\ell d_h > -i\omega_h + u(q_h) - c(q_h) > U(\kappa_\ell).$$

So (26) is violated. A contradiction.

Assume next that (25) is slack while (26) binds. Substitute ω_h by its expression given by (26) into the $h-$buyer's objective function to get

$$U_h^b = \max_{d_h} (\kappa_\ell - \kappa_h) \beta d_h + U(\kappa_\ell) = U(\kappa_\ell),$$

which yields $d_h = 0$ and $u(q_h) - (1+i)\omega_h = U(\kappa_\ell)$. The seller's participation constraint, $\omega_h - c(q_h) \geq 0$, can then be rewritten as

$$-(1+i)c(q_h) + u(q_h) \geq U(\kappa_\ell).$$

Since $d_\ell > 0$, it can be checked from (22) that $\max_q [-(1+i)c(q) + u(q)] < U(\kappa_\ell)$. Hence, there is no q_h that satisfies the constraint above.

Consequently, the solution to (24)-(26) is such that both (25) and (26) bind.

(ii) The solution to (24)-(26) exists and is unique.

Assume $\omega_h > 0$. One can solve for ω_h and d_h from (25) and (26) and get (27)-(28). Substitute ω_h and d_h by their expressions given by (27) and (28) into (24) and differentiate with respect to q_h to show that $q_h = \tilde{q}$ where $\tilde{q}$ is the unique solution to

$$u'(\tilde{q}) - (1+i)c'(\tilde{q}) = 0.$$

Given q_h, (d_h, ω_h) is uniquely determined by (27)-(28). From (27) the condition $\omega_h > 0$ can be reexpressed as

$$u(\tilde{q}) - \frac{\kappa_\ell}{\kappa_h} c(\tilde{q}) > U(\kappa_\ell). \tag{66}$$

If (66) does not hold then $\omega_h = 0$. From Proposition 1, $q_h = \hat{q} \in (0, q_\ell)$ is the unique solution to (17) and $d_h = c(\hat{q})/\beta\kappa_h$. Hence, (q_h, d_h, ω_h) solves (27)-(28) with $\omega_h = 0$. The condition (66) is violated if $\tilde{q} \leq \hat{q}$ or, equivalently, $u'(\hat{q}) - (1+i)c'(\hat{q}) \leq 0$. ■

Proof of Proposition 2.

From Lemma 4 and Eq.(22) the terms of trade $(q_\ell, d_\ell, \omega_\ell)$ and (q_h, d_h, ω_h) are uniquely determined. Therefore, up to the seller's belief system, the equilibrium is unique. The rest of the proof proceeds in two steps.

(a) Condition under which $\omega_\ell > 0$.

Denote $q^c \equiv \min\left[q^*, c^{-1}(\beta\kappa_\ell A)\right]$. From (22)-(23), $\omega_\ell > 0$ iff $i < i_1 \equiv \frac{u'(q^c)}{c'(q^c)} - 1$. Moreover, $i_1 > 0$ iff $c^{-1}(\beta\kappa_\ell A) < q^*$.

(b) Condition under which $\omega_h > 0$.

First, $\omega_h > 0$ when $i < i_1$. Indeed, if $\omega_h = 0$ then $q_h < q_\ell$ (from (17) and Proposition 1). Since $\frac{u'(q_\ell)}{c'(q_\ell)} - 1 = i$ then $\frac{u'(q_h)}{c'(q_h)} - 1 > i$. A contradiction with (29). Second, if $i \geq i_1$ then $\omega_\ell = 0$ (from (a)). From the proof of Lemma 4, $\omega_h > 0$ iff $i < i_2 \equiv \frac{u'(\hat{q})}{c'(\hat{q})} - 1$ where $\hat{q} \in (0, q^c)$ is the unique solution to (17). Since $\hat{q} < q^c \leq q^*$ then $i_2 > i_1 \geq 0$. ■

Proof of Proposition 3. (i) Suppose there exists a monetary equilibrium such that $\omega_h = 0$. From Proposition 1, $q_h < q_\ell$. The condition (29), $u'(q_h)/c'(q_h) \leq 1 + i$, and the fact that $q_h < q_\ell$ imply $u'(q_\ell)/c'(q_\ell) < 1 + i$. Hence, from (23), $\omega_\ell = 0$ and the equilibrium is nonmonetary. A contradiction. So in any monetary equilibrium $\omega_h > 0$ and, from (29), $u'(q_h)/c'(q_h) = 1+i$. From (23), $u'(q_\ell)/c'(q_\ell) \leq 1+i$ and hence $q_h \leq q_\ell$ with an equality if $\omega_\ell > 0$.

Next, I prove $\omega_h > \omega_\ell$. This is immediate if $\omega_\ell = 0$ (since $\omega_h > 0$ in any monetary equilibrium). Consider the case $\omega_\ell > 0$. The incentive-compatibility condition for the $\ell-$type buyer is

$$-(1+i)\omega_\ell + u(q_\ell) - \beta\kappa_\ell d_\ell \geq -(1+i)\omega_h + u(q_h) - \beta\kappa_\ell d_h. \tag{67}$$

Since $c(q_\ell) = \omega_\ell + \beta\kappa_\ell d_\ell$ and $c(q_h) = \omega_h + \beta\kappa_h d_h$, (67) becomes

$$-i\omega_\ell + u(q_\ell) - c(q_\ell) \geq -i\omega_h + u(q_h) - c(q_h) + \beta\left(\kappa_h - \kappa_\ell\right) d_h.$$

Since $q_h = q_\ell$ when $\omega_\ell > 0$, (67) becomes

$$i(\omega_h - \omega_\ell) \geq \beta d_h(\kappa_h - \kappa_\ell).$$

The assumption $\kappa_h > \kappa_\ell$ implies $\omega_h > \omega_\ell$ since $d_h > 0$. (To see that $d_h > 0$ use (28) and the fact that $U(\kappa_\ell) > \max_q\left[-(1+i)c(q) + u\left(q\right)\right]$).

Finally, since $c(q_h) = \omega_h + \beta\kappa_h d_h \leq c(q_\ell) = \omega_\ell + \beta\kappa_\ell d_\ell$ and $\omega_h > \omega_\ell$, $\kappa_h d_h < \kappa_\ell d_\ell$. Hence, $d_h < d_\ell$.

(ii) If $i \geq i_2$ then $\omega_h = 0$. From the proof of Proposition 1, q_h is the unique solution less than q_ℓ to (59). Differentiate (59) to get:

$$\frac{dq_h}{d\kappa_h} = \frac{-\kappa_\ell}{\kappa_h} \frac{c(q_h)}{\kappa_h u'(q_h) - \kappa_\ell c'(q_h)} < 0,$$

since $q_h < q^*$ (i.e., $u'(q_h) > c'(q_h)$) and $\kappa_h > \kappa_\ell$. From (18),

$$\frac{dd_h}{d\kappa_h} = \frac{-u'(q_h)c(q_h)}{\beta\kappa_h\left[\kappa_h u'(q_h) - \kappa_\ell c'(q_h)\right]} < 0,$$

and hence $d\mathcal{V}_h/d\kappa_h < 0$. From (12), $U'(\kappa_\ell) = \left[u'(q_\ell)/c'(q_\ell) - 1\right]\beta A \geq 0$ where q_ℓ satisfies (15). Hence, from (59),

$$\frac{dq_h}{d\kappa_\ell} = \frac{c(q_h) + \kappa_h U'(\kappa_\ell)}{\kappa_h u'(q_h) - \kappa_\ell c'(q_h)} > 0.$$

From (18), $d\mathcal{V}_h/d\kappa_\ell > 0$.

If $i < i_1$ then

$$\mathcal{V}_h \equiv d_h/A = \frac{i\kappa_\ell}{(1+i)\kappa_h - \kappa_\ell},$$

where I have used (28) and the fact that $q_h = q_\ell$ and $U(\kappa_\ell) = u(q_\ell) - (1+i)c(q_\ell) + i\beta\kappa_\ell d_\ell$. It is then straightforward to show that $d\mathcal{V}_h/d\kappa_h < 0$ and $d\mathcal{V}_h/d\kappa_\ell > 0$. If $i \in (i_1, i_2)$ then $\omega_\ell = 0$ and $U'(\kappa_\ell) = \left[u'(q_\ell)/c'(q_\ell) - 1\right]\beta A \geq 0$. From (28),

$$\begin{aligned}\frac{dd_h}{d\kappa_\ell} &= \frac{\left[u'(q_\ell)/c'(q_\ell) - 1\right]A + d_h}{\left[(1+i)\kappa_h - \kappa_\ell\right]} > 0\\ \frac{dd_h}{d\kappa_h} &= \frac{-d_h(1+i)}{\left[(1+i)\kappa_h - \kappa_\ell\right]} < 0.\end{aligned}$$

(iii) From (22), $d_\ell = A$ if $c(q^*) \geq \beta\kappa_\ell A$. Hence, $\mathcal{V}_\ell = 1$. Otherwise, $d_\ell = c(q^*)/\beta\kappa_\ell$ and hence $d\mathcal{V}_\ell/d\kappa_\ell < 0$.

(iv) From the proof of Proposition 2, $i_1 \to \infty$ as $\kappa_\ell \to 0$. Hence, there always exists a monetary equilibrium. From (22), $U(\kappa_\ell) \to \max_q\left\{-ic(q) + [u(q) - c(q)]\right\}$ which from (28) yields $d_h \to 0$. From Proposition 2, $i < i_1$ implies $\omega_h > 0$ and $\omega_\ell > 0$. From (23) and (29), both $q_h = c^{-1}(\omega_h)$ and $q_\ell = c^{-1}(\omega_\ell)$ satisfy $u'(q)/c'(q) = 1 + i$. ■

Proof of Proposition 4. Two cases are distinguished.

(a) $\omega_\ell > 0$. From (23), $\mathtt{d}q_\ell/\mathtt{d}i < 0$ and, using the fact that $\omega_\ell = c(q_\ell) - \beta\kappa_\ell A$, $\mathtt{d}\omega_\ell/\mathtt{d}i < 0$. Rewrite (28) as

$$d_h = \frac{U(\kappa_\ell) - \max_q\left[u\left(q\right) - \left(1+i\right)c(q)\right]}{\beta\left[(1+i)\kappa_h - \kappa_\ell\right]}.$$

Differentiate the equation above to obtain

$$\frac{\mathtt{d}d_h}{\mathtt{d}i} = \frac{\omega_h - \omega_\ell}{\beta\left[(1+i)\kappa_h - \kappa_\ell\right]} > 0,$$

where I used the fact that $\omega_h = c(q_h) - \beta\kappa_h d_h$ and, from (22), $\mathtt{d}U(\kappa_\ell)/\mathtt{d}i = -\omega_\ell = -[c(q_\ell) - \beta\kappa_\ell A]$. From (29), $dq_h/di < 0$. Since $\omega_h = c(q_h) - \beta\kappa_h d_h$ then $d\omega_h/di < 0$.

(b) $\omega_\ell = 0$. Then, q_ℓ is independent of i and $\mathtt{d}U(\kappa_\ell)/\mathtt{d}i = 0$. Differentiating (28),

$$\frac{\mathtt{d}d_h}{\mathtt{d}i} = \frac{\omega_h}{\beta\left(i\kappa_h + \kappa_h - \kappa_\ell\right)} > 0.$$

The rest of the proof is analogous to (a).

Finally, from (29), $q_h \to q^*$ as $i \to 0$. From (28), $d_h \to 0$ as $i \to 0$. ■

Proof of Lemma 5. Consider the bargaining game between a buyer who has made the portfolio choice (ω, a) in the AM and a seller. Recall that the portfolio choice is common knowledge in the match. The outcome of (33)-(34) cannot be pooling (or semi-pooling). Since the argument is analogous to the one in the proof of Lemma 3, I only review it succinctly.

Suppose that the equilibrium of the bargaining game admits a pooling offer $(\bar{q}, \bar{d}, \bar{\tau})$. By definition, $\bar{S}^h(\omega, a) = u(\bar{q}) - \beta\kappa_h\bar{d} - \beta\bar{\tau}/\gamma$ and $\bar{S}^\ell(\omega, a) = u(\bar{q}) - \beta\kappa_\ell\bar{d} - \beta\bar{\tau}/\gamma$. This equilibrium fails the Intuitive Criterion if there exists an unsent offer $(\tilde{q}, \tilde{d}, \tilde{\tau})$ that it is feasible, $\beta\tilde{\tau}/\gamma \leq \omega$ and $\tilde{d} \leq a$, and such that

$$u(\tilde{q}) - \beta\kappa_h\tilde{d} - \frac{\beta}{\gamma}\tilde{\tau} \quad > \quad \bar{S}^h(\omega, a) \tag{68}$$

$$u(\tilde{q}) - \beta\kappa_\ell\tilde{d} - \frac{\beta}{\gamma}\tilde{\tau} \quad < \quad \bar{S}^\ell(\omega, a) \tag{69}$$

$$-c(\tilde{q}) + \beta\kappa_h\tilde{d} + \frac{\beta}{\gamma}\tilde{\tau} \quad \geq \quad 0. \tag{70}$$

Then, one can construct an alternative offer $(\tilde{q}, \tilde{d}, \tilde{\tau})$ with the following properties: $\tilde{\tau} = \bar{\tau}$, $\tilde{d} = \bar{d} - \varepsilon$ for $\varepsilon \in \left(0, \frac{-c(\bar{q})+\beta\kappa_h\bar{d}+\beta\bar{\tau}/\gamma}{\beta\kappa_h}\right) \cap (0, \bar{d}]$, and

$$u(\bar{q}) - \beta\kappa_h\varepsilon < u(\tilde{q}) < u(\bar{q}) - \beta\kappa_\ell\varepsilon.$$

First, such an offer exists since $-c(\bar{q}) + \beta\kappa_h\bar{d} + \beta\bar{\tau}/\gamma > 0$ (i.e., the pooling offer $(\bar{q}, \bar{d}, \bar{\tau})$ is acceptable). Also, $\bar{S}^h \geq 0$ implies $u(\bar{q}) - \beta\kappa_h\varepsilon \geq \beta\kappa_h(\bar{d} - \varepsilon) \geq 0$ so that for any ε there is a $\tilde{q} \geq 0$ satisfying the above inequality. Second, this offer satisfies (68)-(69). Moreover, the inequalities $-c(\bar{q}) + \beta\kappa_h(\bar{d} - \varepsilon) + \beta\bar{\tau}/\gamma > 0$ and $c(\bar{q}) - c(\tilde{q}) > 0$ imply that (70) holds.

The outcome of the bargaining game being separating, a buyer with $\kappa = \kappa_\ell$ cannot do better than his complete-information payoff. Hence, (q, d, τ) is solution to (38)-(39). Let $\tilde{S}^h$ denote the payoff of a κ_h-buyer with portfolio (ω, a) as given by the solution to (41)-(43). Suppose there is an equilibrium where the payoff of a κ_h-buyer is $\hat{S}^h$. First, $\hat{S}^h \leq \tilde{S}^h$ since otherwise either (42) or (43) is violated. If $\hat{S}^h < \tilde{S}^h$ then the κ_h-buyer can deviate and propose the solution to (41)-(43) where the term $S^\ell(\omega, a)$ on the right-hand side of (43) is replaced by $S^\ell(\omega, a) - \xi$ for $\xi > 0$. Provided that ξ is sufficiently small this offer satisfies (68)-(70). So the proposed equilibrium does not satisfy the Intuitive Criterion. Hence, $\hat{S}^h = \tilde{S}^h$. ■

Proof of Lemma 6. The buyer's objective function in (41) is continuous, and it is maximized over a compact set. Hence, by the Theorem of the Maximum, there is a solution to (41)-(44). If $a = 0$ it can easily be checked that $(q_h, \tau_h) = (q_\ell, \tau_\ell)$. So, in the following I focus on the case where $a > 0$.

First, suppose that the incentive-compatibility condition (43) is slack. Then, $q_h = \min\left[q^*, c^{-1}(\beta\kappa_h a + \omega)\right] \geq q_\ell$ and $d_h > 0$ if $\omega < c(q^*)$. Since $c(q_h) = \beta\kappa_h d_h + \beta\tau_h/\gamma$ then (43) becomes

$$u(q_h) - c(q_h) + \beta d_h(\kappa_h - \kappa_\ell) \leq u(q_\ell) - c(q_\ell).$$

Consequently, if $\omega < c(q^*)$ then (43) is violated, which is a contradiction. If $\omega \geq c(q^*)$ then $q_h = q^*$ and the inequality above implies $d_h = 0$ and $\beta\tau_h/\gamma = c(q^*)$.

Second, suppose that the seller's participation constraint (42) is slack. Substitute $u(q_h)$ by its expression given by (43) into the objective function of the $h-$buyer to get

$$\max_{d \leq a} \left[(\kappa_\ell - \kappa_h)\,\beta d + \hat{\mathcal{S}}\left(\omega + \beta\kappa_\ell a\right)\right] = \hat{\mathcal{S}}\left(\omega + \beta\kappa_\ell a\right),$$

and $d_h = 0$. The $h-$buyer gets the same surplus as a $\ell-$buyer, i.e.,

$$\hat{\mathcal{S}}\left(\omega + \beta\kappa_\ell a\right) = \max_{q, \beta\tau/\gamma \leq \omega} \left[u(q) - \frac{\beta}{\gamma}\tau\right] \quad \text{s.t.} \quad -c(q) + \frac{\beta}{\gamma}\tau \geq 0,$$

where I have used that $d_h = 0$ in the maximization problem on the right-hand side of the equality. The equality holds if and only if $\omega \geq c(q^*)$. In that case, $q_h = q^*$ and $\beta\tau_h/\gamma = c(q^*)$, which is consistent with the first case.

Third, suppose $\omega < c(q^*)$ so that both the seller's participation constraint and the incentive-compatibility condition (43) are binding. Since (42) is binding, d_h is given by (48). Substitute d_h by its expression into (43) at equality to get (49). For all $q_h \in [0, q_\ell]$ the left-hand side of (49) is strictly increasing. It is nonpositive at $q_h = 0$ and greater than $u(q_\ell) - c(q_\ell)$ at $q_h = q_\ell$ provided that $c(q_\ell) > \omega$. From (38)-(40) if $\omega < c(q^*)$ then $c(q_\ell) = \min\left[c(q^*), \omega + \beta\kappa_\ell a\right] > \omega$ (since I focus on the case $a > 0$). Hence, there is a unique $q_h \in (0, q_\ell)$ solution to (49). It can be checked that $u(q_h) - c(q_h)$ is decreasing in q_h for any solution to (49). (See Proposition 1 for a related argument.) Hence, the unique solution in $(0, q_\ell)$ delivers a maximum to the problem (41)-(44). Given a unique q_h, d_h is determined by (48). Finally, $c(q_h) = \omega + \beta\kappa_h d_h < c(q_\ell) = \beta\tau_\ell/\gamma + \beta\kappa_\ell d_\ell$ implies $d_h < a$. To see this, recall that $\beta\tau_\ell/\gamma + \beta\kappa_\ell d_\ell = \omega + \beta\kappa_\ell a$ if $\omega + \beta\kappa_\ell a \leq c(q^*)$ and $\beta\tau_\ell/\gamma + \beta\kappa_\ell d_\ell = c(q^*) < \omega + \beta\kappa_\ell a$ otherwise. ■

Proof of Lemma 7. Equations (54) and (55) are the first-order conditions with respect to ω and a of the problem (36). The following cases are distinguished: $\phi > \beta\bar{\kappa}$, $\phi = \beta\bar{\kappa}$ and $\phi < \beta\bar{\kappa}$.

(i) $\phi > \beta\bar{\kappa}$**.**

First, compute the first and second partial derivatives and the cross-partial derivatives of the functions $S^\ell(\omega, a)$ and $S^h(\omega, a)$ where $\omega = \beta z/\gamma$. These expressions will be used to prove that the objective function in (36) is strictly concave with respect to (ω, a) over some relevant range.

From Lemma 5, $(q_\ell, d_\ell, \tau_\ell)$ solves (38)-(39) and $S^\ell(\omega, a) = \hat{\mathcal{S}}(\omega + \beta\kappa_\ell a) = u(q_\ell) - c(q_\ell)$ where $q_\ell = \min\left[q^*, c^{-1}(\omega + \beta\kappa_\ell a)\right]$. Therefore, $S^\ell_a = \beta\kappa_\ell \hat{\mathcal{S}}'_\ell$, $S^\ell_\omega = \hat{\mathcal{S}}'_\ell$, $S^\ell_{\omega\omega} = \hat{\mathcal{S}}''_\ell$, $S^\ell_{a\omega} = \beta\kappa_\ell\hat{\mathcal{S}}''_\ell$ and $S^\ell_{aa} = (\beta\kappa_\ell)^2\,\hat{\mathcal{S}}''_\ell$ where $\hat{\mathcal{S}}_\ell \equiv \hat{\mathcal{S}}(\omega + \beta\kappa_\ell a)$. From Lemma 6, if $\omega < c(q^*)$ then q_h solves (49). Totally differentiating (49),

$$\begin{aligned}
\left[u'(q_h) - \frac{\kappa_\ell}{\kappa_h} c'(q_h)\right] \frac{\mathrm{d}q_h}{\mathrm{d}\omega} &= 1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}'_\ell \\
\left[u'(q_h) - \frac{\kappa_\ell}{\kappa_h} c'(q_h)\right] \frac{\mathrm{d}q_h}{\mathrm{d}a} &= \beta\kappa_\ell\hat{\mathcal{S}}'_\ell,
\end{aligned}$$

where I have used the fact that $u(q_\ell) - c(q_\ell) = \hat{\mathcal{S}}(\omega + \beta\kappa_\ell a)$. Notice that $\frac{\mathrm{d}q_h}{\mathrm{d}\omega} > 0$ for all $\omega < c(q^*)$ and $\frac{\mathrm{d}q_h}{\mathrm{d}a} > 0$ for all (ω, a) such that $\omega + \beta\kappa_\ell a < c(q^*)$. From Lemma 6, the seller's participation constraint (42) holds at equality so that $S^h(\omega, a) = u(q_h) - c(q_h)$. Hence,

$$\begin{aligned}
S^h_\omega(\omega, a) &= [u'(q_h) - c'(q_h)] \frac{\mathrm{d}q_h}{\mathrm{d}\omega} = \Delta(q_h)\left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}'_\ell\right) \\
S^h_a(\omega, a) &= [u'(q_h) - c'(q_h)] \frac{\mathrm{d}q_h}{\mathrm{d}a} = \Delta(q_h)\beta\kappa_\ell\hat{\mathcal{S}}'_\ell
\end{aligned}$$

where

$$\Delta(q) \equiv \frac{u'(q) - c'(q)}{u'(q) - \frac{\kappa_\ell}{\kappa_h} c'(q)} = 1 - \frac{1 - \frac{\kappa_\ell}{\kappa_h}}{u'(q)/c'(q) - \frac{\kappa_\ell}{\kappa_h}}.$$

For all $q \in [0, q^*]$, $\Delta(q) \in [0,1]$ and, since $u'(q)/c'(q)$ is decreasing in q, $\Delta'(q) < 0$. Furthermore,

$$
\begin{aligned}
S^h_{\omega\omega} &= \Delta'(q_h)\frac{\mathrm{d}q_h}{\mathrm{d}\omega}\left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}'_\ell\right) + \Delta(q_h)\hat{\mathcal{S}}''_\ell \\
S^h_{\omega a} &= \Delta'(q_h)\frac{\mathrm{d}q_h}{\mathrm{d}\omega}\beta\kappa_\ell\hat{\mathcal{S}}'_\ell + \Delta(q_h)\beta\kappa_\ell\hat{\mathcal{S}}''_\ell \\
&= \Delta'(q_h)\frac{\mathrm{d}q_h}{\mathrm{d}a}\left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}'_\ell\right) + \Delta(q_h)\beta\kappa_\ell\hat{\mathcal{S}}''_\ell \\
S^h_{aa} &= \Delta'(q_h)\frac{\mathrm{d}q_h}{\mathrm{d}a}\beta\kappa_\ell\hat{\mathcal{S}}'_\ell + \Delta(q_h)\left(\beta\kappa_\ell\right)^2\hat{\mathcal{S}}''_\ell.
\end{aligned}
$$

For all $\omega < c(q^*)$, $S^h_{\omega\omega} < 0$. Consequently, the first leading principal minor of the Hessian matrix associated with (36), $\pi_h S^h_{\omega\omega} + \pi_\ell S^\ell_{\omega\omega}$, is negative for all $\omega < c(q^*)$.

The determinant of the Hessian matrix associated with (36) is

$$
|\mathbb{H}| = \left(\pi_h S^h_{\omega\omega} + \pi_\ell S^\ell_{\omega\omega}\right)\left(\pi_h S^h_{aa} + \pi_\ell S^\ell_{aa}\right) - \left(\pi_h S^h_{\omega a} + \pi_\ell S^\ell_{\omega a}\right)^2.
$$

It can be decomposed as $|\mathbb{H}| = \Gamma_1 + \Gamma_2 + \Gamma_3$ where

$$
\begin{aligned}
\Gamma_1 &= (\pi_\ell)^2\left[S^\ell_{\omega\omega}S^\ell_{aa} - \left(S^\ell_{\omega a}\right)^2\right] \\
\Gamma_2 &= (\pi_h)^2\left[S^h_{\omega\omega}S^h_{aa} - \left(S^h_{\omega a}\right)^2\right] \\
\Gamma_3 &= \pi_h\pi_\ell\left[S^h_{\omega\omega}S^\ell_{aa} + S^\ell_{\omega\omega}S^h_{aa} - 2S^h_{\omega a}S^\ell_{\omega a}\right].
\end{aligned}
$$

Since $S^\ell(\omega, a) = \hat{\mathcal{S}}(\omega + \beta\kappa_\ell a)$, $\Gamma_1 = 0$. After some calculation,

$$
\begin{aligned}
\Gamma_2 &= (\pi_h)^2\left(1 - \frac{\kappa_\ell}{\kappa_h}\right)\Delta\Delta'\hat{\mathcal{S}}''_\ell\beta\kappa_\ell\left(\frac{\mathrm{d}q_h}{\mathrm{d}\omega}\beta\kappa_\ell - \frac{\mathrm{d}q_h}{\mathrm{d}a}\right) \\
\Gamma_3 &= \pi_h\pi_\ell\left(1 - \frac{\kappa_\ell}{\kappa_h}\right)\Delta'\hat{\mathcal{S}}''_\ell\beta\kappa_\ell\left(\frac{\mathrm{d}q_h}{\mathrm{d}\omega}\beta\kappa_\ell - \frac{\mathrm{d}q_h}{\mathrm{d}a}\right)
\end{aligned}
$$

where Δ and Δ' are evaluated at $q = q_h$. Therefore,

$$
|\mathbb{H}| = \left(\frac{\mathrm{d}q_h}{\mathrm{d}\omega}\beta\kappa_\ell - \frac{\mathrm{d}q_h}{\mathrm{d}a}\right)\left(1 - \frac{\kappa_\ell}{\kappa_h}\right)\Delta'\hat{\mathcal{S}}''_\ell\beta\kappa_\ell\pi_h\left(\pi_h\Delta + \pi_\ell\right)
$$

with

$$
\beta\kappa_\ell\frac{\mathrm{d}q_h}{\mathrm{d}\omega} - \frac{\mathrm{d}q_h}{\mathrm{d}a} = \left[u'(q_h) - \frac{\kappa_\ell}{\kappa_h}c'(q_h)\right]^{-1}\beta\kappa_\ell\left(1 - \frac{\kappa_\ell}{\kappa_h}\right) > 0, \quad \forall q_h \le q^*.
$$

Hence, $|\mathbb{H}| > 0$ for all $\omega + \beta\kappa_\ell a < c(q^*)$.

One can now show that there is a unique solution to (36). First, the solution to (36) is such that $\omega + \beta\kappa_\ell a \le c(q^*)$. Suppose $\omega + \beta\kappa_\ell a > c(q^*)$. Then, $\hat{\mathcal{S}}'_\ell = 0$ and $S^h_a(\omega, a) = S^\ell_a(\omega, a) = 0$. But then the first-order condition for a, (55), implies $a = 0$. If $\omega > c(q^*)$ then $q_h = q_\ell = q^*$ and hence $S^h_\omega(\omega, a) = S^\ell_\omega(\omega, a) = 0$.

The first-order condition for ω, (54), implies then $\omega = 0$. A contradiction. So one can restrict (ω, a) to the compact set $\{(\omega, a) \in \mathbb{R}_{2+} : \omega + \beta\kappa_\ell a \leq c(q^*)\}$ and, from the Theorem of the Maximum, a solution to (36) exists and it satisfies the first-order conditions (54)-(55). Since $\mathbb{H}$ is negative definite for all (ω, a) such that $\omega + \beta\kappa_\ell a < c(q^*)$, i.e., the leading principal minors of $\mathbb{H}$ alternate in sign with the first one being negative, the solution to (36) is unique.

(ii) $\phi = \beta\bar{\kappa}$**.**

From the first-order condition for a, (55), $S_a^h(\omega, a) = S_a^\ell(\omega, a) = 0$, which requires $\omega + \beta\kappa_\ell a \geq c(q^*)$. The first-order condition for ω, (54), implies

$$-i + \pi_h \Delta(q_h)\left(1 - \frac{\kappa_\ell}{\kappa_h}\right) \leq 0, \quad \text{“ = ”} \quad \text{if } \omega > 0. \tag{71}$$

where I have used that $\hat{\mathcal{S}}_\ell' = 0$. From (49), q_h is only a function of ω and it solves

$$u(q_h) - c(q_h) + \left(1 - \frac{\kappa_\ell}{\kappa_h}\right)[c(q_h) - \omega] = u(q^*) - c(q^*). \tag{72}$$

For all $\omega \geq c(q^*)$, $\Delta(q_h) = \Delta(q^*) = 0$ and (71) does not hold. Since $\Delta' < 0$ and $\frac{\mathrm{d}q_h}{\mathrm{d}\omega} > 0$ for all $\omega \in (0, c(q^*))$, and since the function on the left-hand side of (71) is continuous in ω, there is a unique $\omega \in [0, c(q^*)]$ solution to (71). Consequently, $a \in [\frac{c(q^*)-\omega}{\beta\kappa_\ell}, \infty)$.

(iii) $\phi < \beta\bar{\kappa}$**.**

Since $S_a^h(\omega, a) \geq 0$ and $S_a^\ell(\omega, a) \geq 0$ there is no solution to the first-order condition for a. ■

Proof of Proposition 5. The proof proceeds in three steps. First, it establishes the existence and uniqueness of the market-clearing price ϕ. Second, it derives the condition for a monetary equilibrium. Third, it characterizes the allocations in the PM.

(i) Existence and uniqueness of ϕ.

Define $A^d(\phi) \equiv \left\{\int_{j\in\mathcal{B}} a(j)dj : a(j) \text{ solution to (36)}\right\}$. From Lemma 7, if $\phi > \beta\bar{\kappa}$ then there is a unique solution (ω, a) to the problem (36). Hence, $A^d(\phi) = \{a\}$. Moreover, since (ω, a) can be restricted to the compact set $\{(\omega, a) \in \mathbb{R}_{2+} : \omega + \beta\kappa_\ell a \leq c(q^*)\}$ and since the objective function in (36) is continuous, the Theorem of the Maximum implies that $A^d(\phi)$ is continuous. Assuming an interior solution, and totally differentiating (54)-(55),

$$\mathbb{H} \cdot \begin{pmatrix} d\omega \\ da \end{pmatrix} = \begin{pmatrix} di \\ d\phi \end{pmatrix}$$

where $\mathbb{H} = [H_{ij}]_{(i,j)\in\{1,2\}^2}$ is the Hessian matrix associated with (36). Since $|\mathbb{H}| > 0$ (see proof of Lemma 7), $\mathbb{H}$ is invertible and

$$\begin{pmatrix} d\omega \\ da \end{pmatrix} = \frac{1}{|\mathbb{H}|}\begin{pmatrix} H_{22} & -H_{12} \\ -H_{21} & H_{11} \end{pmatrix}\begin{pmatrix} di \\ d\phi \end{pmatrix}.$$

Consequently, for all $\phi > \beta\bar{\kappa}$, $da/d\phi = H_{11}/|\mathbb{H}| < 0$ where $H_{11} = \pi_h S^h_{\omega\omega} + \pi_\ell S^\ell_{\omega\omega}$. If the solution to (36) is not interior and $\omega = 0$ then, from (55),

$$\frac{da}{d\phi} = \left[\pi_h S^h_{aa}(0,a) + \pi_\ell S^\ell_{aa}(0,a)\right]^{-1} < 0.$$

So, $A^d(\phi)$ is decreasing provided that $a > 0$.

Next, I establish that $A^d(\phi) = \{0\}$ if $\phi > \beta\bar{\kappa} + i\beta\kappa_\ell$. To see this, rewrite (55) as

$$\frac{-\phi + \beta\bar{\kappa}}{\beta\kappa_\ell} + \pi_h\Delta(q_h)\hat{\mathcal{S}}_\ell' + \pi_\ell\hat{\mathcal{S}}_\ell' \leq 0.$$

From the comparison with (54), if $\frac{\phi-\beta\bar{\kappa}}{\beta\kappa_\ell} > i$ then (55) holds with a strict inequality and $a = 0$. Moreover, it can be checked from (54)-(55) that if $A_d(\phi) = \{0\}$ then $A_d(\phi') = \{0\}$ for all $\phi' > \phi$.

If $\phi = \beta\bar{\kappa}$ then $A^d = \left[\frac{c(q^*)-\hat{\omega}(i)}{\beta\kappa_\ell}, \infty\right)$ where $\hat{\omega}(i)$ is the unique solution to (71)-(72). The equations (54)-(55) being continuous in (ω, a), $\lim_{\phi\to\beta\bar{\kappa}}\omega(\phi) = \hat{\omega}(i)$ and $\lim_{\phi\to\beta\bar{\kappa}}[\omega(\phi) + \beta\kappa_\ell a(\phi)] = c(q^*)$. Consequently, $\lim_{\phi\to\beta\bar{\kappa}} a(\phi) = \frac{c(q^*)-\hat{\omega}(i)}{\beta\kappa_\ell} \in A^d(\beta\bar{\kappa})$.

To summarize: $A^d(\phi)$ is upper-hemi continuous, any selection from $A^d(\phi)$ is decreasing whenever $a > 0$, $A^d(\phi) = \{0\}$ for all $\phi > \beta\bar{\kappa} + i\beta\kappa_\ell$ and $\infty \in A^d(\beta\bar{\kappa})$. Hence, there is a unique $\phi \in [\beta\bar{\kappa}, \beta\bar{\kappa} + i\beta\kappa_\ell]$ such that $A \in A^d(\phi)$. (See Figure 8.)

(ii) Monetary equilibrium.

From Lemma 7, for given ϕ there is a unique ω solution to the buyer's problem. Since $A \in A^d(\phi)$, it satisfies (54) with $a = A$. Hence, $\frac{d\omega}{di} < 0$ whenever $\omega > 0$ and there exists $i_0(A) = \pi_h S^h_\omega(0,A) + \pi_\ell S^\ell_\omega(0,A)$ such that $\omega > 0$ for all $i < i_0$. Since $S^h_\omega(0,A) < \infty$ and $S^\ell_\omega(0,A) < \infty$ for all $A > 0$ then $i_0(A) < \infty$. Furthermore, since $S^h_\omega(0,A) = \Delta(q_h)\left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}_\ell'\right) > 0$ for all $q_h < q^*$ (from Lemma 6) then $i_0(A) > 0$.

(iii) PM allocations.

From (i) ϕ is unique. From Lemma 7, if $\phi > \beta\bar{\kappa}$ then there is a unique solution to (36). Moreover, from Lemma 6, if $\kappa = \kappa_h$ then (q_h, d_h, τ_h) is unique and if $\kappa = \kappa_\ell$ then q_ℓ and $\frac{\beta}{\gamma}\tau_\ell + \beta\kappa_\ell d_\ell$ are uniquely determined. If $\phi = \beta\bar{\kappa}$ then $a(j)$ can vary across buyers but ω is unique and $\omega + \beta\kappa_\ell a(j) \geq c(q^*)$ for all $j \in \mathcal{B}$ (see Lemma 7). Consequently, $q_\ell = q^*$ and, from (49), q_h is independent of $a(j)$ for all $j \in \mathcal{B}$ and it solves (72). ■

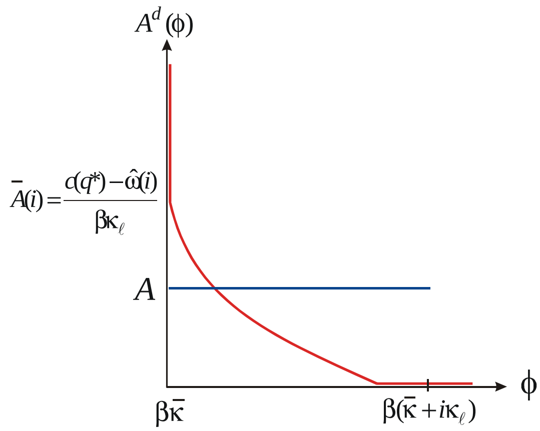

Figure 8: Graph of $A^d(\phi)$

Proof of Proposition 6. (i) From the proof of Proposition 5, $\frac{d\omega}{di} < 0$ whenever $\omega > 0$. Assume $i < i_0(A)$ so that a monetary equilibrium exists. Differentiating (49) I get:

$$\frac{dq_h}{d\omega} = \left[\frac{\frac{u'(q_\ell)}{c'(q_\ell)} - \frac{\kappa_\ell}{\kappa_h}}{u'(q_h) - \frac{\kappa_\ell}{\kappa_h} c'(q_h)} \right] > 0,$$

where I have used the fact that $\frac{d[u(q_\ell) - c(q_\ell)]}{d\omega} = \frac{u'(q_\ell)}{c'(q_\ell)} - 1$. From (48)-(49),

$$\beta\kappa_h \frac{dd_h}{d\omega} = c'(q_h) \left[\frac{\frac{u'(q_\ell)}{c'(q_\ell)} - \frac{u'(q_h)}{c'(q_h)}}{u'(q_h) - \frac{\kappa_\ell}{\kappa_h} c'(q_h)} \right] < 0.$$

Hence, $d\mathcal{V}_h/di > 0$.

(ii) Define $\bar{A}(i) = [c(q^*) - \hat{\omega}(i)] / \beta\kappa_\ell$ where $\hat{\omega}(i)$ is the unique solution to (71)-(72). As shown in the proof of Proposition 5, $A^d(\beta\bar{\kappa}) = [\frac{c(q^*) - \hat{\omega}(i)}{\beta\kappa_\ell}, \infty)$ and $A^d = \{a\}$ with $\beta\kappa_\ell a < c(q^*) - \hat{\omega}(i)$ if $\phi > \beta\bar{\kappa}$ (since a is a decreasing function of ϕ). From market-clearing, if $A < \bar{A}(i)$ then $\phi > \beta\bar{\kappa}$ and $\omega + \beta\kappa_\ell a < c(q^*)$ (See proof of Lemma 7.) From the proof of Proposition 5, $da/di = -H_{21}/\left|\mathbb{H}\right| > 0$ with $H_{21} = \pi_h S^h_{\omega a}(\omega, a) + \pi_\ell S^\ell_{\omega a}(\omega, a)$. So, $A \in A^d(\phi)$ implies that $d\phi/di > 0$. If $A > \bar{A}(i)$ then $\phi = \beta\bar{\kappa}$ and $d\phi/di = 0$.

(iii) From (54), as $i \to 0$, $S^h_\omega, S^\ell_\omega \to 0$ and hence $q_h, q_\ell \to q^*$ and $\omega \to c(q^*)$. From Lemma 6, $d_h \to 0$ and $\tau_h \to c(q^*)$. From the proof of Proposition 6, $\phi = \beta\bar{\kappa}$ for all $A > \bar{A}(i)$ where $\lim_{i \to 0} \bar{A}(i) = 0$. ■

Proof of Proposition 7. From (55),

$$\phi = \beta\bar{\kappa} + \pi_h S^h_a(\omega, a) + \pi_\ell S^\ell_a(\omega, a).$$

Substitute S_a^h and S_a^ℓ by their expressions to get

$$\frac{\phi}{\bar{\kappa}} = \beta \left\{ 1 + \frac{\kappa_\ell}{\bar{\kappa}} \left[\pi_h \Delta(q_h) \hat{\mathcal{S}}_\ell' + \pi_\ell \hat{\mathcal{S}}_\ell' \right] \right\}. \tag{73}$$

Similarly, from (54), and after replacing S_ω^h and S_ω^ℓ by their expressions,

$$i = \pi_h \Delta(q_h) \left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}_\ell' \right) + \pi_\ell \hat{\mathcal{S}}_\ell'.$$

Using the fact that $1 + i = \gamma/\beta$, the equation above can be rewritten as

$$\gamma = \beta \left\{ 1 + \pi_h \Delta(q_h) \left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}_\ell' \right) + \pi_\ell \hat{\mathcal{S}}_\ell' \right\}. \tag{74}$$

From (73) and (74),

$$\begin{aligned} \gamma - \frac{\phi}{\bar{\kappa}} &= \frac{1}{R_m} - \frac{1}{R_a} \\ &= \beta \left\{ \left(1 - \frac{\kappa_\ell}{\kappa_h} \right) \pi_h \Delta(q_h) + \left(1 - \frac{\kappa_\ell}{\bar{\kappa}} \right) \hat{\mathcal{S}}_\ell' \left[\pi_h \Delta(q_h) + \pi_\ell \right] \right\}. \end{aligned}$$

From (54), $\Delta(q_h) > 0$ (since $S_\omega^h > 0$) for all $i > 0$. Moreover, $\kappa_\ell < \bar{\kappa} < \kappa_h$. Hence, $R_a > R_m$. ■

Supplementary appendices

B. Refinements of sequential equilibrium

In this Appendix, I describe two refinements of sequential equilibrium in the context of the bargaining game studied in this paper: the Intuitive Criterion and the undefeated equilibrium. I focus on the model of Section 5 where the real asset is only traded in the PM.

A typical signaling game has the following structure. There are two players: a sender of information and a receiver of information. In the context of the bargaining game in this paper, the sender is the buyer who makes an offer and the receiver is the seller who accepts or rejects the offer. The timing of the game is:

1. Nature draws a type t for the sender according to some (commonly known) probability distribution $\pi(t)$. Here, the set of types is $T = \{\ell, h\}$ and $\pi(\ell) = \pi_\ell$ and $\pi(h) = \pi_h$.

2. The sender (buyer) privately observes the type t, and he sends an offer o to the receiver (seller). Here, an offer is a triple $(q, d, \omega) \in \mathbb{R}^3_+$ where q is the output, d is the transfer of the real asset, and ω is the amount of real balances.

3. The receiver observes the offer o and takes an action r. Here, the set of actions is $\{Y, N\}$. If $r = Y$ then the offer is accepted; If $r = N$ then the offer is rejected.

The payoff of the buyer is $U^b(t, o, r) = -i\omega + [u(q) - \beta\kappa_t d - \omega] \mathbb{I}_{\{r=Y\}}$. The payoff of the seller is $U^s(t, o, r) = [-c(q) + \beta\kappa_t d + \omega] \mathbb{I}_{\{r=Y\}}$. After receiving the offer o, the seller forms a posterior probability assessment over the set of types of the buyer, $\lambda(t\,|o)$. The best response of the seller is

$$\text{BR}(\lambda, o) = \arg\max_{r\in\{Y,N\}} \sum_{t\in\{\ell,h\}} \lambda(t\,|o)\, U^s(t, o, r).$$

In the context of the bargaining game, the best response of the seller can be reexpressed as

$$\text{BR}(\lambda, o) = \arg\max_{r\in\{Y,N\}} \left\{[-c(q) + \beta\left[\lambda(h\,|o)\,\kappa_h + \lambda(\ell\,|o)\,\kappa_\ell\right] d + \omega]\, \mathbb{I}_{\{r=Y\}}\right\}.$$

I adopt the tie-breaking rule according to which $r = Y$ whenever $\text{BR}(\lambda, o) = \{Y, N\}$.

The equilibrium concept for the extensive game with imperfect information is that of sequential equilibrium. It is a pair of strategies and a belief system with the following properties. Strategies are sequentially rational: for each information set of each player i, the strategy of player i is a best response to the other player's strategy, given i's belief at that information set.

Let O denote a strategy for a buyer. It is a mapping from the set of types to the set of feasible offers. Let R denotes a strategy for a seller. It is a mapping from the set of feasible offers to the set $\{Y, N\}$. A (pure strategy) sequential equilibrium is a profile of strategies (O^*,R^*) and a seller's belief system, λ^*, such that the following is true.

1. For all $t \in T$, $\text{O}^*(t) \in \arg\max_{o'} U^b(t, o', \text{R}^*(o'))$
2. For all o, $\text{R}^*(o) \in \text{BR}(\lambda^*(o), o)$
3. λ^* satisfies Bayes' rule whenever possible (and is unconstrained for out-of-equilibrium offers)

In the context of the bargaining game studied in this paper, the buyer's strategy can be simplified by noticing the following. First, $U^b(t, o', \text{R}^*(o')) \leq 0$ for all o' such that $\text{R}^*(o') = \{N\}$. Second, from the tie-breaking rule, $U^b(t, o', \text{R}^*(o')) = 0$ and $\text{R}^*(o') = \{Y\}$ for $o' = (0, 0, 0)$. Hence, with no loss, the buyer can choose an offer among those that are accepted by sellers, i.e.,

$$\text{O}^*(t) \in \arg\max_{o'} U^b(t, o', Y) \quad \text{s.t.} \quad \{Y\} \in BR(\lambda^*(o'), o').$$

The buyer's problem becomes then (19)-(20).

The Intuitive Criterion

The Cho-Kreps (1987) refinement is based on the (intuitive) idea that out-of-equilibrium actions should never be attributed to a type who would not benefit from it under any circumstances. For a subset $K \subseteq T$, let $\text{BR}(K, o)$ denote the set of best responses for the seller to beliefs concentrated on K, i.e.,

$$\text{BR}(K, o) = \bigcup_{\{\lambda : \lambda(K) = 1\}} \text{BR}(\lambda, o).$$

Suppose $K = T = \{\ell, h\}$. Then,

$$\begin{aligned} BR(T, o) &= \{Y\} \quad \text{if} \quad -c(q) + \beta\kappa_\ell d + \omega > 0 \\ &= \{N\} \quad \text{if} \quad -c(q) + \beta d \kappa_h + \omega < 0 \\ &= \{Y, N\} \quad \text{otherwise.} \end{aligned}$$

Consider a proposed equilibrium where the payoff of a buyer of type t is denoted U_t^*. According to Cho and Kreps (1987, p.202), this proposed equilibrium fails the Intuitive Criterion if there exists an unsent offer o' such that:

1. $U_\ell^* > \max_{r \in BR(\{\ell,h\},o')} U^b(\ell, o', r)$

2. $U_h^* < \min_{r \in BR(\{h\},o')} U^b(h, o', r)$

According to the first requirement, the unsent offer o' would reduce the payoff of the $\ell-$type buyer compared to his equilibrium payoff irrespective of the inference the seller draws from o'. Consequently, the seller should attribute the offer o' to an $h-$type buyer. If he does so, the second requirement specifies that the $h-$type buyer should obtain a higher utility with o' compare to his equilibrium payoff.

In the bargaining game, the buyer's equilibrium payoff is bounded below by 0. Hence, the second requirement implies $\min_{r \in BR(\{h\},o')} U^b(h, o', r) > 0$, which requires $\{Y\} \in BR(\{h\}, o')$ (where I am making use of the tie-breaking rule). Since $\{Y\} \in BR(\{\ell, h\}, o')$, the first requirement becomes $U_\ell^* > U^b(\ell, o', Y)$. To summarize, a proposed equilibrium fails the Intuitive Criterion if there is an out-of-equilibrium offer that satisfies (61)-(63).

Undefeated sequential equilibria

Mailath, Okuno-Fujiwara and Postlewaite (1993) criticized the logical foundations of refinements based on forward induction (such as the Intuitive Criterion). They argue that it is difficult to interpret out-of-equilibrium messages as signals. For instance, consider a sequential equilibrium of the bargaining game that is pooling. It has been shown that an $h-$type buyer could make an out-of-equilibrium offer that would make him better-off and that would hurt an $\ell-$type buyer. (See Lemma 1.) By making such an offer the $h-$type buyer hopes to convince the seller that he is an $h-$type. But if the seller finds the Intuitive Criterion appealing he knows that the $h-$type buyer will alter his offer, and hence he should update his belief about the buyer's type if he does send the equilibrium message. "But if the (seller) does this, in the determination of whether a particular type might benefit from sending some disequilibrium message, the relevant comparison is not with the utility that he would receive in the proposed equilibrium but rather the utility that he would receive given that (the seller) is thinking in this way". (Mailath, Okuno-Fujiwara and Postlewaite, 1993, p.250.) They introduce a new refinement, based on the notion of *undefeated equilibrium*, that takes care of some of these concerns.

An equilibrium is composed of a strategy for buyers, $\textsc{o}$, that specifies an offer for each type, an acceptance rule for sellers, $\textsc{r}$, and a belief system for sellers, λ. According to Mailath, Okuno-Fujiwara and Postlewaite (1993, p.254, Definition 2) an equilibrium ($\textsc{o}$', $\textsc{r}$', λ') defeats ($\textsc{o}$,$\textsc{r}$, λ) if there exists an offer o' such that:

1. For all t, $\textsc{o}(t) \neq o'$ and $K \equiv \{t \in T \,|\textsc{o}'(t) = o'\} \neq \varnothing$
2. For all $t \in K$, $U^b[t, o', \textsc{r}'(o')] \geq U^b[t, \textsc{o}(t), \textsc{r}(\textsc{o}(t))]$ with a strict inequality for one t in K
3. $\lambda(t\,|o') \neq p(t)\pi(t)/\sum_{t'} p(t')\pi(t')$ for at least one t in K where $p(t) = 1$ if $t \in K$ and $U^b[t, o', \textsc{r}'(o')] > U^b[t, \textsc{o}(t), \textsc{r}(\textsc{o}(t))]$ and $p(t) = 0$ if $t \notin K$.

So, for a sequential equilibrium to be defeated there must exist an out-of-equilibrium offer that is used in an alternative sequential equilibrium by a subset K of buyers' types (requirement 1). For all buyers with types in K, their payoff at the alternative equilibrium must be greater than the one at the proposed equilibrium with a strict inequality for at least one type (requirement 2). Finally, the belief system in the proposed equilibrium does not update sellers' prior belief conditional on the buyer's type being in K (requirement 3).

C. Undefeated monetary equilibria

The Intuitive Criterion is based on the simple idea that one should not attribute an out-of-equilibrium action to a type that would not benefit from it under any circumstances. While it is an intuitive refinement, its logic as well as some of the properties of the equilibria it selects have not stayed unchallenged.[41] Mailath, Okuno-Fujiwara and Postlewaite (1993) introduced an alternative refinement, *undefeated equilibria*, which addresses some of the shortcomings of refinements based on forward induction. In this section, I check the robustness of the results by investigating the conditions under which the equilibrium selected by the Intuitive Criterion is undefeated.[42] I focus on the model in Section 5 where the real asset is only traded in the PM.

The idea of an undefeated equilibrium is as follows. Consider a proposed sequential equilibrium and an out-of-equilibrium offer $\tilde{o}$. Suppose there is an alternative sequential equilibrium in which a subset of buyers' types choose $\tilde{o}$. Moreover, those buyers prefer the alternative equilibrium to the proposed equilibrium. The test requires seller's beliefs at that action in the original equilibrium to be consistent with the set of buyers who would benefit from the out-of-equilibrium offer $\tilde{o}$ (see the Appendix B for a formal definition). If the beliefs are not consistent, the second equilibrium *defeats* the proposed *equilibrium.*

In the following the attention will be restricted to symmetric equilibria in pure strategies.

Lemma 8 *The separating equilibrium that satisfies the Intuitive Criterion is the only undefeated equilibrium if*

$$\bar{U}_h^b \equiv -(1+i)\omega^p + u(q^p) - \beta\kappa_h d^p < U_h^b \equiv -(1+i)\omega_h + u(q_h) - \beta\kappa_h d_h, \tag{75}$$

where (q_h, d_h, ω_h) *is the solution to (24)-(26) and*

$$(q^p, d^p, \omega^p) = \arg\max_{\omega,q,d\leq A} \{-(1+i)\omega + u(q) - \beta\kappa_h d\} \tag{76}$$

$$s.t. \quad -c(q) + \beta\left(\pi_h\kappa_h + \pi_\ell\kappa_\ell\right)d + \omega = 0, \tag{77}$$

Conversely, if $\bar{U}_h^b > U_h^b$ *then there is an undefeated equilibrium and it is pooling.*

[41] One of the most problematic aspect of the equilibrium selected by the Intuitive Criterion might be its lack of continuity with respect to perturbations of the prior beliefs. In particular, the equilibrium obtained by taking the limit as π_ℓ goes to 0 does not approach the complete information equilibrium.

[42] As Mailath, Okuno-Fujiwara and Postlewaite (1993, p.265) put it,

> "[T]here is no reason that different refinements shouldn't be employed in the analysis of a single game. Various implausibilities may be exhibited in different equilibria of a game, and hence, considering different refinements of the equilibrium set for a single game is like looking at the game from different vantage points".

Proof. First, I establish that among the separating sequential equilibria, the only one that can be undefeated is the one that satisfies the Intuitive Criterion. Consider a sequential equilibrium that is separating and denote $\hat{U}_h^b$ the payoff of the $h-$type buyer at this equilibrium. The offer of the $\ell-$type buyer is his complete information offer, $(q_\ell, d_\ell, \omega_\ell)$. Suppose that the offer $(\hat{q}_h, \hat{d}_h, \hat{\omega}_h)$ of the $h-$type at the proposed equilibrium is different from (q_h, d_h, ω_h) that solves (24)-(26). Since (q_h, d_h, ω_h) is an offer made by the $h-$type only, and $\hat{U}_h^b < U_h^b$, then the proposed equilibrium is defeated by the unique separating equilibrium that satisfies the Intuitive Criterion.

Second, suppose (75) holds. Since, from (76)-(77), $\bar{U}_h^b$ is the highest payoff an $h-$type buyer can reach at a pooling equilibrium, any pooling equilibrium is defeated by the Pareto-efficient separating equilibrium.

Third, suppose that $\bar{U}_h^b > U_h^b$. The pooling equilibrium (q^p, d^p, ω^p) defeats the Pareto-efficient separating equilibrium. To see this, notice that the $h-$type buyer strictly prefers the pooling equilibrium. Moreover, the $\ell-$type buyer also prefers the pooling equilibrium, i.e.,

$$-(1+i)\omega^p + u(q^p) - \beta\kappa_\ell d^p > U(\kappa_\ell). \tag{78}$$

Indeed, from (77) $-c(q^p) + \beta\kappa_h d^p + \omega^p > 0$. Since the solution to (24)-(26) is such the seller's participation constraint and the $\ell-$type buyer incentive-compatibility constraint hold, $\bar{U}_h^b > U_h^b$ implies that the incentive-compatibility constraint is violated, and hence (78) holds. Finally, the pooling equilibrium (q^p, d^p, ω^p) is undefeated among pooling equilibria because the payoff of the $h-$type is maximized. Hence, any out-of-equilibrium offer that would correspond to a different pooling equilibrium can be attributed to an $\ell-$type buyer (so that the payoff of the $\ell-$type buyer associated with this out-of-equilibrium offer would be no greater than $U(\kappa_\ell)$). ■

Define the *lexicographically maximum* (Lex Max) pooling allocation as the solution to (76)-(77).[43] (Notice that the Lex Max pooling allocation corresponds to a sequential equilibrium if the payoff of the $\ell-$type buyer is at least equal to his complete information payoff.) The *lexicographically maximum sequential equilibrium* (LMSE) corresponds to the Lex Max pooling allocation if $\bar{U}_h^b \geq U_h^b$, and to the separating equilibrium given by the Intuitive Criterion otherwise.[44] The LMSE is intuitively appealing because it corresponds to the preferred sequential equilibrium of an $h-$type buyer. Lemma 8 shows that the LMSE is undefeated (Mailath

[43] Consider two sequential equilibria with the associated profile of payoffs for the buyers (u_h, u_ℓ) and (u'_h, u'_ℓ). The first equilibrium lexicographically dominates the second one if $u_h > u'_h$ or $u_h = u'_h$ and $u_\ell > u'_\ell$.

[44] A belief system consistent with the pooling outcome is as follows. For the equilibrium offer, the Bayes rule implies $\lambda(q^p, d^p, \omega^p) = \pi_h$. For all out-of-equilibrium offers that generate a payoff to $\ell-$type buyers greater than $U(\kappa_\ell)$ then $\lambda = 0$. For other out-of-equilibrium offers, $\lambda = \pi_h$.

et al., 1993, Theorem 1) and if it is completely separating, it is the only undefeated (pure strategy) sequential equilibrium (Mailath *et al.*, 1993, Theorem 2).

The determination of undefeated equilibria is illustrated in Figure 9 in the case without fiat money. The separating equilibrium is such that the $\ell-$type buyer gets his complete information payoff (the indifference curves U_ℓ^s and U_ℓ^b are tangent) and the trade in a match with an $h-$type buyers is such that both the participation constraint of the seller and the incentive-compatibility condition for the $\ell-$type bind (U_h^s and U_ℓ^b intersect). The Lex Max pooling allocation is at the tangency point between the indifference curve of the seller given his prior belief, $\bar{U}^s$, and the indifference curve of an $h-$type buyer. If the utility of the $h-$type buyer at separating equilibrium is greater than his utility at the pooling allocation, as in Figure 9, then the separating equilibrium is undefeated.

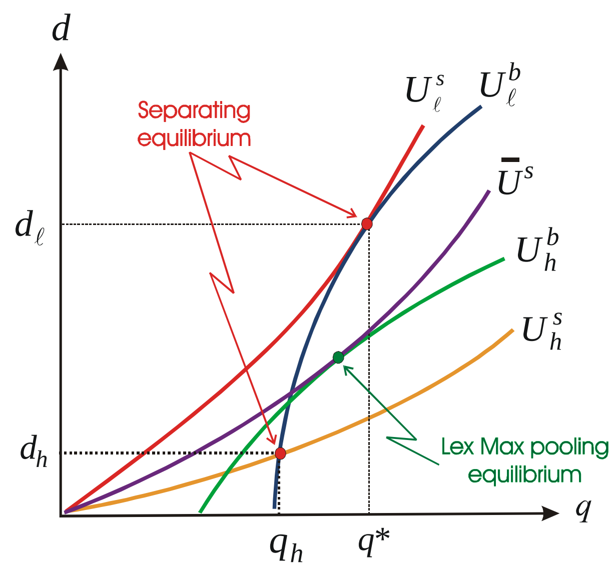

Figure 9: Undefeated equilibria

The next Lemma characterizes the Lex Max pooling allocation. Let $q_m(i)$ denote the value of q that solves $u'(q)/c'(q) = 1 + i$.

Lemma 9 *The Lex Max pooling allocation* (q, d, ω) *that solves (76)-(77) is such that:*

1. *If* $i < \kappa_h/\bar{\kappa} - 1$ *then* $d = 0$, $q = q_m(i)$ *and* $\omega = c(q)$.

2. *If* $i = \kappa_h/\bar{\kappa} - 1$ *then* $q_m(i)$ *and* $\omega + \beta\bar{\kappa}d = c(q)$.

3. *If* $i > \kappa_h/\bar{\kappa} - 1$ *then:*

(a) *If* $u'\left[c^{-1}(\beta\bar{\kappa}A)\right]/c'\left[c^{-1}(\beta\bar{\kappa}A)\right] \geq 1+i$ *then* $d = A$, $q = q_m(i)$ *and* $\omega = c(q) - \beta\bar{\kappa}A$.

(b) *If* $u'\left[c^{-1}(\beta\bar{\kappa}A)\right]/c'\left[c^{-1}(\beta\bar{\kappa}A)\right] \leq \kappa_h/\bar{\kappa}$ *then* $u'(q) = \frac{\kappa_h}{\bar{\kappa}}c'(q)$, $\omega = 0$ *and* $d = c(q)/\beta\bar{\kappa}$.

(c) *If* $u'\left[c^{-1}(\beta\bar{\kappa}A)\right]/c'\left[c^{-1}(\beta\bar{\kappa}A)\right] \in (\kappa_h/\bar{\kappa}, 1+i)$ *then* $q = c^{-1}(\beta\bar{\kappa}A)$, $\omega = 0$ *and* $d = A$.

Proof. Assume the constraint $\omega \geq 0$ is not binding and substitute ω given by (77) into (76) to get

$$\bar{U}_h^b = \max_{q, 0 \leq d \leq A} \{u(q) - (1+i)c(q) + \beta d\left[(1+i)\bar{\kappa} - \kappa_h\right]\}$$

where $\bar{\kappa} = \pi_h\kappa_h + \pi_\ell\kappa_\ell$. If $i < \kappa_h/\bar{\kappa} - 1$ then $d = 0$, $u'(q)/c'(q) = 1+i$ and $\omega = c(q)$. This gives case 1 in the Lemma. If $i = \kappa_h/\bar{\kappa} - 1$ then $u'(q)/c'(q) = 1+i$ and $\omega + \beta\bar{\kappa}d = c(q)$ (the exact combination of ω and d is indeterminate.) This gives case 2 in the Lemma. If $i > \kappa_h/\bar{\kappa} - 1$ then $d = A$, $u'(q)/c'(q) = 1+i$ and $\omega = c(q) - \beta\bar{\kappa}A$ provided that $u'(q)/c'(q) \geq 1+i$ at $q = c^{-1}(\beta\bar{\kappa}A)$, which guarantees that $\omega \geq 0$. This gives case 3(a) in the Lemma. Assume next that $\omega \geq 0$ is binding. Then, from (76)-(77),

$$\bar{U}_h^b = \max_{0 \leq d \leq A} \left[u\left[c^{-1}(\beta\bar{\kappa}d)\right] - \beta\kappa_h d\right].$$

If $d \leq A$ does not bind then $u'(q) = \frac{\kappa_h}{\bar{\kappa}}c'(q)$ and $d = c(q)/\beta\bar{\kappa}$. The constraint $d \leq A$ can then be reexpressed as $c(q) \leq \beta\bar{\kappa}A$ or $u'\left[c^{-1}(\beta\bar{\kappa}A)\right]/c'\left[c^{-1}(\beta\bar{\kappa}A)\right] \leq \kappa_h/\bar{\kappa}$. If $d \leq A$ binds then $q = c^{-1}(\beta\bar{\kappa}A)$. The constraint $\omega = 0$ requires $u'(q)/c'(q) \leq 1+i$. If $c(q) \leq \beta\bar{\kappa}A$ then the condition holds if $i \geq \kappa_h/\bar{\kappa} - 1$. If $d \leq A$ binds then $u'(q)/c'(q) < 1+i$ must be evaluated at $q = c^{-1}(\beta\bar{\kappa}A)$. This gives cases 3(b) and 3(c) in the Lemma.

■

The pooling allocation is such that fiat money is the only means of payment provided that the cost of real balances is sufficiently low, $i < \kappa_h/\bar{\kappa} - 1$. Intuitively, if the two types of buyers are pooled in equilibrium then the $h-$type buyer incurs a cost equal to $(\kappa_h - \bar{\kappa})/\bar{\kappa}$ per unit of the real asset he sells while the expected cost of holding real balances is i. In contrast, in the separating equilibrium both types of buyers use the real asset as means of payment. The quantities traded in the pooling monetary equilibrium are the same as the ones that buyers would trade in a separating monetary equilibrium provided that both buyers hold money.

Proposition 8 *There exists* $\bar{\pi}_h \in (0,1)$ *such that for all* $\pi_h < \bar{\pi}_h$ *the only undefeated equilibrium is the (separating) equilibrium that satisfies the Intuitive Criterion whereas for all* $\pi_h > \bar{\pi}_h$ *any undefeated equilibrium is pooling.*

Proof. From (24)-(26) U_h^b, the utility of an $h-$type buyer at the separating equilibrium, is independent of π_h. From Lemma 9, $\bar{U}_h^b$, the utility of an $h-$type buyer at the Lex Max pooling allocation, is a continuous and (weakly) increasing function of π_h. It is strictly increasing provided that $i > \kappa_h/\bar{\kappa} - 1$, or equivalently, $\pi_h > \frac{\kappa_h-(1+i)\kappa_\ell}{(1+i)(\kappa_h-\kappa_\ell)}$, and it is constant otherwise.

If $\pi_h = 1$ then (q^p, d^p, ω^p) solves (24) subject to (25) without (26) being imposed. From Lemma 4, (26) is binding at the separating equilibrium. Hence, $\bar{U}_h^b > U_h^b$.

If $\pi_h = 0$ then the problem (76)-(77) is analogous to (22) except from the fact that the objective of the $\ell-$type buyer is replaced by the objective of the $h-$type buyer. Hence,

$$-(1+i)\omega^p + u(q^p) - \beta\kappa_\ell d^p \leq U(\kappa_\ell)$$

and (26) holds at (q^p, d^p, ω^p), with a strict inequality if $d^p = 0$ since $d_\ell > 0$. Moreover,

$$0 = -c(q^p) + \beta\kappa_\ell d^p + \omega^p \leq -c(q^p) + \beta\kappa_h d^p + \omega^p,$$

with a strict inequality if $d^p > 0$. From Lemma 4, both (25) and (26) are binding at the optimum so that $\bar{U}_h^b < U_h^b$. Consequently, there is a unique $\bar{\pi}_h \in (0,1)$ such that $\bar{U}_h^b = U_h^b$; for all $\pi_h > \bar{\pi}_h$, $\bar{U}_h^b > U_h^b$; for all $\pi_h < \bar{\pi}_h$, $\bar{U}_h^b < U_h^b$. ■

The equilibrium that satisfies the Intuitive Criterion is the unique undefeated equilibrium provided that the fraction of $h-$type buyers is not too large. Intuitively, $h-$type buyers face a trade-off between the tightness of the seller's participation constraint and the tightness of the $\ell-$type buyer's incentive-compatibility constraint. If the fraction of $\ell-$type buyers is sufficiently small then $h-$type buyers cannot loosen much the seller's participation constraint by separating themselves from the $\ell-$type buyers, i.e., $\pi_\ell\kappa_\ell + \pi_h\kappa_h$ is close to κ_h. However, they would have to tighten significantly the incentive-compatibility condition to separate themselves from the $\ell-$type buyers. Hence, $h-$type buyers are happily pooled with $\ell-$type ones.

Next, I investigate how monetary policy affects the set of undefeated equilibria.

Proposition 9 *For all $i < \kappa_h/\bar{\kappa} - 1$, the unique undefeated equilibrium is the unique (separating) equilibrium that satisfies the Intuitive Criterion.*

Proof. Consider the pooling allocation in (76)-(77). From Lemma 9, for all $i < \kappa_h/\bar{\kappa} - 1$ then $d = 0$ and

$$\bar{U}_h^b(i) = -(1+i)c\left[q_m(i)\right] + u\left[q_m(i)\right].$$

Moreover, $d\bar{U}_h^b/di = -c\,[q_m(i)] < 0$. Consider next the Pareto-efficient separating equilibrium. For all $i < i_2(A)$, defined in the proof of Proposition 2,

$$U_h^b(i) = -(1+i)c\,[q_m(i)] + u\,[q_m(i)] + i\beta\kappa_h d_h(i),$$

and $U_h^b(i) = U_h^b[i_2(A)]$ for all $i > i_2(A)$. Since $d_h(i) > 0$ for all $i > 0$ then $U_h^b(i) > \bar{U}_h^b(i)$ for all $i < \kappa_h/\bar{\kappa} - 1$ and the unique undefeated equilibrium is the separating equilibrium. ■

Provided that the cost of holding real balances is not too high, the unique undefeated equilibrium is separating. Hence, the results in Section 5 are robust to the adoption of the alternative refinement from Mailath, Okuno-Fujiwara and Postlewaite (1993).

A second insights from Proposition 9 is that monetary policy can affect the nature of the (undefeated) equilibrium. Suppose that i is sufficiently large so that no monetary equilibrium exists. From Proposition 8, provided that π_h is sufficiently large, any undefeated equilibrium is pooling. Consider a reduction of the cost of holding real balances. From Proposition 9, provided that i is sufficiently low, the unique undefeated equilibrium is separating. Hence, the reduction of the money growth rate changes the nature of the equilibrium from pooling to separating.[45]

[45] A similar result is found in Jafarey and Rupert (2001) where the nonmonetary equilibrium is always pooling while the monetary equilibrium can be separating.

D. Signaling with more than two types

This appendix shows the robustness of the results if one allows for more than two states for the dividend of the real asset. In particular, there is a unique equilibrium of the bargaining game and it is separating. I consider the version of the model in Section 5 where the real asset is only traded in the PM. I describe the case where the set of buyers' is $T = \{1, ..., K\}$ with $K \geq 3$ and K is finite. The dividend in the state k is κ_k with $0 < \kappa_1 < \kappa_2 < ... < \kappa_K$. The measure of buyers of type k is π_k.

In the PM the seller observes the offer (q, d, ω) made by the buyer, and he forms a belief about the buyer's type. Denote $\lambda(q, d, \omega)$ the probability measure of the buyer's type κ conditional on the offer (q, d, ω) being made. Formally, $\Pr[k \in S] = \int \mathbb{I}_S d\lambda(q, d, \omega)$ where $\mathbb{I}_S(k)$ is an indicator function that is equal to 1 if $k \in S$.

Consider a sequential equilibrium and let U_k^b denote the payoff of a buyer of type κ_k. The proposed equilibrium fails the Intuitive Criterion (Cho and Kreps, 1987, p.202) if there is an unsent offer $(\tilde{q}, \tilde{d}, \tilde{\omega})$ and a set of types $S \subset T$ such that

$$-(1+i)\tilde{\omega} + u(\tilde{q}) - \beta\kappa_k\tilde{d} \quad < \quad U_k^b \quad \forall k \in S \tag{79}$$

$$-(1+i)\tilde{\omega} + u(\tilde{q}) - \beta\kappa_k\tilde{d} \quad > \quad U_k^b \quad \text{for some } k \in T\backslash S \tag{80}$$

$$-c(\tilde{q}) + \tilde{\omega} + \mathbb{E}_{\tilde{\lambda}}[\kappa_k]\beta\tilde{d} \quad \geq \quad 0 \quad \forall\tilde{\lambda} : \text{supp}(\tilde{\lambda}) \subseteq T\backslash S. \tag{81}$$

According to (79), the unsent offer makes buyers with types included in S strictly worse off compared to their equilibrium payoff. According to (80), it makes at least one buyer with type included in $T\backslash S$ strictly better off. According to (81) the offer is acceptable for any belief system that puts no weight on the types in S.

The next lemma shows that there is no offer involving a transfer of the real asset that is pooling.

Lemma 10 *In any equilibrium, there is no pooling offer such that $d > 0$.*

Proof. Suppose there is an equilibrium where a subset $\bar{T} \subseteq T$ of buyers' types (with at least two distinct elements) make the same offer $(\bar{q}, \bar{d}, \bar{\omega})$ with $\bar{d} > 0$. Hence, the equilibrium payoffs are

$$U_k^b \equiv -(1+i)\bar{\omega} + u(\bar{q}) - \beta\kappa_k\bar{d}, \quad \forall k \in \bar{T}.$$

This offer satisfies the seller's participation constraint. Hence,

$$-c(\bar{q}) + \mathbb{E}_{\lambda(\bar{q},\bar{d},\bar{\omega})}[\kappa]\beta\bar{d} + \bar{\omega} \geq 0, \tag{82}$$

where $\lambda(\bar{q},\bar{d},\bar{\omega})$ is determined by Bayes' rule.[46] Let $k=\max\bar{T}$ and $k'=\max\bar{T}\backslash\{k\}$. Suppose that a $k-$type buyer deviates and offers $(\tilde{q},\tilde{d},\tilde{\omega})$ such that $\tilde{\omega}=\bar{\omega}$, $\tilde{d}=\bar{d}-\varepsilon$ where $\varepsilon\in\left(0,\bar{d}+\left[\bar{\omega}-c(\bar{q})\right]/\beta\kappa_k\right)$, and

$$-(1+i)\tilde{\omega}+u\left(\tilde{q}\right)-\beta\kappa_{k'}\tilde{d} \quad < \quad U^b_{k'} \tag{83}$$

$$-(1+i)\tilde{\omega}+u\left(\tilde{q}\right)-\beta\kappa_{k}\tilde{d} \quad > \quad U^b_{k}, \tag{84}$$

or, equivalently,

$$\kappa_{k'}<\frac{u\left(\bar{q}\right)-u\left(\tilde{q}\right)}{\beta\varepsilon}<\kappa_k. \tag{85}$$

First, I establish that the set of offers $(\tilde{q},\tilde{d},\tilde{\omega})$ that satisfy the conditions above is not empty. Since, from Bayes' rule $\mathbb{E}_{\lambda(\bar{q},\bar{d},\bar{\omega})}\left[\kappa\right]<\kappa_k$, (82) implies $c(\bar{q})<\beta\kappa_k\bar{d}+\bar{\omega}$ and $\left(0,\bar{d}+\left[\bar{\omega}-c(\bar{q})\right]/\beta\kappa_k\right)$ is non-empty. The requirement $U^b_k\geq 0$ (in any equilibrium the buyers' payoffs are nonnegative since $(q,d,\omega)=(0,0,0)$ is always available) implies $u\left(\bar{q}\right)-\beta\kappa_k\bar{d}-\bar{\omega}\geq 0$ and hence

$$\frac{u\left(\bar{q}\right)}{\beta\kappa_k}-\varepsilon\geq\bar{d}+\frac{\bar{\omega}}{\beta\kappa_k}-\varepsilon>0$$

(since $\varepsilon<\bar{d}+\left[\bar{\omega}-c(\bar{q})\right]/\beta\kappa_k$.) So for any $\varepsilon\in\left(0,\bar{d}+\left[\bar{\omega}-c(\bar{q})\right]/\beta\kappa_k\right)$ there is a $\tilde{q}\geq 0$ satisfying

$$u\left(\bar{q}\right)-\beta\varepsilon\kappa_k<u\left(\tilde{q}\right)<u\left(\bar{q}\right)-\beta\varepsilon\kappa_{k'}$$

and hence (85).

Second, I show that

$$-(1+i)\tilde{\omega}+u\left(\tilde{q}\right)-\beta\kappa_{k''}\tilde{d}<U^b_{k''}\quad\forall k''<k'. \tag{86}$$

By incentive compatibility,

$$U^b_{k''}\geq-(1+i)\bar{\omega}+u\left(\bar{q}\right)-\beta\kappa_{k''}\bar{d}.$$

From (85), $u\left(\bar{q}\right)>u\left(\tilde{q}\right)+\beta\varepsilon\kappa_k>u(\tilde{q})+\beta\varepsilon\kappa_{k''}$. Since $\tilde{\omega}=\bar{\omega}$ and $\tilde{d}=\bar{d}-\varepsilon$,

$$-(1+i)\tilde{\omega}+u\left(\tilde{q}\right)-\beta\kappa_{k''}\tilde{d}<-(1+i)\bar{\omega}+u\left(\bar{q}\right)-\beta\kappa_{k''}\bar{d}\leq U^b_{k''}.$$

This proves (86).

Finally, I show that any offer $(\tilde{q},\tilde{d},\tilde{\omega})$ disqualifies the proposed equilibrium according to the Intuitive Criterion. From (83) and (86), the set of types S such that (79) is true is $S\supseteq\{\kappa_{k''}:k''\leq k'\}$. From

[46] In any equilibrium buyers make an acceptable offer. Indeed, their payoffs are bounded away from 0 since they can always achieve $\max_{q,\omega}\left[-(1+i)\omega+u\left(q\right)\right]$ s.t. $c(q)=\omega$.

(84), the condition (80) is satisfied for the buyer's type k. The condition $\varepsilon < \bar{d} + [\bar{\omega} - c(\bar{q})] / \beta\kappa_k$ implies $c(\bar{q}) < \beta\kappa_k \left(\bar{d} - \varepsilon\right) + \bar{\omega}$. From (85), $\bar{q} > \tilde{q}$. So,

$$c\left(\tilde{q}\right) < \beta\kappa_k\tilde{d} + \bar{\omega} \leq \beta\mathbb{E}_{\tilde{\lambda}(\tilde{q},\tilde{d},\tilde{\omega})}\left[\kappa\right]\tilde{d} + \bar{\omega}$$

for any $\tilde{\lambda}$ such that supp$\tilde{\lambda}(\tilde{q},\tilde{d},\tilde{\omega}) \subseteq T \backslash S$, since $T \backslash S \subseteq \{k, k+1, ..., K\}$. Consequently, (81) is also satisfied. ■

A buyer reveals his type through his equilibrium offer (unless $d = 0$ in which case the buyer's private information is irrelevant for the seller's payoff). Let (q_k, ω_k, d_k) indicate the offer made by a buyer of type k. The next Lemma shows that the quantity of the asset transferred to the seller is a non-increasing function of the buyer's type. The proof of this result is based on incentive compatibility.

Lemma 11 *If $k > k'$ then $d_k \leq d_{k'}$.*

Proof. Incentive-compatibility requires that for any two types k and k',

$$U_k^b = -(1+i)\omega_k + u\left(q_k\right) - \beta\kappa_k d_k \geq -(1+i)\omega_{k'} + u\left(q_{k'}\right) - \beta\kappa_k d_{k'}. \tag{87}$$

From (87), and interchanging k and k', one can show that

$$\kappa_{k'}\left(d_k - d_{k'}\right) \geq \kappa_k\left(d_k - d_{k'}\right).$$

Hence, if $k > k'$ then $d_k \leq d_{k'}$. ■

So, the result according to which buyers with a high type spend a smaller fraction of their real asset in the PM than buyers with a low type is robust across mechanisms. The next proposition characterizes the buyers' equilibrium offers and payoffs.

Proposition 10 *The equilibrium offer of a κ_1-buyer is the solution to*

$$U_1^b = \max_{(q,d,\omega)} \left\{-(1+i)\omega + u\left(q\right) - \beta\kappa_1 d\right\} \tag{88}$$

$$s.t. \quad -c(q) + \omega + \beta\kappa_1 d = 0. \tag{89}$$

The equilibrium offer of a κ_k-buyer, for $k \in \{2, ..., K\}$, solves

$$U_k^b = \max_{(q,d,\omega)} \left\{-(1+i)\omega + u\left(q\right) - \beta\kappa_k d\right\} \tag{90}$$

$$s.t. \quad -c(q) + \omega + \beta\kappa_k d \geq 0 \tag{91}$$

$$-(1+i)\omega + u\left(q\right) - \beta\kappa_{k-1} d \leq U_{k-1}^b. \tag{92}$$

Proof. The offers $\{(q_k, d_k, \omega_k)\}_{k=1}^{K}$ must be incentive-compatible, i.e.,

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k'}d_k \leq U_{k'}^b \tag{93}$$

for all $k, k' \in T$. First, I establish that (92) implies that the incentive-compatibility condition (93) holds for all $k' < k$ and $k \geq 2$. From (92),

$$U_k^b \equiv -(1+i)\omega_k + u(q_k) - \beta\kappa_k d_k \leq U_{k-1}^b - \beta d_k(\kappa_k - \kappa_{k-1}),$$

and, by successive iterations,

$$U_k^b \leq U_{k-n}^b - \sum_{t=k-n+1}^{k} \beta d_t(\kappa_t - \kappa_{t-1}), \qquad \forall n = 1, .., k-1.$$

By definition of U_k^b,

$$\begin{aligned} -(1+i)\omega_k + u(q_k) - \beta\kappa_{k-n}d_k &= U_k^b - \beta\kappa_{k-n}d_k + \beta\kappa_k d_k \\ &\leq U_{k-n}^b - \sum_{t=k-n+1}^{k} \beta d_t(\kappa_t - \kappa_{t-1}) - \beta\kappa_{k-n}d_k + \beta\kappa_k d_k. \end{aligned}$$

Rearrange the sum on the right-hand side, and use an appropriate change of variable to obtain

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k-n}d_k \leq U_{k-n}^b + \sum_{t=k-n+1}^{k} \beta(d_t - d_{t-1})\kappa_{t-1} - \beta\kappa_{k-n}(d_k - d_{k-n}).$$

Using the fact that $d_k - d_{k-n} = \sum_{t=k-n+1}^{k}(d_t - d_{t-1})$, the inequality becomes

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k-n}d_k \leq U_{k-n}^b + \sum_{t=k-n+1}^{k} \beta(d_t - d_{t-1})(\kappa_{t-1} - \kappa_{k-n}).$$

From Lemma 11, $(d_t - d_{t-1})(\kappa_{t-1} - \kappa_{k-n}) \leq 0$ for all $t \in \{k-n+1, ..., k\}$. Hence, $-(1+i)\omega_k + u(q_k) - \beta\kappa_{k-n}d_k \leq U_{k-n}^b$ for all $n = 1, .., k-1$.

Next, I prove that the incentive-compatibility condition (93) holds for all $k' > k$. The proof is by induction. Suppose

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k+n}d_k \leq U_{k+n} \tag{94}$$

for $n > 0$. This inequality holds at $n = 0$. From (91),

$$-c(q_k) + \omega_k + \beta\kappa_{k+n+1}d_k > -c(q_k) + \omega_k + \beta\kappa_k d_k \geq 0. \tag{95}$$

From (94) and (95), (q_k, d_k, ω_k) satisfies (91)-(92) when k is replaced by $k+n+1$. Hence, one can deduce from (90) that

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k+n+1}d_k \leq U^b_{k+n+1}.$$

So, I have checked that the incentive-compatibility conditions hold for all k, k'. To show that (q_1, ω_1, d_1) is the only possible offer for a κ_1-buyer notice from Lemma 10 that a buyer with the lowest type cannot do better than his complete information payoff and this payoff is always achievable for any belief system λ (since $\mathbb{E}_\lambda[\kappa] \geq \kappa_1$).

Next, I show that the solution (q_k, d_k, ω_k) to (90)-(92) for $k \geq 2$ is the only one that can satisfy the Intuitive Criterion. Suppose that the equilibrium payoff of a $k-$buyer is $\tilde{U}_k < U^b_k$. Replace U^b_{k-1} in (92) by $U^b_{k-1} - \varepsilon$ for some $\varepsilon > 0$ and denote

$$(q^\varepsilon_k, d^\varepsilon_k, \omega^\varepsilon_k) = \arg\max_{(q,d,\omega)} \{-(1+i)\omega + u(q) - \beta\kappa_k d\}$$

$$\text{s.t.} \quad -c(q) + \omega + \beta\kappa_k d \geq 0$$

$$-(1+i)\omega + u(q) - \beta\kappa_{k-1}d \leq U^b_{k-1} - \varepsilon.$$

Denote U^ε_k the value to this problem. Since U^ε_k is continuous in ε there exists $\varepsilon > 0$ such that $\tilde{U}_k < U^\varepsilon_k$. By an argument analogous to the one above,

$$-(1+i)\omega^\varepsilon_k + u(q^\varepsilon_k) - \beta\kappa_{k-n}d^\varepsilon_k < U^b_{k-n}, \quad \forall n = 1, .., k-1.$$

Moreover, if the support of the seller's belief is restricted to $\{k, .., K\}$ then $\mathbb{E}_\lambda[\kappa] \geq \kappa_k$ and, from (91), $(q^\varepsilon_k, d^\varepsilon_k, \omega^\varepsilon_k)$ is acceptable. So the proposed equilibrium violates the Intuitive Criterion.

Finally, I construct a belief system that generates $\{(q_k, d_k, \omega_k)\}_{k=1}^K$ as the solution to the buyers' problems. The belief system is such that $\int \mathbb{I}_{\{k\}} d\lambda(q_k, d_k, \omega_k) = 1$ from Bayes' rule. The beliefs for out-of-equilibrium offers are specified as follows. All out-of-equilibrium offers that generate a payoff strictly greater than U^b_1 to a type-1 buyer are attributed to a κ_1-buyer. By construction these offers are rejected by sellers. Among the remaining offers, all out-of-equilibrium offers that generate a payoff strictly greater than U^b_2 to a type-2 buyer are attributed to a type-2 buyer. These offers satisfy (92) for $k = 2$. Hence, they must violate (91) and, as a consequence, they are rejected by sellers. And so on. The out-of-equilibrium offers that make all players worse-off are attributed to the highest type. ■

The buyer with the lowest type makes his complete information offer. Buyers of type $k > 1$ maximize their expected utility subject to the participation constraint of sellers and the incentive-compatibility condition of buyers of type $k-1$.

The problem (90)-(92) is formally identical to (24)-(26). One can make use of Lemma 4 to provide the following characterization of the equilibrium offers.

Lemma 12 *For all $i > 0$ and all $k \in \{2, ..., K\}$, there is a unique solution (q_k, d_k, ω_k) to (90)-(92) and it solves:*

$$\omega_k = \frac{\kappa_k \left\{ \left[u(q_k) - \frac{\kappa_\ell}{\kappa_h} c(q_k) \right] - U_{k-1}^b \right\}}{(1+i)\kappa_k - \kappa_{k-1}} \tag{96}$$

$$d_k = \frac{U_{k-1}^b - [u(q_k) - (1+i)\, c(q_k)]}{[(1+i)\kappa_k - \kappa_{k-1}]\, \beta} \tag{97}$$

and

$$u'(q_k) - (1+i)\, c'(q_k) \leq 0 \quad \text{“} = \text{''} \quad \text{if } \omega_k > 0. \tag{98}$$

Both the seller's participation constraint (91) and the buyer's incentive-compatibility condition (92) are binding.

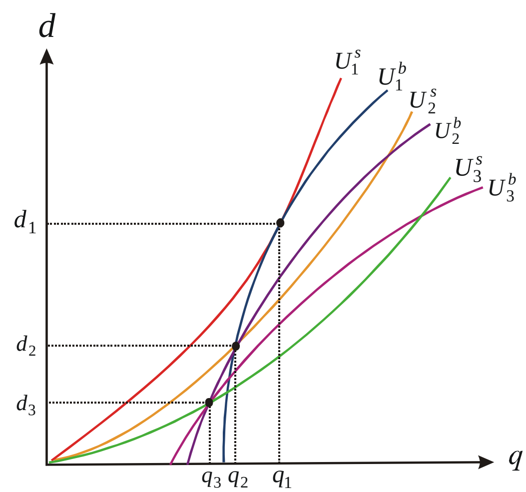

Figure 10: Equilibrium offers (economy without fiat money)

The determination of the equilibrium offers is illustrated in Figure 10 in the case without fiat money. The indifference curve of a seller matched with a buyer of type k is denoted U_k^s. The indifference curve of a buyer of type k is denoted U^b_k. The sellers' indifference curves go through the origin since their participation constraints are binding. Assuming the constraint $d \leq A$ is not binding, the terms of trade (q_1, d_1) are determined at the tangency of the seller's indifference curve and the buyer's indifference curve. In matches where the buyer's type is $k > 1$ then the terms of trade (q_k, d_k) are at the intersection of the seller's indifference curve, U^s_k, and the indifference curve of the buyer of type $k-1$, U^b_{k-1}.

E. Asset pricing under symmetric information

In order to isolate the role of private information, I analyze in this Appendix the economy with symmetric information: buyers and sellers in the PM have the same information about the future dividend of the asset.

Complete information

Consider a match in the PM between a buyer and a seller. The buyer holds ω real balances (expressed in terms of their discounted value in the next AM) and a units of capital. The future dividend of each unit of capital is κ and it is common knowledge in the match. The strategy of a buyer in the PM is a triple $[q(\omega, a, \kappa), d(\omega, a, \kappa), \tau(\omega, a, \kappa)]$ solution to

$$[q(\omega, a, \kappa), d(\omega, a, \kappa), \tau(\omega, a, \kappa)] = \arg\max_{q,\tau,d} \left[u(q) - \beta\kappa d - \frac{\beta}{\gamma}\tau \right] \tag{99}$$

$$\text{s.t.} \quad -c(q) + \beta\kappa d + \frac{\beta}{\gamma}\tau \geq 0, \tag{100}$$

$$\text{s.t.} \quad \frac{\beta}{\gamma}\tau \leq \omega, \quad d \leq a. \tag{101}$$

If $\beta\kappa a + \omega \geq c(q^*)$ then the solution to (99)-(101) is $q = q^*$ and $\beta\kappa d + \omega = c(q^*)$. Otherwise, $q = c^{-1}(\beta\kappa a + \omega)$, $d = \kappa a$ and $\beta\tau/\gamma = \omega$. Let denote $\hat{\mathcal{S}}\left(\beta\kappa a + \omega\right) = u(q) - c(q)$ the buyer's surplus where $q = \min\left[q^*, c^{-1}(\beta\kappa a + \omega)\right]$.

The buyer's portfolio decision in the AM is:

$$\max_{\omega \geq 0, a \geq 0} \left\{ -i\omega - \left(\phi - \beta\bar{\kappa}\right) a + \pi_h \hat{\mathcal{S}}(\omega + \beta\kappa_h a) + \pi_\ell \hat{\mathcal{S}}(\omega + \beta\kappa_\ell a) \right\}, \tag{102}$$

where $\phi - \beta\bar{\kappa}$ in (102) represents the cost of investing in capital: it is equal to the price of capital in the AM minus the discounted expected dividend in the subsequent AM.

The clearing condition for the market for capital goods requires the fixed capital stock to be held by buyers,

$$\int_{j\in\mathcal{B}} a(j)dj = A. \tag{103}$$

(Recall that sellers cannot produce in the AM and hence cannot acquire capital.)

An equilibrium is a list of portfolios, a profile of buyers' strategies in the PM, and the price of capital that satisfy (99)-(103).

The following lemma characterizes the buyer's portfolio choice.

Lemma 13 *Assume $\phi > \beta\bar{\kappa}$. If $(\phi - \beta\bar{\kappa})/\beta\kappa_\ell \neq i$ or $\pi_\ell \hat{\mathcal{S}}'[\kappa_\ell c(q^*)/\kappa_h] < i$ then the buyer's problem (102) admits a unique solution. It satisfies*

$$-i + \pi_h \hat{\mathcal{S}}'(\omega + \beta\kappa_h a) + \pi_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) \quad \leq \quad 0 \quad \text{“} = \text{”} \quad \textit{if } \omega > 0 \tag{104}$$

$$-(\phi - \beta\bar{\kappa}) + \pi_h \beta\kappa_h \hat{\mathcal{S}}'(\omega + \beta\kappa_h a) + \pi_\ell \beta\kappa_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) \quad \leq \quad 0 \quad \text{“} = \text{”} \quad \textit{if } a > 0. \tag{105}$$

If $(\phi - \beta\bar{\kappa})/\beta\kappa_\ell = i$ and $\pi_\ell \hat{\mathcal{S}}'[\kappa_\ell c(q^*)/\kappa_h] \geq i$ then any (ω, a) such that $\pi_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = i$ and $\omega + \beta\kappa_h a \geq c(q^*)$ is solution to (102). If $\phi = \beta\bar{\kappa}$ then any $(\omega, a) \in \{0\} \times [c(q^*)/\beta\kappa_\ell, \infty)$ is solution to (102). Finally, if $\phi < \beta\bar{\kappa}$ then there is no solution to (102).

Proof. Since $\hat{\mathcal{S}}'(\omega + \beta\kappa a) = u'(q)/c'(q) - 1$, where $q = \min\left[q^*, c^{-1}(\omega + \beta\kappa a)\right]$, then $\hat{\mathcal{S}}$ is concave and the buyer's problem (102) is concave as well. The first-order conditions (104) and (105) are then necessary and sufficient. Three cases are distinguished.

(i) $\phi > \beta\bar{\kappa}$. The solution to (102) cannot be such that $\omega + \beta\kappa_\ell a > c(q^*)$. Indeed, if $\omega + \beta\kappa_\ell a > c(q^*)$ then $\hat{\mathcal{S}}'(\omega + \beta\kappa_h a) = \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = u'(q^*)/c'(q^*) - 1 = 0$. But then (104)-(105) imply $\omega = a = 0$. A contradiction. So, one can restrict (ω, a) to the compact set $\{(\omega, a) \in \mathbb{R}_{2+} : \omega + \beta\kappa_\ell a \leq c(q^*)\}$ and from the Theorem of the Maximum a solution to (102) exists. Next, I show that the problem (102) is strictly jointly concave for all (ω, a) such that $\omega + \beta\kappa_h a < c(q^*)$. The Hessian matrix associated with (102) is

$$\mathbb{H} = \begin{pmatrix} \pi_h \hat{\mathcal{S}}''_h + \pi_\ell \hat{\mathcal{S}}''_\ell & \pi_h \beta\kappa_h \hat{\mathcal{S}}''_h + \pi_\ell \beta\kappa_\ell \hat{\mathcal{S}}''_\ell \\ \pi_h \beta\kappa_h \hat{\mathcal{S}}''_h + \pi_\ell \beta\kappa_\ell \hat{\mathcal{S}}''_\ell & \pi_h (\beta\kappa_h)^2 \hat{\mathcal{S}}''_h + \pi_\ell (\beta\kappa_\ell)^2 \hat{\mathcal{S}}''_\ell \end{pmatrix}$$

where $\hat{\mathcal{S}}''_h \equiv \hat{\mathcal{S}}''(\omega + \beta\kappa_h a)$ and $\hat{\mathcal{S}}''_\ell \equiv \hat{\mathcal{S}}''(\omega + \beta\kappa_\ell a)$. For all (ω, a) such that $\beta\kappa_h a + \omega \leq c(q^*)$, $\hat{\mathcal{S}}''_\ell < 0$ and $\hat{\mathcal{S}}''_h < 0$ and

$$|\mathbb{H}| = (\kappa_h - \kappa_\ell)^2 \pi_h \pi_\ell \beta^2 \hat{\mathcal{S}}''_\ell \hat{\mathcal{S}}''_h < 0.$$

So, $\mathbb{H}$ is negative definite and any solution to (104)-(105) such that $\omega + \beta\kappa_h a < c(q^*)$ corresponds to a strict local maximum and hence, from the concavity of the objective, it corresponds to the global maximum.

Suppose next that the solution is such that $\omega + \beta\kappa_h a \geq c(q^*)$. From (104)-(105),

$$\pi_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = \min\left[i, \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}\right],$$

with $\omega = 0$ if $i > \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}$ and $a = 0$ if $i < \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}$. So the solution is unique provided that $i \neq \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}$. If $i = \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}$ then any pair (ω, a) such that $\omega + \beta\kappa_h a \geq c(q^*)$ and $\pi_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = i$ is solution to (102). For such pairs to exist, $\pi_\ell \hat{\mathcal{S}}'\left[\frac{\kappa_\ell}{\kappa_h} c(q^*)\right] \geq i$.

(ii) $\phi = \beta\bar{\kappa}$. The first-order condition for a requires $\hat{\mathcal{S}}'(\omega + \beta\kappa_h a) = \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = 0$ and hence any $a \geq [c(q^*) - \omega] / \beta\kappa_\ell$ is part of a solution. But then (104) implies $\omega = 0$.

(iii) $\phi < \beta\bar{\kappa}$ then the first-order condition for a admits no solution. ■

The following proposition establishes the existence of an equilibrium and the uniqueness of the price of capital and the allocation of the PM output conditional on the realization of κ.[47] Denote q_ℓ and q_h the output levels in the PM when $\kappa = \kappa_\ell$ and $\kappa = \kappa_h$, respectively.

Proposition 11 *There exists an equilibrium and (ϕ, q_ℓ, q_h) is uniquely determined. If $A \geq c(q^*)/\beta\kappa_\ell$ then the equilibrium is nonmonetary and $\phi = \beta\bar{\kappa}$. If $A < c(q^*)/\beta\kappa_\ell$ then $\phi > \beta\bar{\kappa}$ and there is $i_0 > 0$ such that for all $i < i_0$ an equilibrium is monetary.*

Proof. The proof proceeds in four parts. It first characterizes the correspondence

$$A^d \equiv \left\{ \int_{j \in \mathcal{B}} a(j)dj : a(j) \text{ solution to } (102) \right\}.$$

Second, it shows that ϕ is uniquely determined. Third, it establishes that the PM output levels, q_ℓ and q_h, are unique. Finally, it determines the conditions fiat money to be valued.

(i) Existence. Consider first the case $\phi > \beta\bar{\kappa}$. As shown in the proof of Lemma 13 (Part (i)), any solution (ω, a) to (102) lies in the compact set $[0, c(q^*)] \times [0, c(q^*)/\beta\kappa_\ell]$. Since the objective in (102) is continuous, the Theorem of the Maximum guarantees that $A^d(\phi)$ is nonempty and upper-hemi continuous. Since the objective in (102) is concave, $A^d(\phi)$ is convex-valued. From Lemma 13, $A^d(\beta\bar{\kappa}) = [c(q^*)/\beta\kappa_\ell, \infty)$. From (104) and (105), it can be checked that $A^d(\phi) = \{0\}$ for all $\phi > \beta\bar{\kappa} + i\beta\kappa_h$ and $A_d(\phi) = \{a\}$ where a solves

$$\pi_h \beta\kappa_h \hat{\mathcal{S}}'(\beta\kappa_h a) + \pi_\ell \beta\kappa_\ell \hat{\mathcal{S}}'(\beta\kappa_\ell a) = \phi - \beta\bar{\kappa}$$

for all $\phi < \beta\bar{\kappa} + i\beta\kappa_\ell$ (since $\omega = 0$). Moreover, $a \to c(q^*)/\beta\kappa_\ell$ as $\phi \to \beta\bar{\kappa}$. Hence, from this characterization of $A^d(\phi)$, there is a $\phi \in [\beta\bar{\kappa}, \beta\bar{\kappa} + i\beta\kappa_\ell]$ such that $A \in A^d(\phi)$.

(ii) Uniqueness. In order to prove that ϕ is uniquely determined, I show that any selection from A^d is decreasing in ϕ: if $a_1 \in A^d(\phi_1)$ and $a_2 \in A^d(\phi_2)$ for $\phi_2 > \phi_1$ then $a_2 < a_1$ unless $a_2 = a_1 = 0$. Consider

[47] Buyers' portfolios are not always uniquely determined. If $(\phi - \beta\bar{\kappa}) / \beta\kappa_\ell = i$, and provided that $\hat{\mathcal{S}}'(\omega + \beta\kappa_h a) = 0$, real balances and capital are perfect substitutes. If $\phi = \beta\bar{\kappa}$ then buyers hold any quantity of capital above the level that satiates their liquidity needs in the PM, $\hat{\mathcal{S}}' = 0$, and they hold no real balances.

$\phi_2 > \phi_1$, and the associated portfolio choices (ω_1, a_1) and (ω_2, a_2). By revealed preferences,

$$\begin{aligned} -\phi_1 a_1 + \Psi(\omega_1, a_1) &\geq -\phi_1 a_2 + \Psi(\omega_2, a_2) \\ -\phi_2 a_2 + \Psi(\omega_2, a_2) &\geq -\phi_2 a_1 + \Psi(\omega_1, a_1), \end{aligned}$$

where $\Psi(\omega, a) \equiv -i\omega + \beta\bar{\kappa}a + \pi_h \hat{\mathcal{S}}(\omega + \beta\kappa_h a) + \pi_\ell \hat{\mathcal{S}}(\omega + \beta\kappa_\ell a)$. These last two inequalities yield

$$\phi_1(a_1 - a_2) \leq \Psi(\omega_1, a_1) - \Psi(\omega_2, a_2) \leq \phi_2 (a_1 - a_2).$$

Since $\phi_2 > \phi_1$ then $a_1 \geq a_2$. Suppose $a_1 = a_2 > 0$. From (105), $\omega_2 < \omega_1$ (where I have used that $\hat{\mathcal{S}}' < 0$ if $\omega + \beta\kappa a < c(q^*)$). But then, from (104), $\omega_1 = 0$. A contradiction.

(iii) The allocation (q_ℓ, q_h). From (i) and (ii), there exists a unique $\phi \geq \beta\bar{\kappa}$ such that $A \in A^d(\phi)$. If $A \geq c(q^*)/\beta\kappa_\ell$ then $\phi = \beta\bar{\kappa}$. Since $a \geq c(q^*)/\beta\kappa_\ell$ for all $a \in A^d(\beta\bar{\kappa})$ then $q_h = q_\ell = q^*$. If $A < c(q^*)/\beta\kappa_\ell$ then, from Lemma 13, (ω, a) is uniquely determined unless $(\phi - \beta\bar{\kappa})/\beta\kappa_\ell = i$ and $\pi_\ell \hat{\mathcal{S}}' [\kappa_\ell c(q^*)/\kappa_h] \geq i$ in which case $q_\ell = c^{-1}\left[\hat{\mathcal{S}}'^{-1}(i/\pi_\ell)\right]$ and $q_h = q^*$. (See proof of Lemma 13.) For given (ω, a) the problem (99)-(100) determines uniquely q_h and q_ℓ.

(iv) Suppose an equilibrium is nonmonetary. Then, $\omega(j) = 0$ and $a(j)$ is the unique solution to (105) for all $j \in \mathcal{B}$. Hence, $a(j) = A$. From (104) $\omega = 0$ requires

$$-i + \pi_h \hat{\mathcal{S}}'(\beta\kappa_h A) + \pi_\ell \hat{\mathcal{S}}'(\beta\kappa_\ell A) \leq 0.$$

Define $i_0 = \pi_h \hat{\mathcal{S}}'(\beta\kappa_h A) + \pi_\ell \hat{\mathcal{S}}'(\beta\kappa_\ell A)$. By the contrapositive, if $i < i_0$ then the equilibrium is monetary. Provided that $\beta\kappa_\ell A < c(q^*)$, $\hat{\mathcal{S}}'(\beta\kappa_\ell A) > 0$ and $i_0 > 0$. Finally, if $\beta\kappa_\ell A \geq c(q^*)$ then $\phi = \beta\bar{\kappa}$ and $\omega = 0$. ■

If the economy-wide capital stock is large enough to allow agents to trade q^* in the PM for the lowest realization of κ then fiat money is not valued.[48] If the aggregate capital stock is too low relative to agents' liquidity needs (in terms of means of payment) then the price of capital increases above its fundamental value and fiat money can be valued provided that i is sufficiently low.

The expression for the price of capital in equilibrium is obtained from (102) by taking the first order condition for a, i.e.,

$$\phi = \beta\bar{\kappa} + \pi_h \beta\kappa_h \left[\frac{u'(q_h)}{c'(q_h)} - 1\right] + \pi_\ell \beta\kappa_\ell \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right]. \tag{106}$$

[48] This result is in accordance with Lagos and Rocheteau (2006) who show that money is useful in the presence of capital in the Lagos-Wright environment if the first-best level of capital stock provides enough wealth for agents to trade in the PM, i.e., there is no shortage of capital to be used as means of payment.

It has two components. The first component is its fundamental value, $\beta\bar{\kappa}$. The second component is the liquidity value of capital in the PM, the last two terms on the right-hand side of (106). This liquidity value arises because capital can help relaxing buyers' budget constraint in a bilateral match.

To see how monetary policy can affect asset prices, take the first-order condition of (102) with respect to ω, and assume that the solution is interior (so that fiat money is valued),

$$i = \pi_h \left[\frac{u'(q_h)}{c'(q_h)} - 1 \right] + \pi_\ell \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1 \right]. \tag{107}$$

As the cost of holding fiat money increases the marginal liquidity of wealth in the PM increases, which in turn raises ϕ.

The next proposition compares the (gross) rates of return of money and capital, $R_m = \gamma^{-1}$ and $R_a = \bar{\kappa}/\phi$, respectively. Let ρ denote the covariance between the return of capital, κ, and the marginal return of wealth in the PM, $[u'(q)/c'(q) - 1]$, i.e.,

$$\rho = \pi_h (\kappa_h - \bar{\kappa}) \left[\frac{u'(q_h)}{c'(q_h)} - 1 \right] + \pi_\ell (\kappa_\ell - \bar{\kappa}) \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1 \right]. \tag{108}$$

Proposition 12 *In any monetary equilibrium,*

$$R_a = \gamma^{-1} \left\{ \frac{\rho}{\bar{\kappa}(1+i)} + 1 \right\}^{-1} > R_m. \tag{109}$$

Proof. The expression for ϕ given by (106) can be rearranged as

$$\phi = \beta \left\{ \bar{\kappa}(1+i) + \pi_h (\kappa_h - \bar{\kappa}) \left[\frac{u'(q_h)}{c'(q_h)} - 1 \right] + \pi_\ell (\kappa_\ell - \bar{\kappa}) \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1 \right] \right\}. \tag{110}$$

where I have used the fact that, from (104), $i = \left\{ \pi_h \left[\frac{u'(q_h)}{c'(q_h)} - 1 \right] + \pi_\ell \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1 \right] \right\}$ in any monetary equilibrium. Substitute ρ by its expression given by (108) into (110) to get

$$\phi = \beta(1+i)\bar{\kappa} \left\{ \frac{\rho}{\bar{\kappa}(1+i)} + 1 \right\}.$$

Divide by $\bar{\kappa}$ and use the definition $\gamma = \beta(1+i)$ to get (109). In order to show that $R_a > R_m$ it is enough to establish that $\rho < 0$. Notice that $\pi_h (\kappa_h - \bar{\kappa}) + \pi_\ell (\kappa_\ell - \bar{\kappa}) = 0$. From (104), and since $\hat{\mathcal{S}}''(\beta\kappa a + \omega) < 0$ whenever $\hat{\mathcal{S}}'(\beta\kappa a + \omega) > 0$, in any monetary equilibrium $0 \leq \hat{\mathcal{S}}'(\beta\kappa_h a + \omega) < \hat{\mathcal{S}}'(\beta\kappa_\ell a + \omega)$. Since $\frac{u'(q_h)}{c'(q_h)} < \frac{u'(q_\ell)}{c'(q_\ell)}$ then

$$\pi_h (\kappa_h - \bar{\kappa}) \left[\frac{u'(q_h)}{c'(q_h)} - 1 \right] < -\pi_\ell (\kappa_\ell - \bar{\kappa}) \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1 \right]$$

and $\rho < 0$. ■

Capital has a higher rate of return than fiat money in any monetary equilibrium. This result holds even though agents are risk-neutral with respect to their AM consumption. This rate-of-return differential arises because capital is used as a means of payment in the PM where individuals are risk-averse. Capital yields a high dividend in matches when the marginal value of wealth in the PM is low, and a low dividend in matches where the marginal value of wealth is high.[49] In contrast, the value of money is constant and uncorrelated with the marginal utility of wealth in the PM. Finally, as $\kappa_h - \kappa_\ell \to 0$ then $\rho \to 0$ and $R_a = \gamma^{-1}$, i.e., money and capital have the same rate of return.

Incomplete information

I now describe succinctly the case where both buyers and sellers are uninformed about the future value of κ. Buyers choose their portfolios in order to maximize $-i\omega - (\phi - \beta\bar{\kappa})\, a + \hat{\mathcal{S}}(\omega + \beta\bar{\kappa}a)$. If $A < c(q^*)/\beta\bar{\kappa}$ then $\phi > \beta\bar{\kappa}$ and there is $i_0 > 0$ such that for all $i < i_0$ an equilibrium is monetary. Moreover, if a monetary equilibrium exists then $\phi = \beta\bar{\kappa}(1+i)$ and $1+i = u'(q)/c'(q)$ where q is the quantity produced and consumed in bilateral matches in the PM. In this case, $R_a = R_m$, i.e., fiat money and capital have the same rate of return.

[49] This result is analogous to the one in Lagos (2006) who finds that even in the absence of legal restrictions on the use of assets as means of payment his model can be consistent with an equity-premium puzzle, i.e., a too large return differential between bonds and equity.

F. Endogenizing sellers' portfolio choices

This appendix considers the model in Section 6 where both the real asset and fiat money are traded in the AM, and it relaxes the assumption that sellers cannot produce in the AM (and hence cannot accumulate assets.) The utility function of a seller becomes

$$U_t^s = -\ell_t - c(q_t) + \beta x_{t+1}$$

where $\ell_t \in \mathbb{R}_+$ is the disutility of effort of the seller. The production technology in the AM is linear ($y_t = \ell_t$). I assume that both the buyer's and the seller's portfolios are common knowledge in a match in the PM, and I maintain the assumption that buyers have some private information about the future value of the dividend.

Consider first the bargaining problem in the PM. The buyer's portfolio is denoted by $\left(\omega^b, a^b\right)$ while the seller's portfolio is (ω^s, a^s). The offer made by a buyer with private information $\kappa_j \in \{\kappa_\ell, \kappa_h\}$ solves:

$$\max_{q,d,\tau} \left[u(q) - \beta\kappa_j d - \frac{\beta}{\gamma}\tau\right] \tag{111}$$

$$\text{s.t.} \quad -c(q) + \lambda(q,d,\tau)\beta\kappa_h d + [1-\lambda(q,d,\tau)]\,\beta\kappa_\ell d + \frac{\beta}{\gamma}\tau \geq 0 \tag{112}$$

$$-\omega^s \leq \frac{\beta}{\gamma}\tau \leq \omega^b, \quad -a^s \leq d \leq a^b. \tag{113}$$

The novelty with respect to the model in Section 6 is the constraint (113) according to which the seller holds some assets, which he can transfer to the buyer.

The following Lemma rules out pooling offers.

Lemma 14 *In equilibrium, there is no pooling offer such that $d \neq 0$.*

Proof. Suppose first that there is a pooling offer $(\bar{q}, \bar{d}, \bar{\tau})$ such that $\bar{d} > 0$. One can follow the proof of Lemma 5 to show that such an equilibrium would violate the Intuitive Criterion. Suppose next that the pooling offer is such that $\bar{d} < 0$, i.e., the seller transfers some of his real asset to the buyer. The participation constraint of the seller implies $\bar{\tau} > 0$. Since $\bar{d} < 0$ and $\lambda(\bar{q}, \bar{d}, \bar{\tau}) \in (0,1)$ then

$$-c(\bar{q}) + \lambda(\bar{q}, \bar{d}, \bar{\tau})\beta\kappa_h\bar{d} + \left[1-\lambda(\bar{q}, \bar{d}, \bar{\tau})\right]\beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau} < -c(\bar{q}) + \beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau}.$$

Hence, a necessary condition for the pooling offer $(\bar{q}, \bar{d}, \bar{\tau})$ to be acceptable is

$$-c(\bar{q}) + \beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau} > 0.$$

I will establish that there is an unsent offer $(\tilde{q}, \tilde{d}, \tilde{\tau})$ such that

$$u(\tilde{q}) - \beta\kappa_h\tilde{d} - \frac{\beta}{\gamma}\tilde{\tau} \quad < \quad \bar{S}^h \tag{114}$$

$$u(\tilde{q}) - \beta\kappa_\ell\tilde{d} - \frac{\beta}{\gamma}\tilde{\tau} \quad > \quad \bar{S}^\ell \tag{115}$$

$$-c(\tilde{q}) + \beta\kappa_\ell\tilde{d} + \frac{\beta}{\gamma}\tilde{\tau} \quad \geq \quad 0, \tag{116}$$

where $\bar{S}^h \equiv u(\bar{q}) - \beta\kappa_h\bar{d} - \frac{\beta}{\gamma}\bar{\tau}$ and $\bar{S}^\ell \equiv u(\bar{q}) - \beta\kappa_\ell\bar{d} - \frac{\beta}{\gamma}\bar{\tau}$. According to (114), the offer $(\tilde{q}, \tilde{d}, \tilde{\tau})$ would make a buyer in the high-dividend state worse-off relative to his equilibrium payoff; from (115), it would make a buyer in the low-dividend state better off; from (116), it would be accepted by sellers provided that they believe that $\kappa = \kappa_\ell$. If one can find such an offer then the proposed equilibrium fails the Intuitive Criterion.

Consider an out-of-equilibrium offer such that $\tilde{q} = \bar{q}$ and $\tilde{d} = \bar{d} + \varepsilon$ where $\varepsilon > 0$. The conditions (114) and (115) imply

$$\kappa_\ell < \frac{\bar{\tau} - \tilde{\tau}}{\varepsilon\gamma} < \kappa_h. \tag{117}$$

The condition (116) requires then

$$\beta\varepsilon\left[\left(\frac{\bar{\tau} - \tilde{\tau}}{\gamma\varepsilon}\right) - \kappa_\ell\right] \leq -c(\bar{q}) + \beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau}. \tag{118}$$

From (117),

$$\beta\varepsilon\left[\left(\frac{\bar{\tau} - \tilde{\tau}}{\gamma\varepsilon}\right) - \kappa_\ell\right] < \beta\varepsilon(\kappa_h - \kappa_\ell).$$

Hence, for any $\varepsilon > 0$ such that $\varepsilon \leq \left[-c(\bar{q}) + \beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau}\right]/\beta(\kappa_h - \kappa_\ell)$ the inequality (118) is satisfied. For any ε, one can find $\tilde{\tau}$ such that (117) holds. Provided that ε is sufficiently small, the offer $(\tilde{q}, \tilde{d}, \tilde{\tau})$ is feasible. Consequently, the proposed equilibrium with the pooling offer $(\bar{q}, \bar{d}, \bar{\tau})$ violates the Intuitive Criterion. ■

From Lemma 14, the offer made by a buyer in the low-dividend state must satisfy $-c(q) + \beta\kappa_\ell d + \frac{\beta}{\gamma}\tau \geq 0$. Hence, the buyer's payoff in the low-dividend state is bounded above by his complete information payoff. The complete information offer solves

$$(q_\ell, d_\ell, \tau_\ell) = \arg\max_{q,d,\tau}\left[u(q) - \beta\kappa_\ell d - \frac{\beta}{\gamma}\tau\right]$$

$$\text{s.t.} \quad -c(q) + \beta\kappa_\ell d + \frac{\beta}{\gamma}\tau \geq 0$$

$$-\omega^s \leq \frac{\beta}{\gamma}\tau \leq \omega^b, \quad -a^s \leq d \leq a^b.$$

The solution is $q_\ell = q^*$ and $\beta\kappa_\ell d_\ell + \frac{\beta}{\gamma}\tau_\ell = c(q^*)$ if $\beta\kappa_\ell a^b + \omega^b \geq c(q^*)$. Otherwise, $\frac{\beta}{\gamma}\tau_\ell = \omega^b$, $d_\ell = a^b$ and $c(q_\ell) = \beta\kappa_\ell a^b + \omega^b$. Hence, there is always a complete-information offer such that $d_\ell \geq 0$. Since $\lambda(q,d,\tau)\beta\kappa_h d + [1-\lambda(q,d,\tau)]\,\beta\kappa_\ell d \geq \beta\kappa_\ell d$ for all $d \geq 0$, the complete information offer is acceptable for any belief system.

Consider next the offer made by a buyer in the high-dividend state. From Lemma 14, it satisfies $-c(q) + \beta\kappa_h d + \frac{\beta}{\gamma}\tau \geq 0$. If $\omega^b \geq c(q^*)$ then the buyer can achieve his complete-information payoff by offering $q_h = q^*$, $d_h = 0$ and $\frac{\beta}{\gamma}\tau_h = c(q^*)$. Provided that $\omega^b < c(q^*)$, the buyer cannot make the complete information offer since otherwise it would be imitated by a buyer in the low-dividend state. In this case, the buyer makes an offer that maximizes his payoff but that does not provide a buyer in the low-dividend state with strict incentives to imitate it. So, the offer solves

$$(q_h, d_h, \tau_h) = \arg\max_{q,d,\tau}\left[u(q) - \beta\kappa_h d - \frac{\beta}{\gamma}\tau\right]$$
$$\text{s.t.} \quad -c(q) + \beta\kappa_h d + \frac{\beta}{\gamma}\tau \geq 0$$
$$u(q) - \beta\kappa_\ell d - \frac{\beta}{\gamma}\tau \leq \bar{S}^\ell$$
$$-\omega^s \leq \frac{\beta}{\gamma}\tau \leq \omega^b, \quad -a^s \leq d \leq a^b.$$

The solution to this problem is similar to the one described in Lemma 6. The incentive-compatibility constraint cannot be slack since otherwise (q_h, d_h, τ_h) would coincide with the complete information offer. Suppose that the seller's participation constraint is slack. Then, the incentive-compatibility constraint is binding which implies

$$u(q_h) - \beta\kappa_h d_h - \frac{\beta}{\gamma}\tau_h = \bar{S}^\ell - (\kappa_h - \kappa_\ell)\,\beta d_h,$$

and hence

$$\bar{S}^h = \beta \max_{-a^s \leq d \leq a^b} (\kappa_\ell - \kappa_h)\, d + \bar{S}^\ell = \beta\,(\kappa_h - \kappa_\ell)\, a^s + \bar{S}^\ell.$$

The buyer asks for the whole stock of real assets held by the seller. Using the fact that $d_h = -a_s$, the highest achievable payoff for the buyer is

$$\bar{S}^h = \max_{q,d,\tau}\,[u(q) - z]$$
$$\text{s.t.} \quad -c(q) + z \geq 0$$
$$-\omega^s - \beta\kappa_h a^s \leq z \leq \omega^b - \beta\kappa_h a^s.$$

Provided that $\omega^b < c(q^*)$ and $a^b > 0$, it can be checked that $\bar{S}^\ell > \bar{S}^h$ which is a contradiction. So, the seller's participation constraint is binding.

Since sellers get no surplus in the PM trades, their problem in the AM is

$$\max_{\omega,a} \left\{-i\omega - (\phi - \beta\bar{\kappa})\, a\right\}.$$

They accumulate no real balances ($\omega^s = 0$), and they hold the real asset ($a^s > 0$) only if $\phi = \beta\bar{\kappa}$. Sellers hold an asset only if it is priced at its fundamental value.

G. Endogenous capital stock

The model of Section 6 can be readily extended to endogenize the quantity of real assets in the economy. I lay down succinctly such an extension. The approach is similar to the one in Lagos and Rocheteau (2006).

Suppose that buyers can produce capital goods in the first period of their lives. The disutility cost to produce a units of capital is $\psi(a)$ with $\psi(0) = 0$, $\psi' > 0$ and $\psi'' > 0$. Capital goods are one-period lived, and they generate $\kappa \in \{\kappa_\ell, \kappa_h\}$ units of AM-goods in the subsequent period.

The problem of a young buyer in the AM is

$$\max_{\omega,a} \left\{-i\omega - \psi(a) + \pi_h S^h(\omega, a) + \pi_\ell S^\ell(\omega, a) + \beta\bar{\kappa}a\right\}.$$

The determination of the terms of trade in the PM is characterized by Lemmas 5 and 6. Following the proof of Lemma 7, it can be shown that there is a unique solution (ω, a) to the buyer's problem, and hence equilibrium is unique.

Let a^* denote the socially efficient level of capital, i.e., the solution to $\psi'(a) = \beta\bar{\kappa}$. If $a^* \geq c(q^*)/\beta\kappa_\ell$ then $a = a^*$ and $q_\ell = q^*$. Buyers have enough wealth to buy the first best quantity of output in the low-dividend state. A monetary equilibrium exists provided that $i < i_0(a^*)$ where i_0 is defined by

$$i_0 = \pi_h S^h_\omega(0, a^*) = \pi_h \Delta(q_h) \left(1 - \frac{\kappa_\ell}{\kappa_h}\right),$$

where q_h is solution to (48)-(49) with $\omega = 0$. So, in contrast to Lagos and Rocheteau (2006), there exists a monetary equilibrium for any level of a^*.

If $a^* < c(q^*)/\beta\kappa_\ell$ then there exists a threshold $\tilde{\imath}$ defined as

$$\tilde{\imath} = \pi_h S^h_\omega\left(c(q^*) - \beta\kappa_\ell a^*, a^*\right)$$

such that if $i > \tilde{\imath}$ then $a > a^*$. Buyers overaccumulate capital because of the liquidity services it provides in the PM. Denote $\hat{a}$ the solution to $\psi'(a) = \pi_h S^h_a(0, a) + \pi_\ell S^\ell_a(0, a) + \beta\bar{\kappa}$. A monetary equilibrium exists provided that $i < i_0(\hat{a})$ where

$$i_0(\hat{a}) = \pi_h S^h_\omega(0, \hat{a}) + \pi_\ell S^\ell_\omega(0, \hat{a})$$

Following a similar argument as in Proposition 6, it can be shown that if $a^* < c(q^*)/\beta\kappa_\ell$ and $i \in (\tilde{\imath}, i_0(\hat{a}))$ then $da/di > 0$. An increase in inflation induces buyers to accumulate more capital.

Finally, the rate of return of capital is $R_a = \bar{\kappa}/\psi'(a)$. Following the proof of Proposition 7, it can be shown that in any monetary equilibrium $R_a > R_m$, capital dominates fiat money in its rate of return. In contrast, in Lagos and Rocheteau (2006), fiat money and capital have the same rate of return in any monetary equilibrium. Moreover, if $i < \tilde{\imath}$ then $\psi'(a) = \beta\bar{\kappa}$ so that $R_a = \beta^{-1}$, the rate of return of capital is equal to the gross discount rate.