Currency Competition in a Fundamental Model of Money

by Gabriele Camera, Ben Craig, and Christopher J. Waller

FEDERAL RESERVE BANK OF CLEVELAND

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We study how two fiat monies, one safe and one risky, compete in a decentralized trading environment. The equilibrium value of the two currencies, their transaction velocities and agents' spending patterns are endogenously determined. We derive conditions under which agents holding diversified currency portfolios spend the safe currency first and hold the risky one for later purchases. We also examine when the reverse spending pattern is optimal. Traders generally favor dealing in the safe currency, unless trade frictions and the currency risk is low. As risk increases or trading becomes more difficult, the transaction velocity and value of the safe money increases.

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1. Introduction

Centuries ago the comedy writer Aristophanes lamented (in "The Frogs") that "the full-bodied coins that are the pride of Athens are never used while the mean brass coins pass hand to hand." Many observers have since debated on the organization of exchange when several monies, some 'superior' to others in some way, compete to sustain trade.

A long-held notion is that an inferior currency should circulate more widely than a superior money. Those holding both monies would prefer spending the 'bad' money as soon as they can, and keep the 'good' money for future purchases. Others have favored a differing notion: it is the good currency that should circulate more widely. Hayek (1976) argued that this was the logical outcome of currency competition. People would rather spend the good money first, as it has greater purchasing power, and keep the bad money to face future trade opportunities.[1]

These notions are conflicting, yet revolve around rational spending behavior. Thus, a key challenge is to determine what fundamental factors influence the use of competing monies. That is: if two currencies are accepted in trade, when will agents tend to spend the bad and hold the good one for future purchases? When will they do the opposite? A large theoretical literature has offered insight centered around arbitrary transaction costs or institutional restrictions on use of monies (see Giovannini and Turtleboom, 1994). We complement it by studying currency use as a result of decentralized and uncoordinated private decisions, absent currency-specific transaction costs and institutional restrictions.

To do so we consider an economic environment in which money is essential to conduct decentralized trade. There are two intrinsically different monies: a 'bad' money characterized by purchasing power risk and a 'good' safe money. Both have explicit medium-of-exchange roles, and their equilibrium values reflect their ability to facilitate spot trades of consumption goods. This is formalized by modeling trade as a random search process among agents specialized in production/consumption. They hold currency portfolios to buy goods via pairwise trades where prices are determined via bilateral bargaining. In this context currencies compete on a 'level' trading field as currency-specific trade barriers, or direct government action, are absent.

Our main contribution is to show how equilibrium spending patterns and transaction velocities are driven by relative currency risk and trade difficulties.[2] The basic mechanism is this. Changes in relative risk alter the monies' relative values, hence the distribution of market prices. This, in turn alters

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[1] This is reminiscent of some developing economies where a good foreign money circulates more widely than the bad domestic liabilities, or post-WWI Europe where "...the lack of a stable domestic means of payments was a serious inconvenience...and foreign currencies therefore came to be desired...as a means of payment...Thus, in advanced inflation, "Gresham's Law" was reversed: good money tended to drive out bad..." (League of Nations, 1946, p.48).

[2] This is unlike prior search-theoretic work, where currency portfolios were not allowed, with the exception of Head and Shi, (2000) and Craig and Waller (2001).

1

buyers' spending strategies, which affects economy-wide transaction patterns and the relative transaction velocities of the currencies.

Our analysis proceeds in two parts. We first prove that equilibria exist in which agents favor spending the good currency, and hold on to the bad for subsequent trades. This equilibrium, tends to arise if the bad money is quite risky and trade frictions are substantial, and produces the highest velocity for the good money. We then ask if equilibria exist with the opposite pattern: agents prefer spending the bad money and hold the good one for later transactions. While this may appear to be an obvious strategy for the buyer, in fact it is harder to support as an equilibrium. While spending the bad currency makes sense for the buyer, it effectively transfers the risk onto the seller. The seller will not accept the risk without being compensated, via a higher price. This lowers the buyer's current consumption: a very risky money buys so little that the buyer prefers to spend the good money instead. This equilibrium tends to exist if the risk on the bad currency is low and trading is easy.

In equilibrium money holdings are heterogeneous across agents. As a result, agents have differing valuations on additional units of money and this creates price dispersion across sellers. Currency risk affects circulation of the currencies by altering the distribution of relative prices, i.e. real exchange rates. Greater risk induces sellers to charge high prices if paid with the bad currency, which amplifies the dispersion in real exchange rates. As the price charged by sellers increases with currency risk, buyers increasingly spend the good money in a larger fraction of trade encounters. Thus, greater risk on the bad money lowers its transaction velocity while raising the velocity of the good money.

These findings offer insight in the patterns of monetary transaction observed in those developing economies where a foreign money exists alongside the domestic. Our analysis suggests that the level of 'dollarization' can be kept low as long as the domestic currency risk is low and the trading environment is well functioning. However, should currency risk get out of hand or the trading environment break down, a high degree of dollarization will be the outcome.

2. Economic Environment

The model is a divisible-goods version of Kiyotaki and Wright (1989) with multiple holdings of money as in Camera and Corbae (1999), and two distinct currencies. Here we describe its key features.

Time is continuous and unbounded. There is a continuum of infinitely lived agents and good types, both normalized to one. Every agent specializes in consumption and production: he produces one type of good and consumes a subset $x \in (0, 1)$ of good types. Production of quantity q generates disutility $c(q) = q$. Consumption of q units of a desired consumption good generates utility $u(q)$, with $u'(q) > 0$, $u''(q) < 0$, $u'(0) = \infty$ and $0 \leq u'(\infty) \leq 1$ (more on this assumption, later).

Agents engage in decentralized exchange. They are randomly paired over time according to a Poisson process with arrival rate $\alpha > 0$. Barter is ruled out by assuming that in a match there is probability x of single coincidence of wants but a zero chance of double coincidence. The existence of alternative payments systems or financial intermediaries is assumed away, so that intertemporal trade is infeasible. Hence, decentralized spot monetary trade arises as a natural means to expand allocations beyond autarky.

A fraction $M_i \in (0,1)$ of agents hold indivisible fiat money of type $i = g, b$ (g stands for 'good' and b for 'bad'). Individual money holdings are bounded by $N \geq 2$, so that the total supply of monies is $M_g + M_b \in (0, N)$. Agents face the same trading environment, independent of their portfolio holdings. However, the currencies have a key intrinsic difference. Money b has purchasing power risk, while money g does not. A convenient way to model this feature (as proposed by Li, 1995) is to assume existence of a 'government' that randomly taxes agents' holdings of money b. Specifically, with Poisson arrival rate α the agent's entire holdings of money b are taken away with probability $\tau \in (0,1]$ by the government. This captures the idea that currency b is risky and those holding it are prone to sudden losses of purchasing power. Money b is similarly re-injected in the economy. The government buys goods from randomly encountered sellers with probability $\eta \in [0,1]$, paying with one unit of currency b.

The terms of trade are endogenously formed according to a take-it-or-leave-it bargaining protocol. Specifically, buyers offer sellers a trade of d units of currency for q goods, that the seller can accept or reject.

3. Stationary Equilibria with Currency Competition

To identify how economic fundamentals affect the relative circulation of currencies, we study monetary equilibria where both currencies are accepted in trade. Clearly, non-monetary equilibria exist.

At each date agents can be buyers or sellers who maximize utility from consumption by choosing prices and currencies used to settle trades. These choices depend on the agents' portfolio holdings, assumed observable, and the expected distribution of prices in the market. We focus on trade patterns that are sustainable under symmetric and stationary pure Nash strategies. Hence, we look for fixed points in strategy space since equilibrium actions, and beliefs over actions, must be time-invariant and identical across agents. We typify outcomes describing portfolios and price distributions, and patterns of monetary trade. Hence, the strategies of those with 'diversified' portfolios are crucial, as only these buyers can choose which currency to spend. With a large upper bound N, there are many diversified portfolios and thus a multitude of equilibrium transaction patterns. Unfortunately, this impairs analytical clarity as the strategy set expands rapidly. Thus, to simplify the analysis we take two steps.

First, we set $N = 2$ to keep heterogeneity tractable.[3] If we let $m_j \in (0,1)$

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[3] Let $n = 0, 1, 2, ...N$ be an agent's total holdings of the two monies, in combined units.

be the fraction of agents with portfolio j then $j \in J = \{0, g, b, 2g, 2b, gb\}$: everyone has at most one type of money except the m_{gb} fraction who hold one unit of each currency. The advantage of doing this is that equilibria hinge on the behavior of a single set of traders (buyers gb) and in equilibrium currency exchange is absent.[4] Second, we allow buyers to bid only for a seller's goods, not for his goods *and* money (e.g. as in Aiyagari et al., 1996). This is a natural way to model spot monetary transactions, where money is used to buy goods, not 'mixed baskets' of real and nominal objects. The equilibria sustainable under 'mixed' trades are studied in a related paper (Craig and Waller, 2001).

3.1 Distributions and Strategies

In this environment the distribution $\{m_j\}$ must satisfy the following constraints:

$$\begin{aligned} m_0 + m_g + m_b + m_{2g} + m_{2b} + m_{gb} &= 1 \\ m_g + 2m_{2g} + m_{gb} &= M_g \\ m_b + 2m_{2b} + m_{gb} &= M_b \end{aligned} \tag{1}$$

In a stationary equilibrium $\dot{m}_j = 0 \ \forall j \in J$ and the laws of motion depend on the trade pattern (as shown in the Appendix). Since bad money is constantly removed and injected into the economy, stationarity also requires that outflows and inflows of money b are equal, i.e.[5]

$$\tau(m_b + 2m_{2b} + m_{gb}) = \eta \left[m_0 + m_g + m_b \right]. \tag{2}$$

With regard to price formation and trading strategies we omit unnecessary detail (found in Camera and Corbae, 1999) and focus on two key aspects. First, only agents with money can buy, as barter is unfeasible and exchange must be quid-pro-quo. Agents with portfolio $s \in \{0, b, g\}$ can sell, while the proportion $\mu = m_{2g} + m_{2b} + m_{gb}$ of agents holding two currency units, that we call 'rich buyers', can only buy. Since agents without money can only sell we call them 'poor sellers', while agents with one currency unit can buy or sell so they can be either 'poor buyers' or 'rich sellers'. Second, take-it-or-leave-it bargaining implies the optimal offer pair (d, q) leaves the seller with no net surplus. Thus, he is indifferent between accepting or rejecting it and in equilibrium he accepts every offer meeting his reservation value.

To define prices one must specify the equilibrium trade pattern. We focus on one in which it is optimal to always engage in 'small' nominal trades. Here buyers spend a single unit of money per trade, $d = 1$, hence the price is

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The monies' distribution has a support with $\sum_{n=1}^{N+1} n = (N+1)(1+N/2)$ elements. As there are two distinct currencies, there are $\frac{(n+1)!}{n!}$ possible portfolio combinations that allow the agent to hold exactly n units (of either money).

[4]Suppose monies differ in values. Portfolio exchanges (one-for-one or two-for-one) are not mutually beneficial, as one trader is worse off. One-for-one trades between agents with portfolios $2g$ and $2b$ are suboptimal for those having the better valued money (the portfolio gb is worth less). The next section formalizes this intuition.

[5]There are three parameters: τ, η, M_b. We set τ and M_b and let η endogenously adjust to satisfy (2).

$1/q$. While other patterns are possible, this is relevant for a simple reason. We want to determine conditions under which buyers choose to spend one currency over the other. As $N = 2$, buyers with a diversified portfolio face a non-trivial choice only when $d = 1$, i.e. when they wish to spend only part of their money holdings.

In this context, buyers' spending choices are contingent on sellers' money holdings since they affect the seller's reservation value. For instance, we expect sellers to produce different amounts for a unit of good money depending on whether or not they already hold a unit of good money. Hence, we let q_s^i denote the production exchanged by a seller with portfolio s for one unit of money $i = g, b$, in equilibrium.

Note that q_s^i depends only on the seller's portfolio since every buyer makes the same nominal offer $d = 1$ in equilibrium. However, buyers with undiversified holdings $\{g, b, 2g, 2b\}$ cannot choose between monies but those with the diversified portfolio gb can. Hence, to discuss trade strategies we need to formalize the money choice of buyer gb. Conditional on $d = 1$, we let $p_s \in [0, 1]$ denote the probability that this buyer chooses to spend money g when matched to a seller with portfolio s. With probability $1 - p_s$ he spends the bad money. We let the vector $p = (p_0, p_g, p_b)$ describe this buyer's spending strategy, and let p^* denote an equilibrium. Hence, there are eight possible pure strategy vectors p^*, given $d = 1$.

3.2 Value Functions and Reservation Prices

We can now describe the value of holding the different portfolios under the conjectured trade pattern and price mechanism, i.e. when agents expect that $\{q_s^i\}$ define the terms of trade prevailing on the market, and that buyers will adopt the spending strategies p^* and $d = 1$. Given the recursive structure of the model, the stationary value V_j from holding portfolio $j \in J$ is derived using standard dynamic programming techniques. V_j must satisfy

$$\rho V_i = x \sum_{s \in \{0,g,b\}} m_s u(q_s^i) - x(1 - \mu)(V_i - V_0) - \tau(V_i - V_0) 1_{\{i=b\}}$$

$$\rho V_{2i} = x \sum_{s \in \{0,g,b\}} m_s u(q_s^i) - x(1 - \mu)(V_{2i} - V_i) - \tau(V_{2i} - V_0) 1_{\{i=b\}}$$

$$\rho V_{gb} = \max_{p_s \in \{0,1\}} x \sum_{s \in \{0,g,b\}} m_s \left[p_s u(q_s^g) + (1 - p_s) u(q_s^b) + p_s (V_b - V_g) \right]$$
$$- x(1 - \mu)(V_{gb} - V_g) - \tau(V_{gb} - V_g)$$
$$(3)$$

and $V_0 = 0$, because of buyer-take-all. Here $i = g, b$, $1_{\{i=b\}} = 1$ and zero otherwise, and $\rho = r/\alpha$ is the discount factor adjusted by the arrival rate. It measures the severity of the trading frictions in the economy: as ρ goes to zero, frictions vanish.

The first term on the right-hand side of the first two lines is the expected flow utility from consumption. With probability xm_s the buyer meets a seller with portfolio s who can produce his desired consumption good, he spends currency i and enjoys flow utility $u(q_s^i)$. The second term is the change in

5

lifetime utility, as the buyer spends a unit of money with probability $x(1-\mu)$. For those who hold bad currency, $i = b$, the third term is the expected loss due to purchasing power risk: the entire holdings of money b are lost with probability τ. Payoffs from being a seller do not appear because they are zero, due to buyer-take-all bargaining.

For buyers holding gb, the last two lines can be similarly interpreted, once adjusted for the fact that buyers gb can choose spend one currency or the other (the choice p_s). Monies with different values, $V_b \neq V_g$, entail different payoffs and gb buyers take this into account (third line). Not only do they evaluate the expected flow utility from spending money b or g, but they also consider the opportunity cost of spending the good money and being left with the bad one (the term $p_s(V_b - V_g)$, a loss if the good money has greater value). The fourth line accounts for all other expected lifetime utility losses: those due to spending the good money, $x(1-\mu)(V_{gb}-V_g)$, and those due to currency risk, $-\tau(V_{gb} - V_g)$.

It is useful to manipulate the value functions in (3) to show that the values of multi-unit portfolios are a linear combination of the values of single-unit holdings. For $i = g, b$:

$$V_i = \frac{A_i}{1 - \mu} \sum_{s \in \{0,g,b\}} m_s u(q_s^i) \quad \text{and} \quad V_{2i} = (1 + A_i)V_i \tag{4}$$

$$V_{gb} = \frac{A_b}{1 - \mu} \sum_{s \in \{0,g,b\}} m_s p_s \left[u(q_s^g) - u(q_s^b) + V_b - V_g \right] + V_b + A_{gb}V_g \tag{5}$$

where $A_b < A_g < A_{gb}$.[6] It is immediate that, in a monetary equilibrium, the value of any portfolio j is bounded below by zero. Also $V_{2i} \leq 2V_i$, $V_{gb} \leq \max\{V_{2g}, V_{2b}\}$, $V_{gb} \leq V_b + V_g$, and $(V_{2g} - V_g)/V_g = A_g > (V_{2b} - V_b)/V_b = A_b$, i.e. the marginal value of the risky currency declines faster than the good currency.

Since the seller earns no surplus his production cost (in flow disutility) must equal his valuation of the money received. For $i, k = g, b$ and $i \neq k$, in a dual-currency equilibrium the seller's reservation quantities are:

$$q_0^i = V_i, \quad q_i^i = V_{2i} - V_i, \quad \text{and } q_k^i = V_{gb} - V_k \quad \text{with} \quad q_s^i > 0. \tag{6}$$

For example, a buyer trading with a poor seller (who has no money) receives $q_0^i = V_i$ for a unit of money i. This is because the poor seller assigns value $V_i - V_0 = V_i$ to money i. Hence, she is willing to sustain up to V_i disutility from producing in exchange for one unit of money i, beyond which she rejects the trade. The same reasoning applies to the reservation quantities of rich sellers (those with one unit of money), q_i^i and q_k^i. Notice also that, given a currency offer, poor sellers produce more than rich sellers as rich agents value an extra unit of money the least, $V_{2i} - V_i < V_i$.

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[6] $A_b = \frac{x(1-\mu)}{\rho+\tau+x(1-\mu)} < A_g = \frac{x(1-\mu)}{\rho+x(1-\mu)} < A_{gb} = \frac{\tau+x(1-\mu)}{\rho+\tau+x(1-\mu)} < 1$ so that $\lim_{\rho \to 0} A_g = A_{gb} = 1$, $A_b < 1$ if $\tau > 0$.

3.3 Optimal Spending Strategies

To study individual optimality of a trade pattern where $d = 1$ and $p = p^*$, we take three steps.

Given p^*, $d = 1$ is optimal if agents want to spend one unit of money but no more. Hence, rich buyers must receive more surplus from spending one unit rather than both when meeting poor sellers. That is, for holders of portfolios $2i$ and gb, we need:

$$\begin{aligned} u(q_0^i) + V_i - V_{2i} &> u(\tilde{q}) + V_0 - V_{2i} \\ \max\left\{ u(q_0^g) + V_b, u(q_0^b) + V_g \right\} - V_{gb} &> u(\hat{q}) + V_0 - V_{gb} \end{aligned} \tag{7}$$

where $i = g, b$. Here q_0^i satisfies (6), while $\tilde{q}$ and $\hat{q}$ denote the out-of-equilibrium production of poor sellers for, respectively, two units of money i or one of each type. Buyer-take-all implies $\tilde{q} = V_{2i} - V_0$, and $\hat{q} = V_{gb} - V_0$, their flow utility losses must equal their lifetime utility gains, even out-of-equilibrium.

Second, given p^*, $d = 1$ is optimal if the surplus received from spending one unit of money is larger than that from walking away. In short, the seller's reservation price cannot be too high. Since rich agents value extra money the least, it follows that (i) rich buyers trade at a high price whenever poor buyers do and (ii) poor buyers buy from poor (low price) sellers whenever they buy from rich (high price) sellers. With two kinds of poor buyers (holding $i = g, b$) and rich sellers (holding $k = g, b$) all buyers spend always at least one unit of money, if four inequalities hold, summarized by:

$$u(q_k^i) + V_0 - V_i > 0. \tag{8}$$

When trading with rich sellers, whose reservation price $1/q_k^i$ is high, the buyer's loss from spending one money i, $V_0 - V_i$, must exceed the utility from consumption.

Finally, given $d = 1$, buyer gb spends only one of his two currencies. Since we are focusing on pure spending strategies and there are three types of sellers, $s \in \{0, g, b\}$, then the surplus earned from spending one currency must be larger than spending the other:

$$p_s = \begin{cases} 1 & \text{if } u(q_s^g) + V_b - V_{gb} > u(q_s^b) + V_g - V_{gb} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

We now can define a monetary equilibrium with currency competition.

Definition. *A symmetric stationary dual-currency equilibrium with $d = 1$ is $\left\{ V_j, m_j, q_s^i, p_s \right\}_{\forall j, i, s}$ that satisfy (1)-(3), (6)-(9) and $\dot{m}_j = 0$.*

3.3.1 The Role of Preferences

A key decision for buyer gb is what to do in matches with poor sellers. If he spends the good money, he will do so also in matches with richer sellers, where

he faces less favorable terms of trade. Substituting the reservation quantity of the poor seller, $q_0^i = V_i$, in (9) we see that $p_0 = 1$ if

$$u(V_g) - V_g > u(V_b) - V_b.$$

The buyer enjoys $u(V_i)$ flow utility from spending money i, while the seller suffers V_i flow disutility from production. Thus the flow surplus from spending one money i is $S(V_i) = u(V_i) - V_i$. As the buyer captures it entirely, he spends the money i that maximizes $S(V_i)$. Hence the form of preferences and the currencies relative values are key.[7] Two cases may arise, in general.

If $S(V_i)$ is ever-increasing, $p_0 = 1$ only if $V_g > V_b$, that is the good money must have the greatest purchasing power. The buyer will spend the safer money whenever possible since the surplus he receives is increasing in the currency's value. Conversely, $p_0 = 0$ requires $V_g \leq V_b$; the buyer will spend the risky currency money first if its transaction value is higher than the safe money. If $S(V_i)$ is hump-shaped, however, $V_g > V_b$ can sustain $p_0 = 0$. If $S(V_i)$ falls for high V_i, it might be better to spend the bad money although it buys less. Doing so has a higher surplus.

In studying existence of equilibria, this insight on the preference structure is developed using two convenient utility functions. The first, $u(q) = q^\sigma + q$ with $0 < \sigma < 1$, exhibits decreasing relative risk aversion and $S(V_i) = V_i^\sigma$ is ever-increasing. The second, $u(q) = q^\sigma$, is CRRA and $S(V_i) = V_i^\sigma - V_i$ is hump-shaped with unique maximum at $\hat{V} = \sigma^{\frac{1}{1-\sigma}} < 1$.

4. Existence of Equilibria: Two Significant Cases

We proceed by studying the equilibria where $p^* = (1, 1, 1)$ and $p^* = (0, 0, 0)$. These vectors are the extreme cases of the strategy spectrum of buyer gb: he always spends one money type, g or b, regardless of which seller he meets. This provides insight on the factors affecting the competition between currencies. We then build on this intuition via numerical analysis of the remaining equilibria, where the buyer's spending choices vary with the sellers he meets. We call the $p^* = (0, 0, 0)$ case the 'bad-money' equilibrium, as buyers with diversified portfolios always spend the risky money rather than the good.[8] The 'good-money' equilibrium has $p^* = (1, 1, 1)$.

Existence of equilibrium is proved via a constructive approach. Given the conjecture $d = 1$ and $p = p^*$, we solve for equilibrium value functions, prices,

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[7] Technically, the surplus in this match has two components, but only one is affected by the buyer's spending choice. The first is the net flow utility $u(q_0^i) - q_0^i$: it depends on the buyer's spending choice $i = g, b$. The second is the net continuation utility $V_b + V_g - V_{gb} - V_0$: it is independent of the buyer's spending choice.

[8] This is a common occurrence in developing and transitional economies: dollars are used in some transactions and the risky home currency in others. Several models have been proposed to account for this phenomenon (e.g. Chang, 1994, Uribe, 1997, Sibert and Liu, 1998, or Engineer, 2000). Their key element of commonality is that, unlike our model, the foreign currency is assumed to have a relatively higher 'transaction cost' associated with its use.

and distributions, providing parameter conditions sufficient to ensure that the conjectured strategies are individually optimal.

4.1 The 'Good-Money' Equilibrium

Here we determine conditions under which $d = 1$ is individually optimal and buyers with portfolio gb always prefer spending the good money, $p^* = (1, 1, 1)$.

To provide conditions sufficient for existence of a unique stationary distribution we follow a Liapunov function approach (as in Zhou, 1997).[9] One can prove, using (4)-(6), that $V_{gb} = V_g + A_b V_b$. The upshot is currency exchange does not take place in matches between buyers $2b$ and $2g$ since one of them would not swap a unit of his money for another, since $\min\{V_{2b}, V_{2g}\} < V_{gb} < \max\{V_{2b}, V_{2g}\}$. Using (6), it then follows $q_b^g = V_g - (1 - A_b)V_b$ and $q_g^b = q_b^b = A_b V_b$. Thus, in studying the dual-currency equilibrium, where both monies have a positive value, we concentrate on cases where $V_g > (1 - A_b)V_b$, which is necessary for $q_b^g > 0$.[10] We can now state the following:

Proposition 1. *Consider the good-money equilibrium. If ρ and σ are sufficiently small, then*
(i) for $u(q) = q^\sigma + q$, then there exists a unique equilibrium such that $V_g > V_b$.
(ii) for $u(q) = q^\sigma$, the equilibrium does not exist.

Thus, if the good-money equilibrium exists the good money is more valuable. The equilibrium is easily sustained if the trading surplus is monotonically increasing in the transaction's value, otherwise it is not. These results hinge on two elements.

First $d = 1$ must be optimal. The reason is that the currency choice of buyers gb matters only when they do not wish to spend their entire holdings. This requires small ρ and σ (see also Proposition 2 in Camera and Corbae, 1999). The intuition is, when trade frictions are low, sellers charge similar prices so buyers are willing to spend at least one unit rather than searching for a better deal.[11] Furthermore, if σ is low, marginal utility diminishes rapidly so agents spend at least one unit of currency, but not two. Since buyers have an incentive to hold some cash for future purchases, they spend no more than one unit even when prices are low.

Second, our earlier insight about the trade surplus $S(V_i)$, suggests the preference structure matters for existence. If $S'(V_i) > 0$ buyers with portfolio gb would always spend the good money and hold the bad. Since money's value rises as ρ falls, at some point $S'(V_i) < 0$ when surplus is hump-shaped which leads the gb buyer to start spending the bad currency. Thus, low frictions and hump-shaped surplus induce the buyer to deviate by offering the cheap

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[9] The proof is lengthy; it is available from the authors as a technical appendix.

[10] It is easy to show that there exists a non-monetary equilibrium $V_b = V_g = 0$, and a unique equilibrium $V_g > V_b = 0$ that can be thought as the limiting case of currency competition where only the good money circulates. $V_b > V_g = 0$ is never an equilibrium.

[11] The reservation quantity of a rich seller converges to that of a poor seller, as A_g and A_b approach 1. So there is nothing to gain by waiting to meet a poor seller.

bad money.[12]

In equilibrium because the currencies are valued differently, $V_g \neq V_b$ there is a non-degenerate distribution of prices. This implies there is also a non-degenerate distribution of real exchange rates. Let $R_s = q_s^g/q_s^b$ denote the relative prices charged by sellers with portfolio s. This measures the real value of one unit of the good money relative to the bad, for a given seller. Using (6):

$$R_0 = \frac{V_g}{V_b} < \min\left\{R_g = \frac{A_g}{A_b}\frac{V_g}{V_b}, R_b = 1 + \frac{(V_g/V_b) - 1}{A_b}\right\}$$

so that $R_b \leq R_g$ only if $\frac{V_g}{V_b} \leq \frac{1-A_b}{1-A_g}$. Note that the distribution of real exchange rates becomes degenerate as $\tau \to 0$, as $V_g \to V_b$ and $\frac{A_g}{A_b} \to 1$. This seems natural: as fundamental differences in two monies disappear we do not expect their purchasing powers to diverge.[13] On the other hand, as τ rises, there is increased dispersion in real exchange rates since $\frac{A_g}{A_b}$ rises.

4.2 The 'Bad-Money' Equilibrium

We now consider the other end of the strategy spectrum, where $d = 1$ and $p = p^* = (0,0,0)$ are optimal. Expressions (4)-(6) imply $V_{gb} = V_b + A_{gb}V_g$, $q_b^g = A_{gb}V_g$, and $q_g^b = V_b - (1 - A_{gb})V_g$. Hence, in studying the equilibrium we concentrate on $V_b > (1 - A_{gb})V_g$ necessary for $q_g^b > 0$.[14] We can now state the following:

Proposition 2. *Consider the bad-money equilibrium. If ρ and σ are sufficiently small, then*
(i) for $u(q) = q^\sigma + q$, the equilibrium does not exist.
(ii) for $u(q) = q^\sigma$, then there exists a unique equilibrium such that $V_g > V_b$.

Once again the key result is that if a bad-money equilibrium exists the currencies' values differ. If the economy functions smoothly, the good money has the greatest value. As before, there is a non-degenerate distribution of prices and real exchange rates. The intuition behind the need for low ρ and σ is as in the prior proposition. A condition sufficient for existence of the bad-money equilibrium hinges on the structure of the trade surplus but differs from the good-money outcome. A hump-shaped surplus assures that those with diversified portfolios always desire to spend the bad instead of the good money. If the surplus rises in the quantity traded the buyer would spend the

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[12] Technically, for $u(q) = q^\sigma$, $S(V_i)$ is decreasing if V_i is close to 1. For ρ small, $V_g^* > V_b^*$ and V_g^* is close to 1, hence $S(V_g^*) < S(V_b^*)$ so $p_0 = 0$ hence $p^* \neq (1,1,1)$. When $u(q) = q^\sigma + q$, $S(V_i)$ is monotonically increasing, hence $S(V_g^*) > S(V_b^*)$ for all $V_g^* > V_b^* : p_0 = 1$ is optimal, which is necessary for $p^* = (1,1,1)$ to be an equilibrium.

[13] This would not necessarily hold if 'mixed' trades were allowed (see Ayiagari et al, 1996), as the monies' relative values would not *solely* hinge on fundamental differences.

[14] Once again there exists the non-monetary equilibrium $V_b = V_g = 0$, and an equilibrium in which good money does not circulate at all, $V_b > V_g = 0$. However, $V_g > V_b = 0$ is not an equilibrium.

more valuable money. Thus only a hump-shaped surplus induces buyers to offer the bad money, as low ρ and σ ensure that V_g and V_b lie on the decreasing portion of $S(V_i)$. The intuition is that while the good money buys more, the buyer gives up a valuable asset. When the value is sufficiently large, therefore, buyers prefer to get a little less today, by spending the bad money, and hold the good currency for future consumption.

5. Characterization of Equilibria

We now build on the previous results and expand our study to other patterns of monetary exchange via numerical analysis.[15] Specifically, we study equilibria where $d = 1$ and p^* encompasses all possible pure spending strategies, in order to achieve two objectives. We illustrate how trade frictions and relative currency risk affect the pattern of monetary trade. Furthermore, we demonstrate how the currencies' transactions velocity responds in an intuitive way to changes in the relative currency risk.

5.1 A Trade-off Between Exchange Frictions and Currency Risk

To illustrate the importance of trading frictions and currency risk we let ρ and τ vary. For the baseline parameterization, when $u(q) = q^\sigma + q$ only $p^* = (1, 1, 1)$ is an equilibrium. Figure 1 displays the equilibria existing when $u(q) = q^\sigma$, confirming the intuition developed earlier. If the bad currency risk is low and trade is relatively easy to accomplish, then agents prefer to spend the bad currency first. The opposite occurs if the bad currency's risk is high and the economy is not functioning well. The good-money equilibrium arises when the bad currency's risk is high and trading frictions are reasonably low, the reverse or both are high. The bad-money equilibrium occurs when trading frictions are high and risk is very low, the reverse, or both are low.

The intuition for these results is that low trading frictions mean new trade opportunities arise quickly. If the bad currency's risk is also relatively low, then prices charged for paying with bad currency are not much higher than those for paying with good money. By spending the bad currency, buyer gb gets rid of the risk and does not have to wait long to spend the good money. Hence, he spends the risky currency even though he consumes a little less today. When trade frictions are high, the buyer knows that he will not get to consume again for a while, so he wants a substantial amount of consumption when a trading opportunity arises. This leads him to spend the good money to buy more goods. He holds onto the bad currency in the hope of spending it in the future before it is taxed away.

5.2 Currency Risk and Transaction Velocities

How does τ affect the circulation of currencies in the steady state? In general, circulation is affected by the sellers' willingness to accept the currency

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[15]The experiments are as follows. We select a vector p^*, set $d = 1$, and solve for the equilibrium distribution and value functions. We then verify individual optimality of the conjectured strategies. We do so for each of the 8 pure strategy vectors p^*, for $(\tau, \rho) \in [0, 1]^2$ (defined on a grid with increments of size 10^{-7}). Our benchmark (unless otherwise noted) is $x = 0.4$, $\sigma = 0.5$, $\rho = 0.08$, $\alpha = 5$, $M_g = .75$, and $M_b = .25$.

and the buyers' willingness to spend it. By construction, however, sellers always accept both currencies in our equilibria. Hence, for given supplies of the two currencies, changes in their equilibrium circulation are driven by changes in their distribution and the spending pattern. In order to measure the degree of circulation of each currency, we calculate the endogenous transaction velocities.

The transaction velocity is the amount traded per unit time, divided by its stock. When $d = 1$ we define velocities as:

$$v_g \propto \alpha x \{(1 - \mu)(m_g + m_{2g}) + (p_0 m_0 + p_g m_g + p_b m_b) m_{gb}\}$$
$$v_b \propto \alpha x \{(1 - \mu)(m_b + m_{2b}) + [(1 - p_0) m_0 + (1 - p_g) m_g + (1 - p_b) m_b] m_{gb}\}.$$

The first term is the fraction of each currency that changes hands when buyers holding only that currency meet sellers and spend one unit of their holdings. The second term captures how the spending behavior of the buyer with a mixed portfolio affects the relative velocities of each currency. Velocities are affected by the steady-state distribution of money holdings and by the equilibrium strategy vector p^*. In particular, a change in p^* moves v_g and v_b in opposite directions, ceteris paribus.[16] Thus, the confiscation/injection parameters τ and η affects the velocity of each currency via changes in the distribution of money holdings and the buyers' trading strategies.

Figure 2 illustrates the transaction velocities corresponding to the equilibria of Figure 1 for the baseline value of ρ and varying τ. Given that there is more bad than good currency ($M_g = .25$, $M_b = .75$), the transaction velocity for the bad currency is always the highest since more trades are being conducted with it. When $\tau = 0$, $v_b = .74$, and $v_g = .15$. As the risk on the bad currency increases, however, the velocities change as the distribution of money holdings and the transaction pattern change. We can see that, for an equilibrium associated with a given p^*, increases in currency risk lead to small declines in v_b and small increases in v_g. Once the risk gets high enough, buyers with mixed portfolios begin spending the good currency, rather than the bad. Thus more transactions involve good money, so v_b falls and v_g increases. As the spending pattern changes, there are dramatic decreases in v_b and large increases in v_g. When $\tau = 1$, $v_b = .55$, $v_g = .28$ and the ratio v_g/v_b rises to .51 (from .20 at $\tau = 0$). These results seem very intuitive and suggest that as the bad currency becomes increasingly risky, people 'substitute' out of the bad currency into the good currency causing the circulation of the bad currency to fall and the circulation of the good currency to increase.

5.3 Trade Patterns and Availability of Money

We next analyze how varying the relative currency risk τ, and the ratio of the bad to the good money stock affects the equilibrium transaction pattern by varying the relative supplies of currencies when $M_g + M_b = M = 1$. Figure 3 illustrates the equilibria when $u(q) = q^\sigma$, for the baseline parameterization. Its main feature is that the equilibrium transaction pattern is not driven by

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[16]Note that if $p^* = (1, 1, 1)$ and $(m_g + m_{2g}) \approx (m_b + m_{2b})$, then $v_g > v_b$ and vice versa if $p^* = (0, 0, 0)$.

the relative amount of bad currency in the economy. Rather, bad currency risk is the critical parameter. We also observe an interesting spending pattern. Given a value M_b/M, the gb buyer always spends the bad currency for low τ. As the risk rises, he begins spending the good money when buying from sellers who already hold a unit of the bad currency and $p^* = (0, 0, 1)$. This occurs because the b sellers charge a low price for good money in order to acquire a unit of safe currency to diversify their portfolio. As risk continues to increase, the gb buyer starts spending the good money on g sellers and $p^* = (0, 1, 1)$. Finally, when the bad risk is high enough, all sellers charge high prices in terms of the bad currency, i.e. $p^* = (1, 1, 1)$. Hence, buyers with a mixed portfolio always prefer to buy with good money.

Executing a similar exercise for $u(q) = q + q^\sigma$ generates only the equilibrium $p^* = (1, 1, 1)$. We had to decrease σ to 0.15 and ρ to 0.02 in order to find other equilibria. The results appear in Figure 4.[17] Still, despite the fact that there are eight possible vectors p, only two of them are an equilibrium, and are unique: $p^* = (1, 1, 1)$ and $p^* = (0, 1, 1)$. In Figure 4, when the bad currency risk is very low, $p^* = (1, 1, 1)$ is an equilibrium even when good moneys form less than half of the available currency. However, as τ rises, $p^* = (1, 1, 1)$ is an equilibrium only if there is a large supply of good money. This corresponds to the idea of the economy being 'highly dollarized'. If we think of the good currency as dollars, as opposed to the risky domestic currency, then they are the dominant source of currency, and the preferred medium of exchange. On the other hand, if only few dollars are present in the economy, then $p^* = (0, 1, 1)$ is the unique equilibrium. In this situation, agents holding a mixed portfolio only spend the dollar on rich sellers who charge a much higher price when paid with bad currency. Poor sellers offer better prices in terms of bad currency since they need cash. Thus the buyer can afford to spend the bad currency in those trades.

6. Conclusion

We have studied currency competition from first principles in a decentralized trade setting with two fiat monies differing in their purchasing power risk. The currencies' relative risk affects the organization of monetary exchange via its influence on the distribution of prices and real exchange rates in the marketplace. Changes in the currencies relative price affect buyers' desire to spend or hoard the most valuable money. Even small differences in currency risk can be associated with relatively higher circulation of the safer currency, if trade is hard to accomplish.

Our theoretical analysis builds intuition on some aspects of the phenomenon known as "dollarization" whose most basic form is the use of a foreign currency alongside the home (also known as currency substitution). In this context, a relevant issue for policymakers is the extent of the currencies' relative use for internal trade. We have provided insight by focusing on key

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[17]Interestingly, if $u(q) = q^\sigma$, $\sigma = 0.15$, and $\rho = 0.02$ then only $p^* = (1, 1, 1)$ exists.

determinants in the usage patterns of competing monies: their relative purchasing power risk and the frictions of the local trading environment.

We find that a poorly functioning economy with risky home currency is prone to dollarization. Thus our analysis is consistent with the view that the local currency sustains internal trade if the purchasing power risk is kept very low, but once that risk gets too high substantial currency substitution kicks in. The normative aspect of our study is that a low dollarized economy can avoid becoming highly dollarized by implementing policies aimed at reducing currency risk and improving the trading environment so that the economy functions well. At the same time our results serve as a warning that dollarization will be unavoidable if currency risk is not kept under control.

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Appendix

Good-Money Equilibrium

In proving proposition 1 we conjecture $d = 1$ and $p^* = (1, 1, 1)$. Using (4)-(6), it is easy to show that $V_{gb} = A_b V_b + (\frac{A_b}{A_g} - A_b + A_{gb}) V_g$. Substituting for A_b, A_g, and A_{gb}, we obtain $V_{gb} = A_b V_b + V_g$.

The equilibrium V_g and V_b must be a fixed point of the map defined by:

$$V_b = \frac{A_b[m_0 u(V_b) + m_g u(A_b V_b) + m_b u(A_b V_b)]}{1 - \mu} \tag{10}$$

$$V_g = \frac{A_g[m_0 u(V_g) + m_g u(A_g V_g) + m_b u(V_g - (1 - A_b)V_b)]}{1 - \mu}. \tag{11}$$

We make use of the following lemma.

Lemma 1. *If ρ is sufficiently small, there exists a unique fixed point of (10)-(11) that is consistent with the dual-currency equilibrium $p^* = (1, 1, 1)$. Precisely $(V_g, V_b) = (V_g^*, V_b^*)$ where $1 < \frac{V_g^*}{V_b^*} \leq \frac{1 - A_b}{1 - A_g}$.*

Proof of Lemma 1.

There is always a non-monetary equilibrium, since $V_b = V_g = 0$ solve (10)-(11). The limiting case of currency competition, when the bad money has no value and only the good money circulates, is also an equilibrium. There is a unique $V_g > V_b = 0$ that solves (10)-(11). Note that $V_b > V_g = 0$ is not a possible solution.

Our focus is a dual-currency equilibrium, where both monies have a positive value. Thus, we are interested in the existence of a strictly positive fixed point (V_g^*, V_b^*) of the map given by (10)-(11).

Let $V_b = V$. In equilibrium (10) defines the map:

$$[\rho + x(1 - \mu)]V = x[m_0 u(V) + m_g u(A_b V) + m_b u(A_b V)] - \tau V \equiv H(V)$$

$H(V)$ is a strictly concave function on $V \geq 0$, starting at 0, and is hump-shaped. In particular, recalling that $\lim_{q \to \infty} u'(q) \leq 1$, we see that $\lim_{V \to \infty} H'(V) < x(1 - \mu)$. Thus, (10) has two fixed points: $V = 0$ and $V = V_b^* > 0$. Notice that $\frac{\partial V_b^*}{\partial \tau} < 0$, since $\frac{\partial A_b}{\partial \tau} < 0$.

Now let $V_b = V_b^*$. Letting $V_g = V$, in equilibrium (11) defines the map

$$[\rho + x(1 - \mu)]V = x[m_0 u(V) + m_g u(A_g V) + m_b u(V - V_b^* + A_b V_b^*)] \equiv F(V, V_b^*)$$

where we define $F(V, V_b^*)$ for $V \geq V_L = (1 - A_b)V_b^*$ (necessary since $V_g - (1 - A_b)V_b^* = q_b^g \geq 0$, in equilibrium). $F(V, V_b^*)$ is strictly concave in V, $F(V_L, V_b^*) > 0$, $\lim_{V \to {}^+ V_L} \frac{\partial F(V, V_b^*)}{\partial V} = \infty$, and $\lim_{V \to \infty} \frac{\partial F(V, V_b^*)}{\partial V} \leq x(1 - \mu)$. Thus, there can be at most two positive fixed points to the map $[\rho + x(1 - \mu)]V = F(V, V_b^*)$. To see how these fixed points compare to V_b^*, let $F(V_b^*) = F(V, V_b^*)|_{V = V_b^*}$.

Due to strict concavity of $F(V, V_b^*)$, a sufficient condition for $V = V_g^* > V_b^*$ to be a fixed point is

$$F(V_b^*) > [\rho + x(1 - \mu)]V_b^* \iff F(V_b^*) > H(V_b^*). \tag{12}$$

16

Note that

$$F(V_b^*) - H(V_b^*) = xm_g[u(A_g V_b^*) - u(A_b V_b^*)] + \tau V_b^* > 0$$

since $A_b < A_g$. Hence a fixed point $V_g^* > V_b^*$ always exists. As $\tau \to 0$ then $V_g^* \to^+ V_b^*$, since $A_g \to^+ A_b$.

Concavity of $F(V, V_b^*)$ and $F(V_b^*) > [\rho + x(1-\mu)]V_b^*$, implies that if another positive fixed point $V = V_g^{**}$ exists, then $V_g^{**} < V_b^*$ (see Figure A1). However, if ρ is sufficiently small, and V_g^{**} indeed is a fixed point, then $V_g = V_g^{**}$ cannot be an equilibrium. To see why, recall that $q_b^g = V_g - (1 - A_b)V_b^*$. From (8), a buyer g buys from seller b only if $u(q_b^g) > V_g$. Hence, in equilibrium $V_g > \bar{V}$ is necessary, where $\bar{V}$ solves $u(\bar{V} - (1 - A_b)V_b^*) = \bar{V}$. Notice that $\bar{V} > V_L$, since $q_b^g = 0$ when $V_g = V_L$.

Suppose $V = V_g^{**}$ is a fixed point of (11). Concavity of $F(V, V_b^*)$ implies $V_g = V_g^{**}$ cannot be an equilibrium if

$$
\begin{aligned}
F(\bar{V}, V_b^*) > [\rho + x(1-\mu)]\bar{V} \quad \Leftrightarrow \quad & x\left[m_0 u(\bar{V}) + m_g u(A_g \bar{V}) + m_b u(\bar{V} - (1 - A_b)V_b^*)\right] \\
& > [\rho + x(1-\mu)]\bar{V} \\
\Leftrightarrow \quad & x\left[m_0 u(\bar{V}) + m_g u(A_g \bar{V}) + m_b \bar{V}\right] > [\rho + x(1-\mu)]\bar{V} \\
\Leftrightarrow \quad & x\left[m_0 u(\bar{V}) + m_g u(A_g \bar{V})\right] > [\rho + x(m_0 + m_g)]\bar{V}
\end{aligned}
$$

A sufficient condition for this last inequality to hold is ρ sufficiently small. To see why, recall that $\lim_{\rho \to 0} A_g = 1$. Hence, as $\rho \to 0$ the inequality becomes $u(\bar{V}) > \bar{V}$, always satisfied since $u(\bar{V} - (1 - A_b)V_b^*) = \bar{V}$. By continuity and strict concavity of u, it follows that there exists a $\rho_1 > 0$ such that $\forall \rho \in (0, \rho_1)$ then $F(V, V_b^*) > [\rho + x(1-\mu)]V \quad \forall V \in (\bar{V}, V_g^*)$. Since $V_g > \bar{V}$ is necessary for individual optimality, it follows that only $V_g = V_g^*$ can be an equilibrium (see illustration).■

Proof of Proposition 1.
Consider an equilibrium distribution that satisfies (1)-(2), the equations

$$
\begin{aligned}
\dot{m}_{2g} &= m_g(m_g + m_{gb}) - m_{2g}(m_0 + m_b) \\
\dot{m}_{2b} &= x[m_b^2 - m_{2b}(m_0 + m_g)] + \eta m_b - \tau m_{2b} \\
\dot{m}_{gb} &= x[m_g m_{2b} + m_b m_{2g} + 2m_b m_g - m_{gb}(m_0 + m_g)] + \eta m_g - \tau m_{gb}
\end{aligned}
$$

and $\dot{m}_j = 0$ (see our technical appendix). In a technical appendix we show that it exists, under certain conditions.

Case $u(q) = q^\sigma + q$. The equilibrium pair (V_g, V_b) solves:

$$
\begin{aligned}
V_b &= \left\{ \frac{A_b[m_0 + m_g A_b^\sigma + m_b A_b^\sigma]}{1 - \mu - A_b[m_0 + (m_g + m_b)A_b]} \right\}^{\frac{1}{1-\sigma}} \\
V_g &= \left\{ \frac{A_g\left[m_0 + m_g A_g^\sigma + m_b\left(1 - (1 - A_b)\frac{V_b}{V_g}\right)^\sigma\right]}{1 - \mu - A_g\left[m_0 + m_g A_g + m_b\left(1 - (1 - A_b)\frac{V_b}{V_g}\right)\right]} \right\}^{\frac{1}{1-\sigma}}
\end{aligned}
\tag{13}
$$

We note, that $1 - \mu - A_b[m_0 + (m_g + m_b)A_b] > 0$ since $1 - \mu = m_0 + m_g + m_b$, and $A_b < 1$, always. The same is true for $1 - \mu - A_g\left[m_0 + m_g A_g + m_b\left(1 - (1 - A_b)\frac{V_b}{V_g}\right)\right]$.

In particular, from Lemma 1 we know that if $\rho \in (0, \rho_1)$ then there is a unique solution $(V_g, V_b) = (V_g^*, V_b^*)$ to (13). It is such that $1 < \frac{V_g^*}{V_b^*} \leq \frac{1-A_b}{1-A_g}$.

It is just a matter of algebra to verify that the individual optimality conditions (7)-(9) reduce to the (smaller) set of inequalities:

$$\left[\frac{(1+A_g)^\sigma - 1}{1-A_g} \right]^{\frac{1}{1-\sigma}} < V_g < \left(\frac{A_g^\sigma}{1-A_g} \right)^{\frac{1}{1-\sigma}} \tag{14}$$

$$\left[\frac{(1+A_b)^\sigma - 1}{1-A_b} \right]^{\frac{1}{1-\sigma}} < V_b < \left(\frac{A_b^\sigma}{1-A_b} \right)^{\frac{1}{1-\sigma}} \tag{15}$$

$$V_g > V_b \tag{16}$$

$$(1 - A_b) V_b + (A_g V_g)^\sigma > (A_b V_b)^\sigma + (1 - A_g) V_g \tag{17}$$

$$(1 - A_b) V_b + V_g^\sigma > (V_g + A_b V_b)^\sigma \tag{18}$$

Inequalities (14)-(15) tell us that the value of holding a unit of currency must be high enough to prevent rich buyers from spending all of their cash, but not too high, otherwise poor buyers would not buy from rich sellers.

The remaining three inequalities describe the three key conditions for individual optimality of actions taken by the buyer gb. In particular, it is a matter of algebra to show that (i) $p_0 = 1$ and $p_b = 1$ if (16) holds[18], (ii) $p_g = 1$ if (17) holds, and (iii) if (18) holds, buyer gb only spends the good money, and not both, in a match with a seller with no money.

It is straightforward to show that if $1 < \frac{V_g}{V_b} \leq \frac{1-A_b}{1-A_g}$, then (16)-(18) are satisfied as *strict* inequalities. By continuity they are satisfied if $\frac{V_g}{V_b}$ in the right neighborhood of $\frac{1-A_b}{1-A_g}$. Thus, if ρ is small there is a unique pair $(V_g, V_b) = (V_g^*, V_b^*)$ that solves (13), which also satisfies (16)-(18). What remains to be shown is that this equilibrium pair satisfies (14)-(15).

To do so, notice that the intervals defined by the bounds in (14)-(15) are non-empty, and that $\lim_{\rho \to 0} A_g = 1$. Using (13) it is easy to verify that if ρ is sufficiently small then $V_b^* < \left(\frac{A_b^\sigma}{1-A_b} \right)^{\frac{1}{1-\sigma}}$ and $V_g^* < \left(\frac{A_g^\sigma}{1-A_g} \right)^{\frac{1}{1-\sigma}}$. Furthermore, as $\sigma \to 0$ the lower bounds of (14) and (15) approach zero, while V_b^* and V_g^* converge to positive values. Consequently, there exists a small σ and small ρ, such that $(V_g, V_b) = (V_g^*, V_b^*)$ satisfies (13), and (14)-(18), i.e. $d = 1$ and $p^* = (1, 1, 1)$ are individually optimal.

<u>Case $u(q) = q^\sigma$.</u> The equilibrium pair (V_g, V_b) solves

$$V_b = \left\{ \frac{A_b}{1-\mu} \left[m_0 + m_g A_b^\sigma + m_b A_b^\sigma \right] \right\}^{\frac{1}{1-\sigma}}$$

$$V_g = \left\{ \frac{A_b}{1-\mu} \left[m_0 + m_g A_g^\sigma + m_b \left(1 - (1 - A_b) \frac{V_b}{V_g} \right)^\sigma \right] \right\}^{\frac{1}{1-\sigma}}$$

From Lemma 1 we know that if $\rho \in (0, \rho_1)$ then there is a unique solution $(V_g, V_b) = (V_g^*, V_b^*)$, such that $1 < \frac{V_g^*}{V_b^*} \leq \frac{1-A_b}{1-A_g}$.

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[18] For example, substituting q_s^i in $u(q_s^g) + V_b - V_{gb} > u(q_s^b) + V_g - V_{gb}$ implies that $p_b = 1$ if $[V_g - (1 - A_b) V_b]^\sigma > (A_b V_b)^\sigma \Rightarrow V_g > V_b$.

Following a procedure similar to the one described before, it is a matter of algebra to show that (7)-(9) imply:

$$[(1 + A_i)^\sigma - 1]^{\frac{1}{1-\sigma}} < V_i < A_i^{\frac{\sigma}{1-\sigma}}, \quad i = g, b$$
$$V_g^\sigma - V_g > V_b^\sigma - V_b$$
$$[V_g - (1 - A_b)V_b]^\sigma - V_g > (A_b V_b)^\sigma - V_b \tag{19}$$
$$V_b + V_g^\sigma > (V_g + A_b V_b)^\sigma$$

a set of constraints that mirrors (14)-(18). Notice that $V^\sigma - V$ is hump shaped in V, reaching a maximum when $V = \sigma^{\frac{1}{1-\sigma}}$. It is easily verified that $V_b^* > \sigma^{\frac{1}{1-\sigma}}$ if ρ and σ are sufficiently small. The proof of the proposition thus follows from the fact that $V_g^* > V_b^*$, so that when $(V_g, V_b) = (V_g^*, V_b^*)$ the inequality $V_g^\sigma - V_g > V_b^\sigma - V_b$ is violated as soon as ρ and σ fall enough that $V_g^* > \sigma^{\frac{1}{1-\sigma}}$.∎

Bad-Money Equilibrium
In proving Proposition 2 we conjecture $d = 1$ and $p^* = (0, 0, 0)$.

Using (4)-(6) it is easy to show that $V_{gb} = A_{gb}V_g + V_b$. The equilibrium V_g and V_b must be a fixed point of the map defined by:

$$V_g = \frac{A_g[m_0 u(V_g) + m_b u(A_{gb}V_g) + m_g u(A_g V_g)]}{1-\mu} \tag{20}$$

$$V_b = \frac{A_b[m_0 u(V_b) + m_b u(A_b V_b) + m_g u(V_b - (1-A_{gb})V_g)]}{1-\mu} \tag{21}$$

We make use of the following lemma.

Lemma 2. *If ρ is sufficiently small, there exists a unique fixed point of (20)-(21) that is consistent with the dual-currency equilibrium $p^* = (0, 0, 0)$. Precisely $(V_g, V_b) = (V_g^*, V_b^*)$ where $1 < \frac{V_g^*}{V_b^*} < \frac{1 - A_b}{1 - A_{gb}}$.*

Proof of Lemma 2.
As before, $V_b = V_g = 0$ solves (20)-(21). The limiting case of currency competition, when the good money has no value, despite being the safest currency, and only the bad money circulates, is also an equilibrium. There is a unique pair $V_b > V_g = 0$ that solves (20)-(21). Note that $V_g > V_b = 0$ is not a possible solution.

Our focus is a dual-currency equilibrium, where both monies have a positive value. Thus, we are interested in the existence of a strictly positive fixed point $(V_g, V_b) = (V_g^*, V_b^*)$ of the map given by (20)-(21).

Let $V_g = V$. In equilibrium (20) defines the map:

$$[\rho + x(1 - \mu)]V = x[m_0 u(V) + m_b u(A_{gb}V) + m_g u(A_g V)] \equiv F(V)$$

$F(V)$ is a strictly concave function on $V \geq 0$, starting at 0, and is hump-shaped. In particular, recalling that $\lim_{q \to \infty} u'(q) \leq 1$, we see that $\lim_{V \to \infty} F'(V) < x(1 - \mu)$. Thus, (20) has two fixed points: $V = 0$ and $V = V_g^* > 0$.

Now let $V_g = V_g^*$. Letting $V_b = V$, in equilibrium (21) defines the map

$$[\rho + x(1 - \mu)]V = x[m_0 u(V) + m_b u(A_b V) + m_g u(V - (1 - A_{gb})V_g^*)] - \tau V \equiv H(V, V_g^*)$$

where we define $H(V, V_b^*)$ for $V \geq V_L = (1 - A_{gb}) V_g^*$ (necessary since $V_b - (1 - A_{gb})V_g^* = q_g^b \geq 0$, in equilibrium). $H(V, V_g^*)$ is strictly concave in V, $H(V_L, V_g^*) > 0$, $\lim_{V \to^+ V_L} \frac{\partial H(V, V_g^*)}{\partial V} = \infty$, and $\lim_{V \to \infty} \frac{\partial H(V, V_b^*)}{\partial V} \leq x(1 - \mu)$. Thus, there can be at most two positive fixed points to the map $[\rho + x(1 - \mu)]V = H(V, V_b^*)$. To see how these fixed points compare to V_g^*, let $H(V_g^*) = H(V, V_g^*)\big|_{V = V_g^*}$.

Due to strict concavity of $H(V, V_g^*)$, a sufficient condition for $V = V_b^* < V_g^*$ to be the unique fixed point is

$$H(V_g^*) < [\rho + x(1 - \mu)]V_g^* \quad \Leftrightarrow \quad H(V_g^*) < F(V_g^*). \tag{22}$$

and

$$H(V_L, V_g^*) > [\rho + x(1 - \mu)]V_L \quad \Leftrightarrow \quad H\left((1 - A_{gb}) V_g^*, V_g^*\right) > (1 - A_{gb}) F(V_g^*) \tag{23}$$

(see Figure A2). Consider first (22). Rearrange it as

$$m_b u(A_{gb} V_g^*) + m_g u(A_g V_g^*) > \quad m_b u(A_b V_g^*) + m_g u(A_{gb} V_g^*) - \tfrac{\tau}{x} V_g^*$$

It is satisfied by ρ sufficiently small since (i) $A_{gb} > A_g > A_b$ and (ii) $\lim_{\rho \to 0} A_{gb} = \lim_{\rho \to 0} A_g = 1 > \lim_{\rho \to 0} A_b$.

Now consider (23). Rearrange it as

$$m_0 u((1 - A_{gb}) V_g^*) + m_b u(A_b (1 - A_{gb}) V_g^*) - \tfrac{\tau}{x} (1 - A_{gb}) V_g^* >$$
$$(1 - A_{gb})[m_0 u(V_g^*) + m_b u(A_{gb} V_g^*) + m_g u(A_g V_g^*)].$$

Since A_{gb} falls in ρ, $\lim_{\rho \to 0} A_{gb} = 1$ and $\lim_{q \to 0} u'(q) = \infty$, then it follows that the inequality above is satisfied by ρ sufficiently small.

Hence if ρ is sufficiently small then there is a unique $(V_g, V_b) = (V_g^*, V_b^*)$, such that $1 < \frac{V_g^*}{V_b^*} < \frac{1 - A_b}{1 - A_{gb}}$. Notice that $\frac{\partial V_b^*}{\partial \tau} < 0$, since $\frac{\partial A_b}{\partial \tau} < 0$ and $\left| \frac{\partial A_b}{\partial \tau} \right| > \frac{\partial A_{gb}}{\partial \tau} > 0$. Furthermore, as $\tau \to 0$ then $V_b^* \to^- V_g^*$ since $A_{gb} \to^+ A_g$. ∎

Proof of Proposition 2.
Consider an equilibrium distribution that satisfies (1)-(2), the equations

$$\dot{m}_{2g} = m_g m_g - m_{2g} (m_0 + m_b)$$
$$\dot{m}_{2b} = x[m_b^2 + m_b m_{gb} - m_{2b} (m_0 + m_g)] + \eta m_b - \tau m_{2b}$$
$$\dot{m}_{gb} = x[m_g m_{2b} + m_b m_{2g} + 2 m_b m_g - m_{gb} (m_0 + m_b)] + \eta m_g - \tau m_{gb}$$

and $\dot{m}_j = 0$. From a prior discussion (see our technical appendix) we know that it exists, under certain conditions.

<u>Case $u(q) = q^\sigma + q$.</u> The solution $(V_g, V_b) = (V_g^*, V_b^*)$ must satisfy

$$V_g = \left\{ \frac{A_g \left[m_0 + m_b A_{gb}^\sigma + m_g A_g^\sigma \right]}{1 - \mu - A_g \left[m_0 + m_b A_{gb} + m_g A_g \right]} \right\}^{\frac{1}{1 - \sigma}}$$

$$V_b = \left\{ \frac{A_b \left[m_0 + m_b A_b^\sigma + m_g \left(1 - (1 - A_{gb}) \frac{V_g}{V_b} \right)^\sigma \right]}{1 - \mu - A_b \left[m_0 + m_b A_b + m_g \left(1 - (1 - A_{gb}) \frac{V_g}{V_b} \right) \right]} \right\}^{\frac{1}{1 - \sigma}}.$$

20

Note that the denominators in both expressions are positive, given the definitions of A_g, A_b, and A_{gb}. Furthermore, from Lemma 2 we know that if ρ is sufficiently small, there is a unique solution to the system of equations above, such that $1 < \frac{V_g^*}{V_b^*} < \frac{1-A_b}{1-A_{gb}}$.

Once again, it is a matter of algebra to verify that the individual optimality conditions (7)-(9) reduce to the (smaller) set of inequalities (14) and (15) and

$$V_g < V_b \tag{24}$$
$$(A_b V_b)^\sigma + (1 - A_{gb}) V_g > (1 - A_b) V_b + (A_{gb} V_g)^\sigma \tag{25}$$
$$(1 - A_{gb}) V_g + V_b^\sigma > (V_b + A_{gb} V_g)^\sigma \tag{26}$$

Inequalities (14) and (15) have the same interpretation as before. Inequalities (24) and (25) are the conditions needed to ensure that the $p^* = (0,0,0)$ strategy is optimal. The last inequality (26) ensures the gb buyer only spends the bad currency and not both. The key condition is $V_b > V_g$, for this equilibrium to exist when $u(q) = q^\sigma + q$. From Lemma 2, however, we know that (24) is violated if ρ is sufficiently small. It follows that $p^* = (0,0,0)$ and $d = 1$ cannot be an equilibrium if ρ is small.

Case $u(q) = q^\sigma$. The solution $(V_g, V_b) = (V_g^*, V_b^*)$ must satisfy

$$V_g = \left\{ \frac{A_b}{1-\mu} \left[m_0 + m_b A_{gb}^\sigma + m_g A_g^\sigma \right] \right\}^{\frac{1}{1-\sigma}}$$
$$V_b = \left\{ \frac{A_b}{1-\mu} \left[m_0 + m_b A_b^\sigma + m_g \left(1 - (1 - A_{gb}) \frac{V_g}{V_b} \right)^\sigma \right] \right\}^{\frac{1}{1-\sigma}}$$

By Lemma 2, if ρ is sufficiently small, then the solution to these equations is unique and such that $1 < \frac{V_g^*}{V_b^*} < \frac{1-A_b}{1-A_{gb}}$. It takes just some algebra to show that the conditions specified in (8)-(9) reduce to

$$[(1 + A_i)^\sigma - 1]^{\frac{1}{1-\sigma}} < V_i < A_i^{\frac{\sigma}{1-\sigma}} \qquad \text{for } i = g, b$$
$$V_g + V_b^\sigma > (V_b + A_{gb} V_g)^\sigma$$
$$V_g^\sigma - V_g < V_b^\sigma - V_b$$
$$(A_g V_g)^\sigma - V_g < (V_b - V_g(1 - A_{gb}))^\sigma - V_b$$
$$(A_{gb} V_g)^\sigma - V_g < (A_b V_b)^\sigma - V_b$$

The inequalities in the first three lines are satisfied when ρ and σ are sufficiently small since A_{gb} and V_g^* approach 1 as ρ and σ approach zero, while A_b and V_b^* converge to values less than one (since $\tau > 0$). The inequality in the fourth line is satisfied when $\frac{V_g}{V_b} < \frac{1-A_b}{1-A_{gb}}$. The inequality in the last line is satisfied when $1 < \frac{V_g}{V_b}$. Since if ρ and σ are sufficiently small the unique solution is $(V_g, V_b) = (V_g^*, V_b^*)$ such that $1 < \frac{V_g^*}{V_b^*} < \frac{1-A_b}{1-A_{gb}}$, then the bad-money equilibrium exists and is unique.∎
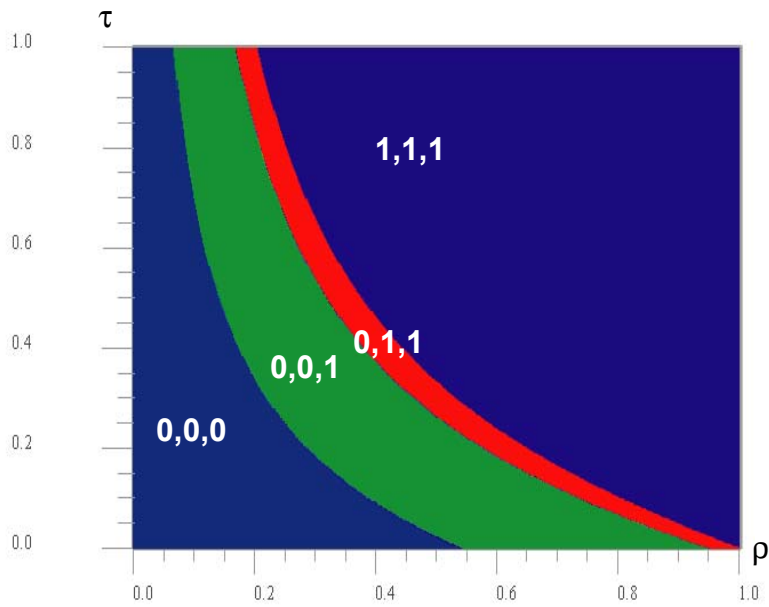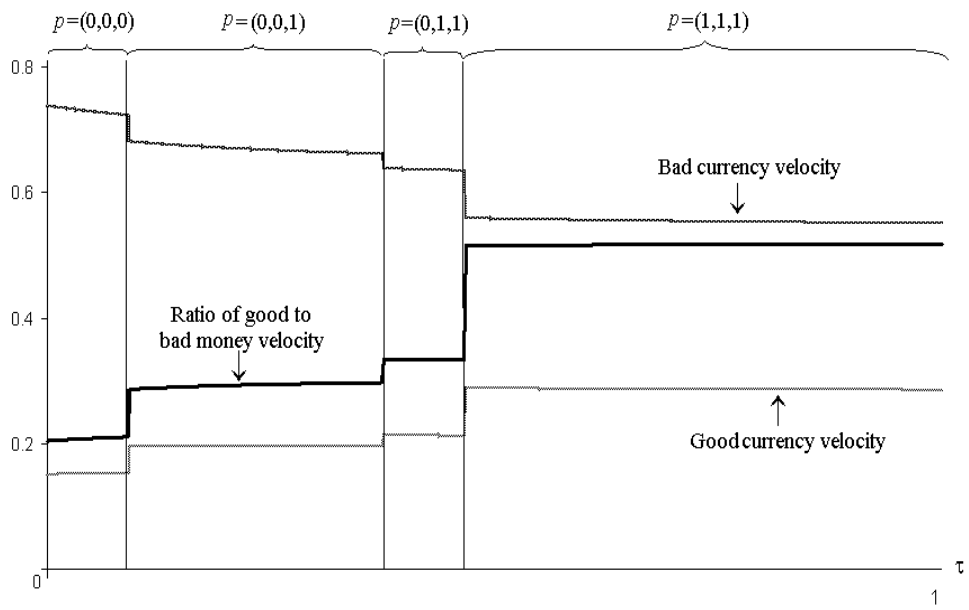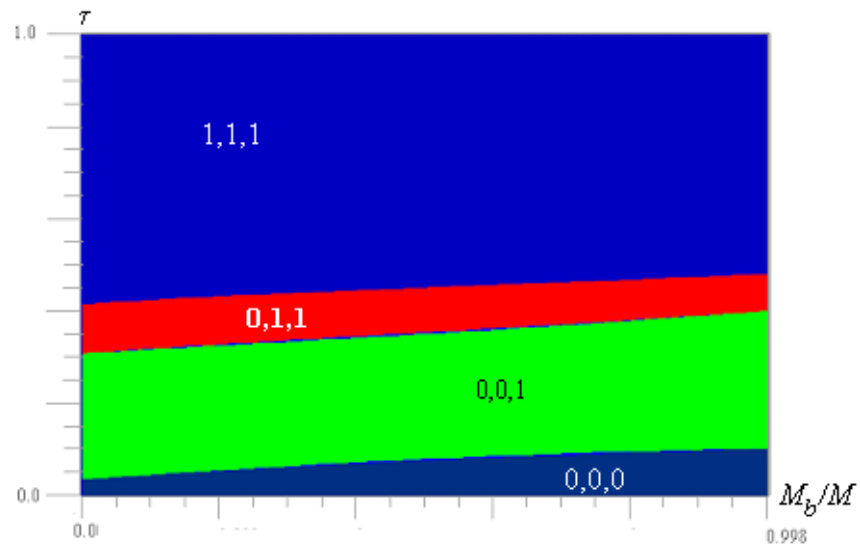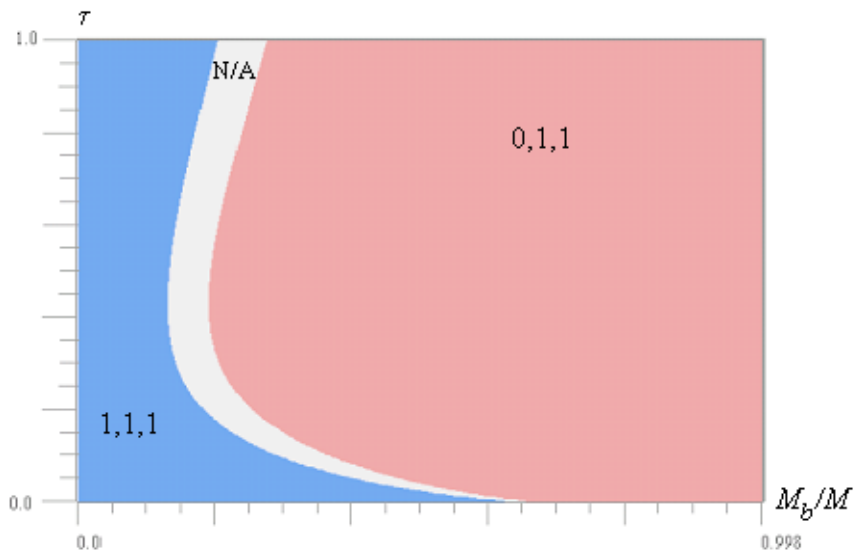
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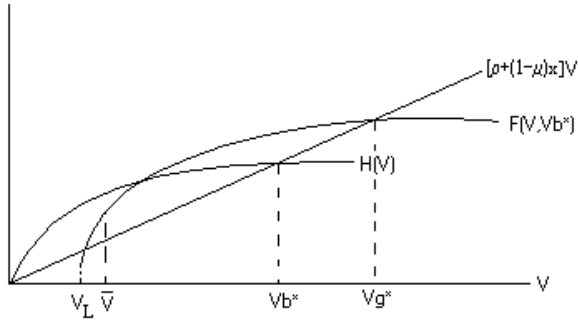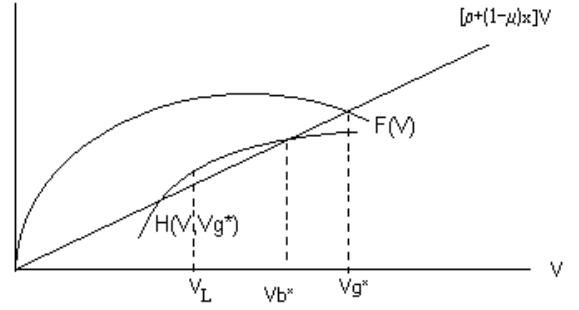

Figure 2

Figure 3

Figure 4

Figure A1

Figure A2

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