Crashes and Recoveries in Illiquid Markets

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We study the dynamics of liquidity provision by dealers during an asset market crash, described as a temporary negative shock to investors' aggregate asset demand. We consider a class of dynamic market settings where dealers can trade continuously with each other, while trading between dealers and investors is subject to delays and involves bargaining. We derive conditions on fundamentals, such as preferences, market structure and the characteristics of the market crash (e.g., severity, persistence) under which dealers provide liquidity to investors following the crash. We also characterize the conditions under which dealers' incentives to provide liquidity are consistent with market efficiency.

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JEL Codes: C78, D83, E44, G1.

1 Introduction

Liquidity in financial markets is often provided by dealers who trade assets from their own inventories. Even in markets where liquidity provision by dealers may be inconspicuous in normal times, it becomes critical during times of large financial imbalances. During market crashes, for instance, it can take a long time for an investor to find a counterpart for trade, either because of the technological limitations of order-handling systems or, as is the case in over-the-counter markets, due to the decentralized nature of the trading process.[1] These situations appear to be very costly to investors, who concede striking price discounts to unwind their positions (e.g., the 23% price drop of the Dow Jones Industrial Average on October 19, 1987). Some have argued that the social cost could be even larger because of the risk that the financial crisis propagates to the macroeconomy (see, e.g., Borio (2004)). It is commonly believed that liquidity provision by dealers plays a crucial role in mitigating these costs.

In this paper we study the equilibrium and the socially optimal inventory policies of dealers during a market crash, which we model as a temporary negative shock to investors' willingness to hold the asset. We derive conditions under which dealers will find it in their interest to provide liquidity in the aftermath of a crash, as well as conditions under which their incentives to provide liquidity are consistent with market efficiency. We also study how liquidity provision by dealers depends on the market structure, e.g., dealers' degree of market power or the extent of the trading frictions, and the characteristics of the crash, e.g., severity and persistence of the shock to investors' demands.

Our work is related to a recent literature that studies trading frictions in asset markets.[2] In particular, the market setting we consider is similar to that of Duffie et al. (2005) (DGP hereafter). Investors rebalance their asset holdings periodically in response to random changes in their utility from holding assets, and they must engage in a time-consuming process to contact dealers and bargain over the terms of trade. Dealers get no direct utility from holding assets, and they can trade continuously in a perfectly competitive interdealer market. DGP focused on steady states, so their analysis is silent about liquidity provision by dealers.

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[1] For a description of the various trading problems that arose during the market crashes of October 1987 and October 1997, see the report of The Presidential Task Force on Market Mechanisms (1988) and the US Securities and Exchange Commission Staff Legal Bulletin No. 8 (September 9, 1998).

[2] Examples include, Gârleanu (2006), Longstaff (2005), and Vayanos and Weill (2007). Conceptually, our analysis is also related to the inventory models of Stoll (1978) and Ho and Stoll (1983) (see Chapter 2 in O'Hara (1997) for a review of this earlier market-microstructure literature).

Weill (2007) studies the timing of liquidity provision by dealers in a dynamic version of DGP. He asks under which conditions dealers will, and ought to, *lean against the wind* in the immediate aftermath of the crash. Weill (2007) and the literature spurred by DGP, however, keep the framework tractable by imposing a stark restriction on asset holdings, namely, that investors can only hold either 0 or 1 unit of the asset. Lagos and Rocheteau (2007) study a version of DGP where investors can hold unrestricted asset positions and find that, as result of this restriction on asset holdings, existing search-based theories of financial liquidity neglect a critical aspect of investor behavior in illiquid markets, namely the fact that market participants can mitigate trading frictions by adjusting their asset positions so as to reduce their trading needs. This effect of trading frictions on the demand for liquidity has been pointed out in a different context by Constantinides (1986).

In this paper, we go beyond previous studies by allowing both dealers and investors to hold unrestricted asset positions. This turns out to generate new implications for both the demand and the supply of liquidity. Absent extraneous upper bounds on asset holdings, in times of crisis, investors with high utility for the asset may absorb the selling pressure coming from investors with low utility by holding positions that are large relative to what they would hold during normal times. In other words, by removing the typical restrictions on investors' asset holdings, we find that investors may provide liquidity to other investors in times of crisis, much like dealers do. These new effects on the supply and demand of liquidity imply that, in contrast to Weill (2007), dealers may sometimes *not* find it in their interest to provide liquidity during a crash. Also, it may sometimes be efficient for them not to lean against the wind. Whether or not dealers will provide liquidity, and whether or not they ought to, depends on fundamentals, including the details of market structure and the characteristics of the crash.

Our stylized description of a market crash consists of an aggregate negative preference shock to investors' asset demands, followed by a (possibly stochastic) recovery path.[3] We find that the amount of liquidity provided by dealers following the crash varies nonmonotonically with the magnitude of trading frictions. When frictions are small, investors choose to take more

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[3]This scenario could represent, for instance, an international shock such as the 1997 Asian crisis or the 1998 Russian sovereign default, domestic turbulence such as that triggered by the September 11 terrorist attack, or even some company-specific shock, such as the collapse of Enron. Our "crash" follows the spirit of Grossman and Miller's (1988) crash dynamics. In Grossman and Miller, dealers provide liquidity in order to share risk with outside investors. In our model, dealers have no such utility motive for holding assets; instead, they allow investors to trade faster. In related work, Bernardo and Welch (2004) use the feature of nonsequential access of investors to market makers to describe a market crash as a financial run.

extreme positions because they know that they can rebalance their asset holdings very quickly. Specifically, investors with higher-than-average utility for assets become more willing to hold larger-than-average positions and absorb more of the selling pressure coming from investors whose demands for the asset are lower than normal. In some cases, the former end up supplying so much liquidity to other investors, that dealers don't find it profitable to step in. If, on the contrary, trading frictions are large enough, dealers do not accumulate inventories either, but for a different reason: Trading frictions reduce investors' demand for liquidity. Indeed, in order to reduce their exposure to the trading frictions, investors choose to take less extreme asset positions. In fact, it is possible that they demand so little liquidity that dealers don't find it profitable to accumulate inventories following a crash. Thus, if one considers a spectrum of asset markets ranging from those with very small frictions, such as the New York Stock Exchange (NYSE), to those with large trading frictions, such as the corporate bond market, one would expect to see dealers accumulate more asset inventories during a crash in markets which are in the intermediate range of the spectrum.

We also find that, from the standpoint of investors, an increase in dealers' bargaining strength is equivalent to an increase in trading frictions. Hence, just as with trading frictions, dealers are less likely to accumulate inventories if their bargaining strength is either very small or very large. This finding contrasts with the commonly held view that the market power of dealers (e.g., NYSE specialists) is what gives them incentives to provide liquidity. In our model, an increase in the dealers' bargaining strength may reduce the aggregate amount of inventory they accumulate, because investors endogenously take less extreme positions and demand less liquidity. Similarly, a market reform that reduces dealers' market power, as observed in equity markets in the 90's, can raise dealers' incentives to provide liquidity during a market crash.

Our model can rationalize why dealers intervene in some crises and withdraw in others. In line with Hendershott and Seasholes's (2006) empirical evidence on the inventory strategies of NYSE specialists, in our model, dealers' incentives to provide liquidity are driven by anticipated capital gains. Therefore, dealers are more likely to accumulate inventories when the crisis is severe and expected to be short-lived: A large price drop and the expectation of a quick rebound make it more profitable for dealers to buy low early in the crash and sell high later, as demand for the asset recovers. From a normative standpoint, we find that the equilibrium asset allocation across investors and the dealers' inventory policies are socially efficient if and only if dealers' bargaining strength is equal to zero. Given the nonmonotonic equilibrium relationship between

dealers' asset inventories and their bargaining strength, this means that dealers may fail to build up inventories in situations where it would be socially efficient to do so, and vice-versa.

The rest of the paper is organized as follows. Section 2 lays down the environment. Section 3 characterizes investors' and dealers' behavior and Section 4 defines equilibrium. Section 5 characterizes the socially optimal allocation. Sections 6 and 7 provide two alternative descriptions of a market crash and determine the conditions under which dealers act as providers of liquidity. Section 8 concludes. Appendix A contains all proofs and Appendix B contains supplementary material.

2   The environment

Time is continuous and the horizon infinite. There are two types of infinitely-lived agents: a unit measure of investors and a unit measure of dealers. There is one asset and one perishable good, which we use as a numéraire. The asset is durable, perfectly divisible and in fixed supply, $A \in \mathbb{R}_+$. The numéraire good is produced and consumed by all agents. The instantaneous utility function of an investor is $u_i(a) + c$, where $a \in \mathbb{R}_+$ represents the investor's asset holdings, $c \in \mathbb{R}$ is the net consumption of the numéraire good ($c < 0$ if the investor produces more than he consumes), and $i \in \{1, ..., I\}$ indexes a preference shock. The utility function $u_i(a)$ is strictly increasing, concave, continuously differentiable and satisfies the Inada condition that $u_i'(0) = \infty$. We also assume that it is either bounded below or above. Investors receive idiosyncratic preference shocks that occur with Poisson arrival rate δ. Conditional on the preference shock, the investor draws preference type i with probability π_i, and $\sum_{i=1}^{I} \pi_i = 1$. These preference shocks capture the notion that investors value the services provided by the asset differently over time, and will generate a need for investors to periodically change their asset holdings.[4] The instantaneous utility of a dealer is $v(a) + c$, where $v(a)$ is increasing,

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[4] Our specification associates a certain utility to the investor as a function of his asset holdings. This is a feature that we have borrowed from DGP. The utility the investor gets from holding a given asset position could be simply the value from enjoying the asset itself, as would be the case for real assets such as cars or houses. Alternatively, we can also think of the asset as being physical capital. Then, if each investor has linear utility over a single consumption good (as is the case in most search models), we can interpret $u_i(\cdot)$ as a production technology that allows the agent to use physical capital to produce the consumption good. The idiosyncratic component "i" can then be interpreted as a productivity shock that induces agents with low productivity to sell their capital to agents with high productivity in an OTC market. As yet another possibility, one could adopt the preferred interpretation of DGP, namely that $u_i(a)$ is in fact a reduced-form utility function that stands in for the various reasons why investors may want to hold different quantities of the asset, such as differences in liquidity needs, financing or financial-distress costs, correlation of asset returns with endowments (hedging needs), or relative tax disadvantages (as in Michaely and Vila (1996)). By now, several papers that build on

concave and continuously differentiable. All agents discount at the same rate $r > 0$.
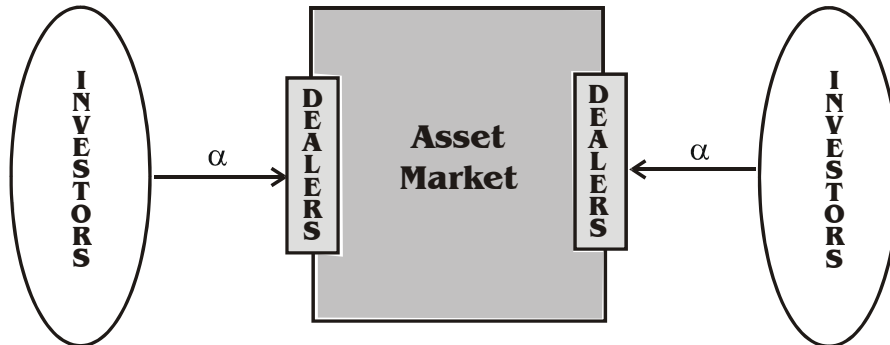

Figure 1: Trading arrangement

There is a competitive market for the asset. Dealers can continuously buy and sell in this market at price $p(t)$, while investors can only access the market periodically and indirectly, through a dealer. Specifically, we assume that investors contact a randomly chosen dealer according to a Poisson process with arrival rate α. Once the investor and the dealer have made contact, they negotiate the quantity of assets that the dealer will acquire (or sell) in the market on behalf of the investor and the intermediation fee that the investor will pay the dealer for his services. After completing the transaction, the dealer and the investor part ways.[5] The trading arrangement is illustrated in Figure 1.

3   Dealers, investors, and bargaining

In this section we describe the decision problems faced by investors and dealers, and the determination of the terms of trade in bilateral meetings between them. Investors readjust their asset

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the work of DGP have formalized the "hedging needs" interpretation. Examples include Duffie, Gârleanu and Pedersen (2006), Gârleanu (2006) and Vayanos and Weill (2007). (See also Lo, Mamaysky and Lang (2004).) Notice that investors in DGP, and therefore the investors in our paper, are akin to the liquidity traders which are commonplace in the large body of the finance microstructure literature that uses asymmetric information instead of search frictions to rationalize bid-ask spreads, such as Glosten and Milgrom (1985) and Easley and O'Hara (1987).

[5] In actual financial markets, there are *position traders* who hold asset inventories in the hope of making capital gains. There are also *pure spread traders* who don't hold inventories but instead profit exclusively from "buying low and selling high." Stoll (1978), for example, calls the former *dealers* and the latter *brokers*. In our model, the agents that we refer to as *dealers* engage in both of these activities. The analysis would remain unchanged if we were to assume that these activities are carried out by two different types of agents with continuous direct access to the asset market.

holdings infrequently, at the random times when they meet dealers. In between those times, an investor enjoys the utility flow associated with his current asset position. A dealer's problem consists of continuously managing his own asset position by trading in the asset market. At random times, the dealer contacts an investor who wishes to buy or sell some quantity of assets. At these times, the dealer executes the desired purchase or sale in the asset market on behalf of the investor and receives a fee for his services.[6]

We begin with the determination of the terms of trade in bilateral trades between dealers and investors. Consider a meeting at time t between a dealer who is holding inventory a_d and an investor of type i who is holding inventory a. Let a' denote the investor's post-trade asset holding and ϕ be the intermediation fee.[7] The pair (a', ϕ) is taken to be the outcome corresponding to the Nash solution to a bargaining problem where the dealer has bargaining power $\eta \in [0, 1]$. Let $V_i(a, t)$ denote the expected discounted utility of an investor with preference type i who is holding a quantity of asset a at time t. Then, the utility of the investor is $V_i(a', t) - p(t)(a' - a) - \phi$ if an agreement (a', ϕ) is reached, and $V_i(a, t)$ in case of disagreement. Therefore, the investor's gain from trade is $V_i(a', t) - V_i(a, t) - p(t)(a' - a) - \phi$. Analogously, let $W(a_d, t)$ denote the maximum attainable expected discounted utility of a dealer who is holding inventory a_d at time t. Then, the utility of the dealer is $W(a_d, t) + \phi$ if an agreement (a', ϕ) is reached and $W(a_d, t)$ in case of disagreement, so the dealer's gain from trade is equal to the fee, ϕ.[8] The outcome of the bargaining is given by

$$[a_i(t), \phi_i(a, t)] = \arg \max_{(a', \phi)} [V_i(a', t) - V_i(a, t) - p(t)(a' - a) - \phi]^{1-\eta} \phi^{\eta}.$$

Hence, the investor's new asset holding solves

$$a_i(t) = \arg \max_{a'} \left[V_i(a', t) - p(t)a' \right], \tag{1}$$

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[6] In principle, the dealer may fill the investor's order partially or in full by trading out of, or for his own inventory of the asset. For example, if at some time t the dealer contacts an investor who wishes to buy some quantity a' and the dealer's inventory is $a_d(t) > a'$, then in that instant, the dealer may fill the buy order by giving the investor a' from his inventory and charging him $p(t)a'$ plus the fee, and instantaneously buying back $a_d(t) - a'$ for his own account in the asset market. Alternatively, the dealer may instead choose not to trade out of his inventory and simply buy a' in the market on behalf of the investor at cost $p(t)a'$ (and charge him this cost plus the intermediation fee). Clearly, the dealer is indifferent between these modes of execution because he has continuous access to the asset market and all the transactions he is involved in are instantaneous.

[7] In our formulation we assume that the investor pays the dealer a fee. However, the bargaining problem can be readily reinterpreted as one in which the dealer pays the investor a bid price which is lower than the market price if the investor wants to sell, and charges an ask price which is higher than the market price if the investor wants to buy. See Lagos and Rocheteau (2007) for details.

[8] The outcome of the bilateral trade does not affect the dealer's continuation payoff because he has continuous access to the asset market and his trades are executed instantaneously (see footnote 6).

and that the intermediation fee is

$$\phi_i(a,t) = \eta\left\{V_i\left[a_i(t),t\right] - V_i(a,t) - p(t)\left[a_i(t) - a\right]\right\}. \tag{2}$$

According to (1), the investor's post-trade asset holding is the one he would have chosen if he were trading in the asset market himself, rather than through a dealer. According to (2), the intermediation fee is set so as to give the dealer a share η of the gains associated with readjusting the investor's asset holdings.[9]

The value function corresponding to a dealer who is holding asset position a_t at time t satisfies

$$W(a_t,t) = \sup_{q(s),a_d(s)} \mathbb{E}\left\{\int_t^T e^{-r(s-t)}\left\{\upsilon[a_d(s)] - p(s)q(s)\right\}ds + e^{-r(T-t)}[\bar{\phi}(T) + W(a_d(T),T)]\right\},$$

subject to the law of motion $\dot{a}_d(s) = q(s)$, the short-selling constraint $a_d(s) \geq 0$, and the initial condition $a_d(t) = a_t$. Here, $a_d(s)$ represents the stock of assets that the dealer is holding and $q(s)$ is the quantity that he trades for his own account at time s. The expectations operator, $\mathbb{E}$, is taken with respect to T, which denotes the next random time at which the dealer meets an investor, where $T - t$ is exponentially distributed with a mean of $1/\alpha$. Since the intermediation fee determined in a bilateral meeting depends on the investor's preference type and asset holdings, and given that the investor is a random draw from the population of investors, at time T the dealer expects to extract the average fee $\bar{\phi}(T) = \int \phi_j(a_i,T)dH_T(j,a_i)$, where H_T denotes the distribution of investors across preference types and asset holdings at time T. The dealer enjoys flow utility $\upsilon[a_d(s)]$ from carrying inventory $a_d(s)$, and gets utility $p(s)q(s)$ from changing this inventory.

Since intermediation fees are independent of the dealer's asset holdings, we can write

$$W(a_t,t) = \max_{q(s)}\left\{\int_t^\infty e^{-r(s-t)}\left\{\upsilon[a_d(s)] - p(s)q(s)\right\}ds\right\} + \Phi(t), \tag{3}$$

subject to $\dot{a}_d(s) = q(s)$, $a_d(s) \geq 0$ and $a_d(t) = a_t$. The function $\Phi(t)$ is the expected present discounted value of future intermediation fees from time t onward and satisfies $\Phi(t) = \mathbb{E}\{e^{-r(T-t)}[\bar{\phi}(T) + \Phi(T)]\}$, where the expectation is with respect to T. This formulation makes it clear that dealers trade assets in two ways: continuously, in the competitive market, or at random times, in bilateral negotiations with investors. Since dealers have quasi-linear preferences

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[9]Our choice of notation for the bargaining solution in (1) and (2) emphasizes the fact that the terms of trade depend on the investor's preference type but are independent of the dealer's inventories. In addition, the investor's post-trade asset holding is independent of his pre-trade holding, while the intermediation fee is not.

and they can trade instantaneously and continuously in the competitive asset market, their optimal choice of asset holdings is independent from what happens in bilateral negotiations with investors. The following lemma describes the solution dealer's inventory accumulation problem which is in the first term on the right-hand side of (3).

Lemma 1 *Suppose that $p(t)$ is a given, piecewise continuously differentiable price path. An inventory path, $a_d(t)$, solves the dealer's inventory accumulation problem if and only if*

1. for all t such that $p(t)$ is differentiable, $a_d(t)$ satisfies

$$v'[a_d(t)] + \dot{p}(t) \leq rp(t) \text{ with } \text{equality } \text{ if } a_d(t) > 0; \tag{4}$$

2. $a_d(t) = 0$ for any t for which the price has a negative jump, i.e., if

$$p(t^+) - p(t^-) < 0, \text{ then } a_d(t) = 0; \tag{5}$$

3. $a_d(t)$ satisfies the transversality condition

$$\lim_{t \to \infty} e^{-rt} p(t) a_d(t) = 0. \tag{6}$$

There is no bounded inventory path, $a_d(t)$, that solves the dealer's inventory accumulation problem if $p(t)$ has a positive jump, $p(t^+) - p(t^-) > 0$.

The last part of Lemma 1, states that if the asset price had a positive jump, a dealer could improve his utility from any bounded inventory path by buying assets just before the jump and re-selling just after.[10] The opposite trading strategy implies that the short-selling constraint must be binding whenever the price jumps down. According to (4), whenever the price path is differentiable and a dealer finds it optimal to hold strictly positive inventory, the flow cost of buying the asset, $rp(t)$, must equal the direct utility flow from holding the asset, $v'[a_d(t)]$, plus the capital gain, $\dot{p}(t)$. As it is well known from Mangasarian's results (see Theorem 13, Chapter 3 of Seierstad and Sydsaeter (1987)), together with the other first-order conditions, the transversality condition (6) is sufficient for optimality. Here, we show that it is necessary as well.[11]

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[10]Note that, because there is a finite measure of assets, and agents face short-selling constraints, dealers' asset holdings will have to be bounded in an equilibrium. This observation, together with the last part of the lemma will imply that a price paths with upward jumps cannot be part of an equilibrium.

[11]The necessity of such transversality conditions for general formulations of infinite-horizon optimal control problems has been regarded as a delicate issue since Halkin's (1974) counterexample. See Benveniste and Scheinkman (1982) for fairly general results.

We now proceed with an analysis of an investor's problem. The value function corresponding to an investor with preference type i who is holding a assets at time t, $V_i(a,t)$, satisfies

$$
V_i(a,t) = \mathbb{E}_i\left[\int_t^T e^{-r(s-t)}u_{k(s)}(a)ds + \right.
$$
$$
\left. e^{-r(T-t)}\{V_{k(T)}[a_{k(T)}(T),T] - p(T)[a_{k(T)}(T) - a] - \phi_{k(T)}(a,T)\}\right], \qquad (7)
$$

where T denotes the next time the investor meets a dealer, and $k(s) \in \{1,...,I\}$ denotes the investor's preference type at time s. The expectations operator, $\mathbb{E}_i$, is taken with respect to the random variables T and $k(s)$, and is indexed by i to indicate that the expectation is conditional on $k(t) = i$. Over the interval of time $[t,T]$ the investor holds a assets and enjoys the discounted sum of the utility flows associated with this holding a (the first term on the right-hand side of (7)). The length of this interval of time, $T - t$, is an exponentially distributed random variable with mean $1/\alpha$. The flow utility is indexed by the preference type of the investor, $k(s)$, which follows a compound Poisson process. At time T the investor contacts a random dealer and readjusts his holdings from a to $a_{k(T)}(T)$. In this event the dealer purchases a quantity $a_{k(T)}(T) - a$ of the asset in the market (or sells if this quantity is negative) at price $p(T)$ on behalf of the investor. At this time the investor pays the dealer an intermediation fee, $\phi_{k(T)}(a,T)$. Both the fee and the asset price are expressed in terms of the numéraire good.

Substituting the terms of trade (1) and (2) into (7), we get

$$
V_i(a,t) = \mathbb{E}_i\left[\int_t^T e^{-r(s-t)}u_{k(s)}(a)ds + \right. \qquad (8)
$$
$$
\left. e^{-r(T-t)}\{(1-\eta)\max_{a'}\left[V_{k(T)}(a',T) - p(T)(a' - a)\right] + \eta V_{k(T)}(a,T)\}\right].
$$

From the last two terms on the right-hand side of (8), it is apparent that the investor's payoff is the one he would get in an economy in which he meets dealers according to a Poisson process with arrival rate α, and instead of bargaining, he readjusts his asset holdings and extracts the whole surplus with probability $1-\eta$; whereas with probability η he cannot readjust his holdings (and enjoys no gain from trade). Therefore, from the investor's standpoint, the stochastic trading process and the bargaining solution are payoff-equivalent to an alternative trading mechanism in which the investor has all the bargaining power in bilateral negotiations with dealers, but he only gets to meet dealers according to a Poisson process with arrival rate

$\kappa = \alpha(1 - \eta)$. Consequently, we can rewrite (8) as

$$V_i(a, t) = \mathbb{E}_i \left[\int_t^{\tilde{T}} u_{k(s)}(a) \, e^{-r(s-t)} \, ds + e^{-r(\tilde{T}-t)} \{ p(\tilde{T})a + \max_{a'} [V_{k(\tilde{T})}(a', \tilde{T}) - p(\tilde{T})a'] \} \right], \quad (9)$$

where the expectations operator, $\mathbb{E}_i$, is now taken with respect to the random variables $\tilde{T}$ and $k(s)$, where $\tilde{T} - t$ is exponentially distributed with mean $1/\kappa$. From (9), the problem of an investor with preference shock i, who gains access to the market at time t, consists of choosing $a' \in \mathbb{R}_+$ in order to maximize

$$\mathbb{E}_i \left[\int_t^{\tilde{T}} e^{-r(s-t)} u_{k(s)}(a') \, ds \right] - \left\{ p(t) - \mathbb{E}_t \left[e^{-r(\tilde{T}-t)} p(\tilde{T}) \right] \right\} a',$$

or equivalently,

$$\mathbb{E}_i \left[\int_t^{\tilde{T}} e^{-r(s-t)} \left\{ u_{k(s)}(a') - [rp(s) - \dot{p}(s)] \, a' \right\} ds \right]. \quad (10)$$

If the investor had continuous access to the asset market, he would choose his asset holdings so as to continuously maximize $u_i(a) - [rp(t) - \dot{p}(t)] \, a$, his flow utility net of the flow cost of holding the asset. But since the investor can only trade infrequently, his objective is to maximize (10) instead. Intuitively, the investor chooses his asset holdings in order to maximize the present value of his utility flow net of the present value of the cost of holding the asset from time t until the next time $\tilde{T}$ when he can readjust his holdings. The following lemma offers a simpler, equivalent formulation of the investor's problem.

Lemma 2 *Let*

$$U_i(a) = \frac{(r + \kappa) \, u_i(a) + \delta \sum_{j=1}^I \pi_j u_j(a)}{r + \delta + \kappa} \quad (11)$$

$$\xi(t) = (r + \kappa) \left[p(t) - \kappa \int_0^\infty e^{-(r+\kappa)s} p(t + s) ds \right], \quad (12)$$

and assume that $p(t)e^{-rt}$ is decreasing. Then a bounded process $a(t)$ solves the investor's problem if and only if

1. $a(t) = a_i(t)$, when the investor contacts the market with current type i, with

$$U_i'[a_i(t)] = \xi(t) \quad (13)$$

2. $a(t)$ satisfies the transversality condition

$$\lim_{t \to \infty} \mathbb{E}\left[p(\theta_t)a(\theta_t)e^{-r\theta_t}\right] = 0, \tag{14}$$

where θ_t denotes the investor's last contact time with a dealer before t.

The assumption that $p(t)e^{-rt}$ is decreasing is without loss of generality, because it will be true in an equilibrium (this follows from the dealer's first-order conditions (4) and (5)). Intuitively, $U_i(a)$ is the flow expected utility the investor enjoys from holding a assets until his next opportunity to rebalance his holdings, and $\xi(t)$ is the cost of buying the asset minus the expected discounted resale value of the asset (expressed in flow terms). Notice that we do not need to know the path for the price of the asset, $p(t)$, to solve for the investor's optimal asset holdings. It is sufficient to know $\xi(t)$. The following lemma establishes the relationship between $\xi(t)$ and $p(t)$.

Lemma 3 *Condition (12) implies*

$$rp(t) - \dot{p}(t) = \xi(t) - \frac{\dot{\xi}(t)}{r + \kappa}. \tag{15}$$

Lemma 3 allows us to rewrite (4) as

$$v'[a_d(t)] + \frac{\dot{\xi}(t)}{r + \kappa} \leq \xi(t) \quad \text{with} \quad \text{an} \quad \text{equality} \quad \text{if} \quad a_d(t) > 0. \tag{16}$$

Equations (13) and (16) illustrate the main differences between dealers and investors in our setup. Relative to investors, dealers get an extra return from holding the asset, captured by $\dot{\xi}(t)/(r + \kappa)$. This reflects a dealer's ability to make capital gains by exploiting his continuous access to the asset market. Another difference is the fact that the utility function for investors on the left-hand side of (13) is a weighted-average of the marginal utility flows that the investor enjoys until the next time he is able to readjust his asset holdings.

4 Equilibrium

In this section, we study the determination of the asset price, define equilibrium, and show how to characterize it. Since each investor faces the same probability to access the market irrespective of his asset holdings, and since these probabilities are independent across investors, we appeal to the law of large numbers to assert that the flow supply of assets by investors

12

is $\alpha [A - A_d(t)]$, where $A_d(t)$ is the aggregate stock of assets held by dealers. (Note that $A_d(t) = a_d(t)$, since there is a unit measure of identical dealers facing the same strictly concave optimization problem). The measure of investors with preference shock i who are trading in the market at time t is $\alpha n_i(t)$, where $n_i(t)$ is the measure of investors with preference type i at time t. Therefore, the investors' aggregate demand for the asset is $\alpha \sum_{i=1}^{I} n_i(t) a_i(t)$, and the net supply of assets by investors is $\alpha[A - A_d(t) - \sum_{i=1}^{I} n_i(t) a_i(t)]$. The net demand from dealers is $\dot{A}_d(t)$, the change in their inventories. Therefore, market clearing requires

$$\dot{A}_d(t) = \alpha \left[A - A_d(t) - \sum_{i=1}^{I} n_i(t) a_i(t) \right]. \tag{17}$$

The measure $n_i(t)$ satisfies $\dot{n}_i(t) = \delta \pi_i - \delta n_i(t)$ for all i, and therefore,

$$n_i(t) = e^{-\delta t} n_i(0) + (1 - e^{-\delta t}) \pi_i, \qquad \text{for } i = 1, .., I. \tag{18}$$

If we use (13) to substitute $a_i(t)$ from (17), it becomes apparent that this market-clearing condition determines $\xi(t)$. The intermediation fees along the equilibrium path are given by (2). Using (9), (11) and (12), (2) reduces to

$$\phi_i(a, t) = \eta \left[\frac{U_i[a_i(t)] - U_i(a) - \xi(t)[a_i(t) - a]}{r + \kappa} \right]. \tag{19}$$

Definition 1 *An equilibrium is a collection of bounded asset holdings $[\{a_i(t)\}_{i=1}^{I}, A_d(t)]$, together with piecewise continuously differentiable trajectories for prices and intermediation fees, $[\xi(t), p(t), \phi_i(a, t)]$, that satisfy , (4)–(6), (12)–(14), (17) and (19).*

We do not list the distribution of asset holdings across investors in the preceding definition because it does not affect the dealer's problem, the investor's problem, nor any of the variables which are relevant to our analysis. To characterize the equilibrium, we begin by establishing two important properties of any equilibrium price path.

Lemma 4 *In an equilibrium,*

$$\lim_{t \to \infty} e^{-rt} p(t) = 0, \qquad and \tag{20}$$

$$p(t) = \int_t^\infty e^{-r(s-t)} \left[\xi(s) - \frac{\dot{\xi}(s)}{r + \kappa} \right] ds. \tag{21}$$

13

The no-bubble condition (20) follows from adding up the transversality conditions (6) and (14) across all agents, which after observing that agents' holdings must add up to $A > 0$ imply $\lim_{t\to\infty} \mathbb{E}\left[p(\theta_t)e^{-r\theta_t}A\right] = 0$. This, in turn, can be shown to imply (20). With (20), (21) follows from (15).

If we combine (13), (16) and (17) and assume an interior solution for dealers' inventories, the model can be reduced to a system of two first-order differential equations

$$\dot{A}_d(t) = \alpha \left\{ A - A_d(t) - \sum_{i=1}^{I} n_i(t) U_i'^{-1}[\xi(t)] \right\}, \tag{22}$$

$$\dot{\xi}(t) = (r + \kappa) \left\{ \xi(t) - v'[A_d(t)] \right\}, \tag{23}$$

with $n_i(t)$ given by (18). This system is nonlinear and nonautonomous. The steady-state equilibrium is such that $U_i'(a_i) = v'(a_d) = \xi = rp$, where ξ is the unique solution to

$$v'^{-1}(\xi) + \sum_{i=1}^{I} \pi_i U_i'^{-1}(\xi) = A. \tag{24}$$

Consider the limit as the trading frictions vanish, i.e., as α approaches ∞. From (15), $\xi(t) = rp(t) - \dot{p}(t)$, so the investor's cost of investing in the asset is the flow cost $rp(t)$ minus the capital gain $\dot{p}(t)$, the same as the dealer's. From (11), $U_i(a)$ tends to $u_i(a)$, so (11) implies that the investor's optimal choice of assets satisfies $u_i'(a_i) = rp(t) - \dot{p}(t)$. This is the asset demand of an investor in a frictionless Walrasian market.

A very tractable special case of (22) and (23) obtains when $n_i = \pi_i$ for all i, i.e., when the distribution of preference types across investors is time-invariant, since the system is then homogenous. (Note that this does not imply that the joint distribution of assets and preference types across investors is constant, so the economy need not be in a steady state.) Linearizing (22) and (23) in the neighborhood of the unique steady-state, $(\bar{A}_d, \bar{\xi})$, the steady state can be verified to be a saddle-point. For some initial condition $A_d(0)$ in the neighborhood of the steady state there is a unique trajectory, the saddle-path, that brings the economy to its steady state. This trajectory also satisfies (6), so it is an equilibrium. Lemma 5 establishes that for a given initial condition, such a path is the unique equilibrium. Figure 2 depicts the dynamics of the system with a phase diagram.

Lemma 5 *Suppose that $n_i(0) = \pi_i$ for all i, and that the initial condition $a_d(0) = A_d(0)$ is close to the steady-state value $\bar{A}_d$. Then, there is a unique dynamic equilibrium, and it converges to the steady state.*
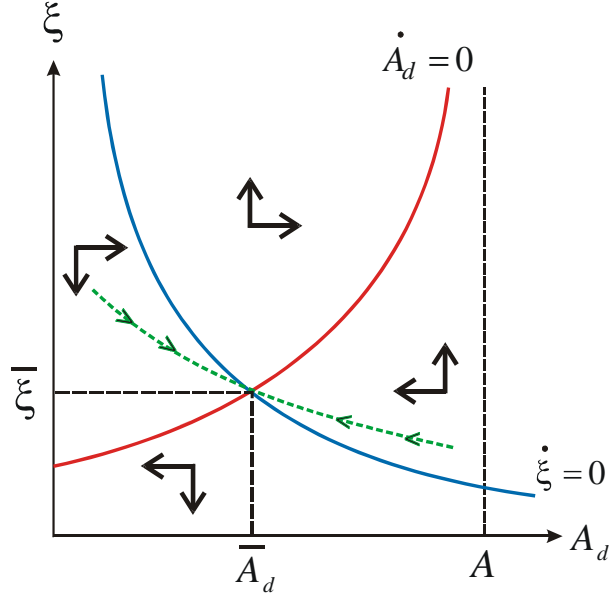
Figure 2: Dealers and inventories: Phase diagram

As dealers' marginal utility for the asset decreases, the $\xi-$isocline shifts downward. As $v'(a_d)$ tends to 0, a case we will focus on in the following sections, the $\xi-$isocline approaches the horizontal axis for all $A_d > 0$ and the vertical axis for $A_d = 0$. The steady state is then at the intersection of the A_d-isocline and the vertical axis, and there is a saddle-path that brings the economy to the steady state.

5   Efficiency

In this section we characterize the efficient allocation. We carry out an elementary variational experiment to identify the social gains associated with liquidity provision by dealers, and provide a more formal treatment of the social planner's problem in Appendix A.

Use $\mathrm{m}(\tau, t)$ to denote the marginal utility that an investor enjoys at time t, from the asset position he acquired at time $\tau \leq t$. Let

$$\mathrm{M}(\tau, t) \equiv (r + \alpha) \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} \mathrm{m}(\tau, s) ds \right],$$

i.e., $\mathrm{M}(\tau, t)$ is the flow expected present value of an investor's marginal utility for the assets

he acquired at time τ, from time $t \geq \tau$ until his next contact time with dealers, T.[12] Let Δ represent the length of a small time interval, then $\mathrm{M}(\tau, t)$ solves the recursion

$$\mathrm{M}(\tau, t) = (r + \alpha)\mathrm{m}(\tau, t)\Delta + (1 - r\Delta - \alpha\Delta)\mathbb{E}_t\left[\mathrm{M}(\tau, t + \Delta)\right]. \tag{25}$$

At each point in time $t > 0$, a quantity $A_d(t)$ of assets is held by dealers, and the remaining $A - A_d(t)$ is held by investors. Because there is a continuum of investors establishing contact with dealers at Poisson intensity α, the law of large numbers implies that, during any small time interval $[t, t + \Delta]$, there is a quantity $A_d(t) + \alpha\Delta\left[A - A_d(t)\right]$ of assets that can be reallocated between those investors who are in contact with dealers, and between investors and dealers.

Holding $A_d(t)$ fixed, an efficient allocation of the remaining $\alpha\Delta\left[A - A_d(t)\right]$ assets must equalize the marginal value $\mathrm{M}(t, t)$ of all investors who are currently contacting dealers and holding assets. Otherwise, one could improve welfare by reallocating assets from investors with low marginal valuations to investors with high marginal valuations. This means that,

$$\mathrm{M}(t, t) = \lambda(t), \tag{26}$$

for some $\lambda(t) \geq 0$, which represents the shadow price that the planner assigns to assets in the hands of dealers at time t (assuming investors hold some assets).

We now provide a necessary condition for dealers' inventory holdings, $A_d(t)$, to be part of an efficient allocation. Start from an allocation such that (26) holds at each time, and perturb it as follows: (i) keep the same allocation during $[0, t)$, (ii) take a marginal asset from some positive measure of "early" investors at time t and give them to dealers until time $t + \Delta$. (iii) If an early investor recontacts the market at time $t + \Delta$, give the asset back to him. If he does not recontact the market at time $t + \Delta$, give the asset to some other "late" investor who contacted the market at time $t + \Delta$. (iv) Continue with the initial asset allocation after $t + \Delta$. (Since dealers' asset holdings at $t + \Delta$ are the same as in the initial allocation, the quantity of assets available in the market stays the same, and it is feasible to continue with the initial allocation after $t + \Delta$.)

We can break up the net utility of this perturbation as follows. First, during $[t, t + \Delta]$ assets are held by dealers, with a marginal utility $\upsilon'(t)$, instead of the early investors, with a marginal

utility of $\mathrm{m}(t,t)$.[13] This represents a net flow utility of $(r+\alpha)\left[v'(t)-\mathrm{m}(t,t)\right]\Delta$. Second, there is a fraction $\alpha\Delta$ of early investors who re-establish contact with dealers at $t+\Delta$ and receive their asset back, with a net utility of zero from $t+\Delta$ onwards. For the fraction $1-\alpha\Delta$ of early investors who do not re-establish contact with dealers, there is an expected discounted cost of

$$\mathbb{E}_t\left[e^{-r\Delta}\mathrm{M}(t,t+\Delta)\right] \simeq (1-r\Delta)\mathbb{E}_t\left[\mathrm{M}(t,t+\Delta)\right]. \tag{27}$$

This represents the discounted marginal value that is lost because early investors hold one unit less of assets until their next respective contact times with dealers. Lastly, since the asset is transferred to some late investors at time $t+\Delta$, there is an expected gain of

$$(1-r\Delta)\mathbb{E}_t\left[\mathrm{M}(t+\Delta,t+\Delta)\right]. \tag{28}$$

As before, equation (28) is the discounted marginal value that is gained because late investors hold one more unit of assets until their next respective contact time with dealers. This discussion shows that the net utility of the perturbation is

$$(r+\alpha)\left[v'(t)-\mathrm{m}(t,t)\right]\Delta + (1-\alpha\Delta)(1-r\Delta)\mathbb{E}_t\left[\mathrm{M}(t+\Delta,t+\Delta)-\mathrm{M}(t,t+\Delta)\right]. \tag{29}$$

The second term represents the gain from liquidity provision. The discounting factor, $(1-r\Delta)$, appears because the gain occurs later in time. The probability factor, $(1-\alpha\Delta)$, appears because the gain occurs only if the early investors do not manage to re-establish contact with dealers. The last factor, $\mathbb{E}_t\left[\mathrm{M}(t+\Delta,t+\Delta)-\mathrm{M}(t,t+\Delta)\right]$, is positive when the marginal utility of the early investor, $\mathrm{M}(t,t+\Delta)$, is, on average, smaller than the marginal utility of the late investor, $\mathrm{M}(t+\Delta,t+\Delta)$. This means that liquidity provision can raise welfare by improving intertemporal matching, i.e., by creating a mutually beneficial match between two investors who contact dealers at different points in time.

To a first-order approximation, equation (29) can be rearranged as follows[14]

$$(r+\alpha)\Delta\left[v'(t)-\lambda(t)\right] + (1-r\Delta-\alpha\Delta)\left\{\mathbb{E}_t\left[\lambda(t+\Delta)\right]-\lambda(t)\right\}. \tag{30}$$

Divide (30) through by Δ and take Δ to zero to find that increasing the amount of inventories held by dealers does not improve welfare if

$$v'(t) + \frac{1}{r+\alpha}\lim_{\Delta\to 0}\frac{\mathbb{E}_t\left[\lambda(t+\Delta)-\lambda(t)\right]}{\Delta} \leq \lambda(t). \tag{31}$$

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[13]Note that, since agents have quasi-linear preferences, one must give equal weights to all agents' marginal utilities for the assets.

[14]Use (25) to rewrite (29) as $(r+\alpha)v'(t)\Delta - \mathrm{M}(t,t) + (1-r\Delta-\alpha\Delta)\mathbb{E}_t\left[\mathrm{M}(t+\Delta,t+\Delta)\right]$. The expression (30) then follows from (26).

Considering the opposite perturbation of decreasing dealers' inventories, we find that (31) holds with equality whenever $A_d(t) > 0$. In the environment of the previous section, with no aggregate uncertainty and where $v'(0) = \infty$, we can derive these first-order conditions formally using the Maximum Principle.

Lemma 6 *An efficient allocation* $\left[\{a_i(t)\}_{i=1}^I, a_d(t) \right]$ *satisfies*

$$\frac{(r + \alpha) u_i'[a_i(t)] + \delta \sum_{j=1}^I \pi_j u_j'[a_i(t)]}{r + \alpha + \delta} = \lambda(t), \tag{32}$$

$$v'[a_d(t)] + \frac{\dot{\lambda}(t)}{r + \alpha} = \lambda(t), \tag{33}$$

the resource constraint (17), and the transversality condition

$$\lim_{t \to \infty} e^{-rt} \lambda(t) = 0, \tag{34}$$

for some $\lambda(t) \geq 0$. *In addition, if* $a_d(t)$ *satisfies*

$$\lim_{t \to \infty} e^{-rt} \lambda(t) a_d(t) = 0, \tag{35}$$

then $\left[\{a_i(t)\}_{i=1}^I, a_d(t) \right]$ *is an optimal path.*

If we identify the equilibrium "price," $\xi(t)$, with the planner's shadow price of assets, $\lambda(t)$, and compare (4) and (11) with (32) and (33), it becomes apparent that they would be identical if $\kappa = \alpha$, i.e., if η were equal to zero. The following proposition formalizes this observation.

Proposition 1 *Equilibrium is efficient if and only if* $\eta = 0$.

Whenever $\eta > 0$, an inefficiency arises from a holdup problem due to ex-post bargaining. Whenever they trade, investors anticipate the fact that they will have to pay fees for rebalancing their asset holdings in the future. These intermediation fees increase with the surplus that the trade generates. As a consequence, investors will tend to avoid positions that could lead to large rebalancing in the future.

6 Crash and deterministic recovery

In this section, we describe the dynamic adjustment of the asset price and the allocation of assets between dealers and investors following a market crash. We think of a market crash as

18

a sudden rise in selling pressure, and model it as a one-time unexpected shock that modifies the distribution of investors across preference types, $\{n_i(t)\}_{i=1}^I$, in a way that causes the total demand for the asset to fall unexpectedly.[15] We suppose that the economy is in the steady state at the time this shock hits, which we take to be $t = 0$. The total quantity of assets demanded by investors is lowest at $t = 0$, and then gradually recovers over time as the initial distribution of preference types, $\{n_i(0)\}_{i=1}^I$, reverts back to the invariant distribution, $\{\pi_i\}_{i=1}^I$.

In order to highlight the intermediation role of dealers, we assume that they start off with no inventory, $a_d(0) = 0$, and that they get no utility from holding the asset, i.e., $v(a) = 0$. In this formulation, dealers will only buy assets for their own account in an attempt to make capital gains over some holding period. Hence, $A_d = 0$ in the steady state, since dealers cannot make capital gains if the asset price is constant. For investors, we adopt $u_i(a) = \varepsilon_i a^{1-\sigma}/(1-\sigma)$, which implies $U_i(a) = \bar{\varepsilon}_i a^{1-\sigma}/(1-\sigma)$, with $\bar{\varepsilon}_i = \frac{(r+\kappa)\varepsilon_i + \delta\bar{\varepsilon}}{r+\kappa+\delta}$ and $\bar{\varepsilon} = \sum_{k=1}^I \pi_k \varepsilon_k$. The following lemma summarizes the key properties of the investor's and the dealer's optimization problems.

Lemma 7 *(a) An investor with preference type i who gains access to the market at time t, demands*

$$a_i(t) = \left[\frac{\bar{\varepsilon}_i}{\xi(t)} \right]^{1/\sigma} . \tag{36}$$

(b) A dealer's asset holdings satisfy

$$[rp(t) - \dot{p}(t)] a_d(t) = 0. \tag{37}$$

The second part of Lemma 7 formalizes the notion that if dealers do not enjoy any direct benefits from holding the asset, then they will only hold it to try to obtain capital gains. Dealers hold no inventories over periods when the price is growing at a rate lower than the rate of time preference. Conversely, they are willing to take long positions in the asset only if $\dot{p}(t)/p(t) = r$. (Naturally, $\dot{p}(t)/p(t) > r$ would be inconsistent with equilibrium.) We can use (15) to express the dealer's optimality condition as

$$\left[\xi(t) - \frac{\dot{\xi}(t)}{r+\kappa} \right] A_d(t) = 0, \tag{38}$$

with $\dot{\xi}(t)/\xi(t) \leq r+\kappa$, where $A_d(t) \geq 0$ denotes dealers' aggregate inventories. (Notice that individual dealers need not hold the same inventories here.)

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[15]This is the same notion of market crash used by Weill (2006). We study a different notion of market crash in the following section.

With (18) and (36), the market-clearing condition (22) can be written as

$$\dot{A}_d(t) = \alpha \left\{ A - A_d(t) - \xi(t)^{-1/\sigma} \left[\bar{E} - e^{-\delta t} \left(\bar{E} - E_0 \right) \right] \right\}, \tag{39}$$

where $\bar{E} = \sum_{i=1}^{I} \pi_i \bar{\varepsilon}_i^{1/\sigma}$ and $E_0 = \sum_{i=1}^{I} n_i(0) \bar{\varepsilon}_i^{1/\sigma}$. Intuitively, $\xi(t)^{-1/\sigma} \left[\bar{E} - e^{-\delta t} \left(\bar{E} - E_0 \right) \right]$ is the total quantity of assets demanded by investors at time t. This way of writing the investors' aggregate demand reveals two sources of time variation. First, investors' aggregate demand will change in response to changes in the effective cost of purchasing the asset, $\xi(t)$. The second component, $\left[\bar{E} - e^{-\delta t} \left(\bar{E} - E_0 \right) \right]$, captures changes in aggregate demand due to composition effects coming from variations in the distribution of investors over the various preference types. The constant $\bar{E}$ is a measure of investors' willingness to hold the asset in the steady state, i.e., when $n_i(t) = \pi_i$, while E_0 reflects the investors' willingness to hold the asset at time 0, when the aggregate shock hits. Thus, $E_0/\bar{E}$ is a measure of the magnitude of the composition shock to aggregate demand for the asset. In line with our market crash interpretation, we maintain $E_0/\bar{E} < 1$ throughout the analysis, i.e., lower preference types receive larger population weights at time 0 relative to the steady state.

The dealers' first-order condition, (38), and the market-clearing condition, (39), are a pair of differential equations that can be solved for $\xi(t)$ and $A_d(t)$. If $A_d(t) > 0$ for all t in some interval $[t_1, t_2]$, then (38) implies $\xi(t) = e^{(r+\kappa)(t-t_2)} \xi(t_2)$, and given this path for $\xi(t)$, (39) is a first-order differential equation that can be readily solved for the path $A_d(t)$. Similarly, if $A_d(t) = 0$ over some interval, then (39) immediately implies a path for $\xi(t)$. In order to fully characterize the equilibrium path one needs to determine the time intervals over which dealers accumulate inventories as well as the continuity of the trajectory. The following proposition provides the salient features of the equilibrium path following a market crash.

Proposition 2 *The unique equilibrium path, $\{\xi(t), A_d(t)\}$, has the following features:*

1. *It converges to the steady state, $\{\bar{\xi}, \bar{A}_d\} = \{(\bar{E}/A)^{\sigma}, 0\}$.*

2. *There exists a time $T \in [0, \infty)$ such that $A_d(t) > 0$ for all $t \in (0, T)$ and $A_d(t) = 0$ for all $t \geq T$.*

3. *Let $p_0(t)$ denote the equilibrium asset price that would obtain if dealers were constrained to hold no inventories. Dealers intervene, i.e., $T > 0$, if and only if, at the time of the*

crisis, $t = 0$, $\frac{\dot{p}_0(0)}{p_0(0)} > r$, which is equivalent to

$$\frac{\sum_{i=1}^{I} n_i(0) \left[(r + \kappa) \varepsilon_i + \delta \bar{\varepsilon}\right]^{1/\sigma}}{\sum_{i=1}^{I} \pi_i \left[(r + \kappa) \varepsilon_i + \delta \bar{\varepsilon}\right]^{1/\sigma}} < \frac{\delta \sigma}{r + \kappa + \delta \sigma}. \tag{40}$$

According to Proposition 2, the equilibrium path following a market crash is characterized by a switching time $T \in [0, \infty)$ such that dealers hold the asset for all $t \in (0, T)$ and do not hold it for $t \geq T$. It is possible that $T = 0$, in which case dealers do not hold inventories at all. The last part of the proposition establishes that dealers will intervene if and only if $\dot{p}_0(0)/p_0(0) > r$, i.e., if and only if the rate of growth of the asset price that would result at the time of the crisis if they *did not* intervene, exceeds the rate of time preference.

If dealers intervene, the time period during which they hold the asset is an interval, which starts at the outset of the crisis, i.e., at $t = 0$. (See Figure 4 for an illustration.) Thus, dealers never find it beneficial to delay the acquisition of the asset: If they will buy at all, they start buying from the very beginning, when the investors' selling pressure is strongest. The economic reasoning behind this result is that since dealers get no direct utility from holding the asset, they are only willing to take long positions if the capital gains associated with those positions are large enough, i.e., if the growth rate of the asset price is greater than the discount rate. It is possible to show that, in the absence of dealers' intervention, the price of the asset, $p_0(t)$, grows at a decreasing rate. Hence, if dealers don't have incentives to hold inventories at $t = 0$, they never will. In contrast, Weill (2006) finds that dealers do not necessarily start accumulating inventories right after the crash, and that for some parameter values, delaying the intervention of dealers is socially optimal.[16]

Notice that the left side of (40) equals $E_0/\bar{E}$. Thus, the last part of Proposition 2 states that the condition for dealers to participate, i.e., $\dot{p}_0(0)/p_0(0) > r$, can be expressed in terms of

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[16] The key assumption in Weill (2006) that lies behind this result is that investors' utility function is of the Leontief form, $u(a) = \min\{a, 1\}$, so they are effectively restricted to hold zero or one unit of the asset. In contrast, here we allow investors to hold any nonnegative position. To reconcile our results with Weill's, we can nest Weill's specification with ours by assuming an investor's utility function is

$$u(a) = \frac{\left[\sigma + (1-\sigma)a^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}}}{1 - \sigma},$$

for some $(\sigma, \theta) \in \mathbb{R}_+ \times (0, 1)$. Our isoelastic utility function is obtained as $\theta \to 1^-$, and Weill's Leontief utility function is obtained as $\theta \to 0^+$. Numerical calculations (available upon request) suggest that, for θ close to zero, we would recover Weill's result that dealers do not necessarily start accumulating inventories at the time of the crash.

the exogenous severity of the crash, as measured by $1 - E_0/\bar{E}$, i.e., the magnitude of the initial drop in the investors' willingness to hold the asset. If this drop is larger than the threshold $1 - \frac{\delta\sigma}{r+\kappa+\delta\sigma}$, then dealers will step in to take up the slack resulting from the reduction in investors' demand. Conversely, dealers will not intervene if (40) is not satisfied. Condition (40) depends on all the fundamentals of the economy, e.g., preferences (σ), the extent of trading frictions and the market-power of dealers (κ), the change in the distribution of valuations that triggers the crisis ($\{n_i(0)\}_{i=1}^{I}$) and the frequency of the preference shocks (δ). As shown in the next corollary, there exist parametrizations for which condition (40) does not hold.

Corollary 1 *The set of parameter values under which dealers do not accumulate inventories (i.e., $T = 0$) is nonempty.*

Corollary 1 contrasts with Theorem 1 in Weill (2006), which establishes that there is *always* a period of time during which dealers *lean against the wind* before the investors' selling pressure subsides.[17] A sufficient condition for condition (40) to fail is that $(r + \kappa)/\delta$ be sufficiently large. Suppose that preference shocks are very persistent (δ very small). In this case the recovery is slow, the growth rate of the asset price is small, and dealers find that the prospective capital gains are smaller than the opportunity cost of holding the asset. It is also instructive to consider the limiting case as α goes to infinity and the economy approaches the frictionless Walrasian benchmark. In this case, dealers no longer have the advantage of trading continuously vis-à-vis investors, and their ability to realize capital gains vanishes (recall our discussion of (16)). Put differently, as frictions vanish, the market provides dealers no incentive to buy assets early in the crisis, and they do not intervene regardless of the severity of the crisis.

Below, we will show that there are also parametrizations for which condition (40) is satisfied and dealers buy assets at the beginning of the crisis, hold them for a while and sell them off as the investors' selling pressures subside. In these cases, dealers choose positive asset positions (foregoing interest on their stock of the numéraire good) even though they get no utility from holding these assets. The reason why dealers may be willing to carry assets is that they have continuous access to the market while investors do not: This trading advantage allows dealers to "time the market" continuously in order to capture capital gains that investors cannot realize. Without dealers, or if dealers were unable to hold inventories, these capital gains would remain

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[17]This difference in results is also due to the fact that investors' asset holdings are unrestricted here but subject to a unit upper bound in Weill (2006) (see footnote 16).

unexploited. In equilibrium, competition among dealers ends up equalizing these capital gains to the opportunity cost of holding assets, i.e., $\dot{p}/p = r$. This logic is consistent with the frictionless limit we discussed above.

Next, we use numerical examples to illustrate and explain how the key parameters influence the dealers' incentives to hold inventories. In what we will consider to be the benchmark example, we set $\sigma = 1/2$ and assume that the preference shock can either be $\varepsilon_1 = 0$ or $\varepsilon_2 = 1$, with equal probability. This means that the invariant distribution has an equal measure of investors with low and high valuations. We also set $r = 0.05$ and $\alpha = \delta = 1$, so that on average, investors get one preference shock and one chance to trade per "period." We also set $\eta = 0$ so that the equilibrium allocation of the benchmark parametrization corresponds to the solution to the planner's problem. We consider an economy which is at its steady state, and at time 0 is subject to a shock that causes the fraction of investors with the low preference shock to rise from $\pi = 1/2$ to $n_1(0) = 0.95$.

The shaded (green) regions in Figure 3 illustrate the combinations of parameter values for which condition (40) is satisfied so that dealers hold inventories after the crash. In each panel, we let the two parameters in the axes vary and keep the rest fixed at their benchmark values. All panels have α—our index of the degree of the trading frictions—on the horizontal axis. Markets with large α are very liquid markets where trades get executed very fast.

Figure 3 allows us to address the following normative question: Could it be socially efficient for dealers to accumulate inventories, even though they are pure speculators who don't derive any direct utility from holding assets? The answer is: yes. Recall (Proposition 1) that the equilibrium allocations of an economy with $\eta = 0$ correspond to the Pareto-optimal allocations. The third panel of Figure 3 shows that there are parameterizations involving $\eta = 0$ where dealers indeed choose to intervene. As explained in Section 5, the planner allocates assets to dealers in order to exploit an intertemporal trade-off between the marginal utility of investors in the market at the current date and in the future. The average marginal valuation of the asset across investors is low at the outset of the crisis and higher later on. The planner uses dealers' inventories to smooth these marginal valuations over time. Specifically, the planner may choose to put assets in the hands of dealers in the early stages of the crisis (when the opportunity cost of not allocating them to investors is relatively low) to be able to transfer these assets *without delays* to investors in the later stages of the crisis, when the marginal valuation of the average investor is high. Therefore, depending on fundamentals, it can be optimal to have dealers act
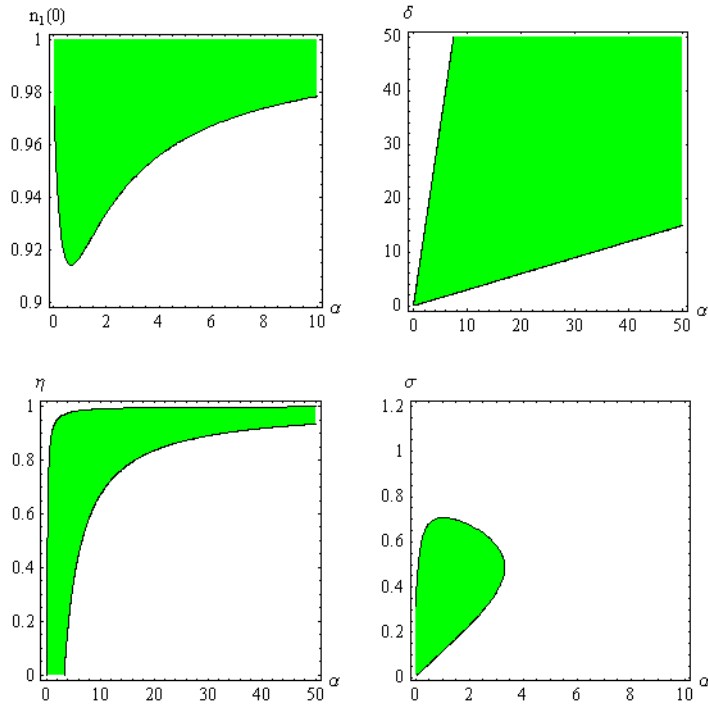
23

Figure 3: Parametrizations for which dealers "lean against the wind"

as a "buffer stock." The opportunity cost of having dealers carry an asset they don't value for a while is the price the planner pays to provide *immediacy* to the future higher-than-average-valuation "cohorts" of investors that will gain access to the asset market at later dates. Let us now turn to the effects of fundamentals on dealers' likelihood to intervene during the crisis. In turn, we will consider the effects of the characteristics of the crisis, market structure and investors' preferences.

Characteristics of the crisis. The first panel in Figure 3 shows that for any given α, dealers intervene if $n_1(0)$ is large enough, i.e., if the crash is sufficiently severe. To explain this result we resort to the connection to the planner's problem (but there is an equivalent explanation in terms of the dealers' incentives in the equilibrium). In the early stages of the crisis, the "cohorts" of investors that contact the marketplace involve a very large fraction of low-valuation investors who have relatively low individual demands for the asset. If the planner chooses not to use the dealers' inventories, then in these early stages he will be reallocating more assets

24

to the few high-marginal-valuation investors. Such an allocation will imply a very low shadow price of assets (denoted $\lambda(t)$ in Section 5) in the early stages of the crisis. Conversely, the shadow price of the asset will be relatively large at later dates, as the fraction of high-valuation investors increases toward its steady-state level, since at that point there will be many more high-valuation investors who are willing to hold relatively large quantities of the asset. To larger values of $n_1(0)$ correspond larger discrepancies between the marginal utilities of earlier and later cohorts of investors, among which the planner can reallocate assets (this discrepancy is measured by the term $\lambda(t + \Delta) - \lambda(t)$ of (30)). Dealers offer the planner a way to smooth these differences in intertemporal marginal utilities across cohorts of investors, and are used as a buffer stock for large values of $n_1(0)$, i.e., whenever the crash is severe.

The second panel in Figure 3 shows that, given α, dealers find it optimal to intervene if the recovery is fast enough (i.e., if δ is large enough), so that they would not have to hold the asset for very long. However, the figure also shows that dealers won't intervene if δ is too large. This is because δ not only measures the speed of the recovery but also the arrival intensity of idiosyncratic preference shocks. With a very large δ, the average type of an investor over his holding period, e.g., $\bar{\varepsilon}_i$, becomes very close to the mean, $\bar{\varepsilon}$. In this case, the economy becomes very similar to an economy without idiosyncratic preference shocks, so there is little need to reallocate assets across investors (see our discussion of Corollary 1).

Market structure. We identify the structure of the market with two parameters: α, the extent of the trading frictions, and η, dealers' bargaining strength. The first panel in Figure 3 shows that, for a given size of the aggregate shock, dealers provide liquidity if trading frictions are neither too severe nor too small. For large α, investors face short delays to rebalance their asset holdings, $1/\alpha$ on average. This increases their willingness to take more extreme positions. In particular, investors with higher-than-average utility become more willing to hold larger-than-average positions and absorb more of the selling pressure. In some cases, when α is large enough, they end up supplying so much liquidity to other investors that dealers don't find it profitable to step in. Conversely, if α is very small, then $\bar{\varepsilon}_i$ becomes close to $\bar{\varepsilon}$, and all investors choose very similar asset holdings regardless of their preference type. In this case, the economy becomes similar to an economy without idiosyncratic preference shocks, and dealers are not needed to reallocate assets across investors.

The third panel in Figure 3 reveals that for any given α, dealers are more likely to hold

inventories if their bargaining power is neither too large nor too small. Since α and $(1 - \eta)$ enter the equilibrium conditions as a product, an economy with large η is, from an investor's standpoint, payoff equivalent to an economy where investors access the market very infrequently, i.e., an economy with small α. Recall that if $\eta = 0$, the economy is constrained-efficient. Therefore, the third panel shows that there are parametrizations for which dealers intervene in equilibrium although the planner would not have them intervene, as well as parametrizations for which the opposite is true.

Preferences. The fourth panel of Figure 3 illustrates the role that σ, the curvature of the investor's utility function, plays in the dealer's decision to hold the asset. First, $\sigma < 1$ is a necessary condition for dealers to intervene. In the case of the most severe crisis possible, i.e., $n_1(0) = 1$ (no investor values the asset at $t = 0$), one can show that dealers intervene if and only if $\sigma < 1$ regardless of the value of α. If $\sigma = 1$, the trajectory of the price is

$$p(t) = \frac{\bar{\varepsilon}}{rA} + \frac{e^{-\delta t}}{(r + \delta)A} \sum_{i=1}^{I} n_i(0)(\varepsilon_i - \bar{\varepsilon}),$$

which is independent of α. In fact, in this case $p(t)$ coincides with the price that would prevail in a frictionless Walrasian market.[18] But as we argued earlier, in a Walrasian market dealers would hold no assets since arbitrage by investors would prevent the asset price from growing faster than the discount rate. Therefore, dealers never hold inventories for $\sigma = 1$. For lower values of $n_1(0)$, the dealers' incentives to hold the asset are nonmonotonic in σ. In particular, for a range of values of α they only hold it if σ is in some intermediate range, but not if it is too big or too small. Suppose that σ is very big, so that marginal utility is very steep. One could think that there is more room for dealers to smooth differences in intertemporal marginal utilities across cohorts. However, if σ is very large, then the individual asset demand of an investor with high valuation tends to be very close to the asset demand of an investor with low valuation, and this reduces the benefit from transferring assets between them. In the extreme case $\sigma \to \infty$, $a_i(t) = A$ for all i and all t. But of course this means that shocking the invariant distribution from $\{\pi_i\}_{i=1}^{I}$ to $\{n_i(0)\}_{i=1}^{I}$ has no effect on asset holdings. So effectively, there *is no shock* and thus no gain from liquidity provision, even if $\{\pi_i\}_{i=1}^{I}$ first-order stochastically

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[18]With log preferences an investor's demand is linear in $\bar{\varepsilon}_i$, so the aggregate demand for the asset only depends on $\bar{\varepsilon}$, i.e., it is independent of α, see Lagos and Rocheteau (2007). For a related result under a CARA utility function, see Gârleanu (2006). Also, note that for $\sigma = 1$, condition (40) reduces to $\sum_{i=1}^{I} n_i(0)\varepsilon_i < 0$, indicating that dealers never intervene.

dominates $\{n_i(0)\}_{i=1}^I$. Alternatively, one can interpret $1/\sigma$ as the elasticity of asset demand, a_i, with respect to the preference shock, ε_i. As $\sigma \to \infty$, asset demand becomes inelastic to the preference shock. In this case, the planner's shadow price ($\lambda(t)$ in the notation of Proposition 6) is constant over time, so there is no need nor scope for him to reallocate assets over time.

It is also instructive to look at the opposite extreme of very low σ. For example, consider what happens as $\sigma \to 0$ so that investors' preferences become linear. Suppose that $\varepsilon_1 < \varepsilon_2 < ... < \varepsilon_I$.[19] From (11) it follows that $a_i \to 0$ for $i \in \{1, ..., I-1\}$, i.e., only investors with the highest marginal utility, ε_I, hold the asset. Furthermore, $\xi(t) \to \bar{\varepsilon}_I$ for all t and, from (21), $p(t) \to p = \bar{\varepsilon}_I/r$ for all t. Thus, the price of the asset is constant and equal to its steady-state level. There is clearly no incentive for dealers to buy the asset, regardless of the initial shock to the population weight of investors with high valuation. In this extreme case, investors' desired holdings change dramatically in response to preference shocks, but marginal utility is constant at all times among those who demand the asset, so a planner would have no need to use dealers to "store" the assets in order to smooth the marginal utilities of cohorts of investors at various points in time.

We can summarize the discussion above as follows. Dealers provide liquidity by accumulating asset inventories if: (i) the market crash is abrupt and the recovery is fast; (ii) trading frictions are neither too severe nor too small; (iii) dealers' market power is not too large; (iv) idiosyncratic preference shocks are not too persistent and investors' asset demand is not too inelastic with respect to preference shocks.
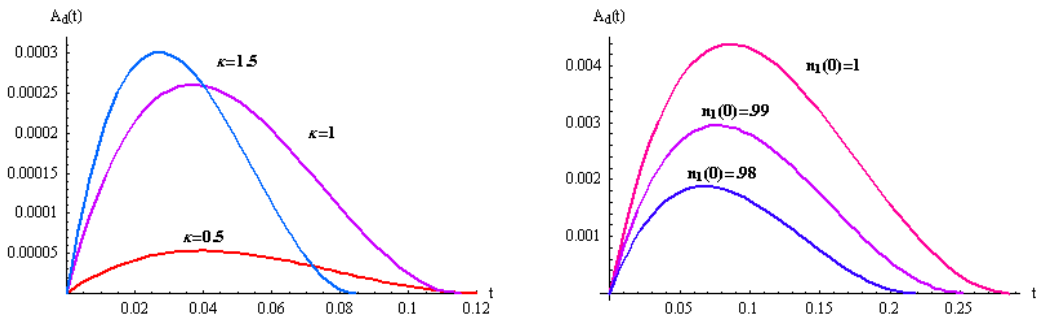

Figure 4: Dealers' asset holdings

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[19]This utility specification is the same one used in Duffie et al. (2005) and Weill (2006), except that they assume a unit upper bound on investors' holdings.

While Figure 3 illustrates the conditions under which dealers accumulate inventories, it is not informative about the extent of dealers' intervention, e.g., what quantity of the asset do dealers accumulate, and how long is the holding period? To answer these questions, Figure 4 plots the trajectory for dealers' inventories for the parameter values of our benchmark example.[20] In both panels one can clearly identify T, namely the switching time at which $A_d(t)$ becomes zero after a period over which dealers have held assets. The first panel illustrates the relationship between market structure (κ) and dealers' inventory policy. Trading frictions have a nonmonotonic effect on T: the length of the holding period is increasing in κ for low values of κ (because investors take more extreme positions, which increases the discrepancy between their marginal utility at different dates), and decreasing for large values of κ (because investors need less liquidity from dealers when trading frictions are mild). The second panel of Figure 4 describes dealers' inventory behavior as a function of the severity of the crash. As $n_1(0)$ decreases, the holding period shrinks and the quantity of assets held by dealers at any point in time becomes smaller. So in a more severe crash, dealers provide more liquidity and for a longer period of time.

The following proposition compares the trajectory for $\xi(t)$, the equilibrium effective cost of holding the asset, to the trajectory of $\xi_0(t)$, the effective cost of holding the asset that would result if dealers were constrained to hold no inventories.

Proposition 3 *If condition (40) holds, then there exists $\underline{t}$ such that $\xi(t) > \xi_0(t)$ for all $t \in [0, \underline{t})$ and $\xi(t) < \xi_0(t)$ for all $t \in (\underline{t}, T)$.*

According to Proposition 3, the presence of dealers mitigates the effect of the market crash on the effective cost of holding the asset. By accumulating inventories right after the crash, dealers prevent ξ from falling too much: $\xi(0)$ is higher than it would have been had dealers not stepped in to buy assets. This is illustrated in Figure 5.

7   Crash and stochastic recovery

In the previous section, our operational definition of a "market crash" was a shock to the distribution of investors across valuations which caused the investors' total demand for the asset to fall. The recovery path corresponded to the transitional dynamics leading to the

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[20]Together with Proposition 2, Lemma 13, which is stated and proved in Appendix A, provides a full characterization of the equilibrium path following a market crash, including closed-form expressions for the paths of $\xi(t)$ and $A_d(t)$.
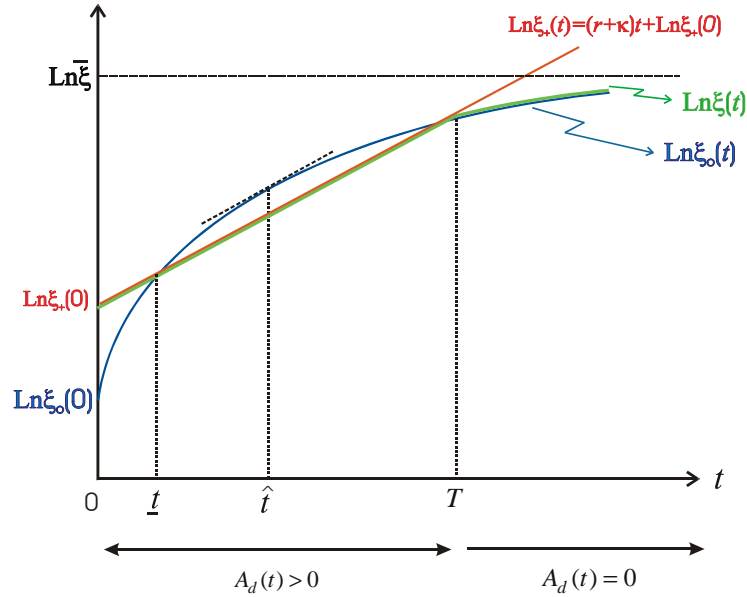
Figure 5: Paths for $\xi_0(t)$ and $\xi(t)$

steady state, so it was deterministic and it started immediately after the shock. It may be argued that during actual market crashes, the investors' behavior and the dealers' decisions about whether to intervene and when to intervene, may be affected by uncertainty about the duration of the crisis. For this reason, in this section we study the dealers' incentives to provide liquidity in the aftermath of a crisis with an uncertain recovery.

We consider the following scenario. At time zero all investors receive an unanticipated multiplicative shock that temporarily scales down their marginal utility from holding the asset. This constitutes the crash. Subsequently, the economy awaits a "recovery shock" that follows a Poisson process with arrival rate ρ which causes all investors to simultaneously revert back to their pre-crisis willingness to hold the asset. Formally, we let T_ρ be an exponentially distributed random variable with mean $1/\rho$, where T_ρ denotes the time at which the economy reverts to normal. An investor with preference type i gets utility $u_i(a)$ from holding a for all $t < 0$ and all $t \geq T_\rho$. For $t \in [0, T_\rho)$, the investor gets utility $Ru_i(a)$, with $R < 1$. Thus, a small R indicates that the crash is severe, and a small ρ that it is expected to be long-lived.[21] We assume that the stochastic process that describes the recovery is independent of the one

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[21] One virtue of this formulation is that it disentangles the speed of the recovery, ρ, and the frequency of the idiosyncratic preference shocks, δ. In the previous section, both were captured by δ.

that governs an investor's transitions between preference types. Furthermore, in this section we assume $\{n_i(0)\}_{i=1}^I = \{\pi_i\}_{i=1}^I$, i.e., that the initial distribution of preference types is the invariant distribution.

We discuss the equilibrium dynamics using Figure 6. (Appendix B provides an analytical solution of the model.) We let $A_d^\ell(t)$ be the dealers' inventories at time t, conditional on $t < T_\rho$, i.e., given that the recovery has not occurred until time t. We denote $\xi^\ell(t)$ the effective cost of holding the asset before the recovery takes place. Similarly, we use the superscript "h" to denote variables after the recovery has occurred. The isocline $\dot{A}_d^\ell = 0$ is located to the right of the isocline $\dot{A}_d = 0$ implied by (22). This is because, for any given ξ, dealers need to hold more of the asset in order to clear the market. The isocline $\dot{\xi}^\ell = 0$ is downward-sloping and located underneath the saddle-path leading to the long-run steady state, $(\bar{\xi}, 0)$. The equilibrium unfolds as follows. The economy starts at $A_d^\ell(0) = 0$, and at the time of the crash, ξ jumps down to the saddle-path leading to $(\bar{\xi}^\ell, \bar{A}_d^\ell)$. (This saddle-path is represented by a dotted line in the figure.) The economy then evolves along this saddle-path until the random recovery shock occurs. In the meantime, along this path, dealers' inventories increase and $\xi^\ell(t)$ decreases. At the random time when the recovery occurs, say t_ρ, the system jumps to the saddle-path leading to $(\bar{\xi}, 0)$. This saddle-path, denoted $\xi = \psi(A_d)$, is represented by a dashed line in the figure. At the time the recovery shock occurs, the cost of holding the asset jumps from ξ^ℓ to ξ^h, and dealers begin selling their inventories gradually until they are completely depleted.

The following proposition provides a condition under which $A_d^\ell(t) > 0$ for all $t > 0$ before the recovery occurs, i.e., a condition for dealers to lean against the wind during a crisis of random duration. It is convenient to define $\tilde{V}_i^\ell(a)$ as the expected sum of discounted utility flows from holding asset position a for an investor of preference type i until the next time he contacts a dealer, and $U_i^\ell(a) = (r + \kappa)\tilde{V}_i^\ell(a)$.[22]

Proposition 4 *Let p_0^ℓ be the asset price during the crisis, and p_0^h be the price after the stochastic recovery, that would obtain if dealers were constrained to hold no inventories. Dealers hold inventories during the crisis if and only if $\frac{\rho(p_0^h - p_0^\ell)}{p_0^\ell} > r$, which is equivalent to*

$$\sum_{i=1}^I \pi_i U_i^{\ell\prime-1}\left(\frac{\rho\bar{\xi}}{r + \kappa + \rho}\right) < A. \tag{41}$$

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[22] We report the expression for $U_i^\ell(a)$ in the proof of Lemma 15 in Appendix B.
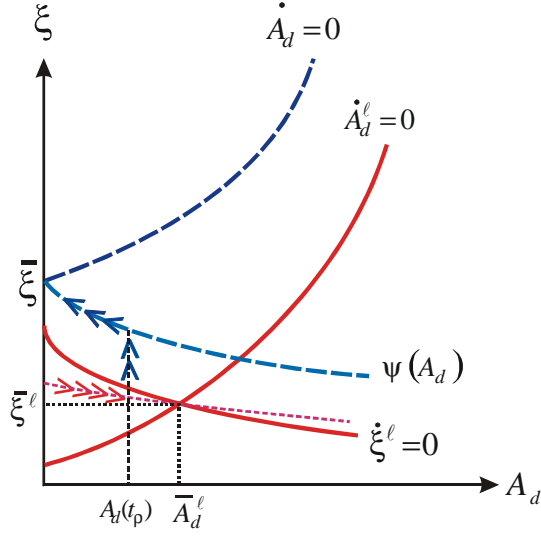
Figure 6: Stochastic recovery: Phase diagram

Proposition 4 provides a condition on fundamentals such that dealers find it beneficial to buy assets during the crisis. Analogously to what we found for the case of a deterministic recovery, dealers intervene if and only if $\rho(p_0^h - p_0^\ell)/p_0^\ell > r$, i.e., if and only if the expected capital gain that they would obtain by buying the asset during the crash and re-selling it once the economy recovers, in an economy where dealers do *not* intervene, exceeds the rate of time preference.

Condition (41) need not always hold, as the following two limiting cases show. Consider first the frictionless limit $\alpha \to \infty$. Then, $U_i^\ell(a) \to Ru_i(a)$, and the left side of (41) approaches ∞. If investors can access the market as frequently as dealers, there is no role for dealers to provide liquidity by buying assets. Next, consider the case where $\rho \to 0$, i.e., the crisis becomes permanent. Again, the left side of (41) can be shown to approach ∞. If the shock is permanent, dealers cannot expect to make capital gains, and therefore they do not invest in the asset. We summarize these findings as follows.

Corollary 2 *The set of parameter values under which dealers do not accumulate inventories is nonempty.*

Corollary 2 is the analogue of Corollary 1 for a crisis of random duration. Next, we show that there exist parameterizations for which condition (41) is satisfied. To this end, let $u(a) =$

$a^{1-\sigma}/(1-\sigma)$ and $u_i(a) = \varepsilon_i u(a)$. Then, during the crisis, an investor's flow expected utility $U_i^\ell(a) = \hat{\varepsilon}_i u(a)$, where $\hat{\varepsilon}_i$ is given in the following corollary.

Corollary 3 *Let $u_i(a) = \frac{\varepsilon_i a^{1-\sigma}}{1-\sigma}$. Dealers hold inventories during a crash if and only if*

$$\frac{\sum_{i=1}^{I} \pi_i \hat{\varepsilon}_i^{1/\sigma}}{\sum_{i=1}^{I} \pi_i \bar{\varepsilon}_i^{1/\sigma}} < \left(\frac{\rho}{r + \kappa + \rho}\right)^{\frac{1}{\sigma}}, \tag{42}$$

where

$$\hat{\varepsilon}_i = \frac{r+\kappa}{r+\kappa+\rho}\left[\frac{(r+\kappa+\rho)\varepsilon_i + \delta \sum_{j=1}^{I}\pi_j\varepsilon_j}{r+\kappa+\delta+\rho}\right]R + \frac{\rho}{r+\kappa+\rho}\frac{(r+\kappa+\rho)\bar{\varepsilon}_i + \delta\sum_{j=1}^{I}\pi_j\bar{\varepsilon}_j}{r+\kappa+\delta+\rho}.$$

Condition (42) is a condition on fundamentals, including the degree of trading frictions, preferences and the properties of the crash. The shaded (green) regions in Figure 7 illustrate the combinations of parameter values for which condition (42) is satisfied so that dealers hold inventories in times of crisis. The benchmark parametrization is: $\sigma = 0.5$, $r = 0.05$, $\pi_1 = \pi_2 = 0.5$, $\alpha = \delta = 1$, $\rho = 1$, $R = 0.02$ and $\eta = 0$. In each panel, we let the two parameters in the axes vary and keep the rest fixed at their benchmark values. All panels have α—our index of the degree of the trading frictions—on the horizontal axis.

Characteristics of the crisis. The first panel confirms one of our findings from Section 6: dealers are more likely to accumulate asset inventories when the market crash is severe (R low). If the crash is severe, dealers expect a larger capital gain when the economy recovers and hence they have an incentive to buy assets during the crash. According to the second panel of Figure 7, for dealers to buy the asset, the crash must be anticipated to be short-lived (ρ must be sufficiently large). From the planner's standpoint, if ρ is small, the opportunity cost of having dealers hold assets (i.e., the utility foregone by investors) is high. Thus, for ρ low enough, the planner would not use dealers' inventories to reallocate the asset across investors over time.

Market structure. As before, we identify the market structure with the parameters α and η. The third panel of Figure 7 shows that dealers accumulate the asset if their bargaining power is neither too large nor too small. If η is close to 1, investors only enjoy a small gain from rebalancing their asset holdings. As a consequence, when in contact with a dealer they put more weight on their average preferences in order to reduce their need to readjust their asset holdings in the future. As discussed above, if idiosyncratic preference shocks become less

relevant, there is less scope for dealers to help reallocate the asset over time. To understand why dealers have lower incentives to provide liquidity when η is small, recall that a reduction in η is similar from the point of view of investors' payoffs to an increase in α: If trading frictions are reduced, there is less need for the buffer stock of assets provided by dealers.

Preferences.   The fourth panel shows that the curvature of investors' utility function must be sufficiently small for dealers to accumulate asset inventories. As before, if σ is high, investors' demand for the asset is relatively inelastic with respect to the idiosyncratic preference shock, which reduces the usefulness of dealers.

To conclude, we study how the characteristics of the crash and the structure of the market affect the amount of liquidity provided by dealers. For our baseline parameter values, in Figure 8 we plot the maximum quantity of assets that dealers are willing to accumulate during the crash, namely, $\bar{A}_d^\ell = \lim_{t\to\infty} A_d^\ell(t)$.[23] The first panel confirms the nonmonotonic relationship between dealers' provision of liquidity and the degree of frictions that prevail in the market. The second panel shows that dealers' willingness to provide liquidity increases with the severity of the crisis. According to the third panel, the relationship between the maximum amount of liquidity that dealers are willing to provide and the expected duration of the crisis $(1/\rho)$ is nonmonotonic. If the crash is very persistent (ρ small) dealers are not willing to accumulate large positions since the expected discounted capital gain of these inventories is small. If the crash is anticipated to be short-lived (ρ large), dealers will not accumulate too much inventories because the crash reduces investors' asset demand only by a small amount. The fourth panel shows that dealers' inventories decrease as investors' intertemporal elasticity of substitution $(1/\sigma)$ gets smaller. In particular, as investors' utility function becomes linear ($\sigma \to 0$) dealers are willing to accumulate the entire stock of assets in the economy ($\bar{A}_d^\ell \to A$).

We can summarize the results in this section as follows. Dealers are more likely to provide liquidity during a crash with stochastic recovery if: (i) the market crash is abrupt and expected to be short-lived; (ii) dealers' market power is above some minimum value but not too close to one; (iii) trading delays are neither too long nor too short; (iv) investors' asset demand is sufficiently elastic with respect to idiosyncratic preference shocks. The amount of liquidity provided by dealers, as proxied by the maximum quantity of assets they are willing to accumulate, increases with the severity of the crash but is nonmonotonic with respect to the duration of the

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[23]In Appendix B we report a closed-form expression for $\bar{A}_d^\ell$.

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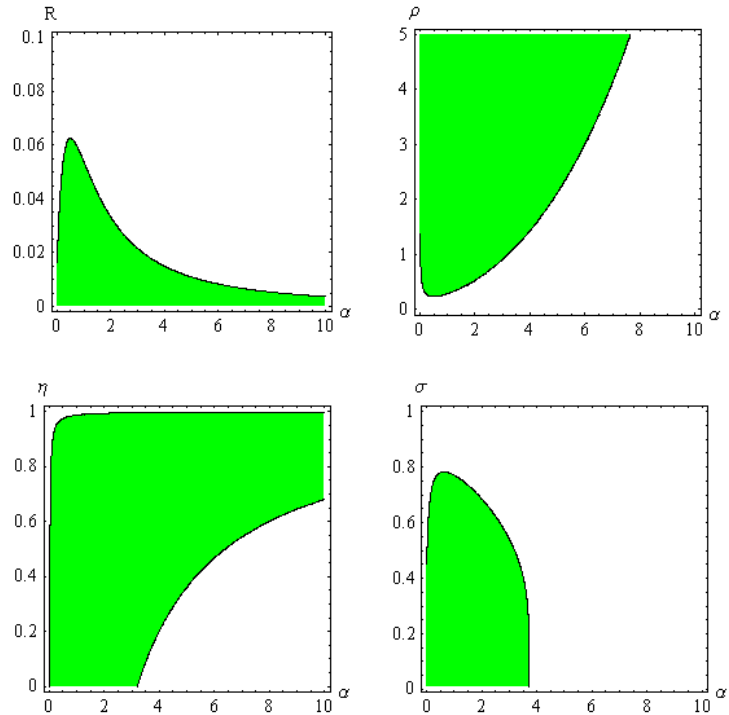
Figure 7: Parametrizations for which dealers "lean against the wind"

crisis and the extent of the trading frictions.

8   Conclusion

We have studied the equilibrium dynamics of an asset market in the presence of the types of trading frictions that are characteristic of many financial markets during times of crisis and of many other markets, e.g., over-the-counter markets, even in normal times. In particular, we have analyzed the recovery path of the market in the aftermath of an aggregate shock to investors' preferences, which we interpret as a crash or could also be thought of as a "flight-to-liquidity shock." In principle, dealers can mitigate the effects of such aggregate shocks on the asset price by providing liquidity during these times of market distress. However, there is evidence that sometimes they do, and sometimes they don't.

We have established conditions on fundamentals, such as the extent of the trading frictions, the degree of market power of dealers, and the severity and expected duration of the crash, under

Figure 8: Dealers' provision of liquidity

which dealers will find it profitable to step in to accumulate asset inventories during times when investors' demand collapses, thereby preventing the asset price from falling as much as it would have had they not intervened. We have found that dealers are more likely to accumulate asset inven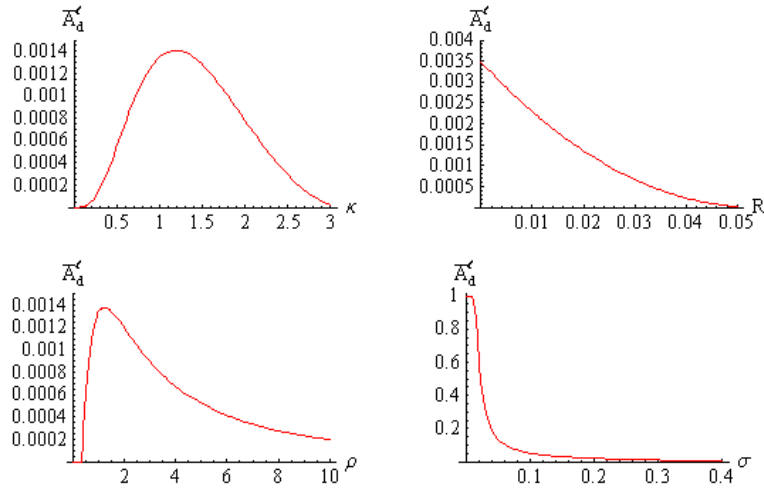tories during a market crash if execution delays are neither too long nor too short. This suggests that a regulation that increases the capacity of dealers to execute a large volume of orders, thereby reducing trading delays, may in fact reduce dealers' incentives to provide liquidity during a market crash.[24] Similarly, dealers are less likely to accumulate inventories in times of crisis if they have high bargaining power. This suggests that a market reform that reduces dealers' rents can improve liquidity during times when selling pressures intensify. Finally, since dealers' incentives to accumulate inventories are based on their expected capital gains, our theory predicts that dealers will provide liquidity when the crash is abrupt and short-lived.

From a normative standpoint, we have established necessary and sufficient conditions for liquidity provision by dealers to be efficient. We have found that there exist parametrizations for which dealers accumulate asset inventories when it is socially inefficient for them to do so, as well as parametrizations for which the opposite is true.

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[24]The regulatory developments in the securities markets since the October 1987 crisis are reviewed in Lindsey and Pecora (1998). According to Lindsey and Pecora (1998, p.290) "most exchanges now have excess capacity of approximately three times that needed for an average trading session."

35

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A   Proofs

Proof of Lemma 1. Consider any feasible inventory path $a(t)$. Let $W_0^t(a)$ denote the dealer's expected discounted utility from following a plan a from time 0 to t. Let $t_1 < t_2 \ldots$ be the successive jumps of the price path, and let K be the last jump before some time t, we can write, with the convention that $t_0 = 0$,

$$
\begin{aligned}
W_0^t(a) &= \int_0^t \left\{ v\left[a(s)\right] - p(s)\dot{a}(s) \right\} e^{-rs}\, ds \\
&= \int_0^t v\left[a(s)\right] e^{-rs}\, ds - \sum_{k=1}^{K} \int_{t_{k-1}}^{t_k} p(s)\dot{a}(s) e^{-rs}\, ds + \int_{t_K}^{t} p(s)\dot{a}(s) e^{-rs}\, ds \\
&= \int_0^t v\left[a(s)\right] e^{-rs}\, ds - \sum_{k=1}^{K} \int_{t_{k-1}}^{t_k} a(s)\left[rp(s) - \dot{p}(s)\right] e^{-rs}\, ds \\
&\quad - \int_{t_K}^{t} a(s)\left[rp(s) - \dot{p}(s)\right] e^{-rs} ds - \sum_{k=1}^{K} \left[e^{-rt_k} p(t_k^-) a(t_k) - e^{-rt_{k-1}} p(t_{k-1}^+) a(t_{k-1}) \right] \\
&\quad - \left[e^{-rt} p(t) a(t) - e^{-rt_K} p(t_K^+) a(t_K) \right] \\
&= \int_0^t \left\{ v\left[a(s)\right] - a(s)\left[rp(s) - \dot{p}(s)\right] \right\} e^{-rs}\, ds \\
&\quad + \sum_{k=1}^{K} a(t_k) e^{-rt_k} \left[p(t_k^+) - p(t_k^-) \right] - e^{-rt} p(t) a(t),
\end{aligned}
$$

where the second equality follows from integration by part over each interval $[t_{k-1}, t_k]$, and the last equality by collecting time-t_k terms.

We first establish the "only if" part of the lemma. Consider any bounded solution $a(t)$ to the dealer's problem and suppose that the price has a positive jump up at some t_k. Then, for ε small enough, consider the perturbation $a(t) + \Delta(t)$ where $\Delta(t) = 0$ for $t < t_k - \varepsilon$, $\Delta(t) = 1 + (t - t_k)/\varepsilon$ for $t \in [t_k - \varepsilon, t_k]$, $\Delta(t) = 1 - (t - t_k)/\varepsilon$ for $t \in [t_k, t_k + \varepsilon]$, and $\Delta(t) = 0$ thereafter. Then, using the above, the net utility of this perturbation, $W_0^\infty(a) - W_0^\infty(a + \Delta)$ is

$$
\int_{t_k - \varepsilon}^{t_k + \varepsilon} \left\{ v[a(s) + \Delta(s))] - v[a(s)] e^{-rs} - \Delta(s)\left[rp(s) - \dot{p}(s)\right] \right\} ds + e^{-rt_k} \left[p(t_k^+) - p(t_k^-) \right].
$$

Because $a(s)$ and $\Delta(s)$ are bounded, the first term goes to zero as ε goes to zero, showing that the net utility of the perturbation converges to $e^{-rt_k} \left[p(t_k^+) - p(t_k^-) \right] > 0$, a contradiction that proves that the price can only have a negative jump. If the price has a negative jump $p(t_k^+) - p(t_k^-) < 0$ then, as long as $a(t_k) > 0$ the reverse perturbation could improve the

39

dealer's utility. Therefore, if there is a negative jump, then $a(t_k) = 0$. Now suppose that, at some differentiability point s, $v'[a(s)] \geq rp(s) - \dot{p}(s)$. Then, using the expression for W_0^t, one easily shows that a dealer could improve his utility by accumulating more inventories around s. Therefore, $v'[a(s)] \leq rp(s) - \dot{p}(s)$. If the inequality is strict and $a(s) > 0$, then accumulating less inventory around s would improve the dealer's utility. Therefore, if $a(s) > 0$, then $rp(s) - \dot{p}(s) = 0$. In order to establish the necessity of the transversality condition (6), we calculate the net-utility of scaling down an optimal path $a(t)$ by $1 - \varepsilon$, for some small $\varepsilon > 0$. We find

$$W_0^t(a) - W_0^t\left[(1-\varepsilon)a\right] = \int_0^t \left\{v[a(s)] - v[a(s)(1-\varepsilon)] - \varepsilon a(s)\left[rp(s) - \dot{p}(s)\right]\right\} e^{-rs}\,ds - \varepsilon a(t)p(t)e^{-rt}.$$

Take limits as $t \to \infty$ on both sides, to find

$$\begin{aligned}
W_0^\infty(a) - W_0^\infty(a(1-\varepsilon)) &= \int_0^\infty \left\{v[a(s)] - v[a(s)(1-\varepsilon)] - \varepsilon a(s)\left[rp(s) - \dot{p}(s)\right]\right\} e^{-rs}\,ds \\
&\quad - \varepsilon \lim_{t\to\infty} a(t)p(t)e^{-rt}.
\end{aligned} \tag{43}$$

Now divide by ε and let ε go to zero, to get

$$\begin{aligned}
W_0^\infty(a) - W_0^\infty(a(1-\varepsilon)) &= \int_0^\infty \varepsilon a(s)\left\{v'[a(s)] - [rp(s) - \dot{p}(s)]\right\} e^{-rs}\,ds - \lim_{t\to\infty} e^{-rt}p(t)a(t) \\
&= -\lim_{t\to\infty} e^{-rt}p(t)a(t),
\end{aligned} \tag{44}$$

where we have used the first-order condition (4). (Precise arguments for taking these limits are provided in the last paragraph of the proof.) Because, $a(t)$ is an optimal path, the net utility calculated above must be positive, meaning that the limit of $e^{-rt}p(t)a(t)$ must be non-positive. Since $a(t)$ is positive, $e^{-rt}p(t)a(t)$ must converge to zero, and we are done. The "if" part of the Lemma follows from Theorem 13, Chapter 3, in in Seierstad and Sydaester (1987).

Lastly, we show that we can take limits in (43) and (44). The left-hand side of (43) converges by definition of the inter-temporal utility. Because of concavity and because of the first-order condition (4), the first term on the right-hand side is positive and increasing, and thus converges to some limit. Now note that $p(t)e^{-rt}$ is positive and decreasing: indeed it can only jump down and, by the first-order condition (4), its derivative is negative. Hence, $p(t)e^{-rt}$ is bounded. Because $a(t)$ is bounded, it follows that $e^{-rt}p(t)a(t)$ is also bounded. Taken together, this implies that the first-term on the right-hand side of (43) has some finite limit, and so does the second term. For (44), note that

$$\frac{v[a(s)] - v[a(s)(1-\varepsilon)]}{\varepsilon} - a(s)\left[rp(s) - \dot{p}(s)\right] \searrow 0,$$

40

which allows us to apply the dominated convergence theorem.

Proof of Lemma 2. When evaluating an investor's intertemporal utility we can ignore price jumps: this is because the probability that an investor contact the market at a jump time is equal to zero. We let the random flow utility of an investor at time t be $u(a, t)$, where we use the time argument "t" as a short-hand for the investor's current preference shock.

Notation. Considering an individual investor, we let $T_1 < T_2 < \ldots$ be the sequence of his contact times with dealers, with the convention that $T_0 = 0$. Also, we let N_t be the number of contact times during $[0, t]$. Then, for any asset plan, a, we calculate the intertemporal utility

$$V_0^t(a) \equiv \int_0^t u[a(s), s] e^{-rs} \, ds - \sum_{n=1}^{N_t} p(T_n) e^{-rT_n} \left[a(T_n) - a(T_{n-1}) \right],$$

between 0 and t, along a realization of the contact time and type processes. This utility can be decomposed as

$$V_0^t = U_0^t - B_0^t + p(T_1) e^{-rT_1} a(0) - p(\theta_t) a(\theta_t) e^{-r\theta_t},$$

where

$$
\begin{aligned}
U_0^t(a) &= \int_0^t u\left[a(s), s\right] e^{-rs} \, ds, \\
B_0^t(a) &= \sum_{n=1}^{N_t-1} a(T_n) \left[p(T_n) e^{-rT_n} - p(T_{n+1}) e^{-rT_{n+1}} \right].
\end{aligned}
$$

We consider portfolio plans a that are bounded, and such that the intertemporal utility $\mathbb{E}\left[V_0^\infty(a)\right]$ is well defined. We first establish:

Result 1. $\mathbb{E}\left[U_0^t(a)\right]$, $\mathbb{E}\left[B_0^t(a)\right]$, and $\mathbb{E}\left[p(\theta_t) e^{-r\theta_t} a(\theta_t)\right]$ converge to finite limits.

Let's start with $\mathbb{E}\left[U_0^t(a)\right]$. When the investor's utility is bounded below, then the result follows from the assumption that the portfolio plan, a, is bounded. When the investor's utility is unbounded below and bounded above, we can assume without loss of generality that it is negative. Then $\mathbb{E}[U_0^t]$ is decreasing and thus converges either to some finite or some infinite limit. The limit, in turn, must be finite because

$$
\begin{aligned}
\mathbb{E}[U_0^t] = \mathbb{E}[V_0^t] + \mathbb{E}[B_0^t] - \mathbb{E}[p(T_1) e^{-rT_1} a(0)] &\geq \mathbb{E}[V_0^t] + \mathbb{E}[B_0^t] - \mathbb{E}[p(T_1) e^{-rT_1} a(0)] \\
&\geq \mathbb{E}[V_0^t] - \mathbb{E}[p(T_1) e^{-rT_1} a(0)],
\end{aligned}
$$

41

where the inequality follows because $p(t)e^{-rt}$ is decreasing and B_0^t is therefore positive. Because $\mathbb{E}[V_0^\infty]$ is well defined, the right-hand side of the inequality is bounded below, implying that $\mathbb{E}[U_0^t]$ has a finite limit. It then immediately follows that

$$\mathbb{E}\left[B_0^t + p(\theta_t)e^{-r\theta_t}a(\theta_t)\right] = \mathbb{E}\left[V_0^t\right] - \mathbb{E}\left[U_0^t\right] - \mathbb{E}[p(T_1)e^{-rT_1}a(0)],$$

also converges to some finite limit. Note that B_0^t is increasing because $a(t) \geq 0$ and $p(t)e^{-rt}$ is decreasing, implying that $E[B_0^t]$ has a limit. This limit must be finite because the above equality implies that $\mathbb{E}[B_0^t] \leq \mathbb{E}[V_0^t] - \mathbb{E}[U_0^t] - \mathbb{E}[p(T_1)e^{-rT_1}a(0)]$. It then follows that $\mathbb{E}[p(\theta_t)e^{-r\theta_t}a(\theta_t)]$ also has a finite limit, which completes this part of the proof.

Result 2. An investor's intertemporal utility is

$$\mathbb{E}\left[V_0^\infty\right] = (r+\kappa)^{-1}\mathbb{E}\left[\sum_{n=1}^\infty e^{-rT_n}\left\{U\left[a(T_n), T_n\right] - \xi(T_n)a(T_n)\right\}\right] - \lim_{t\to\infty}\mathbb{E}\left[p(\theta_t)e^{-r\theta_t}a(\theta_t)\right], \quad (45)$$

where

$$U\left[a(T_n), T_n\right] = (r+\kappa)\mathbb{E}\left[\int_{T_n}^{T_{n+1}} u\left[a(s), s\right]e^{-r(s-T_n)}\,ds \,\bigg|\, T_n\right].$$

To show that result, write

$$\begin{aligned}
\mathbb{E}\left[B_0^\infty\right] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{n=1}^\infty a(T_n)\left[p(T_n)e^{-rT_n} - p(T_{n+1})e^{-rT_{n+1}}\right]\,\bigg|\, T_n\right]\right] \\
&= (r+\kappa)^{-1}\mathbb{E}\left[\sum_{n=1}^\infty a(T_n)\xi(T_n)e^{-rT_n}\right],
\end{aligned}$$

by definition of $\xi(T_n)$. In addition note that, when u is bounded below, we can without loss of generality assume that it is positive, and we have

$$u[a(s), s]e^{-rs}\mathbb{I}_{\{s\leq\theta_t\}} \leq u[a(s), s]e^{-rs}\mathbb{I}_{\{s\leq t\}} \leq u[a(s), s],$$

and $u[a(s), s]\mathbb{I}_{s\leq\theta_t} \nearrow u[a(s), s]$ as t goes to infinity. The same reasoning go through with opposite inequalities when u is negative. Therefore, an application of the dominated convergence theorem implies that

$$\begin{aligned}
\mathbb{E}\left[U_0^\infty\right] &= \mathbb{E}\left[\int_0^{\theta_t} u(a(s), s)e^{-rs}\,ds\right] = \mathbb{E}\left[\sum_{n=1}^\infty \int_{T_n}^{T_{n+1}} u(a(s), s)e^{-rs}\,ds\right] \\
&= (r+\kappa)\mathbb{E}\left[\sum_{n=1}^\infty e^{-rT_n}U(a(T_n), T_n)\right],
\end{aligned}$$

42

where the last equality follows by taking expectations of each term in the sum with respect to T_n.

Result 3. The flow inter-contact time utility is $U\left[a(T_n), T_n\right] = (r + \kappa)^{-1} U_{i(T_n)}\left[a(T_n)\right]$, where $U_i(a)$ is defined in equation (11) of the lemma. To see why, denote,

$$\tilde{V}_i(a, t) = \mathbb{E}_i\left[\int_0^{\tilde{T}} e^{-rs} u_{k(t+s)}(a')\, ds \,\Big|\, k(t) = i\right].$$

By the Markovian nature of the process $k(t)$, $\tilde{V}_i(a, t)$ only depends on t through the condition $k(t) = i$ which is already captured by the subscript i. Therefore, hereafter we will slightly abuse notation and write $\tilde{V}_i(a)$ for $\tilde{V}_i(a, t)$. Denote $\hat{T}$ the length of the period of time before the investor receives a preference shock. By definition, $\hat{T}$ is exponentially distributed with mean $1/\delta$. The value of an investor can then be written recursively as follows,

$$\tilde{V}_i(a) = \mathbb{E}\left[\mathbb{I}_{(\tilde{T} < \hat{T})} \int_0^{\tilde{T}} e^{-rs} u_i(a) ds\right] + \mathbb{E}\left[\mathbb{I}_{(\hat{T} < \tilde{T})} \int_0^{\hat{T}} e^{-rs} u_i(a) ds + \mathbb{I}_{(\hat{T} < \tilde{T})} e^{-r\hat{T}} \tilde{V}_{k(\hat{T})}(a)\right], \quad (46)$$

where $k(\hat{T})$ indicates the new realization of the preference shock at time $\hat{T}$. Using the fact that $\tilde{T}$ and $\hat{T}$ are independent random variables, one can rewrite the first term on the right-hand side of (46) as

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{I}_{(\tilde{T} < \hat{T})} \int_0^{\tilde{T}} e^{-rs} u_i(a) ds\right] &= \iint \mathbb{I}_{(\tilde{t} < \hat{t})} \kappa e^{-\kappa \tilde{t}} \delta e^{-\delta \hat{t}} \int_0^{\tilde{t}} e^{-rs} u_i(a) ds\, d\hat{t}\, d\tilde{t} \\
&= \int \kappa e^{-\kappa \tilde{t}} \int_{\tilde{t}}^\infty \delta e^{-\delta \hat{t}} d\hat{t} \int_0^{\tilde{t}} e^{-rs} u_i(a) ds\, d\tilde{t} \\
&= \frac{u_i(a)}{r} \int \kappa e^{-[\kappa + \delta]\tilde{t}} \left(1 - e^{-r\tilde{t}}\right) d\tilde{t} \\
&= \frac{\kappa u_i(a)}{(\kappa + \delta)(\kappa + \delta + r)}.
\end{aligned}
\quad (47)
$$

Similarly, the second term on the right-hand side of (46) can be reexpressed as

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{I}_{(\hat{T} < \tilde{T})} \int_0^{\hat{T}} e^{-rs} u_i(a) ds\right] &= \iint \mathbb{I}_{(\hat{t} < \tilde{t})} \kappa e^{-\kappa \tilde{t}} \delta e^{-\delta \hat{t}} \int_0^{\hat{t}} e^{-rs} u_i(a) ds\, d\hat{t}\, d\tilde{t} \\
&= \frac{\delta u_i(a)}{(\kappa + \delta)(\kappa + \delta + r)}
\end{aligned}
\quad (48)
$$

Since the realizations of the preference shocks are independent and identically distributed, the

distribution of $k(\hat{T})$ is given by $\{\pi_i\}_{i=1}^{I}$. Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{I}_{(\hat{T}<\tilde{T})}e^{-r\hat{T}}\tilde{V}_{k(\hat{T})}(a)\right] &= \iint \mathbb{I}_{(\hat{t}<\tilde{t})}\kappa e^{-\kappa\tilde{t}}\delta e^{-\delta\hat{t}}e^{-r\hat{t}}d\hat{t}d\tilde{t}\sum_{k=1}^{I}\pi_k\tilde{V}_k(a) \\
&= \frac{\delta}{\delta+r+\kappa}\sum_{k=1}^{I}\pi_k\tilde{V}_k(a). \quad (49)
\end{aligned}
$$

Adding (47), (48) and (49), one finds

$$
\tilde{V}_i(a) = \frac{u_i(a)}{\kappa+\delta+r} + \frac{\delta}{\delta+r+\kappa}\sum_{k=1}^{I}\pi_k\tilde{V}_k(a). \quad (50)
$$

After carrying out some calculations, (50) yields

$$
\tilde{V}_i(a) = \frac{U_i(a)}{r+\kappa}, \quad (51)
$$

where $U_i(a)$ is as in (11).

Result 4. The expected discounted price at the time the investor regains direct access to the asset market is:

$$
\mathbb{E}[e^{-r\tilde{T}}p(t+\tilde{T})] = \kappa\int_0^\infty e^{-(r+\kappa)s}p(t+s)ds. \quad (52)
$$

Result 5. The "only if" part of the lemma. First, it is clear from (45) that an optimal portfolio strategy should maximize each term $U[a(T_n), T_n] - \xi(T_n)a(T_n)$, implying the investor's first-order condition. As for the necessity of the transversality condition, consider an optimal asset holding plan and scale it down by $(1-\varepsilon)$, for some small enough ε. Using (45), the net change in intertemporal utility can be written

$$
\begin{aligned}
\Delta_\varepsilon &= (r+\kappa)^{-1}\left[\sum_{n=1}^{\infty}e^{-rT_n}\{U[a(T_n), T_n] - U[a(T_n)(1-\varepsilon), T_n] - \varepsilon\xi(T_n)a(T_n)\}\right] \quad (53) \\
&\quad -\varepsilon\lim_{t\to\infty}\mathbb{E}\left[a(\theta_t)e^{-r\theta_t}p(\theta_t)\right]. \quad (54)
\end{aligned}
$$

Divide by ε and note that

$$
\frac{1}{\varepsilon}\left[U(a(T_n), T_n) - U(a(T_n)(1-\varepsilon), T_n)\right] - \xi(T_n)a(T_n) \searrow \left(U_a(a(T_n), T_n) - \xi(T_n)\right)a(T_n) = 0,
$$

because of the first-order condition in the lemma. Convergence is monotonic because of concavity. This last property allows us to apply the dominated convergence theorem, and we find that

$$\mathbb{E}\left[\sum_{n=1}^{\infty} \frac{1}{\varepsilon}\Big[U(a(T_n), T_n) - U(a(T_n)(1-\varepsilon), T_n)\Big] - \xi(T_n)a(T_n)\right] \to 0,$$

and thus

$$\lim_{\varepsilon \to 0} \frac{\Delta_\varepsilon}{\varepsilon} = -\lim_{t \to \infty} \mathbb{E}\left[a(\theta_t)e^{-r\theta_t}p(\theta_t)\right] \geq 0. \tag{55}$$

Since $a(t) \geq 0$, it follows that $\lim_{t \to \infty} \mathbb{E}\left[a(\theta_t)e^{-r\theta_t}p(\theta_t)\right] = 0$, and we are done.

Result 6. For the "if" part, we consider a plan a that satisfies the first-order conditions and compare it to some other plan a'. We find

$$\mathbb{E}[V_0^\infty(a) - V_0^\infty(a')]$$
$$= \mathbb{E}\left[\sum_{n=1}^{\infty} e^{-rT_n}\left(U(a(T_n), T_n) - U(a'(T_n), T_n) - \xi(T_n)\left(a(T_n) - a'(T_n)\right)\right)\right]$$
$$+ \lim_{t \to \infty} \mathbb{E}\left[p(\theta_t)e^{-r\theta_t}a'(\theta_t)\right]$$
$$\geq \mathbb{E}\left[\sum_{n=1}^{\infty} e^{-rT_n}\left(U_a(a(T_n), T_n) - \xi(T_n)\right)\left(a(T_n) - a'(T_n)\right)\right] \geq 0,$$

where the first inequality follows because of concavity, and the second inequality follows because of the first-order condition in the lemma and because $a'(\theta_t) \geq 0$. ∎

Proof of Lemma 3. (a) To obtain (15), rewrite (12) as

$$\xi(t) = (r + \kappa)\,p(t) - \kappa e^{(r+\kappa)t}\int_t^\infty (r + \kappa)\,e^{-(r+\kappa)s}p(s)ds \tag{56}$$

and differentiate with respect to t.

Proof of Lemma 4. First note that the dealer's first-order conditions imply that the price can only have negative jumps and that $d/dt(e^{-rt}) = \dot{p}(t) - rp(t) \leq 0$. Hence, $p(t)e^{-rt}$ is decreasing and positive, and thus has a limit. Now we know that

$$\mathbb{E}\left[p(\theta_t)e^{-r\theta_t}a(\theta_t)\right] \to 0,$$

where $a(t)$ denotes the asset holding of some investor and θ_t the last contact time of that investor before t. Note that the cdf of θ_t is

$$\Pr(\theta_t \leq s) = \Pr(N_t - N_s = 0) = e^{-\kappa(t-s)}.$$

So θ_t has an atom at zero, and its cdf is $\kappa e^{-\kappa(t-s)}$. Another thing we know is that

$$p(t)e^{-rt}a(t) \to 0,$$

where $a(t)$ denotes a dealer's asset holdings. In particular, if one integrates $p(t)e^{-rt}a(t)$ against the cdf of θ_t, one finds that

$$\mathbb{E}\left[p(\theta_t)e^{-r\theta_t}a(\theta_t)\right] \to 0,$$

as t goes to infinity, because θ_t goes to infinity almost surely. Now consider some time s. The sum of asset holdings across investors and dealers must be equal to A, i.e.,

$$\int a^j(s)\,dj = A,$$

where j indexes all agents in the economy. Now, we can also write

$$
\begin{aligned}
A\mathbb{E}\left[p(\theta_t)e^{-r\theta_t}\right] &= \mathbb{E}\left[Ap(\theta_t)e^{-r\theta_t}\right] = \mathbb{E}\left[\int_j a^j(\theta_t)\,dj \times p(\theta_t)e^{-r\theta_t}\right] \\
&= \mathbb{E}\left[\int_j p(\theta_t)e^{-r\theta_t}a^j(\theta_t)\,dj\right] = \int_j \mathbb{E}\left[p(\theta_t)e^{-r\theta_t}a^j(\theta_t)\right]dj.
\end{aligned}
$$

As shown above, the last expression goes to zero as t goes to infinity. Therefore, because $A > 0$,

$$\mathbb{E}\left[p(\theta_t)e^{-r\theta_t}\right] \to 0, \tag{57}$$

as t goes to infinity. Because we know that $p(t)e^{-rt}$ converges to some limit, it follows that $p(t)e^{-rt}$ converges to zero. Indeed, suppose that the limit is strictly positive. Then there is some $\varepsilon > 0$ and t_ε such that $p(t)e^{-rt} > \varepsilon$ for all $t \geq t_\varepsilon$ and

$$\mathbb{E}\left[p(\theta_t)e^{-r\theta_t}\right] \geq \mathbb{E}\left[p(\theta_t)e^{-r\theta_t}\mathbb{I}_{\{t \geq t_\varepsilon\}}\right] \geq \varepsilon\Pr(\theta_t \geq t_\varepsilon) = \varepsilon\left(1 - e^{-\kappa(t-t_\varepsilon)}\right) \to \varepsilon$$

as t goes to infinity, which is a contradiction. To arrive at (21), integrate (15) forward using the transversality condition (6). ∎

Proof of Lemma 5. The proof consists of showing that from any initial condition close to the steady state, only the trajectory that follows the saddle path to the steady state is consistent with individual maximization. Consider Figure 1 and focus on trajectories below the saddle path. These trajectories eventually lead to $\xi(t) \leq 0$ or to $A_d(t) = 0$. The former are inconsistent with the investor's optimization (note that (11) would be violated since $U_i' > 0$).

46

The latter are inconsistent with the dealer's maximization. To see this, integrate (4) forward to obtain

$$p\left(t\right) = e^{rt}\lim_{s\to\infty} e^{-rs}p\left(s\right) + \int_0^\infty e^{-rs}v'\left[A_d\left(s+t\right)\right]ds. \tag{58}$$

If we multiply through by e^{-rt}, take limits as $t \to \infty$, and use the transversality condition (6), this expression implies $\lim_{t\to\infty}\int_0^\infty e^{-r(s+t)}v'\left[A_d\left(s+t\right)\right]ds = 0$, which is violated along trajectories where $A_d\left(t\right)$ equals zero in the limit, or in finite time. Trajectories above the saddle path are also inconsistent with the dealer's optimization. First, note that $\xi\left(t\right)$ diverges to $+\infty$ along any such trajectory. From (11), this implies that $a_i\left(t\right)$ converges to zero for each i. In turn, using (17), this implies that $A_d\left(t\right)$ converges to A. Again, (6) and (58) imply $p\left(t\right) = \int_0^\infty e^{-rs}v'\left[A_d\left(s+t\right)\right]ds$, hence $\lim_{t\to\infty}p\left(t\right) = v'\left(A\right)/r$, a constant. But then (12) implies $\lim_{t\to\infty}\xi\left(t\right) = v'\left(A\right) < \infty$, i.e., a contradiction that indicates that these paths violate the first-order necessary conditions of the dealer's problem. Thus, trajectories that lie above the saddle path are not solutions to the dealer's asset accumulation problem. Conversely, the trajectory that follows the saddle path satisfies the equilibrium conditions (6), (11), (16) and (17), as well as (6). ∎

Proof of Lemma 6. We study the problem of a social planner who maximizes the sum of all agents' utilities, subject to the trading technology. As before, $H_t(i,a)$ denotes the distribution of investors across preference types and asset holdings at time t. Since at any point in time all investors access the market according to independent stochastic processes with identical distributions, the quantity of assets that the measure α of randomly-drawn investors make available to the planner is $\alpha \int a dH_t(a,i) = \alpha\left[A - A_d\left(t\right)\right]$. So the quantity of assets available to be reallocated among agents who are in the market depends on the distribution $H_t(i,a)$ only through its mean, $A - A_d\left(t\right)$. Consequently, $H_t(i,a)$ is not a state variable for the planner's problem. Notwithstanding, in order to allocate assets across investors, the planner needs to know $n_i(t) = \int \mathbb{I}_{\{j=i\}}dH_t(j,a)$, i.e., the measure of investors of preference type i at date t.

Let $\tilde{\mathcal{V}}_i(a)$ denote the expected discounted utility of an investor of type i who holds a stock of assets a until the next time his portfolio can be changed, i.e.,

$$\tilde{\mathcal{V}}_i(a) = \mathbb{E}_i\left[\int_t^{t+T} u_{k(s)}(a)e^{-r(s-t)}ds\right]. \tag{59}$$

The value function $\tilde{\mathcal{V}}_i(a)$ satisfies

$$\tilde{\mathcal{V}}_i(a) = \frac{\left(r+\alpha\right)u_i(a) + \delta\sum_{j=1}^I \pi_j u_j(a)}{\left(r+\alpha+\delta\right)\left(r+\alpha\right)}. \tag{60}$$

(The calculations leading to (60) parallel the derivation of $\tilde{V}_i(a)$ in the proof of Lemma 2.) Since general goods enter linearly in the utility function of all agents, the utilities from production and consumption of those goods net out to 0 and can therefore be ignored by the planner. Thus, the planner only maximizes the direct utilities that dealers and investors enjoy from holding assets. At each date the planner chooses $q(t)$, the change in the quantity of assets held by dealers and $a_i(t)$, the quantity of assets allocated to an investor of type i when he readjusts his portfolio, in order to maximize

$$\int \tilde{\mathcal{V}}_i(a)dH_0(a,i) + \int_0^\infty e^{-rt}\left\{v[a_d(t)] + \alpha\sum_{i=1}^I n_i(t)\tilde{\mathcal{V}}_i[a_i(t)]\right\}dt \tag{61}$$

$$\text{s.t.}\qquad q(t) = \alpha\left[A - a_d(t) - \sum_{i=1}^I n_i(t)a_i(t)\right], \tag{62}$$

and subject to the law of motion $\dot{a}_d(t) = q(t)$, (18), and the initial conditions $n_i(0)$ and $a_i(0)$ for $i = 1,...,I$. The first term in (61) captures the utility of all investors before the first time their portfolios can be reallocated. It is a constant and can therefore be ignored in choosing the optimal allocation. Hence, the planner's current-value Hamiltonian reduces to

$$v[a_d(t)] + \alpha\sum_{i=1}^I n_i(t)\tilde{\mathcal{V}}_i[a_i(t)] + \mu(t)q(t), \tag{63}$$

where $\mu(t)$ is the co-state variable associated with the law of motion for $a_d(t)$. (The nonnegativity constraints on $a_i(t)$ and $a_d(t)$ are slack at all times since $u_i'(0) = v'(0) = \infty$.) From the Maximum Principle (e.g., Theorem 12 in Seierstad and Sydsæter, 1987), the necessary conditions for an optimum are

$$\alpha n_i(t)\left\{\tilde{\mathcal{V}}_i'[a_i(t)] - \mu(t)\right\} = 0, \tag{64}$$

which using (60) can be rewritten as

$$\frac{(r+\alpha)u_i'[a_i(t)] + \delta\sum_{j=1}^I \pi_j u_j'[a_i(t)]}{r+\alpha+\delta} = (r+\alpha)\mu(t), \tag{65}$$

and

$$v'[a_d(t)] + \dot{\mu}(t) = (r+\alpha)\mu(t). \tag{66}$$

Next, we show that the optimal path must also satisfy the transversality condition

$$\lim_{t\to\infty} e^{-rt}\mu(t) = 0. \tag{67}$$

48

We begin by noticing that for every path of the controls, the functional

$$\mathcal{U}\left[q\left(\cdot\right),\left\{a_i\left(\cdot\right)\right\}_{i=1}^I\right] \;=\; \int_0^\infty e^{-rt}\left\{v[a_d(t)] + \alpha\sum_{i=1}^I n_i(t)\check{\mathcal{V}}_i[a_i(t)]\right\}dt \;+$$

$$\int_0^\infty e^{-rt}\left\{\mu\left(t\right)\left[q\left(t\right) - \dot{a}_d\left(t\right)\right]\right\}dt \tag{68}$$

with $a_d\left(t\right) = A - q\left(t\right)/\alpha - \sum_{i=1}^I n_i(t)a_i\left(t\right)$, yields the same value as the planner's objective function (61) (ignoring the constant term in (61)). Integration by parts implies that

$$\int_0^\infty e^{-rt}\mu\left(t\right)\dot{a}_d\left(t\right)dt \;=\; e^{-rt}\mu\left(t\right)a_d\left(t\right)\big|_{t=0}^{t=\infty} - \int_0^\infty e^{-rt}\left[\dot{\mu}\left(t\right) - r\mu\left(t\right)\right]a_d\left(t\right)dt,$$

and substituting this expression into (68) yields

$$\mathcal{U}\left[q\left(\cdot\right),\left\{a_i\left(\cdot\right)\right\}_{i=1}^I\right] \;=\; \int_0^\infty e^{-rt}\left\{v[a_d(t)] + \left[\dot{\mu}\left(t\right) - r\mu\left(t\right)\right]a_d\left(t\right) + \alpha\sum_{i=1}^I n_i(t)\check{\mathcal{V}}_i[a_i(t)]\right\}dt \;+$$

$$\int_0^\infty e^{-rt}\mu\left(t\right)q\left(t\right)dt - e^{-rt}\mu\left(t\right)a_d\left(t\right)\big|_{t=0}^{t=\infty}.$$

Suppose that $q\left(t\right)$ and $\left\{a_i\left(t\right)\right\}_{i=1}^I$ are optimal paths for the controls, then along this optimal trajectory, the implied path for the state variable $a_d\left(t\right)$ is $A - q\left(t\right)/\alpha - \sum_{i=1}^I n_i(t)a_i\left(t\right)$. Consider the admissible paths $\hat{q}\left(t,\varepsilon\right)$ and $\left\{\hat{a}_i\left(t,\varepsilon\right)\right\}_{i=1}^I$, where $\hat{q}\left(t,\varepsilon\right) = q\left(t\right) + \varepsilon\Delta_q\left(t\right)$ and $\hat{a}_i\left(t,\varepsilon\right) = a_i\left(t\right) + \varepsilon\Delta_i\left(t\right)$, for some arbitrary $\varepsilon \in \mathbb{R}$. The implied path for the state is $\hat{a}_d\left(t,\varepsilon\right) = a_d\left(t\right) - \varepsilon\Delta_d\left(t\right)$, where $\Delta_d\left(t\right) = \Delta_q\left(t\right)/\alpha + \sum_{i=1}^I n_i(t)\Delta_i\left(t\right)$. (An "admissible path" is a path which is piece-wise continuously differentiable and satisfies (62), together with the initial conditions $\hat{a}_i\left(0,\varepsilon\right) = a_i\left(0\right)$ and $\hat{a}_d\left(0,\varepsilon\right) = a_d\left(0\right)$.) Let $J\left(\varepsilon\right) = \mathcal{U}\left[\hat{q}\left(\cdot,\varepsilon\right),\left\{\hat{a}_i\left(\cdot,\varepsilon\right)\right\}_{i=1}^I\right]$. Since the paths $q\left(t\right)$ and $\left\{a_i\left(t\right)\right\}_{i=1}^I$ are optimal, we must have $\frac{\partial J(\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=0} = 0$, or equivalently,

$$0 \;=\; \int_0^\infty e^{-rt}\left\{-\left\{v'[a_d(t)] + \dot{\mu}\left(t\right) - r\mu\left(t\right)\right\}\Delta_d\left(t\right) + \alpha\sum_{i=1}^I n_i(t)\check{\mathcal{V}}_i'[a_i(t)]\Delta_i\left(t\right)\right\}dt \;+$$

$$\int_0^\infty e^{-rt}\left\{\mu\left(t\right)\Delta_q\left(t\right)\right\}dt + e^{-rt}\mu\left(t\right)\Delta_d\left(t\right)\big|_{t=0}^{t=\infty}.$$

If we substitute $\Delta_q\left(t\right) = \alpha\Delta_d\left(t\right) - \alpha\sum_{i=1}^I n_i(t)\Delta_i\left(t\right)$ and notice that $\Delta_d\left(0\right) = 0$ (because $\hat{a}_d\left(t,\varepsilon\right)$ is admissible), we find that this last expression is equivalent to

$$0 \;=\; -\int_0^\infty e^{-rt}\left\{v'[a_d\left(t\right)] + \dot{\mu}\left(t\right) - \left(r + \alpha\right)\mu\left(t\right)\right\}\Delta_d\left(t\right)dt \;+$$

$$\int_0^\infty e^{-rt}\left[\sum_{i=1}^I\left\{\alpha n_i(t)\left[\check{\mathcal{V}}_i'[a_i\left(t\right)] - \mu\left(t\right)\right]\right\}\Delta_i\left(t\right)\right]dt + \lim_{t\to\infty}e^{-rt}\mu\left(t\right)\Delta_d\left(t\right).$$

49

But then (64) and (66) imply that $\frac{\partial J(\varepsilon)}{\partial \varepsilon}\big|_{\varepsilon=0} = 0$ only if $\lim_{t\to\infty} e^{-rt}\mu(t)\Delta_d(t) = 0$, and since $\Delta_d(t)$ is arbitrary, (67) is a necessary condition for optimality. If we rescale the co-state by defining $\lambda(t) \equiv (r+\alpha)\mu(t)$, it becomes clear that (65), (66) and (67) correspond to (32), (33) and (34), respectively. Finally, the Mangasarian condition (35) is sufficient because the Hamiltonian is jointly concave (see Theorem 13 in Seierstad and Sydsæter, 1987). ∎

Proof of Proposition 1. We wish to show that the planner's optimality conditions and the equilibrium conditions are identical when $\eta = 0$. First, note that the planner's law of motion (62) is always the same as the market-clearing condition (17). Then let $\lambda(t) = \xi(t)$ and note that the planner's optimality conditions (32) and (33) are identical to the equilibrium conditions (11) and (4) if and only if $\eta = 0$. To conclude, we must show that (34) is equivalent to (6), but given $\lambda(t) = \xi(t)$, it suffices to show that $\lim_{t\to\infty} e^{-rt}\xi(t) = 0$ if and only if $\lim_{t\to\infty} e^{-rt}p(t) = 0$. From (12),

$$
\begin{aligned}
\lim_{t\to\infty} e^{-rt}\xi(t) &= \lim_{t\to\infty} e^{-rt}(r+\kappa)\int_0^\infty e^{-(r+\kappa)s}\left\{rp(t) - \kappa[p(t+s) - p(t)]\right\}ds \\
&= (r+\kappa)\int_0^\infty e^{-(r+\kappa)s}\left\{r\lim_{t\to\infty} e^{-rt}p(t) - \kappa\lim_{t\to\infty} e^{-rt}[p(t+s) - p(t)]\right\}ds \\
&= r\lim_{t\to\infty} e^{-rt}p(t). \quad\blacksquare
\end{aligned}
$$

Proof of Lemma 7. For part *(a)*, note that the investor's asset demand (36) is immediate from (11) given the functional form assumptions. The Hamiltonian corresponding to the dealer's problem is $-p(t)q(t) + \chi(t)q(t) + \varsigma(t)a_d(t)$, where $\chi(t) \geq 0$ is the costate variable and $\varsigma(t) \geq 0$ is the multiplier on the constraint $a_d(t) \geq 0$. The Maximum Principle (e.g., Theorem 12 in Seierstad and Sydsæter, 1987) delivers $\chi(t) = p(t)$ and $\varsigma(t) = rp(t) - \dot{p}(t)$, together with the complementary slackness condition $\varsigma(t)a_d(t) = 0$. This implies $[rp(t) - \dot{p}(t)]a_d(t) = 0$, which together with the fact that $rp(t) - \dot{p}(t) \geq 0$ implies part *(b)*. ∎

Before proving Proposition 2, we establish several intermediate results (Lemmas 8–13) which will aid us in the proofs.

We begin with a characterization of the equilibrium trajectories of $A_d(t)$ and $\xi(t)$ over arbitrary time-intervals:

Lemma 8 *(i) Consider a time-interval $[t_1, t_2]$ such that $A_d(t) > 0$ for all $t \in (t_1, t_2)$. Then,*

$$A_d(t) = \alpha \left\{ A \frac{1-e^{-\alpha t}}{\alpha} - \frac{\bar{E}}{\left[e^{-(r+\kappa)t_2}\xi(t_2)\right]^{1/\sigma}} \left[\frac{1-e^{-\left(\alpha-\frac{r+\kappa}{\sigma}\right)t}}{\alpha-\frac{r+\kappa}{\sigma}}\right] e^{-\frac{r+\kappa}{\sigma}t} \right.$$
$$\left. + \frac{\bar{E}-E_0}{\left[e^{-(r+\kappa)t_2}\xi(t_2)\right]^{1/\sigma}} \left[\frac{1-e^{-\left(\alpha-\frac{r+\kappa+\delta\sigma}{\sigma}\right)t}}{\alpha-\frac{r+\kappa+\delta\sigma}{\sigma}}\right] e^{-\frac{r+\kappa+\delta\sigma}{\sigma}t} \right\} \tag{69}$$

and $\xi(t) = \xi^+(t)$, where

$$\xi^+(t) = e^{(r+\kappa)(t-t_2)}\xi^+(t_2) \tag{70}$$

for all $t \in (t_1, t_2)$.

(ii) Consider a time-interval during which $A_d(t) = 0$. Then, $\xi(t) = \xi_0(t)$ for all t in such interval, where

$$\xi_0(t) = \left[1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(t-\hat{t})}\right]^\sigma \bar{\xi}, \tag{71}$$

with $\hat{t} = (1/\delta)\ln\left[\frac{r+\kappa+\delta\sigma}{r+\kappa}\left(1-\frac{E_0}{\bar{E}}\right)\right]$.

Proof. *(i)* Consider an interval (t_1, t_2) such that $A_d(t) > 0$ for all t in that interval. From (38), $\dot{\xi}(t)/\xi(t) = r + \kappa$ which gives (70). Substituting this expression into (39), implies that $A_d(t)$ satisfies

$$\dot{A}_d(t) + \alpha A_d(t) = \alpha A - \alpha \frac{\bar{E}e^{-\left(\frac{r+\kappa}{\sigma}\right)t} - \left(\bar{E}-E_0\right)e^{-\left(\frac{r+\kappa+\delta\sigma}{\sigma}\right)t}}{\left[e^{-(r+\kappa)t_2}\xi(t_2)\right]^{1/\sigma}},$$

and (69) is the solution to this first-order differential equation. In the case of resonance where $\frac{r+\kappa}{\sigma} = \alpha$, the solution becomes

$$A_d(t) = \alpha \left\{ A \frac{1-e^{-\alpha t}}{\alpha} - \frac{\bar{E}}{\left[e^{-(r+\kappa)t_1}\xi(t_1)\right]^{1/\sigma}} te^{-\alpha t} - \frac{\bar{E}-E_0}{\delta\left[e^{-(r+\kappa)t_1}\xi(t_1)\right]^{1/\sigma}} \left[e^{-(\alpha+\delta)t} - e^{-\alpha t}\right] \right\}.$$

There is a second nongeneric case of resonance where $\frac{r+\kappa+\delta\sigma}{\sigma} = \alpha$. In this case, the solution becomes

$$A_d(t) = \alpha \left\{ A \frac{1-e^{-\alpha t}}{\alpha} - \frac{\bar{E}}{\delta\left[e^{-(r+\kappa)t_1}\xi(t_1)\right]^{1/\sigma}} \left(e^{-\frac{r+\kappa}{\sigma}t} - e^{-\alpha t}\right) + \frac{\bar{E}-E_0}{\left[e^{-(r+\kappa)t_1}\xi(t_1)\right]^{1/\sigma}} te^{-\alpha t} \right\}.$$

To avoid repetitive derivations, we restrict our analysis to the generic case, where $\alpha - \frac{r+\kappa}{\sigma} \neq 0$ and $\alpha - \frac{r+\kappa+\delta\sigma}{\sigma} \neq 0$.

(ii) Consider a time interval (t_1, t_2) such that $A_d(t) = 0$. From (39), $A_d(t) = \dot{A}_d(t) = 0$ implies $\xi(t) = \xi_0(t)$ with $\xi_0(t)$ given by (71). ∎

The following lemma establishes a key continuity property of equilibrium prices and allocations.

Lemma 9 *In any equilibrium, $A_d(t)$ and $\xi(t)$ are continuous for all t.*

Proof. To establish the continuity of $A_d(t)$ we proceed in three steps. *(i)* From Lemma 8 it is immediate that $A_d(t)$ and $\xi(t)$ are both continuous on every open interval (t_1, t_2) over which $A_d(t) > 0$ for all $t \in (t_1, t_2)$ or $A_d(t) = 0$ for all $t \in (t_1, t_2)$. *(ii)* We establish that if $A_d(t) > 0$ for all $t \in (t_1, t_2)$, and $A_d(t) = 0$ for all $t \in (t_2, t_3)$, then $A_d(t)$ must be continuous at t_2. Assume this is not the case, i.e., suppose that $\lim_{t \uparrow t_2} A_d(t) > 0$, but $A_d(t_2) = 0$. If dealers are reducing their asset holdings discretely at t_2, by market clearing, it must be that the investors who are in the market at t_2 are increasing their holdings discretely. But since their demands are continuous decreasing functions of $\xi(t)$, this can only happen if $\xi(t)$ has a downward jump at t_2. (Since there is only a measure 0 of investors in the market at any point in time, investors' demand would have to be infinite at t_2 and $\xi(t_2) = 0$.) Rearranging (56) from the proof of Lemma 3, we get

$$p(t) - \frac{\xi(t)}{r + \kappa} = \kappa e^{(r+\kappa)t} \int_t^\infty e^{-(r+\kappa)s} p(s) ds.$$

Thus, since the right-hand side is continuous in t, any pointwise downward jump in $\xi(t)$ corresponds a pointwise downward jump in $p(t)$. Since $\lim_{t \uparrow t_2} A_d(t) > 0$, we have $\lim_{t \uparrow t_2} a_d(t) > 0$ for at least some dealer(s). Focus on any such dealer's problem as t_2 approaches. In the proposed equilibrium, $p(t_2^-) - p(t_2) > 0$, and $a_d(t_2^-) - a_d(t_2) = a_d(t_2^-) > 0$, so in the interval $(t_2^-, t_2]$, the dealer's utility from trading inventories is $p(t_2) a_d(t_2^-)$, the proceeds of his asset sale at t_2 (recall that, $\dot{p}(t)/p(t) = r$ while $a_d(t) > 0$, so he is getting zero utility from trading inventories on (t_1, t_2)). But this dealer could have attained a payoff $p(t_2^-) a_d(t_2^-) > p(t_2) a_d(t_2^-)$ by selling off his inventory an instant before the price jumped downward. Thus, we conclude that the equilibrium path $A_d(t)$ cannot exhibit this type of discontinuity. In this part we have considered the case where the discontinuity is from the left, i.e., $\lim_{t \uparrow t_2} A_d(t) > A_d(t_2) = 0$. The case where $\lim_{t \uparrow t_2} A_d(t) = A_d(t_2) > \lim_{t \downarrow t_2} A_d(t) = 0$ is handled similarly. *(iii)* By an argument analogous to the one in step *(ii)*, one can show that if $A_d(t) = 0$ for all $t \in (t_1, t_2)$, and $A_d(t) > 0$ for all $t \in (t_2, t_3)$, then $A_d(t)$ must be continuous at t_2. (The measure of assets held by investors in the market is $\alpha dt A$ where $dt \to 0$ which prevents dealers' inventories from jumping upward.) Together, steps *(i)*–*(iii)* imply that any equilibrium path $A_d(t)$ must be continuous for all t. To conclude, we establish that $\xi(t)$ must be continuous for all t. First, we show that the continuity of $A_d(t)$ implies that $\xi(t)$ cannot have a downward jump at t_2. The continuity of $A_d(t)$ means that $A_d(t_2) = 0$, which together with the nonnegativity constraint

$A_d(t) \geq 0$ implies $\dot{A}_d(t_2^-) \leq 0 \leq \dot{A}_d(t_2^+)$. Since $A_d(t_2^+) = A_d(t_2^-) = 0$, (39) yields

$$\left[\bar{E} - e^{-\delta t_2^+} \left(\bar{E} - E_0 \right) \right] \frac{1}{\xi(t_2^+)} \leq A \leq \left[\bar{E} - e^{-\delta t_2^-} \left(\bar{E} - E_0 \right) \right] \frac{1}{\xi(t_2^-)},$$

and therefore, $\xi(t_2^-) \leq \xi(t_2^+)$. The fact that $\xi(t)$ cannot have an upward jump, i.e., that $\xi(t_2^-) < \xi(t_2^+)$ cannot be part of an equilibrium, follows from a no-arbitrage argument like the one in step *(iii)*. Hence, $\xi(t_2^-) = \xi(t_2^+)$. ∎

The following lemma shows that there is no equilibrium in which dealers hold positive inventories at all dates.

Lemma 10 *There is no equilibrium with $A_d(t) > 0$ for all $t < \infty$.*

Proof. Otherwise, it follows from the dealer's first-order condition that $rp(t) = \dot{p}(t)$ and therefore that $p(t)e^{-rt} = p(0)$. Since $p(0) > 0$, this violates the no-bubble condition (20) of Lemma 4. ∎

Lemma 11 shows that the $\hat{t}$ defined in part *(c)* of the statement of Proposition 13 has the property that dealers will hold inventories for all $t < \hat{t}$.

Lemma 11 *In any equilibrium, $\{t : t \leq \hat{t}\} \subseteq \{t : A_d(t) > 0\}$, where $\hat{t} = \ln \left[\frac{r+\kappa+\delta\sigma}{r+\kappa} \left(1 - \frac{E_0}{\bar{E}} \right) \right]^{1/\delta}$.*

Proof. Suppose the contrary, i.e., that $A_d(t) = 0$ for all $t \in (t', t'')$, with $t'' < \hat{t}$. Then $\xi(t) = \left[1 - \frac{r+\kappa}{r+\kappa+\delta\sigma} e^{-\delta(t-\hat{t})} \right]^{\sigma} \bar{\xi}$ for all $t \in (t', t'')$ (by part *(ii)* of Lemma 8). Thus,

$$\frac{\dot{\xi}(t)}{\xi(t)} = \frac{\delta\sigma(r+\kappa)}{(r+\kappa+\delta\sigma) e^{-\delta(\hat{t}-t)} - (r+\kappa)}$$

for all $t \in (t', t'')$. But note that $\dot{\xi}(t)/\xi(t) > r + \kappa$ for all $t < \hat{t}$, so the proposed path for $A_d(t)$ violates the dealer's first-order condition (38) on (t', t''). ∎

Lemma 12 establishes that the equilibrium asset holdings of dealers after a crash follow a very precise pattern: if dealers hold positive inventories, they will do so from the outset of the crash, over a connected interval of time of finite length $\bar{t}$, and will hold no inventories thereafter.

Lemma 12 *In any equilibrium, $\{t : A_d(t) > 0\} = [0, T)$ where $0 \leq T < \infty$.*

Proof. We first show that if $A_d(t') = 0$, then $A_d(t) = 0$ for all $t \geq t'$. (Note that this immediately implies that $\{t : A_d(t) > 0\} = [0, \bar{t})$, with $\bar{t} \geq 0$ but possibly infinite.) We proceed by contradiction. Suppose that $A_d(t)$ is part of an equilibrium, with $A_d(t) = 0$ for all $t \in (t' - \Delta^-, t']$ and $A_d(t) > 0$ for all $t \in (t', t' + \Delta^+)$, for some Δ^-, $\Delta^+ > 0$. Then, from (71) (part *(ii)* of Lemma 8), $\xi(t) = \xi_0(t)$ for all $t \in (t' - \Delta^-, t')$, where $\xi_0(t) = \left[1 - \frac{r+\kappa}{r+\kappa+\delta\sigma} e^{-\delta(t-\hat{t})}\right]^\sigma \bar{\xi}$, and from (70) (part *(i)* of Lemma 8), $\xi(t) = \xi^+(t)$ for all $t \in (t', t' + \Delta^+)$, where $\xi^+(t) = e^{(r+\kappa)(t-t')}\xi^+(t')$. From Lemma 9 we know that $\xi(t)$ must be continuous, so $\xi^+(t) = e^{(r+\kappa)(t-t')}\xi_0(t')$ on $(t', t' + \Delta^+)$. From Lemma 11 we know that for $A_d(t) = 0$ on $t \in (t' - \Delta^-, t')$ to be part of an equilibrium, it must be that $t' > t' - \Delta^- \geq \hat{t}$, so $\dot{\xi}_0(t)/\xi_0(t) = \frac{\delta\sigma(r+\kappa)}{(r+\kappa+\delta\sigma)e^{-\delta(\hat{t}-t)} - (r+\kappa)} \leq r+\kappa$ for all $t \geq t' - \Delta^-$ (with strict inequality for $t > t' - \Delta^-$). But then the fact that $\xi_0(t') = \xi^+(t')$ and $\dot{\xi}_0(t)/\xi_0(t) < r+\kappa = \dot{\xi}^+(t)/\xi^+(t)$ for all $t > t'$ implies that $\xi^+(t) > \xi_0(t)$ for all $t > t'$. Since $\xi(t)$ must be continuous, this would imply an equilibrium with $A_d(t) > 0$ for all $t > t'$. But this is a contradiction, since we know by Lemma 10 that such a path for $A_d(t)$ is inconsistent with the dealer's transversality condition. Thus, if dealers hold inventories at all in equilibrium, they must do so from $t = 0$ and for an uninterrupted period of time, up to some time $T \geq 0$. Finally, the fact that $T < \infty$ follows by appealing to Lemma 10 once again. Figure 5 illustrates the main idea of this proof. ∎

Lemma 13 *Following a market crash:*

(a). If dealers do not intervene, the equilibrium is $A_d(t) = 0$ and $\xi(t) = \xi_0(t)$ for all t, with

$$\xi_0(t) = \left[1 - \frac{r+\kappa}{r+\kappa+\delta\sigma} e^{-\delta(t-\hat{t})}\right]^\sigma \bar{\xi}$$

and $\hat{t} = (1/\delta) \ln\left[\frac{r+\kappa+\delta\sigma}{r+\kappa}\left(1 - \frac{E_0}{\bar{E}}\right)\right]$.

(b). If dealers intervene, the equilibrium is

$$\xi(t) = \begin{cases} \xi_+(t) & \text{for } t < T \\ \xi_0(t) & \text{for } t \geq T \end{cases} \qquad A_d(t) = \begin{cases} A_d^+(t) & \text{for } t < T \\ 0 & \text{for } t \geq T, \end{cases}$$

where $\xi_+(t) = e^{(r+\kappa)(t-T)}\xi_0(T)$,

$$A_d^+(t) = \alpha \left\{ \frac{1 - e^{-\alpha t}}{\alpha} + \frac{e^{-\frac{r+\kappa}{\sigma}(t-T)}}{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(T-\hat{t})}} \left[\frac{r+\kappa}{r+\kappa+\delta\sigma} \frac{1 - e^{-\left(\alpha - \frac{r+\kappa+\delta\sigma}{\sigma}\right)t}}{\alpha - \frac{r+\kappa+\delta\sigma}{\sigma}} e^{-\delta(t-\hat{t})} - \frac{1 - e^{-\left(\alpha - \frac{r+\kappa}{\sigma}\right)t}}{\alpha - \frac{r+\kappa}{\sigma}} \right] \right\} A,$$

and $T \geq \hat{t}$ is the unique positive root of

$$\int_0^T e^{\alpha s} \left[1 - e^{\frac{r+\kappa}{\sigma}(T-s)} \frac{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(s-\hat{t})}}{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(T-\hat{t})}}\right] ds = 0.$$

54

Proof of Lemma 13 and Proposition 2. From Lemma 12, we know that an equilibrium must have $A_d(t) > 0$ for all $t \in [0, T)$ and $A_d(t) = 0$ for $t \geq T$, with $0 \leq T < \infty$, so we construct such an equilibrium to establish parts *(a)* and *(b)* of Lemma 13. For part *(a)*, note that $A_d(t) = 0$ and $\xi(t) = \xi_0(t)$ for $t \geq T$ by part *(ii)* of Lemma 8. Thus in particular, this is true if $T = 0$ (i.e., if dealers do not intervene). For part *(b)*, note that again, $A_d(t) = 0$ and $\xi(t) = \xi_0(t)$ for $t \geq T$. For $t < T$, we have $A_d(t)$ and $\xi(t) = \xi^+(t)$, given by (69) and (70), respectively, in the proof of Lemma 8. Since $\xi(t)$ must be continuous (Lemma 9), $\xi^+(T) = \xi_0(T)$, so $\xi^+(t) = e^{(r+\kappa)(t-T)}\xi_0(T)$, which in the statement of Lemma 13 is denoted $\xi_+(t)$. The expression for $A_d(t)$ for $t < T$, i.e., (69), reduces to $A_d^+(t)$ in the statement of Lemma 13 after setting $t_2 = T$ and $\xi(T) = \xi_0(T) = \left[1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(T-\hat{t})}\right]^\sigma \bar{\xi}$, using $E_0/\bar{E} = 1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{\delta\hat{t}}$, and rearranging terms. So far we have described the full equilibrium for a given switching date T. To determine T, we use the fact that $A_d(t)$ must be continuous, which implies $A_d(T) = 0$, a condition to be solved for T. To derive this condition, we start with (39), which leads to

$$\int_0^t e^{\alpha s}\left[\dot{A}_d(s) + \alpha A_d(s)\right]ds = \alpha\int_0^t e^{\alpha s}\left\{A - \xi(s)^{-1/\sigma}\left[\bar{E} - e^{-\delta s}\left(\bar{E} - E_0\right)\right]\right\}ds$$

and in turn to

$$A_d(t) = \alpha\int_0^t e^{-\alpha(t-s)}\left\{A - \frac{\bar{E}}{\xi(s)^{1/\sigma}}\left[1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(s-\hat{t})}\right]\right\}ds. \tag{72}$$

For $t \leq T$, $\xi(s) = \xi_+(s) = e^{(r+\kappa)(s-T)}\xi_0(T) = e^{(r+\kappa)(s-T)}\left[1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(T-\hat{t})}\right]^\sigma\bar{\xi}$, and substituting this into (72) yields

$$A_d(t) = \alpha A\int_0^t e^{-\alpha(t-s)}\left[1 - e^{\frac{r+\kappa}{\sigma}(T-s)}\frac{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(s-\hat{t})}}{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(T-\hat{t})}}\right]ds. \tag{73}$$

Thus, $A_d(T) = 0$ if and only if $\Gamma(T) = 0$, where

$$\Gamma(T) \equiv \int_0^T e^{\alpha s}\left[1 - e^{\frac{r+\kappa}{\sigma}(T-s)}\frac{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(s-\hat{t})}}{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(T-\hat{t})}}\right]ds.$$

This is the same map we used to define T in the statement of Lemma 13, so the proof of part *(b)* of the lemma is complete. Finding T reduces to finding the zeroes of the map Γ. Note that $\Gamma(0) = 0$ and $\Gamma(T) \to -\infty$ as $T \to \infty$, so $\lim_{T\downarrow 0}\Gamma'(T) > 0$ is sufficient to guarantee the existence of some $T \in (0, \infty)$ such that $\Gamma(T) = 0$ (i.e., dealers intervene). If in addition, we can show that $\Gamma'(T) < 0$ for $T > 0$, then this will guarantee that the root is unique. Conversely, if

$\lim_{T\downarrow 0} \Gamma'(T) \le 0$, then $\Gamma'(T) < 0$ for $T > 0$ implies there exists no $T > 0$ such that $\Gamma(T) = 0$ (i.e., dealers do not intervene). Differentiating, we find

$$\Gamma'(T) = \frac{r+\kappa}{\alpha\sigma}\left[e^{\delta(T-\hat{t})} - \frac{r+\kappa}{r+\kappa+\delta\sigma}\right]\left(e^{\alpha T} - 1\right)\left[1 - e^{\delta(T-\hat{t})}\right].$$

From Lemma 11 we know that $T \ge \hat{t}$, so for $T > 0$, $\Gamma'(T)$ has the same sign as

$$-\left[1 - e^{-\delta(T-\hat{t})}\right],$$

which is negative. As $T \to 0$, this expression is positive if and only if $\hat{t} > 0$, which amounts to condition (40) in the statement of Proposition 2. Hence, a $T > 0$ such that $A_d(T) = 0$ exists (i.e., dealers intervene) if and only if (40) holds, and when such a T exists, it is unique. To link this condition to $\dot{p}_0(t)/p_0(t)$, recall that $\dot{p}(t)/p(t) > r$ if and only if $\dot{\xi}(t)/\xi(t) > r+\kappa$ (e.g., from (15)). Then, from (39), if dealers do not intervene, $\dot{\xi}(t)/\xi(t) = \frac{\delta\sigma(r+\kappa)}{(r+\kappa+\delta\sigma)e^{-\delta(\hat{t}-t)}-(r+\kappa)}$, which is decreasing in t and equal to $r+\kappa$ at $\hat{t} = (1/\delta)\ln[\frac{r+\kappa+\delta\sigma}{r+\kappa}(1 - E_0/\bar{E})]$. Thus, $\lim_{t\downarrow 0}\dot{\xi}(t)/\xi(t) = \frac{\delta\sigma(r+\kappa)}{(r+\kappa+\delta\sigma)e^{-\delta\hat{t}}-(r+\kappa)} > r+\kappa \Leftrightarrow \hat{t} > 0$, and this last condition is equivalent to (40). Finally, notice that the uniqueness of the equilibrium follows from Lemma 12 and the uniqueness of the switching time T such that $\Gamma(T) = 0$. The convergence to the steady state is immediate from the equilibrium prices and allocations described in parts *(a)* and *(b) of Lemma* 13. ∎

Proof of Proposition 3. First, note that from parts (c) and (d) of Proposition 13, $\xi_+(s) - \xi_0(s) \ge 0$ if and only if

$$1 - e^{\frac{r+\kappa}{\sigma}(T-s)}\frac{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(s-\hat{t})}}{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(T-\hat{t})}} \ge 0.$$

We first establish that $\xi_0(0^+) < \xi(0^+)$. We proceed by contradiction. Suppose $\xi_0(0^+) \ge \xi(0^+)$. From the proof of Lemma 12 we know that $\dot{\xi}_0(t)/\xi_0(t)$ is decreasing, with $\dot{\xi}_0(t)/\xi_0(t) \to 0$ as $t \to \infty$. In addition, under condition (40), $\dot{\xi}_0(t)/\xi_0(t) > r+\kappa$ at $t = 0^+$. Therefore, there is a unique $T > 0$ such that $\xi_0(T) = e^{(r+\kappa)T}\xi(0^+)$. For all $s \in (0,T)$, $\xi_0(s) > \xi(s) = \xi_+(s)$ and therefore

$$1 - e^{\frac{r+\kappa}{\sigma}(T-s)}\frac{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(s-\hat{t})}}{1 - \frac{r+\kappa}{r+\kappa+\delta\sigma}e^{-\delta(T-\hat{t})}} < 0,$$

which together with (73) implies $A_d(t) < 0$ for all $t \in (0,T)$, a contradiction. Thus, $\xi_0(0^+) < \xi(0^+)$. Finally, the fact that $\ln\xi_0(0^+) < \ln\xi(0^+)$, $\ln\xi_0(T) = \ln\xi(T)$ and that there is a $\hat{t} \in (0,T]$ defined as in Proposition 13 such that $\frac{d}{dt}\ln\xi_0(t) \ge r+\kappa = \frac{d}{dt}\ln\xi(t)$ if and only if

$t \in [0, \hat{t}]$, implies there is a unique $\underline{t} < T$ with $\xi_0(\underline{t}) = \xi(\underline{t})$, and the property that $\xi(t) > \xi_0(t)$ for all $t \in (0, \underline{t})$ and $\xi(t) < \xi_0(t)$ for all $t \in (\underline{t}, T)$. See Figure 5 for an illustration. ∎

B   Stochastic recovery

Suppose the recovery occurs at some time t_ρ, i.e., t_ρ is the realization of the random variable T_ρ. We begin by describing the equilibrium of the economy after the recovery, taking as given dealers' inventories at the time the recovery occurs, $A_d(t_\rho)$. Once we have solved for the equilibrium from the time of the recovery onward, we solve for the equilibrium price and allocations before the recovery and then piece both sets of paths together to characterize the full equilibrium from the outset of the crash at $t = 0$.

Consider first the economy after the recovery. Let $V_i^h(a, t, t_\rho)$ denote the value function corresponding to an investor who has preference type i and is holding portfolio a at time t, conditional on the recovery having occurred at time $t_\rho \leq t$. The investor's value function is

$$V_i^h(a, t, t_\rho) = \mathbb{E}_i \left\{ \int_t^{\tilde{T}} u_{k(s)}(a) e^{-r(s-t)} ds + \right.$$
$$\left. e^{-r(\tilde{T}-t)} \{ p^h(\tilde{T}, t_\rho) a + \max_{a'} [V_{k(\tilde{T})}^h(a', \tilde{T}, t_\rho) - p^h(\tilde{T}, t_\rho) a'] \} \right\}, \qquad (74)$$

where $p^h(t, t_\rho)$ denotes the asset price. Notice that (74) is identical to (9) except for the fact that $p^h(\tilde{T}, t_\rho)$ replaces $p(\tilde{T})$. Therefore, the investor's problem is the same as in Lemma 2 where $p^h(t, t_\rho)$ replaces $p(t)$. The dealer solves

$$W^h(a_d, t, t_\rho) = \max_{q(s)} \int_t^\infty -e^{-r(s-t)} p^h(s, t_\rho) q(s) \, ds \qquad (75)$$

subject to $\dot{a}_d(s) = q(s)$, $a_d(s) \geq 0$ for all $s \geq t$, and the initial condition $a_d(t_\rho) = a_d$. The problem (75) is analogous to (3).

For all $t \geq t_\rho$, the equilibrium is characterized by the pair of differential equations (22) and (38), together with the initial condition $A_d(t_\rho)$. The following lemma characterizes the equilibrium path that the economy follows after the recovery has taken place.

Lemma 14 *Suppose the economy recovers at some time $t_\rho \geq 0$. Then, there exists a unique equilibrium path $\{\xi(t), A_d^h(t)\}$ for $t \geq t_\rho$ such that:*
(a) For all $t \in (t_\rho, T]$,

$$\xi(t) = \bar{\xi} e^{-(r+\kappa)(T-t)} \qquad (76)$$

$$A_d^h(t) = e^{-\alpha(t-t_\rho)} A_d(t_\rho) + \alpha \int_{t_\rho}^t e^{-\alpha(t-s)} \left[A - \sum_{i=1}^I \pi_i U_i'^{-1} [\xi(s)] \right] ds, \qquad (77)$$

where $T < \infty$ is the unique solution to $A_d^h(T) = 0$.

(b) For all $t \geq T$, $\{\xi(t), A_d^h(t)\} = (\bar{\xi}, 0)$, where $\bar{\xi}$ solves $\sum_{i=1}^{I} \pi_i U_i'^{-1}(\bar{\xi}) = A$.

Proof. Note that if $A_d(T) = 0$ for some $T \geq t_\rho$, then (22) and (38) imply $\{\xi(t), A_d(t)\} = (\bar{\xi}, 0)$ for all $t \geq T$. Thus, let $T = \inf\{t \geq t_\rho : A_d(t) = 0\}$. Next we show that $T < \infty$ by establishing that $A_d(t) > 0$ for all $t \geq t_\rho$ is inconsistent with equilibrium. Note that if $A_d(t) > 0$ for all $t \geq t_\rho$, (22) and (38) imply, after a change of variable,

$$A_d(t) = e^{-\alpha(t-t_\rho)} A_d(t_\rho) + \alpha \int_0^{t-t_\rho} e^{-\alpha u} \left[A - \sum_{i=1}^{I} \pi_i U_i'^{-1} [\xi(t-u)] \right] du$$

with $\xi(s) = e^{(r+\kappa)(s-t_\rho)} \xi(t_\rho)$. Thus, $\lim_{t\to\infty} A_d(t) = A > 0$. From (37), $p(t) = e^{r(t-t_\rho)} p(t_\rho)$ which implies $\lim_{t\to\infty} e^{-rt} p(t) = e^{-rt_\rho} p(t_\rho) > 0$. The dealer's transversality condition is violated, so $A_d(t) > 0$ for all $t \geq t_\rho$ cannot be part of an equilibrium. We conclude that $T < \infty$ and this establishes part (b) of the lemma. For part (a), first note that the same arguments we used in Lemma 9 can be applied here to establish that $\xi(t)$ and $A_d(t)$ are continuous for all $t \in (t_\rho, \infty)$. (The only difference is that Lemma 9 is proven with $U_i(a) = \bar{\varepsilon}_i \frac{a^{1-\sigma}}{1-\sigma}$, but this is immaterial for the results.) In particular, this means that $\xi(t)$ and $A_d(t)$ are continuous at $t = T > t_\rho$. For any $t \in (t', t'') \subset (t_\rho, T]$, (22) and (38) imply that $\{\xi(t), A_d(t)\}$ are given by (77) and (76), where (76) uses $\xi(T) = \bar{\xi}$, which follows from the continuity of $\xi(\cdot)$. We use the continuity of $A_d(\cdot)$, to determine T: using (77) and (76), $A_d(T) = 0$ can be written as

$$A_d(t_\rho) + \alpha \int_{t_\rho}^{T} e^{\alpha(s-t_\rho)} \left[A - \sum_{i=1}^{I} \pi_i U_i'^{-1} \left[e^{(r+\kappa)(s-T)} \bar{\xi} \right] \right] ds = 0. \tag{78}$$

The left-hand side of (78) is equal to $A_d(t_\rho) \geq 0$ at $T = t_\rho$ and goes to $-\infty$ as $T \to \infty$. Differentiate the left-hand side of (78) with respect to T to get:

$$\alpha(r+\kappa) \int_{t_\rho}^{T} e^{\alpha(s-t_\rho)} \sum_{i=1}^{I} \pi_i \frac{e^{(r+\kappa)(s-T)} \bar{\xi}}{U_i'' [\bar{a}_i(s)]} ds < 0,$$

where $\bar{a}_i(s) = U_i'^{-1} \left[e^{-(r+\kappa)(T-s)} \bar{\xi} \right]$. So there is a unique T that satisfies $A_d(T) = 0$. To conclude, the uniqueness of the equilibrium follows from the fact that the saddle path leading to the steady state depicted in Figure 9 is the only path that satisfies all the equilibrium conditions. Any other path is inconsistent with the dealer's optimization: paths above the saddle path violate the transversality condition while those below would imply an upward jump in $\xi(t)$ at $t = T$ (see Figure 9). ∎

59

According to Lemma 14 the equilibrium path of the economy starting from t_ρ is such that $A_d^h(t) > 0$ for all t in the interval (t_ρ, T) and $A_d^h(t) = 0$ for all $t \geq T$. Furthermore, $T > t_\rho$ unless $A_d(t_\rho) = 0$. According to (76), the investor's effective cost of holding the asset, $\xi(t)$, increases at rate $r + \kappa$ while dealers hold inventories, meanwhile according to (77), the stock of assets held by dealers decreases monotonically until it is fully depleted at time T. (To see this, notice from (76) that $\xi(t) < \bar{\xi}$ for all $t < T$. As a consequence, $A - \sum_{i=1}^I \pi_i U_i'^{-1}[\xi(t)] < 0$ for all $t < T$ and from (22) $\dot{A}_d(t) < 0$.) The condition $A_d^h(T) = 0$ can be rewritten as

$$A_d(t_\rho) + \alpha \int_0^{T-t_\rho} e^{\alpha s} \left\{ A - \sum_{i=1}^I \pi_i U_i'^{-1} \left[e^{-(r+\kappa)[(T-t_\rho)-s]}\bar{\xi} \right] \right\} ds = 0. \tag{79}$$

From (79) the time that it takes for dealers' inventories to be depleted, $T - t_\rho$, is an implicit function of the stock of inventories in dealers' hands at the recovery time, $A_d(t_\rho)$. Equivalently, (24) provides a relationship between the effective cost of holding the asset at the recovery time, $\xi(t_\rho) = \bar{\xi}e^{-(r+\kappa)(T-t_\rho)}$, and dealers' initial inventories, $A_d(t_\rho)$. We represent this relationship by the function ψ such that $\xi(t_\rho) = \psi[A_d(t_\rho)]$.[25] Notice that $\psi' < 0$, so $\xi(t_\rho)$ is decreasing in $A_d(t_\rho)$, and $\psi(0) = \bar{\xi}$. Intuitively, the larger the stock of inventories that dealers are holding at the time of the recovery, the lower the effective cost of holding the asset at the recovery time, and the longer it will take to deplete dealers' inventories once the recovery has occurred.

Figure 9 shows the phase diagram of the dynamic system $[A_d(t), \xi(t)]$ following the recovery. From (22) we see that the A_d−isocline is upward-sloping and intersects the vertical axis at the steady-state point. The equilibrium trajectory of the economy is indicated in the figure by arrows along the saddle-path, namely, $\xi(t) = \psi[A_d(t)]$. The initial condition $A_d(t_\rho)$ determines the starting point on the saddle path. The trajectories marked with dotted lines that do not follow the saddle path are solutions to the differential equations (22) and (38) but they either fail to satisfy the transversality condition or the requirement that the equilibrium path $\xi(t)$ be continuous.

Next, we analyze the economy before the arrival of the recovery shock. Let $V_i^\ell(a, t)$ denote the value function corresponding to an investor who has preference type i and is holding portfolio

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[25]From (79), the reciprocal of ψ is defined as

$$\psi^{-1}(\xi) = -\alpha \int_0^{\frac{1}{(r+\kappa)} \ln(\bar{\xi}/\xi)} e^{\alpha s} \left\{ A - \sum_i \pi_i U_i'^{-1} \left[e^{(r+\kappa)s}\xi \right] \right\} ds$$
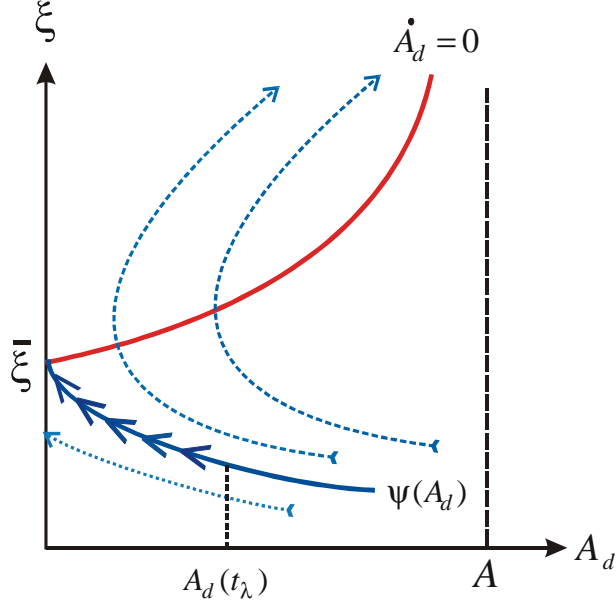
Figure 9: Dynamics after the recovery

a at time $t < T_\rho$. Then, the investor's value function satisfies

$$V_i^\ell(a,t) = \tilde{V}_i^\ell(a) + \mathbb{E}_i \left\{ \mathbb{I}_{\{T_\rho \leq \tilde{T}\}} e^{-r(\tilde{T}-t)} \max_{a'} \left[V_{k(\tilde{T})}^h(a, \tilde{T}, T_\rho) - p^h(\tilde{T}, T_\rho)(a' - a) \right] \right.$$

$$\left. + \mathbb{I}_{\{\tilde{T} < T_\rho\}} e^{-r(\tilde{T}-t)} \max_{a'} \left[V_{k(\tilde{T})}^\ell(a', \tilde{T}) - p^\ell(\tilde{T})(a' - a) \right] \right\}, \tag{80}$$

where the indicator function $\mathbb{I}_{\{T_\rho \leq \tilde{T}\}}$ equals one if $T_\rho \leq \tilde{T}$ and zero otherwise, and

$$\tilde{V}_i^\ell(a) \equiv \mathbb{E}_i \left\{ \int_t^{\tilde{T}} \left[R + \mathbb{I}_{\{s > T_\rho\}}(1 - R) \right] u_{k(s)}(a) e^{-r(s-t)} ds \right\}.$$

This Bellman equation is a natural generalization of (9), for example, they coincide if we set $R = 1$ and let $\rho \to 0$. The function $\tilde{V}_i^\ell(a)$ is the expected discounted sum of utility flows that an investor enjoys from holding a quantity a of the asset until he gains effective access to the market at Poisson rate κ. The term $\left[R + \mathbb{I}_{\{s > T_\rho\}}(1 - R) \right]$ indicates that the investor's instantaneous utility is scaled down by R until the economy recovers. It will be convenient to define $U_i^\ell(a) = (r + \kappa)\tilde{V}_i^\ell(a)$. If $\rho = 0$ then $U_i^\ell(a)$ reduces to $RU_i(a)$. Alternatively, as $\rho \to \infty$ (the economy recovers almost surely in the next instant), $U_i^\ell(a) \to U_i(a)$.

61

The following Lemma gives a formulation of the investor's problem which is analogous to the one in Lemma 2.

Lemma 15 *An investor of preference type i who holds portfolio a and gains direct effective access to the market at time t before the recovery has taken place, solves*

$$\max_{a_i^\ell} \left[U_i^\ell(a_i^\ell) - \xi^\ell(t) a_i^\ell \right] \tag{81}$$

where

$$
\begin{aligned}
\xi^\ell(t) &= (r + \kappa) \left[p^\ell(t) - \int_0^\infty \kappa e^{-(r+\kappa+\rho)\tau_\kappa} p^\ell(t + \tau_\kappa) d\tau_\kappa \right. \\
&\quad \left. - \int_0^\infty \int_{\tau_\rho}^\infty \rho e^{-\rho\tau_\rho} \kappa e^{-(r+\kappa)\tau_\kappa} p^h\left(t + \tau_\kappa, t + \tau_\rho\right) d\tau_\kappa d\tau_\rho \right].
\end{aligned}
\tag{82}
$$

Proof. The first term on the right-hand side of (80), $\tilde{V}_i^\ell(a)$, satisfies the following flow Bellman equation,

$$(r + \kappa) \tilde{V}_i^\ell(a) = R u_i(a) + \delta \sum_{j=1}^I \pi_j \left[\tilde{V}_j^\ell(a) - \tilde{V}_i^\ell(a) \right] + \rho \left[\tilde{V}_i(a) - \tilde{V}_i^\ell(a) \right], \tag{83}$$

where $\tilde{V}_i(a) = U_i(a)/(r + \kappa)$. The investor's portfolio problem before the recovery is

$$\max_a \left\{ \tilde{V}_i^\ell(a) - p^\ell(t) a - \mathbb{E}\left[e^{-r(\tilde{T}-t)} \left[\mathbb{I}_{(\tilde{T}<T_\rho)} p^\ell(\tilde{T}) + \mathbb{I}_{(\tilde{T}\geq T_\rho)} p^h(\tilde{T}, T_\rho) \right] \right] a \right\}. \tag{84}$$

Use $U_i^\ell(a) = (r + \kappa) \tilde{V}_i^\ell(a)$ to rewrite (84) as

$$\max_a \left[U_i^\ell(a) - \xi^\ell(t) a \right],$$

where

$$\xi^\ell(t) = [r + \kappa] \left\{ p^\ell(t) - \mathbb{E}\left[\mathbb{I}_{\{\tilde{T}<T_\rho\}} e^{-r(\tilde{T}-t)} p^\ell(\tilde{T}) + \mathbb{I}_{\{T_\rho \leq \tilde{T}\}} p^h(\tilde{T}, T_\rho) \right] \right\}$$

and

$$U_i^\ell(a) = \frac{r+\kappa}{r+\kappa+\rho} \left[\frac{(r+\kappa+\rho) R u_i(a) + \delta \sum_{j=1}^I \pi_j R u_j(a)}{r+\kappa+\delta+\rho} \right] + \frac{\rho}{r+\kappa+\rho} \frac{(r+\kappa+\rho) U_i(a) + \delta \sum_{j=1}^I \pi_j U_j(a)}{r+\kappa+\delta+\rho}.$$

Using the fact that $\tilde{T} - t$ and $T_\rho - t$ are two independent exponentially distributed random variables, the expected value of the resale price is

$$\mathbb{E}\left[e^{-r(\tilde{T}-t)}\left[\mathbb{I}_{(\tilde{T}<T_\rho)}p^\ell(\tilde{T}) + \mathbb{I}_{(\tilde{T}\geq T_\rho)}p(T_\rho,\tilde{T})\right]\right] =$$

$$\int_0^\infty \int_0^\infty e^{-r\tau_\kappa}\left[\mathbb{I}_{(\tau_\kappa<\tau_\rho)}p^\ell(t+\tau_\kappa) + \mathbb{I}_{(\tau_\kappa\geq\tau_\rho)}p(t+\tau_\kappa,t+\tau_\rho)\right]\kappa e^{-\kappa\tau_\kappa}\rho e^{-\rho\tau_\rho}d\tau_\rho d\tau_\kappa$$

$$= \int_0^\infty e^{-r\tau_\kappa}\left[e^{-\rho\tau_\kappa}p^\ell(t+\tau_\kappa) + \int_0^{\tau_\kappa}\rho e^{-\rho\tau_\rho}p(t+\tau_\kappa,t+\tau_\rho)\right]\kappa e^{-\kappa\tau_\kappa}d\tau_\rho d\tau_\kappa.$$

Change the order of integration of the second term to arrive at (82). ∎

According to Lemma 15, an investor maximizes his effective utility function, $U_i^\ell(a)$, minus the effective cost of investing in the asset, $\xi^\ell(t)\,a$. Just as $U_i^\ell(a)$ takes into account both idiosyncratic and aggregate preference shocks, $\xi^\ell(t)$ takes into account the expected capital gain that will be realized the next time the investor gains access to the market, which may be before or after the economy recovers. As before the last two terms on the right-hand side of (82) represent the expected resale price of the asset. From Lemma 15 it follows that during the crisis, an optimal portfolio choice $a_i^\ell(t)$ satisfies

$$U_i^{\ell\prime}[a_i^\ell(t)] = \xi^\ell(t). \tag{85}$$

We now turn to analyze a dealers's problem. At any time t before the recovery has occurred, the dealer solves

$$\max_{q(s)}\mathbb{E}\left[\int_t^{T_\rho} -e^{-r(s-t)}p^\ell(s)q(s)ds + e^{-r(T_\rho-t)}W^h\left[a_d\left(T_\rho\right),T_\rho,T_\rho\right]\right], \tag{86}$$

subject to $\dot{a}_d\left(s\right) = q\left(s\right)$, $a_d\left(s\right) \geq 0$ for all $s \geq t$ and the initial condition $a_d\left(t\right)$. Lemma 16 simplifies the dealer's problem.

Lemma 16 *At any every time t before the recovery has occurred, the dealer solves*

$$\max_{a_d(t+s)\geq 0}\int_0^\infty e^{-(r+\rho)s}\left\{-rp^\ell(t+s) + \dot{p}^\ell(t+s) + \rho\left[p^h(t+s,t+s) - p^\ell(t+s)\right]\right\}a_d(t+s)ds$$

given an initial condition $a_d(t)$.

Proof. Integration by parts and the fact that $\lim_{t\to\infty}e^{-rt}p^h(s,t_\rho)a_d\left(t\right) = 0$ (by Lemma 14) implies that (75) can be written as

$$W^h\left(a_d,t,t_\rho\right) = W^h\left(0,t,t_\rho\right) + p^h(t,t_\rho)a_d, \tag{87}$$

where $W^h(0, t, t_\rho) = \max_{a_d(s) \geq 0} \int_t^\infty e^{-r(s-t)} \left[\dot{p}^h(s, t_\rho) - r p^h(s, t_\rho) \right] a_d(s) \, ds$. Integration by parts and (87) allow us to rewrite the dealer's problem (86) as

$$
\max_{a_d(s) \geq 0} \mathbb{E} \left\{ \int_t^{T_\rho} e^{-r(s-t)} \left[\dot{p}^\ell(s) - r p^\ell(s) \right] a_d(s) ds + e^{-r(T_\rho - t)} \left[p^h(T_\rho, T_\rho) - p^\ell(T_\rho) \right] a_d(T_\rho) \right\}.
$$

After a change of variables, defining $\tau_\rho = T_\rho - t$ and noticing that τ_ρ is an exponentially distributed random variable with mean $1/\rho$, this last expression becomes

$$
\max_{a_d(t+s) \geq 0} \{ \int_0^\infty \rho e^{-\rho \tau_\rho} \int_0^{\tau_\rho} e^{-rs} \left[\dot{p}^\ell(t+s) - r p^\ell(t+s) \right] a_d(t+s) ds d\tau_\rho +
$$

$$
\int_0^\infty \rho e^{-(r+\rho)\tau_\rho} \left[p^h(t+\tau_\rho, t+\tau_\rho) - p^\ell(t+\tau_\rho) \right] a_d(t+\tau_\rho) d\tau_\rho \}
\tag{88}
$$

for $s \geq t$, with $a_d(t)$ given. Since (88) is the same as

$$
\max_{a_d(t+s) \geq 0} \{ \int_0^\infty \int_0^\infty \rho e^{-\rho \tau_\rho} \mathbb{I}_{\{s < \tau_\rho\}} e^{-rs} \left[\dot{p}^\ell(t+s) - r p^\ell(t+s) \right] a_d(t+s) ds d\tau_\rho +
$$

$$
\int_0^\infty \rho e^{-(r+\rho)\tau_\rho} \left[p^h(t+\tau_\rho, t+\tau_\rho) - p^\ell(t+\tau_\rho) \right] a_d(t+\tau_\rho) d\tau_\rho \},
$$

we can change the order of integration in the first term and integrate with respect to τ_ρ to arrive at the dealer's problem as formulated in the statement of the lemma. ∎

From Lemma 16 we see that the flow of profit of dealers during the crisis has three components: the opportunity cost of holding the asset, $r p^\ell(t+s)$, the capital gain while the economy remains in the crisis state, $\dot{p}^\ell(t)$, and the expected capital gain $p^h(t+s, t+s) - p^\ell(t+s)$ if the economy recovers (which occurs with Poisson intensity ρ). Clearly, $\dot{p}^\ell(t) + \rho \left[p^h(t, t) - p^\ell(t) \right] > r p^\ell(t)$ is inconsistent with equilibrium (the dealer's problem would have no solution). Let $a_d^\ell(t)$ denote the solution to the dealer's problem. The dealer's necessary conditions are immediate from Lemma 16: as long as the economy is in the crisis state,

$$
\left\{ -r p^\ell(t) + \dot{p}^\ell(t) + \rho \left[p^h(t, t) - p^\ell(t) \right] \right\} a_d^\ell(t) = 0
\tag{89}
$$

for all t, with $a_d^\ell(t) \geq 0$ and $-r p^\ell(t) + \dot{p}^\ell(t) + \rho \left[p^h(t, t) - p^\ell(t) \right] \leq 0$. The following lemma, which is analogous to Lemma 3 allows us express the dealer's first-order conditions (89) in terms of investors' effective cost of buying the asset before the recovery and after the recovery has occurred. We use (76) to define

$$
\xi^h(t, t_\rho) = \psi \left[A_d(t_\rho) \right] e^{-(r+\kappa)(t_\rho - t)}.
\tag{90}
$$

64

Notice that given $\xi^h\left[t, t_\rho, A_d(t_\rho)\right]$ we can use (21) to find the path for the asset price after the recovery.[26]

Lemma 17 *Condition (82) implies*

$$-rp^\ell(t) + \dot{p}^\ell(t) + \rho\left[p^h(t,t) - p^\ell(t)\right] = -\xi^\ell(t) + \frac{\dot{\xi}^\ell(t) + \rho\left[\xi^h(t,t) - \xi^\ell(t)\right]}{r + \kappa}.$$

Proof. Let

$$P^\ell(t) = \int_t^\infty \kappa e^{-(r+\kappa+\rho)(s-t)} p^\ell(s)\, ds$$

$$P^h(t) = e^{(r+\kappa+\rho)t} \int_t^\infty \int_z^\infty \rho e^{-\rho z} \kappa e^{-(r+\kappa)s} p^h(s,z)\, ds\, dz,$$

which correspond to the second and third terms in (82), respectively, after a change of variables. Then (82) can be written more compactly as

$$\xi^\ell(t) = (r + \kappa)\left[p^\ell(t) - P^\ell(t) - P^h(t)\right], \tag{91}$$

and therefore,

$$\dot{\xi}^\ell(t) = (r + \kappa)\left[\dot{p}^\ell(t) - \dot{P}^\ell(t) - \dot{P}^h(t)\right]. \tag{92}$$

Note that

$$\dot{P}^\ell(t) = (r + \kappa + \rho)\, P^\ell(t) - \kappa p^\ell(t) \tag{93}$$

and

$$\dot{P}^h(t) = (r + \kappa + \rho)\, P^h(t) - \rho \int_t^\infty \kappa e^{-(r+\kappa)(s-t)} p^h(s,t)\, ds. \tag{94}$$

From the investor's problem ((12) and Lemma 3), we know that

$$\xi^h(t, t_\rho) = (r + \kappa)\left[p^h(t, t_\rho) - \int_t^\infty \kappa e^{-(r+\kappa)(s-t)} p^h(s, t_\rho) ds\right],$$

which evaluated at $t_\rho = t$ implies

$$\int_t^\infty \kappa e^{-(r+\kappa)(s-t)} p^h(s,t) ds = p^h(t,t) - \frac{\xi^h(t,t)}{r + \kappa}. \tag{95}$$

---

[26] Specifically, $p^h(t, t_\lambda) = \int_t^\infty e^{-r(s-t)} \left[\xi^h(s, t_\lambda) - \frac{\dot{\xi}^h(s,t_\lambda)}{r+\kappa}\right] ds$, where hereafter, $\dot{\xi}^h(s, t_\lambda)$ is used to denote $\partial \xi^h(s, t_\lambda)/\partial s$ and $\dot{p}^h(t, t_\lambda)$ to denote $\partial p^h(t, t_\lambda)/\partial t$.

Substitute (95) back into (94) to get

$$\dot{P}^h(t) = (r + \kappa + \rho) P^h(t) - \rho \left[p^h(t,t) - \frac{\xi^h(t,t)}{r + \kappa} \right]. \tag{96}$$

Next, substitute (93) and (96) into (92) to arrive at

$$\frac{\dot{\xi}^\ell(t)}{r + \kappa} = \dot{p}^\ell(t) - (r + \kappa + \rho) \left[P^\ell(t) + P^h(t) \right] + \kappa p^\ell(t) + \rho \left[p^h(t,t) - \frac{\xi^h(t,t)}{r + \kappa} \right],$$

which after using (91) to substitute $\left[P^\ell(t) + P^h(t) \right]$ and rearranging reduces to

$$-rp^\ell(t) + \dot{p}^\ell(t) + \rho \left[p^h(t,t) - p^\ell(t) \right] = -\xi^\ell(t) + \frac{\dot{\xi}^\ell(t) + \rho \left[\xi^h(t,t) - \xi^\ell(t) \right]}{r + \kappa},$$

the expression in the statement of the lemma. ∎

Lemma 17 allows us to write (89) as

$$\left\{ \dot{\xi}^\ell(t) + \rho \xi^h(t,t) - (r + \kappa + \rho) \xi^\ell(t) \right\} a_d^\ell(t) = 0.$$

To summarize, we have shown that once the economy has recovered from the crisis, say at some time t_ρ, it will evolve along a deterministic path $\left\{ A_d^h(t), \xi^h(t, t_\rho) \right\}$ given by (77) and (90). Before it has recovered from the crisis, it follows a path $\left\{ A_d^\ell(t), \xi^\ell(t) \right\}$ which, using (90), satisfies

$$\left\{ \dot{\xi}^\ell(t) + \rho \psi[A_d^\ell(t)] - (r + \kappa + \rho) \xi^\ell(t) \right\} A_d^\ell(t) = 0 \tag{97}$$

and the market clearing condition

$$\dot{A}_d^\ell(t) = \alpha \left\{ A - A_d^\ell(t) - \sum_{i=1}^{I} \pi_i U_i^{\ell\prime - 1}[\xi^\ell(t)] \right\}. \tag{98}$$

We can now define an equilibrium to be a stochastic process $\{\xi(t), A_d(t)\}$, such that for $t < T_\rho$, $\{\xi(t), A_d(t)\} = \left\{ \xi^\ell(t), A_d^\ell(t) \right\}$ satisfying (97) and (98), and for $t \geq T_\rho$, $\{\xi(t), A_d(t)\} = \left\{ \xi^h(t, T_\rho), A_d^h(t) \right\}$ satisfying (77) and (90).

Let $(\bar{\xi}^\ell, \bar{A}_d^\ell)$ denote the steady-state associated with (97) and (98); it is characterized by

$$\bar{\xi}^\ell \geq \frac{\rho}{r + \kappa + \rho} \psi(\bar{A}_d^\ell) \qquad \text{`` = " if } \bar{A}_d^\ell > 0 \tag{99}$$

$$A = \bar{A}_d^\ell + \sum_{i=1}^{I} \pi_i U_i^{\ell\prime - 1}(\bar{\xi}^\ell) \tag{100}$$

As the random time of recovery, T_ρ, becomes very large, $\{\xi^\ell(t), A_d^\ell(t)\}$ approach their steady state values $(\bar\xi^\ell, \bar A_d^\ell)$ as given by (99) and (100). Assuming $\bar A_d^\ell > 0$, it can be checked from (97) and (98) that the steady state is a saddle point and that there is a unique trajectory that brings the system to its steady state.

Proof of Proposition 4 Dealers accumulate inventories if and only if $\bar A_d^\ell > 0$. From (99) and (100), $\bar A_d^\ell$ is determined by the condition $\Gamma(\bar A_d^\ell) = 0$, where

$$\Gamma(A_d) = A_d + \sum_{i=1}^I \pi_i U_i^{\ell\prime-1} \left[\frac{\rho}{r+\kappa+\rho} \psi(A_d) \right] - A.$$

Since $\Gamma'(A_d) > 0$ and $\lim_{A_d \to \infty} \Gamma(A_d) = \infty$, there is a unique $\bar A_d^\ell > 0$ such that $\Gamma(\bar A_d^\ell) = 0$ iff $\Gamma(0) < 0$. Using the fact that $\psi(0) = \bar\xi$ we know that $\Gamma(0) = \sum_{i=1}^I \pi_i U_i^{\ell\prime-1} \left(\frac{\rho}{r+\kappa+\rho} \bar\xi \right) - A$, so $\Gamma(0) < 0$ is equivalent to (41). ■

Derivation of $\bar A_d^\ell$. From (79), normalizing t_ρ to 0 and assuming that $A_d(t_\rho) = \bar A_d^\ell$, we get

$$\bar A_d^\ell + \alpha \int_0^T e^{\alpha s} \left[A - \sum_{i=1}^I \pi_i U_i^{\prime-1} [\xi(s)] \right] ds = 0.$$

With the functional form $u_i(a) = \varepsilon_i a^{1-\sigma}/(1-\sigma)$ we have $U_i^{\prime-1}[\xi(s)] = [\bar\varepsilon_i/\xi(s)]^{1/\sigma}$ and $\xi(s) = \bar\xi e^{-(r+\kappa)(T-s)}$. Hence,

$$\bar A_d^\ell + \alpha \int_0^T e^{\alpha s} \left[A - \sum_{i=1}^I \pi_i \left[\frac{\bar\varepsilon_i}{\bar\xi} \right]^{1/\sigma} e^{(\frac{r+\kappa}{\sigma})(T-s)} \right] ds = 0.$$

Notice that $\sum_{i=1}^I \pi_i (\bar\varepsilon_i/\bar\xi)^{1/\sigma} = A$. After some calculations, we arrive at

$$\frac{\bar A_d^\ell - A}{A} + \frac{\gamma}{\gamma-\alpha} e^{\alpha T} - \frac{\alpha}{\gamma-\alpha} e^{\gamma T} = 0, \tag{101}$$

where $\gamma \equiv \frac{r+\kappa}{\sigma}$. The steady-state condition (100) yields

$$A = \bar A_d^\ell + \sum_{i=1}^I \pi_i \left(\frac{\hat\varepsilon_i}{\bar\xi^\ell} \right)^{1/\sigma}.$$

Combined with (99), $\bar\xi^\ell = \frac{\rho}{r+\kappa+\rho} \bar\xi e^{-(r+\kappa)T}$,

$$A = \bar A_d^\ell + e^{\gamma T} \sum_{i=1}^I \pi_i \left[\frac{r+\kappa+\rho}{\rho} \frac{\hat\varepsilon_i}{\bar\xi} \right]^{1/\sigma}. \tag{102}$$

67

Using the fact that $\bar{\xi}^{\frac{1}{\sigma}} = \sum_{i=1}^{I} \pi_i (\bar{\varepsilon}_i)^{1/\sigma} / A$, (102) becomes

$$e^{\gamma T} \Omega = \frac{A - \bar{A}_d^{\ell}}{A}, \tag{103}$$

where $\Omega \equiv \left(\frac{r+\kappa+\rho}{\rho} \right)^{1/\sigma} \sum_{i=1}^{I} \pi_i (\hat{\varepsilon}_i)^{1/\sigma} / \sum_{i=1}^{I} \pi_i (\bar{\varepsilon}_i)^{1/\sigma}$. Substitute (103) into (101) to obtain

$$T = \frac{1}{\alpha - \gamma} \ln \left[\frac{\alpha}{\gamma} + \left(\frac{\gamma - \alpha}{\gamma} \right) \Omega \right]. \tag{104}$$

Finally, substitute the expression for T given by (104) into (103) to obtain

$$\bar{A}_d^{\ell} = A \left\{ 1 - \Omega \left[\frac{\alpha}{\gamma} + \left(\frac{\gamma - \alpha}{\gamma} \right) \Omega \right]^{\frac{\gamma}{\alpha - \gamma}} \right\}.$$