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Information and Liquidity: A Discussion

by Guillaume Rocheteau

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I extend and discuss the model of asset liquidity by Lester, Postlewaite, and Wright (2007, 2008). I consider a model with decentralized trades in which claims on a real and divisible asset serve as means of payment. A recognizability problem is introduced by assuming that the claims on the asset can be counterfeited at a positive cost. This formalization nests the models by Lagos and Rocheteau (2008) and Geromichalos, Licari, and Suarez-Lledo (2007) in which there is no recognizability problem, and Lester, Postlewaite, and Wright (2007), in which counterfeits can be produced at no cost. Even though no counterfeiting occurs in equilibrium, the recognizability problem affects the composition of trades: buyers consume less and spend a lower fraction of their asset holdings in matches where sellers are uninformed. Both the asset price and its liquidity (as measured by its transaction velocity) depend on the recognizability of the asset. The asset is more liquid and its return is lower if either the sellers' ability to recognize counterfeits or the cost of producing counterfeits increases.

Key words: Money, counterfeiting, search, bargaining.

JEL code: D82, D83, E40

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1 Introduction

Lester, Postlewaite, and Wright (2007, 2008) propose a theory of liquidity based on the (lack of) recognizability of an asset. They adopt the environment in Lagos and Wright (2005), where trades alternate in centralized and decentralized markets and the lack of a double coincidence of wants makes the use of a medium of exchange necessary. Following Lagos and Rocheteau (2008) and Geromichalos, Licari, and Suarez-Lledo (2007), they assume that two assets can play this role: fiat money, an intrinsically useless object, and a real asset. The novelty is the introduction of a recognizability problem with respect to the real asset in order to overcome the rate-of-return-equality principle. The real asset (or claims on the asset) can be counterfeited at no cost once the buyer has learnt whether the seller he is matched with has the technology to detect counterfeits.[1] Because producing a counterfeit when the seller is uninformed is a dominant strategy, the real asset cannot be used as means of payment in a fraction of the matches.

The assumption that counterfeits can be produced at no cost makes the moral hazard problem very severe. In particular, one wonders how much this assumption drives the result that no trade takes place in uninformed matches. In Lester, Postlewaite, and Wright's (2007) words, this assumption is convenient because "*it allows a simple solution to the bargaining problem, despite the asymmetric information*", but it is also extreme, "*and one might like to see generalizations or alternative specifications where agents sometimes trade for and bargain over assets whose quality is unknown*". They go on conjecturing that such extensions are considerably more difficult. The objective of this paper is to provide such an extension in which a single real and perfectly divisible asset can be counterfeited at a fixed cost $k > 0$. I will show that the model remains tractable, and it nests the models of Lagos and Rocheteau (2008), Geromichalos, Licari, and Suarez-Lledo (2007) when k is sufficiently large, and the model of Lester, Postlewaite, and Wright (2007) as k approaches 0.[2]

As suggested above, the intricate part is the determination of the terms of trade in matches

[1] Lester, Postlewaite, and Wright (2008) consider different games with indivisible goods and assets and a positive cost of producing counterfeits and explore different assumptions in terms of timing. Kim and Lee (2008) developed a related model with a similar problematic. Freeman (1985) uses the assumption that claims on capital can be counterfeited in the context of an overlapping-generations model with fiat money and capital.

[2] In Lagos and Rocheteau (2008) the real asset, interpreted as capital, can be produced. Geromichalos, Licari, and Suarez-Lledo (2007) extend the model to the case where the asset is in fixed supply.

where sellers are uninformed. Typically, bargaining games with incomplete information have a large number of (sequential) equilibria. Following most of the literature, I adopt a simple bargaining game where the buyer makes a take-it-or-leave-it offer, and I show that this trading mechanism is socially optimal. Using the methodology from Inn and Wright (2008) for signaling games with unobservable choices I am able to select a unique equilibrium. The equilibrium is such that agents trade in all matches, but the amount of asset a buyer transfers in matches where sellers are uninformed is bounded above by a quantity that is increasing with the cost of producing counterfeits.[3] The model provides microfoundations for the liquidity constraints found in Kiyotaki and Moore (2005) and Lagos (2006). As the cost of producing a counterfeit goes to 0, agents stop trading the asset in uninformed matches, as in Lagos. If the fraction of informed matches goes to 0 and the cost of producing counterfeits is not too large, then agents only spend a fraction of their asset holdings in all matches, as in Kiyotaki and Moore.

The model predicts that no counterfeiting occurs in equilibrium. This result is reminiscent to the no-counterfeiting proposition in Nosal and Wallace (2007, Proposition 1). Still, the threat of counterfeiting affects the equilibrium outcome. If the cost of producing counterfeits is not too large, then the quantities traded in uninformed matches are inefficiently low. Moreover, there is a distribution of terms of trade: the propensity to spend the asset and the buyer's consumption are larger in informed matches.

Because the asset has a role as a means of payment, its price can exhibit a liquidity premium. The rate of return of the asset is then less than the discount rate. The liquidity premium increases, and the rate of return of the asset decreases, as the asset becomes easier to recognize. This suggests that the model has the potential to explain the rate-of-return differentials between assets based on their recognizability properties. Moreover, I show that the price of the asset can fall discontinuously as the cost of producing counterfeits decreases. So a change in regulation that makes it easier to counterfeit an asset can manifest itself as a sharp decline in the asset price.

Finally, I study how the ease of recognizing counterfeits and the cost of producing them af-

[3] This result differs from the one in Lester, Postlewaite, and Wright (2008) in the case of indivisible goods. Their key result is that there exist parameter values for which sellers do not accept objects that they do not recognize. I establish that this result does not go through to the divisible asset case: in equilibrium agents trade in all matches, but the quantities traded in uninformed matches are smaller than the ones in informed matches.

fects the liquidity of the asset, as measured by its velocity. The asset becomes more liquid if its recognizability improves. If it is easier to recognize counterfeits, then the frequency of larger trades increases. If it is more costly to produce counterfeits, then the quantity traded in uninformed matches increases and the distribution of trade sizes becomes less dispersed. In both cases, the higher recognizability of the asset raises society's welfare.

2 Environment[4]

Time is discrete, starts at $t = 0$, and continues forever. Each period has two subperiods, a morning where trades occur in a decentralized market (DM), followed by an afternoon where trades take place in competitive markets (CM). There is a continuum of infinitely lived agents divided into two types, called *buyers* and *sellers*, who differ in terms of when they produce and consume. The labels *buyers* and *sellers* indicate agents' roles in the DM market. The measures of buyers and sellers are equal to 1. There are two consumption goods, one produced in the DM and the other in the CM. Consumption goods are perishable.

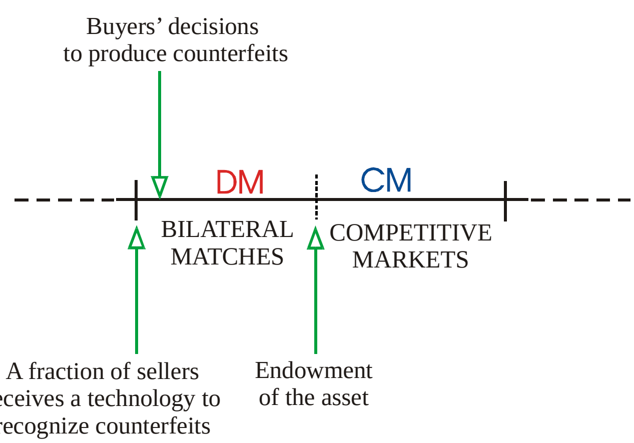

Figure 1: Timing (Representative period)

Buyers and sellers are treated symmetrically in the CM: they can both produce and consume. In the DM, however, buyers only consume, while sellers only produce. This heterogeneity will

[4] The environment is similar to the one in Rocheteau and Wright (2005). It differs from Lester, Postlewaite, and Wright (2007) in that the probability of a match for each agent in the DM is one, and agents' role in the DM is determined ex-ante. These changes are inessential for the results.

generate a temporal double-coincidence problem. The lifetime expected utility of a buyer from date 0 onward is

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}\left[x_t-\ell_t+u(q_t)\right], \tag{1}$$

where x_t is the CM consumption of period t, ℓ_t is the CM disutility of work, q_t is the DM consumption, and $\beta \in (0,1)$ is a discount factor. The utility function $u(q)$ is twice continuously differentiable, $u(0)=0$, $u'(0)=\infty$, $u'(q)>0$, and $u''(q)<0$. The production technology in the CM is linear, with labor as the only input, $y_t=\ell_t$.

The lifetime expected utility of a seller from date 0 onward is

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}\left[x_t-\ell_t-c(q_t)\right], \tag{2}$$

where q_t is the DM production. The cost function $c(q)$ is twice continuously differentiable, $c(0)=c'(0)=0$, $c'(q)>0$, $c''(q)\geq 0$ and $c(q)=u(q)$ for some $q>0$. Let q^* denote the solution to $u'(q^*)=c'(q^*)$.

At the beginning of the CM, each buyer is endowed with $A>0$ units of a one-period-lived real asset. Each unit of the period$-t$ asset yields one unit of CM-output, which is delivered in the CM of $t+1$, and which fully depreciates subsequently.[5] The *fundamental price* of the asset, as defined by its discounted sum of dividends, is simply β.

In the CM, there is a competitive market where agents can trade goods and the real asset. In the DM, each seller is matched bilaterally and at random with a buyer. Terms of trade are determined according to a simple bargaining game: The buyer makes an offer, which the seller accepts or rejects. Agents are free to transfer DM-output and the assets held by the matched agents.[6] In order to guarantee that there is an essential role for a medium of exchange, there is no public record of individuals' trading histories, and agents cannot commit.

In the DM, buyers have the technology to produce a counterfeited asset at some fixed cost $k>0$. The marginal cost of producing counterfeits is zero. The counterfeited asset yields no output, and

[5] It would be straightforward to consider a long-lived asset but it is not needed for the argument. This would be interesting, however, to obtain a feedback effect from the price of the asset to its liquidity value.

[6] One can think of agents trading pieces of paper representing the ownership of the asset or simply agents trading the physical asset itself.

it fully depreciates in the CM.[7] In the DM, each seller receives with probability η a technology allowing him to distinguish genuine assets from counterfeits.[8] The buyer chooses whether or not to produce a counterfeit after he learns the ability of the seller he is matched with to recognize counterfeits.

3 Counterfeiting and bargaining

I first study the game between a buyer holding a units of the asset and an uninformed seller. I restrict my attention to equilibria where sellers do not to hold assets: it will be checked later that it is an optimal strategy. In order to write the payoffs of this game, I characterize some properties of the buyers' and sellers' value functions in the CM.

The expected utility of a buyer holding a_t units of the asset at the beginning of the CM in period t is given by

$$W^b(a_t) = \max_{x_t, \ell_t, a'} \left\{ x_t - \ell_t + \beta V^b \left(a' \right) \right\} \tag{3}$$

$$\text{s.t.} \quad x_t + \phi_t a' = \ell_t + a_t + \phi_t A, \tag{4}$$

where ϕ_t is the price of the real asset in terms of period-t CM output, and $V^b(a)$ is the value function of the buyer in the DM. According to (4), the buyer finances his net consumption $(x_t - \ell_t)$ and his net purchase of the asset $(\phi_t(a' - A))$ from the output generated by his a_t units of the asset upon entering the CM. Substitute $x_t - \ell_t = a_t + \phi_t(A - a')$ from (4) into (3) to obtain

$$W^b(a_t) = a_t + \phi_t A + \max_{a' \geq 0} \left\{ -\phi_t a' + \beta V^b \left(a' \right) \right\}. \tag{5}$$

The buyer's utility in the CM is linear in his wealth. Moreover, a buyer's choice of portfolio is independent of his initial portfolio when he entered the period. Following a similar reasoning, the

[7] While the formalism is different, there is a clear analogy with Kiyotaki and Moore (2005, p.320). They assume that in order to finance an investment opportunity, an agent can sell his land and capital holdings. "But, crucially, the transfer of ownership of capital is not instantaneous; delivery is overnight. (...) After receiving goods from an agreed sale of capital (in the evening before the overnight transfer of ownership), the agent can abscond and start a new life the next day with a fresh identity and clear record.We assume that he cannot take all his capital with him, though; he can steal at most a fraction $1 - \theta \in (0, 1)$." For related moral hazard considerations, see the corporate finance model of liquidity and asset pricing by Holmstrom and Tirole (2001).

[8] In my model, I will treat η and k as independent parameters. It would also be plausible to assume that the ease of recognizing an asset is related to the cost of producing a counterfeit.

expected utility of a seller at the beginning of period t is given by

$$W^s(a_t) = a_t + \max_{a' \geq 0} \left\{-\phi_t a' + \beta V^s\left(a'\right)\right\}. \tag{6}$$

The game in a match where the seller is uninformed is represented in Figure 2. I adopt the same timing as in Lester, Postlewaite, and Wright (2008), according to which the counterfeiting decision precedes the bargaining stage. At the beginning of the game, the buyer chooses to buy the good either with the genuine asset ($\chi = g$) or the counterfeits that he produces at a fixed cost ($\chi = c$). Then, he makes an offer $(q, d) \in \mathbb{R}_+ \times [0, a]$, where q is the production/consumption of the DM good, and d is the transfer of the asset. The seller accepts ($r = Y$) or rejects ($r = N$) the offer. The payoff of the buyer is

$$-k\mathbb{I}_{\{\chi=c\}} + \left[u(q) + W^b\left(a - d\mathbb{I}_{\{\chi=g\}}\right)\right]\mathbb{I}_{\{r=Y\}} + W^b(a)\mathbb{I}_{\{r=N\}},$$

where $\mathbb{I}_{\{\chi=c\}}$ is an indicator function that equals one if $\chi = c$. Using the linearity of W^b and eliminating the constant terms, the buyer's payoff can be redefined as

$$U^b(\chi, q, d, r) = -k\mathbb{I}_{\{\chi=c\}} + \left\{u(q) - d\mathbb{I}_{\{\chi=g\}}\right\}\mathbb{I}_{\{r=Y\}} \tag{7}$$

Similarly, the payoff of the seller is

$$U^s(\chi, q, d, r) = \left\{-c(q) + d\mathbb{I}_{\{\chi=g\}}\right\}\mathbb{I}_{\{r=Y\}}. \tag{8}$$

A (behavioral) strategy for a buyer is composed of the probability θ to hand over a genuine claim ($\chi = g$) and the distributions $F(q, d; \chi)$ from which to draw the offer, conditional on the choice $\chi \in \{c, g\}$. A strategy for the seller is a mapping $p : \mathbb{R}_+ \times [0, a] \to [0, 1]$, which assigns an acceptance probability to any feasible offer.

In order to check the sequential rationality of the seller's acceptance rule, one needs to specify the seller's belief regarding the buyer's action, χ, conditional on the offer (q, d) being made. Sequential equilibrium imposes little discipline on those beliefs, which can lead to a plethora of equilibria.[9] A

[9] A sequential equilibrium of an extensive game with imperfect information is an assessment composed of a profile of behavioral strategies and a belief system such that strategies are sequentially rational given the belief system, and the belief system is consistent with the strategies. For a definition of the consistency requirement see Osborne and Rubinstein (1994, Definition 224.2). Alternatively, one could use the notion of a perfect extended Bayesian equilibrium from Fudenberg and Tirole (1991, p.348) .

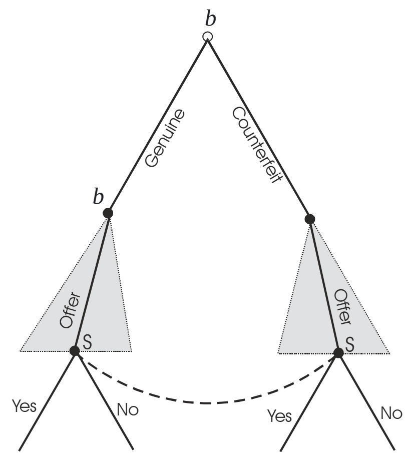

Figure 2: Original game (uninformed match)

simple way to refine the set of equilibria is to consider the reverse-ordered game, where the buyer makes first an offer (q, d), and then he chooses whether to produce a counterfeit.[10] (See Figure 3.) The order of the buyers' moves does not affect the payoffs, and it does not convey any information to the seller. Moreover, here both timings are equally plausible: the buyer could make an offer first and then decide whether or not to forge a claim on his assets (before the seller has accepted or rejected the offer). The benefit derived from considering this reverse-ordered game is that subgame perfection is sufficient to solve the game, and it predicts a unique outcome.

In the reverse-ordered game, a strategy for the buyer is a distribution of offers $F(q, d)$ and a decision rule $\theta : \mathbb{R}_+ \times [0, a] \to [0, 1]$ that assigns a probability of handing over a genuine claim for all feasible offers. Consider a subgame following the offer (q, d).[11] Sequential rationality requires that the players' strategies that are restricted to this subgame form a Nash equilibrium. The seller's belief that he faces a genuine buyer is denoted $\lambda(q, d)$. The consistency of sellers' beliefs with strategies implies $\lambda(q, d) = \theta(q, d)$. Given θ, the decision of sellers to accept or reject an offer satisfies

[10] This methodology, called the *reordering invariance refinement*, was developed by In and Wright (2008) for signaling games with unobservable choices. It is based on the invariance condition of strategic stability from Kohlberg and Mertens (1986). It states that the solution of a game should also be the solution of any game with the same reduced normal form.

[11] It should be noticed that this subgame corresponds essentially to the game studied in Lester, Postlewaite, and Wright (2008), in which the size of the asset transfer and the size of the consumption good are exogenously given.

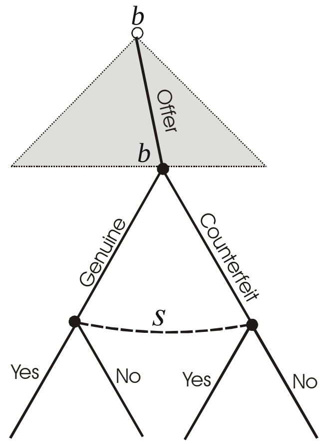

Figure 3: The reverse-ordered game

$$
-c(q)+\theta d \begin{array}{l} >0 \\ <0 \\ =0 \end{array} \Longrightarrow p \begin{array}{l} =1 \\ =0 \\ \in[0,1] \end{array} \tag{9}
$$

Given p, the buyer produces a counterfeit if $-k+pu(q) > p\left[u(q)-d\right]$, he does not produce a counterfeit if $-k+pu(q) < p\left[u(q)-d\right]$, and he is indifferent otherwise. Hence, the decision to produce a counterfeit is given by

$$
k \begin{array}{l} > \\ < \\ = \end{array} pd \Longrightarrow \theta \begin{array}{l} =1 \\ =0 \\ \in[0,1] \end{array} \tag{10}
$$

According to (10), by producing a counterfeit a buyer incurs the cost k, but he saves a valuable asset if the trade is accepted by the seller. A Nash equilibrium of the subgame following an offer (q,d) is a pair $(p,\theta)\in[0,1]^2$ that satisfies (9) and (10). Given a set of Nash equilibria $[\theta(q,d),p(q,d)]_{(q,d)\in\mathbb{R}_+\times[0,a]}$ for the subgames following all feasible offers (q,d), the offer made by the buyer at the beginning of the game solves

$$
(q,d)\in\arg\max_{d\leq a}\left\{-k\left[1-\theta(q,d)\right]+\left[u(q)-d\theta(q,d)\right]p(q,d)\right\}. \tag{11}
$$

Definition 1 *An equilibrium of the reverse-ordered game is a list $\langle (q,d),\theta(q,d),p(q,d)\rangle$, such that: (i) for all feasible offers $(q,d)\in\mathbb{R}_+\times[0,a]$, $[\theta(q,d),p(q,d)]$ solves (9)-(10); (ii) Given $[\theta(\cdot,\cdot),p(\cdot,\cdot)]$, (q,d) solves (11).*

Proposition 1 ***(Endogenous liquidity constraints)***

There is a unique outcome of the reverse-ordered game, and it is such that the offer made by the buyer is

$$(q^u, d^u) = \arg\max \left[u(q) - d\right] \tag{12}$$

$$s.t. \quad -c(q) + d \geq 0 \tag{13}$$

$$d \leq \min(k, a), \tag{14}$$

and the decision to counterfeit is $\theta(q^u, d^u) = 1$.

The solution to the reverse-ordered game is such that the buyer makes a take-it-or-leave-it offer, subject to a liquidity constraint that prevents him from spending more than k units of his asset holdings. From (12)-(14), $q^u = \min\left\{q^*, c^{-1}\left[\min(k, a)\right]\right\}$, and $d^u = c(q^u)$. The constraint $d \leq k$ is reminiscent of the one in Kiyotaki and Moore (2005) according to which agents with investment opportunities can only liquidate a fraction of their capital stock. In contrast to Kiyotaki and Moore, however, this constraint is endogenous, and depends on the ease with which the asset can be counterfeited.

Surprisingly, even though the possibility of counterfeiting affects the equilibrium outcome, no counterfeiting occurs in equilibrium. Buyers hand over genuine claims in all matches irrespective of the cost of producing counterfeits. This result is analogous to the no-counterfeiting proposition in Nosal and Wallace (2007). Intuitively, sellers will reject with a positive probability offers such that buyers would have an incentive to produce a counterfeit. This provides a discipline mechanism that prevents counterfeiting from occurring. A result deriving from this mechanism is the illiquidity of the asset in the sense that only a fraction of it would be spent in a trade when its quality is not recognized.

The reverse-ordered game can be used to specify the sellers' beliefs (called reordering invariant beliefs in In and Wright, 2008) following any offer made by a buyer, $\lambda(q, d) = \theta(q, d)$. In the proof of Proposition 1 it is shown that

$$\lambda(q,d) \begin{array}{l} = 1 \text{ if } d < k \\ = \min\left[\frac{c(q)}{d}, 1\right] \text{ if } d \geq k \end{array} \tag{15}$$

Any offer such that $d < k$ is attributed to a genuine buyer. This is consistent with the fact that any strategy in which the buyer produces a counterfeit ($\chi = c$) and makes an offer (q, d) such that $d < k$ is strictly dominated by the strategy that would consist of making the same offer but not producing a counterfeit ($\chi = g$). If $d > \max[k, c(q)]$, then the seller attributes the offer to a counterfeiter with positive probability.

The proof of Proposition 1 also provides the probability according to which a seller accepts an offer.

$$p(q,d) \begin{array}{l} = \min\left[1, \frac{k}{d}\right] \text{ if } d \geq c(q) \\ = 0 \text{ if } d < c(q) \end{array} \tag{16}$$

Small trades ($d < k$) are accepted with probability one, provided that the transfer of the asset compensates the seller for his production cost. Above a threshold ($d > k$) the probability for a trade to be accepted decreases with the size of the trade. This finding is related to the notion that larger trades are more costly and take more time to implement than smaller ones (Easley and O'Hara, 1987). The standard argument is that informed traders want to maximize the value of their inside information by trading large quantities. Here, larger trades have a lower chance to go through because they are attributed to counterfeiters with a higher probability.

The reasoning for why sellers accept a trade with probability less than one when $d \geq c(q)$ and $k < d$ is similar to the logic in Williamson and Wright (1994) in the context of indivisible assets. Suppose that sellers accept the trade with probability $p = 1$. Then, buyers have a strict incentive to produce a counterfeit since $k < d$, and hence accepting the trade cannot be part of an equilibrium. If, on the other hand, sellers reject the trade, $p = 0$, then buyers do not produce counterfeits, $\theta = 1$, which would have made it profitable for sellers to accept the trade. So an equilibrium of the subgame will involve mixed strategies.

The key result in Lester, Postlewaite, and Wright (2008) is that there are parameter values for which sellers do not accept objects that they do not recognize. In my model, sellers accept with a positive probability all offers for which they enjoy a positive surplus if the asset is genuine. Moreover, in equilibrium only offers that are accepted with probability one are made.

I conclude this section with a proposition establishing that the bargaining game I consider corresponds to an optimal mechanism.

Proposition 2 ***(Optimal mechanism)***

The optimal trading mechanism chosen by a social planner who maximizes the expected match surplus net of the cost of counterfeiting generates the same outcome as in (12)-(14).

Proposition 2 generalizes the result according to which the buyer-takes-all bargaining game is a socially efficient trading mechanism in monetary economies to an environment where the asset is subject to a recognizability problem. Moreover, the illiquidity of the asset in uninformed matches is not an artifact of a suboptimal trading mechanism. A social planner would also require buyers not to spend more than the fraction k/a of their asset holdings.

4 Asset pricing

I now incorporate the game described in Section 3 into the general equilibrium structure. The game tree is depicted in Figure 4. First, the buyer makes a portfolio choice, $a \in \mathbb{R}_+$. Then, with probability η, he is matched with a seller who has the ability to recognize a counterfeited asset and, with complement probability $1-\eta$, he is matched with an uninformed seller. The quantity of (genuine) assets held by the buyer, a, is common knowledge in the match.[12] If the seller is uninformed, the buyer decides whether or not to produce a counterfeit. The game ends with a take-it-or-leave-it offer by the buyer. Recall that the subgame can be replaced by the reverse-ordered game described in Figure 3 if the seller is uninformed.

In matches in which the seller can distinguish counterfeited assets from genuine ones, the buyer does not produce a counterfeit, and the terms of trade are

$$(q^i, d^i) = \arg\max\left[u(q) - d\right] \text{ s.t. } -c(q) + d \geq 0 \text{ and } d \leq a. \tag{17}$$

The solution is $q^i = \min\left[c^{-1}(a), q^*\right]$ and $d^i = c(q^i)$. The terms of trade in uninformed matches, (q^u, d^u), solve (12)-(14).

The value function of the buyer in the DM is then

$$V^b(a) = \eta\left\{u(q^i) + W^b(a - d^i)\right\} + (1-\eta)\left\{u(q^u) + W^b(a - d^u)\right\}.$$

[12] This assumption is innocuous since the quantity of assets held by the buyer does not affect the sellers' beliefs or their decisions to accept or reject offers. So even if the buyer's portfolio was not observed by the seller, one could still construct an equilibrium with the same strategies as in (15) and (16).

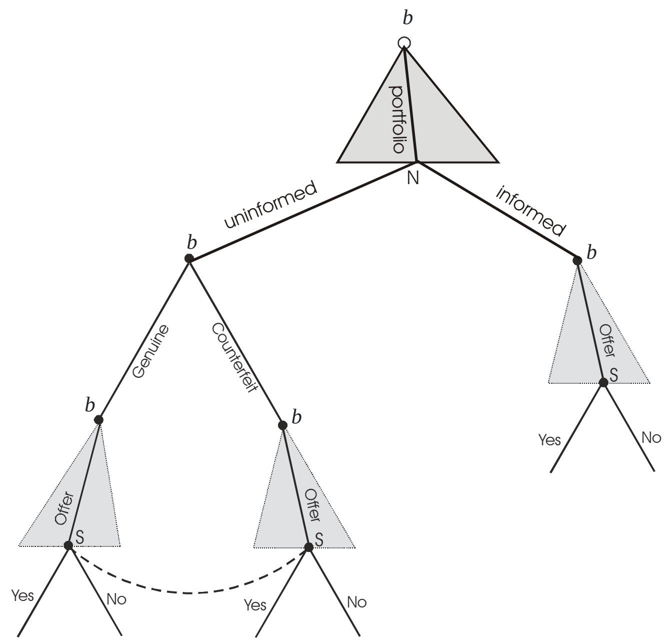

Figure 4: Game tree (whole game)

Using the linearity of W^b, and the results $d^i = c(q^i)$ and $d^u = c(q^u)$, the buyer's expected utility in the DM solves

$$V^b(a) = \eta S^i(a) + (1-\eta)S^u(a;k) + W^b(a), \tag{18}$$

where $S^\ell(a) = u(q^\ell) - c(q^\ell)$ is the buyer's surplus in a match of type $\ell \in \{i, u\}$. See Figure 5 for a graphical representation of the surplus functions. According to (18) the expected utility of the buyer when he enters the DM is his expected surplus from a bilateral match and the continuation value in the next CM.

Substitute $V^b(a)$ by its expression in (18) into (5) to rewrite the buyer's portfolio problem as

$$\max_{a\geq 0}\left\{-\left(\frac{\phi-\beta}{\beta}\right)a + \eta S^i(a) + (1-\eta)S^u(a;k)\right\}. \tag{19}$$

The buyer chooses his portfolio so as to maximize his expected surplus in the DM, net of the cost of holding the asset. The cost of investing in the asset is the difference between the price of the asset and its fundamental value (β). The objective function in (19) is continuous, concave, and differentiable except at $a = k$ if $k < c(q^*)$. The necessary and sufficient conditions for an optimum

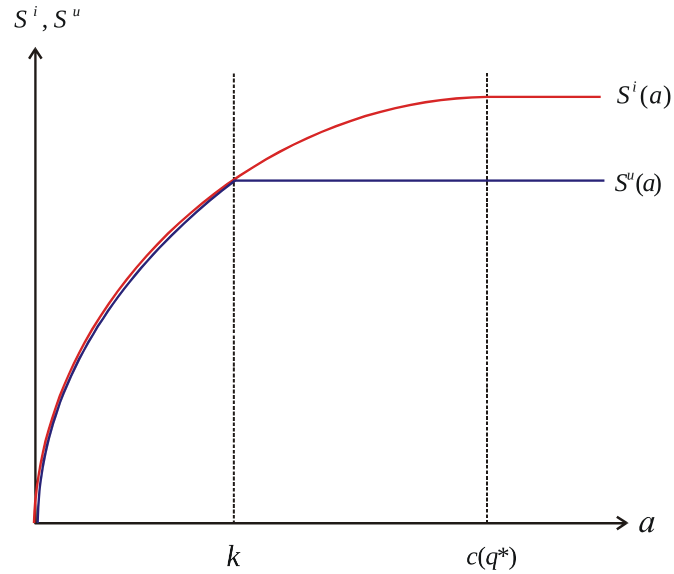

Figure 5: Buyers' surpluses in informed (S^i) and uninformed (S^u) matches.

are

$$\eta \frac{\partial S^i(a)}{\partial a} + (1-\eta)\frac{\partial S^u(a;k)}{\partial a} = \frac{\phi - \beta}{\beta} \tag{20}$$

if $k \geq c(q^*)$ or $a \neq k$, and

$$\eta \left. \frac{\partial S^i(a)}{\partial a}\right|_{a=k} \leq \frac{\phi-\beta}{\beta} \leq \left. \frac{\partial S^i(a)}{\partial a}\right|_{a=k} \tag{21}$$

where I used the result according to which the right-hand derivative of $\frac{\partial S^u(a;k)}{\partial a}$ evaluated at $a = k$ is equal to 0, and the left-hand derivative is equal to $\frac{\partial S^i(a)}{\partial a}$.

Let $\mathcal{L}(a) = \frac{u'[c^{-1}(a)]}{c'[c^{-1}(a)]} - 1 = \frac{\partial S^i(a)}{\partial a}$ if $a \leq c(q^*)$. It represents the liquidity value of an additional unit of asset in an informed match.

Lemma 1 *If $\phi > \beta$, then the solution to (19) is the singleton*

$$a^b\left(\frac{\phi-\beta}{\beta}\right) = \begin{array}{l} \left\{\mathcal{L}^{-1}\left(\frac{\phi-\beta}{\beta}\right)\right\} \;\; \textit{if } \frac{\phi-\beta}{\beta} > \mathcal{L}(k) \\ \left\{\mathcal{L}^{-1}\left(\frac{\phi-\beta}{\eta\beta}\right)\right\} \;\; \textit{if } \frac{\phi-\beta}{\beta} \in (0, \eta\mathcal{L}(k)) \\ \{k\} \;\; \textit{if } \frac{\phi-\beta}{\beta} \in [\eta\mathcal{L}(k); \mathcal{L}(k)] \end{array}. \tag{22}$$

If $\phi = \beta$, then the solution to (19) is the interval

$$a^b\left(\frac{\phi-\beta}{\beta}\right) = [c(q^*), \infty). \tag{23}$$

If $\phi < \beta$, then there is no solution to (19).

If the price of the asset is greater than its fundamental value ($\phi > \beta$), then the demand for the asset is unique. Buyers adjust their portfolios so that the liquidity gain from holding the asset is exactly equal to the cost as measured by the difference between the asset price and its fundamental value. Moreover, the demand for the asset decreases with its price. If $\phi = \beta$, then buyers accumulate enough assets to satiate their liquidity needs in informed matches.

The asset demand is represented in Figure 6. The level of assets is on the horizontal axis while the cost of holding the asset, the liquidity premium, is on the vertical axis. If $a < k$, then the liquidity constraint is not binding in uninformed matches and the liquidity premium is $\mathcal{L}(a)$. If $a > k$, then the liquidity constraint ($d \leq k$) binds and the liquidity premium is $\eta\mathcal{L}(a) < \mathcal{L}(a)$. The asset demand is flat for all prices in the interval $[\beta + \eta\mathcal{L}(k), \beta + \mathcal{L}(k)]$. This flat portion is a consequence of the nondifferentiability of the buyer's expected surplus at $a = k$ if $k < c(q^*)$. (See Figure 5.)

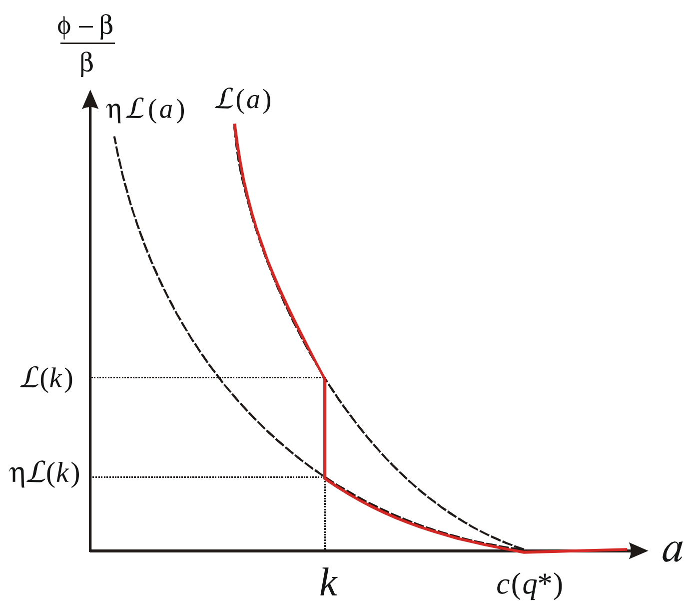

Figure 6: Asset demand

It should be noticed that sellers get no surplus in the DM, and hence they have no strict incentives to hold the asset. Their portfolio problem reduces to $\max_{a\geq 0}\left\{-\left(\frac{\phi-\beta}{\beta}\right)a\right\}$. If $\phi>\beta$, then $a=0$, and if $\phi=\beta$, then any $a\geq 0$ is a solution. Finally, the clearing of the asset market requires

$$A\in a^b\left(\frac{\phi-\beta}{\beta}\right). \tag{24}$$

Definition 2 *An equilibrium is a list $\langle (q^i,d^i),(q^u,d^u),\phi\rangle$ that solves (12)-(14), (17) and (24).*

Proposition 3 ***(Existence and uniqueness of equilibrium)***

There exists an equilibrium and it is such that $\phi\geq\beta$. Provided that $A\geq c(q^*)$ or $A\neq k$, it is unique.

The equilibrium is generically unique.[13] If $A=k<c(q^*)$, then there is a range of prices that are consistent with market clearing. (See Figure 6.)

Proposition 4 ***(Allocations)***

1. *If $\min(k,A)\geq c(q^*)$, then $q^u=q^i=q^*$ and $d^u=d^i=c(q^*)$.*
2. *If $A\geq c(q^*)>k$, then $q^u<q^i=q^*$ and $d^u=k<d^i=c(q^*)$.*
3. *If $c(q^*)>A>k$, then $q^u<q^i<q^*$ and $d^u=k<d^i=A$.*
4. *If $c(q^*)>k>A$, then $q^u=q^i<q^*$ and $d^u=d^i=A$.*

The model nests Lagos and Rocheteau (2008) and Lester, Postlewaite, and Wright (2007). In the former, k is infinite so that only the cases 1 and 4 in Proposition 4 are relevant. In the latter, k is 0 so that only the cases 2 and 3 are relevant.

The first-best allocation is achieved if the stock of the asset is larger than the cost incurred by sellers of producing q^* and if the cost of producing counterfeits is sufficiently large. If the cost of producing counterfeits is sufficiently low, then there is a distribution of terms of trade across matches. Buyers consume less and transfer a smaller fraction of their asset holdings when sellers are uninformed.

[13] The distribution of asset holdings across buyers was not introduced in the definition of an equilibrium. If $A\leq c(q^*)$, this distribution is degenerate and uniquely determined. If $A>c(q^*)$, then the distribution is not uniquely pinned down, but this indeterminacy is payoff-irrelevant.

The next Proposition investigates the effects of the recognizability of the asset on its price and return. Let $R = 1/\phi$ denote the gross rate of return of the asset.

Proposition 5 ***(Asset prices and recognizability)***

1. *If $A \geq c(q^*) > k$, then $\phi = \beta$ and $R = \beta^{-1}$.*

2. *If $c(q^*) > A > k$, then $\phi = \beta\left[1 + \eta\mathcal{L}(A)\right] > \beta$ and $R < \beta^{-1}$.*

3. *If $c(q^*) > k > A$, then $\phi = \beta\left[1 + \mathcal{L}(A)\right] > \beta$ and $R < \beta^{-1}$.*

The asset price exhibits a liquidity premium, and the rate of return of the asset is less than the discount rate, if there is a shortage of the asset. Moreover, this liquidity premium is affected by the recognizability of the asset (η) provided that the cost of producing counterfeits is sufficiently low. An increase in the ability of sellers to recognize genuine claims raises the asset price. In contrast, if the cost of producing counterfeits is not too low, then the asset price is independent of its recognizability.

The model can be used to study the effects of a change in regulation that affects the recognizability of the asset. Suppose that $c(q^*) > k > A$. If the cost of producing counterfeits is reduced below A, then the asset price falls discretely since the liquidity constraint in uninformed matches becomes binding. See Figure 7. This abrupt reduction in the liquidity value of the asset can be interpreted as a bubble bursting. Moreover, the size of the price drop, $\beta(1-\eta)\mathcal{L}(k)$, increases with the difficulty of recognizing counterfeits.

Next, I turn to the relationship between the liquidity of the asset and its recognizability. Following Wallace (2000), liquidity is defined as the transaction velocity of the asset,

$$\mathcal{V} = \frac{\eta d^i + (1-\eta)d^u}{A}. \tag{25}$$

So $\mathcal{V}$ is the total transfer of assets in the DM as a fraction of the aggregate stock.[14]

[14] As it was pointed out to me by Randy Wright, Kiyotaki and Wright (1989) have argued that velocity is not necessarily a good measure of moneyness. They showed that objects can have a high transactions velocity even though they are accepted in bilateral trade with low probability, because the endogenous stock of the object is low. In my model, the stock of the asset is kept fixed, so that transaction velocity captures well the extent to which a good is used as means of payment.

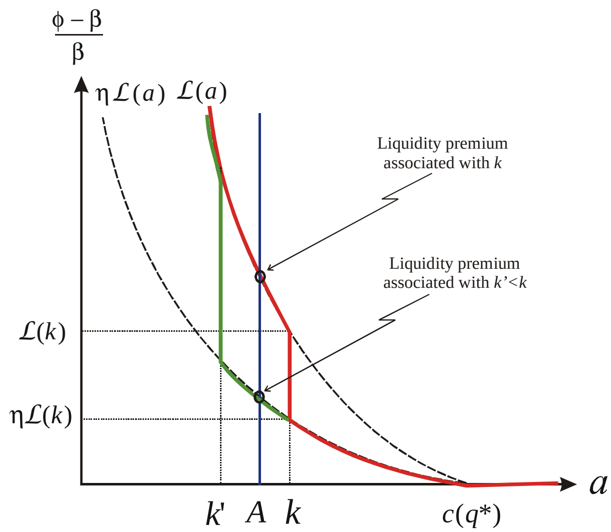

Figure 7: Reduction of the cost of producing counterfeits from $k > A$ to $k' < A$

Proposition 6 ***(Liquidity and recognizability)***

1. *If* $\min(k, A) \geq c(q^*)$, *then* $\mathcal{V} = \frac{c(q^*)}{A}$
2. *If* $A \geq c(q^*) > k$, *then* $\mathcal{V} = \frac{\eta c(q^*)+(1-\eta)k}{A}$
3. *If* $c(q^*) > A > k$, *then* $\mathcal{V} = \eta + \frac{(1-\eta)k}{A}$
4. *If* $c(q^*) > k > A$, *then* $\mathcal{V} = 1$

Provided that the aggregate stock of the asset is neither too large nor too small, the liquidity of the asset depends on both η and k. The sellers' ability to recognize genuine claims affects the composition of trades (the extensive margin) while the cost of producing counterfeits affects the size of the trade in uninformed matches (the intensive margin). As η goes up, the liquidity of the asset improves because there are more informed trades, and the transfer of asset is larger in those trades. If k increases, then the liquidity of the asset improves because buyers can trade a larger quantity in uninformed matches.

Consider the case where $c(q^*) > A > k$. My model provides some microfoundations for the trading restrictions in Kiyotaki and Moore (2005) and Lagos (2006). If η tends to 0 (i.e., all matches are uninformed), then buyers can only spend a fraction, $\frac{k}{A}$, of their asset holdings in all matches, as in Kiyotaki and Moore. If k tends to 0 (i.e., counterfeits are costless to produce), then buyers can only spend their assets in a fraction, η, of the matches, as in Lagos. This latter limiting case is the one studied in Postlewaite, Lester, and Wright (2007). If $\eta \in (0,1)$ and $k > 0$, then the imperfect recognizability of the asset affects both the intensive and the extensive margins of the trading activity.

Finally, I turn to the normative implications of the model. Social welfare, measured by the sum of the utilities of buyers and sellers in the DM, is equal to

$$\mathcal{W} = \frac{\eta[u(q^i) - c(q^i)] + (1-\eta)[u(q^u) - c(q^u)]}{1-\beta}.$$

Proposition 7 ***(Asset recognizability and welfare)***

1. *If* $\min(k, A) \geq c(q^*)$, *then* $\mathcal{W} = \frac{u(q^*)-c(q^*)}{1-\beta}$ *is maximum.*

2. *If* $\min(k, A) = k < c(q^*)$, *then* $\frac{\partial W}{\partial k} > 0$ *and* $\frac{\partial W}{\partial \eta} > 0$.

3. *If* $\min(k, A) = A < c(q^*)$, *then* $\mathcal{W} < \frac{u(q^*)-c(q^*)}{1-\beta}$ *but* $\frac{\partial W}{\partial k} = \frac{\partial W}{\partial \eta} = 0$.

If the asset is sufficiently abundant and the cost of producing counterfeits is sufficiently high, then agents trade the surplus-maximizing output in all matches and social welfare is maximum. In contrast, if there is a shortage of assets to serve as means of payment, or the cost of producing counterfeits is low, then the quantities traded in some or all matches are too low. If $k < A$, then improving the recognizability of the asset by raising either k or η improves welfare. This result can justify the information campaigns in the U.S., which help individuals distinguish counterfeits from genuine currency, or periodic redesign of the currency. If $c(q^*) > k > A$, then improving the recognizability of the asset does not affect welfare: the inefficient trade is due to a shortage of the asset and not a lack of its recognizability.

5 Concluding remarks

In the following, I discuss several aspects of the model and possible extensions.

Multiple assets Lester, Postlewaite, and Wright (2007) study a model with two assets, fiat money and capital. One objective is to explain the rate-of-return differential between fiat money and real assets. Fiat money can be introduced in my model in a standard way. Agents are endowed with a stock of fiat currency at the beginning of time, and the quantity of money grows (or shrinks) through lump-sum money transfers in the CM. The same methodology as in Section 3 can be used to determine the terms of trade in uninformed matches. I conjecture the following results. A monetary equilibrium exists if the quantities traded in the nonmonetary equilibrium are inefficiently low, i.e., $\min(A, k) < c(q^*)$, and the inflation rate is not too large. If $A < k$, then there is rate-of-return equality, whereas if $k < A$, then the real asset has a higher rate of return than fiat money. In this latter case, an increase in inflation raises the asset price and lowers its return. The rate-of-return differential between fiat money and the real asset arises because, as in Lester, Postlewaite, and Wright (2007), the real asset is illiquid at the margin in uninformed matches. The optimal monetary policy corresponds to the Friedman rule, and it achieves the first-best allocation.

Lester, Postlewaite, and Wright (2007) assume a strong asymmetry between the two assets: fiat money is perfectly recognizable (either its quality can be assessed in all trades, or the cost of producing a counterfeit is very large) while real assets can be counterfeited at no cost. A natural extension would be to consider the coexistence of multiple assets that are all imperfectly recognizable, and study how payment arrangements are shaped by the differences with respect to the assets' recognizability properties. It would also be worthwhile to conduct a mechanism design exercise in which a social planner would choose the recognizability of each asset, as captured by the cost of producing a counterfeit, k, in order to maximize society's welfare, net of the cost of making an asset more or less recognizable. One could then check the conditions under which it is socially optimal to have fiat money as the only recognizable asset.

The no-counterfeiting result A striking result of my model is that buyers never produce counterfeits irrespective of the recognizability of the asset (k and η). If one thinks of the asset as currency, this result appears counterfactual. Historically, counterfeiting has been a severe problem in the 19th century United States (Mihm, 2007). Even nowadays, counterfeiting of currency persists even though its extent is very limited. In Li and Rocheteau (2008) we provide extensions of a related model in which counterfeiting can materialize in equilibrium.[15] If counterfeits can circulate across periods, then they can acquire some value and be produced. Also, the choice of the bargaining protocol is important for the no-counterfeiting result. If sellers are able to set the terms of trade in some matches, then counterfeiting can also emerge in equilibrium.

Private information and liquidity The papers by Lester, Postlewaite, and Wright (2007, 2008) offer a theory of asset liquidity based on a physical property of the asset, its recognizability.[16] A related but different approach promoted by the finance literature (Kyle, 1985; Glosten and Milgrom, 1985) consists of explaining the illiquidity of assets (as measured, e.g., by bid-ask spreads) by informational asymmetries about the fundamental value of the asset. In Rocheteau (2008) I use a monetary model similar to the one developed in this paper, in which a real asset is characterized by a stochastic stream of dividends. The asset cannot be produced or counterfeited ($k \to \infty$). However, agents holding the asset enjoy some private information about its future dividend. I show that the liquidity of the asset (velocity and liquidity premium) depends on its intrinsic characteristics as defined by the properties of its dividend process. Moreover, the model delivers a pecking-order theory of payments, according to which agents have a strict preference for cash in some states, i.e., they use their risky asset as a last resort.

Endogenous information Lester, Postlewaite, and Wright (2008) endogenize the recognizability parameter η by assuming that sellers can invest in a technology that can recognize genuine assets. This extension is key since it allows them to relate the illiquidity of the asset to fundamentals and policy. In my model, since buyers have all the bargaining power, sellers would never invest in such a technology. This difficulty can be overcome if I assume that buyers are the ones who can invest in a technology that makes their assets easier to identify. An alternative would be to extend the bargaining protocol to let sellers make take-it-or-leave-it offers in some meetings.

[15] We use a model with indivisible assets where agents can use lotteries to determine the terms of trade in bilateral matches.

[16] By so doing, they conform to the Wallace (1996) dictum according to which the moneyness of an asset should not be a primitive. Other models have emphasized the portability or the divisibility (Wallace, 2000) of an asset to explain its liquidity.

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Appendix: Proofs

Proof of Proposition 1 I describe the Nash equilibria of the subgames following any feasible offer $(q,d) \in \mathbb{R}_+ \times [0,a]$. Then, I evaluate the buyer's payoff at the different Nash equilibria in order to solve (11).

1. $(p,\theta) = (1,1)$. From (9) and (10), $-c(q)+d \geq 0$ and $k \geq d$. The offer is feasible if $d \leq a$. The supremum of the buyer's payoff among all offers (q,d) such that $(p,\theta)=(1,1)$ is $U^* > 0$ given by (12)-(14).

2. $(p,\theta) = (0,1)$. It requires $-c(q)+d \leq 0$ and $k \geq 0$. The buyer's payoff is 0.

3. $(p,\theta) = (0,0)$. From (10) $\theta = 0$ if $k \leq 0$, which is ruled out by assumption.

4. $(p,\theta) \in (0,1)^2$. From (9) and (10), $\theta = \frac{c(q)}{d}$ and $p = \frac{k}{d}$. Moreover, $p \in (0,1)$ and $\theta \in (0,1)$ imply $c(q) < d$ and $d \in (k,a]$. Such an equilibrium exists if $k < a$. The supremum of the buyer's payoff among all offers such that such $(p,\theta) \in (0,1)^2$ is

$$\begin{aligned} U &= \sup\left\{-k + \frac{k}{d}u(q) : d \in (k,a],\, c(q) = \theta d \text{ and } \theta \in (0,1)\right\} \\ &= \sup\left\{-k + \theta k \frac{u(q)}{c(q)} : c(q) \in (\theta k, \theta a] \text{ and } \theta \in (0,1)\right\}. \end{aligned}$$

   Since $u'' < 0$, $c'' \geq 0$ and $u(0) = c(0)$ then $u(q)/c(q)$ is decreasing in q. Consequently,

$$\begin{aligned} U &= \sup\left\{-k + u \circ c^{-1}(\theta k) : \theta \in (0,1)\right\} \\ &= -k + u \circ c^{-1}(k) \\ &\leq U^* \end{aligned}$$

   The supremum is not achieved since it requires $(p,\theta) = (1,1)$.

5. $(p,\theta) \in \{1\} \times (0,1)$. From (10), $k = d$. Such an equilibrium exists if $k \leq a$. From (9) $p = 1$ if $-c(q) + \theta k \geq 0$. The supremum of the buyer's payoff among such offers is

$$\sup\{-k + u(q) : c(q) \leq \theta k \text{ and } \theta \in (0,1)\} = -k + u \circ c^{-1}(k) \leq U^*.$$

   It requires $\theta = 1$ and hence the supremum is not achieved.

6. $(p,\theta) \in \{0\} \times (0,1)$. From (10), $k = 0$, which is ruled out by assumption.

7. $(p,\theta) \in (0,1) \times \{1\}$. From (9) and (10), $d = c(q)$ and $pd \leq k$. Moreover, the offer is feasible if $d \leq a$. The supremum of the buyer's payoff among such offers is

$$U = \sup\{p\left[u(q) - c(q)\right] : c(q) \leq a,\, pc(q) \leq k \text{ and } p \in (0,1)\}$$

Suppose first that $k \geq a$. Then,

$$\begin{aligned} U &= \sup\{p\left[u(q) - c(q)\right] : c(q) \leq a, \text{ and } p \in (0,1)\} \\ &= \sup\{u(q) - c(q) : c(q) \leq a\} \end{aligned}$$

Suppose next that $k < a$. If $c(q^*) \leq k$ then $U = u(q^*) - c(q^*)$. Otherwise, the solution must be such that $pc(q) = k$, and hence

$$U = \sup\left\{k\left[\frac{u(q)}{c(q)} - 1\right] : q \in \left[c^{-1}(k), c^{-1}(a)\right]\right\} = u\left[c^{-1}(k)\right] - k.$$

To summarize, $U = U^*$ but the supremum is not achieved since it requires $p = 1$.

8. $(p,\theta) \in (0,1) \times \{0\}$. From (9), $q = 0$. The buyer's payoff is $-k < 0$.

From the cases above, the equilibria of the subgames following an offer (q,d) are given by (see also Figure 8):

$$\theta(q,d) \begin{array}{l} = 1 \text{ if } d < k \\ = \min\left[\frac{c(q)}{d}, 1\right] \text{ if } d > k \\ \in \left[\min\left[\frac{c(q)}{k}, 1\right], 1\right] \text{ if } d = k \end{array}$$

and

$$p(q,d) \begin{array}{l} = \min\left[1, \frac{k}{d}\right] \text{ if } d > c(q) \\ = 0 \text{ if } d < c(q) \\ \in \left[0, \min\left[1, \frac{k}{d}\right]\right] \text{ if } d = c(q) \end{array}$$

From case 1 and the fact that $[\theta(q,d), p(q,d)]$ is uniquely determined for all (q,d) such that $d < k$ and $c(q) < d$ then

$$\sup\{-k\left[1 - \theta(q,d)\right] + \left[u(q) - d\theta(q,d)\right]p(q,d)\} = U^*.$$

The supremum is achieved if and only if $p(q^u, d^u) = 1$ where (q^u, d^u) denotes the solution to (12)-(14). Moreover, $\theta(q^u, d^u) = 1$.

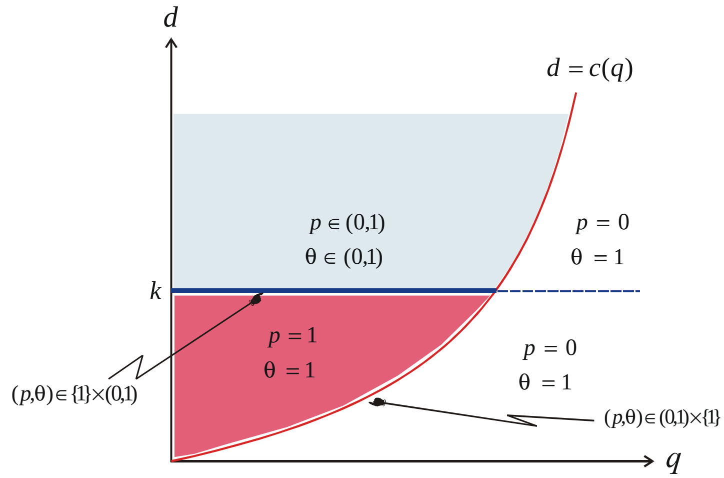

Figure 8: Equilibria of the subgame following (q, d)

Proof of Proposition 2 Consider a social planner whose objective is to maximize the total surplus of a match net of the cost of counterfeiting subject to the buyers' and sellers' incentive-feasibility constraints. The game is as follows. First, each buyer chooses whether or not to produce a counterfeit. Second, the planner proposes a trade $(q, d) \in \mathbb{R}_+ \times [0, a]$, and the matched agents accept or reject the trade simultaneously. The planner chooses the probability $p \in [0, 1]$ that the seller accepts the proposed trade, the probability $\theta \in [0, 1]$ that the buyer produces a counterfeit, and the trade (q, d). It should be clear that it is optimal from the planner's standpoint to have buyers accept the trade with probability one.[17] However, the planner might want sellers to reject

[17]Buyers who produced a counterfeit would never find it optimal to reject an offer such that $q > 0$. Moreover, the probability of a buyer to accept a trade does not affect the sellers' participation constraint, which is given by (27).

some offers in order to reduce buyers' incentives to produce counterfeits. The planner's problem is

$$\max_{q,d,p,\theta} \{p\,[u(q)-c(q)]-(1-\theta)k\} \tag{26}$$

$$\text{s.t.} \quad -c(q)+\theta d \geq 0 \tag{27}$$

$$u(q)-d \geq 0 \tag{28}$$

$$k-pd \geq 0 \tag{29}$$

$$[-c(q)+\theta d]\,(1-p)=0 \tag{30}$$

$$(k-pd)(1-\theta)=0 \tag{31}$$

$$d \leq a. \tag{32}$$

According to (26) the planner's objective is the match surplus multiplied by the probability that the trade goes through minus the counterfeiting cost. Sellers and genuine buyers are willing to participate to the trade if (27) and (28) hold. According to (29) the buyer does not have strict incentives to produce counterfeits ($\theta > 0$). According to (30) the probability that the trade does not go through can be less than one if the seller's participation constraint (27) holds at equality. According to (31), some counterfeiting can occur if $k = pd$. From (29), a buyer is willing to produce a counterfeit since $-k + pu(q) \geq -k + pd = 0$. The last condition (32) is the feasibility constraint according to which a buyer cannot transfer more than the quantity of assets he holds. The problem (26)-(32) has a solution since the objective function is continuous and it is maximized over a compact and nonempty set.

Suppose (29) is not binding. The planner's objective is increasing in both p and θ and an increase in θ relaxes the seller's participation constraint (27). The solution is such that $p = \theta = 1$ and (30)-(31) hold. The planner's problem reduces to

$$\max_{q,d\leq a} [u(q)-c(q)] \quad \text{s.t.} \quad -c(q)+d \geq 0 \text{ and } u(q)-d \geq 0$$

The solution is $q = \min\left[q^*, c^{-1}(a)\right]$ and $d \in [c(q), u(q)] \cap [0, a]$. The no-counterfeiting constraint (29) holds if $d \leq k$. This requires $k \geq \min\left[c(q^*), a\right]$. The recognizability problem does not affect the planner's choice if either the cost of producing a counterfeit is greater than the seller's disutility of producing the first-best level of output, or the cost of producing a counterfeit is greater than the quantity of genuine assets held by the buyer.

Suppose next that (29) is binding. If $p = 1$ and $\theta < 1$ then one can raise θ to increase the planner's objective without violating (27)-(32). Suppose $p < 1$. From (30)-(31), $c(q) = \theta d$ and $pd = k$. Then, $pc(q) = \theta k$. The planner's objective can be rewritten as $pu(q) - k$. Using that $q = c^{-1}\left(\frac{\theta k}{p}\right)$ the planner's problem becomes

$$\max_{p,\theta} \left\{ pu \circ c^{-1}\left(\frac{\theta k}{p}\right) - k \right\}$$

which does not have a solution such that $p < 1$. Hence, the only possible solution implies $p = \theta = 1$. From (29), $d = k$. It is feasible if $k \leq a$. The planner's problem reduces to

$$\max_{q} \left[u(q) - c(q)\right] \quad \text{s.t.} \quad q \in \left[u^{-1}(k), c^{-1}(k)\right].$$

The solution is $q = \min\left[c^{-1}(k), q^*\right]$. Hence, if $k < \min\left[c(q^*), a\right]$ then $q = c^{-1}(k)$ and $d = k$.

Proof of Lemma 1 The objective function in (19) is concave and differentiable everywhere except at $a = k$ if $k < c(q^*)$. To see this, from (12)-(14),

$$\begin{aligned} q^u &= \min\left[q^*, c^{-1}(a), c^{-1}(k)\right] \\ d^u &= c(q^u). \end{aligned}$$

From (17),

$$\begin{aligned} q^i &= \min\left[q^*, c^{-1}(a)\right] \\ d^i &= c(q^i). \end{aligned}$$

The surplus functions can be expressed as

$$S^i(a) = \begin{array}{l} u \circ c^{-1}(a) - a \quad \text{if} \quad a < c(q^*) \\ u(q^*) - c(q^*) \quad \text{if} \quad a \geq c(q^*) \end{array} \tag{33}$$

$$S^u(a;k) = \begin{array}{l} u \circ c^{-1}(a) - a \quad \text{if} \quad a < \min\left[k, c(q^*)\right] \\ u \circ c^{-1}(k) - k \quad \text{if} \quad k < \min\left[a, c(q^*)\right] \\ u(q^*) - c(q^*) \quad \text{if} \quad c(q^*) \leq \min(a,k) \end{array} \tag{34}$$

It is immediate from (33) that $S^i(a)$ is differentiable for all $a > 0$; $\frac{\partial S^i(a)}{\partial a} = \max\left[\frac{u'\left[c^{-1}(a)\right]}{c'\left[c^{-1}(a)\right]} - 1, 0\right]$. From (34), $\frac{\partial S^u(a)}{\partial a} = \frac{u'\left[c^{-1}(a)\right]}{c'\left[c^{-1}(a)\right]} - 1$ if $a < \min\left[c(q^*), k\right]$ and $\frac{\partial S^u(a)}{\partial a} = 0$ if $a > \min\left[c(q^*), k\right]$. If

$k < c(q^*)$ then $\left.\frac{\partial^- S^u(a;k)}{\partial a}\right|_{a=k} = \frac{u'\left[c^{-1}(k)\right]}{c'\left[c^{-1}(k)\right]} - 1 > 0$ and $\left.\frac{\partial^+ S^u(a;k)}{\partial a}\right|_{a=k} = 0$ so that $S^u(a;k)$ is not differentiable at $a = k$. (I used the notations $\frac{\partial^+ S^u(a;k)}{\partial a}$ for the right-hand derivative of $S^u(a;k)$ and $\frac{\partial^- S^u(a;k)}{\partial a}$ for the left-hand derivative). Necessary and sufficient conditions for an optimum are

$$-\frac{\phi-\beta}{\beta} + \eta\frac{\partial S^i(a)}{\partial a} + (1-\eta)\frac{\partial^+ S^u(a;k)}{\partial a} \quad \leq \quad 0 \tag{35}$$

$$-\frac{\phi-\beta}{\beta} + \eta\frac{\partial S^i(a)}{\partial a} + (1-\eta)\frac{\partial^- S^u(a;k)}{\partial a} \quad \geq \quad 0, \tag{36}$$

Recall that $\frac{\partial^+ S^u(a;k)}{\partial a} = \frac{\partial^- S^u(a;k)}{\partial a}$ except at $a = k$ if $k < c(q^*)$.

Consider the case $\phi > \beta$. For all $a \geq c(q^*)$ then $\frac{\partial S^i(a)}{\partial a} = \frac{\partial^- S^u(a;k)}{\partial a} = 0$ and hence (36) is violated. For all $a < c(q^*)$ the objective function in (19) is strictly concave and hence the solution to (19) is unique. If $a \neq k$ then the objective function is differentiable and a solves

$$\eta\frac{\partial S^i(a)}{\partial a} + (1-\eta)\frac{\partial S^u(a;k)}{\partial a} = \frac{\phi-\beta}{\beta},$$

with $\frac{\partial S^i(a)}{\partial a} = \mathcal{L}(a)$ and $\frac{\partial S^u(a;k)}{\partial a} = \mathcal{L}(a)$ if $a < k$ and $\frac{\partial S^u(a;k)}{\partial a} = 0$ if $a > k$. So if $a > k$ then $a = \mathcal{L}^{-1}\left(\frac{\phi-\beta}{\eta\beta}\right)$ and if $a < k$ then $a = \mathcal{L}^{-1}\left(\frac{\phi-\beta}{\beta}\right)$. If $a = k$ then (35)-(36) become

$$\eta\frac{\partial S^i(k)}{\partial k} = \eta\mathcal{L}(k) \leq \frac{\phi-\beta}{\beta} \leq \frac{\partial S^i(k)}{\partial k} = \mathcal{L}(k).$$

Consider the case $\phi = \beta$. If $a < c(q^*)$ then $\frac{\partial S^i(a)}{\partial a} > 0$ and (35) is violated. For all $a \geq c(q^*)$ then $\frac{\partial S^i(a)}{\partial a} = \frac{\partial^+ S^u(a;k)}{\partial a} = \frac{\partial^- S^u(a;k)}{\partial a} = 0$. Hence, any solution to (35)-(36) is such that $a \geq c(q^*)$.

Finally, if $\phi < \beta$ then there is no $a \geq 0$ such that (35) holds.

Proof of Proposition 3 From Lemma 1, the correspondence $a^b\left(\frac{\phi-\beta}{\beta}\right)$ is decreasing and continuous. Moreover, $a^b(\infty) = \{0\}$ and $a^b(0) \ni \infty$. Hence, there is a solution $\phi \geq \beta$ to (24). The correspondence $a^b\left(\frac{\phi-\beta}{\beta}\right)$ is strictly decreasing except when $\frac{\phi-\beta}{\beta} \in [\eta\mathcal{L}(k);\mathcal{L}(k)]$ in which case $a^b = k$. The interval $[\eta\mathcal{L}(k);\mathcal{L}(k)]$ exists and is non-degenerate if $k < c(q^*)$. Hence, the solution to (24) is unique unless $A = k < c(q^*)$.

Proof of Propositions 4 and 5

1. If $A \geq c(q^*)$ then from (23) and (24), $\phi = \beta$. From (12)-(14) and (17), $q^u = q^i = q^*$ and $d^u = d^i = c(q^*)$, where I have used that $a \geq c(q^*)$ and $k \geq c(q^*)$.

2. If $A \geq c(q^*)$ but $c(q^*) > k$ then $d^u = k$ and $q^u = c^{-1}(k) < q^*$ (from (12)-(14)). From (17), $q^i = \min\left[q^*, c^{-1}(A)\right] = q^*$ and $d^i = c(q^*)$.

3. If $c(q^*) > A$ then, from (22) and (24), all buyers hold A and $\phi > \beta$. Then, $d^i = A$ and $q^i = \min\left[q^*, c^{-1}(A)\right] = c^{-1}(A) < q^*$. From (22), if $A > k$ then $\frac{\phi-\beta}{\beta} = \eta\mathcal{L}(A)$. From (12)-(14), $q^u = \min\left[q^*, c^{-1}(k), c^{-1}(A)\right] = c^{-1}(k) < c^{-1}(A) = q^i < q^*$ and $d^u = k$.

4. From (22), if $c(q^*) > k > A$ then $\frac{\phi-\beta}{\beta} = \mathcal{L}(A)$. From (12)-(14) and (17), $q^i = \min\left[q^*, c^{-1}(A)\right] = q^u = \min\left[q^*, c^{-1}(k), c^{-1}(A)\right] = c^{-1}(A) < q^*$ and $d^i = d^u = A$.

Proof of Proposition 6 The proof uses (12)-(14), (17) to determine d^i and d^u. (See also Proposition 4.)

1. If $\min(k, A) \geq c(q^*)$ then $d^i = d^u = c(q^*)$.

2. If $A \geq c(q^*) > k$ then $d^i = c(q^*)$ and $d^u = k$.

3. If $c(q^*) > A > k$ then $d^i = A$ and $d^u = k$.

4. If $c(q^*) > k > A$ then $d^i = d^u = A$.