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# AUCTIONS WITH BUDGET-CONSTRAINED BUYERS: A NONEQUIVALENCE RESULT

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## Abstract

Anecdotal evidence of concern about the limited financial resources of small firms abounds in government auctions. Recent empirical work also provides evidence of the importance of capital constraints. In this paper, we show that the first-price sealed-bid auction yields higher expected revenue than the second-price sealed-bid auction if buyers face wealth constraints. Differences in the extent to which wealth constraints bind in the different auction formats generate the revenue nonequivalence.

## Introduction

Sellers of goods and services use a wide array of sales mechanisms, including one-on-one bargaining, oral and sealed-bid auctions, and posted-price schemes. Auctions are frequently used to sell goods ranging from real estate and works of art to mineral extraction rights and timber harvesting rights. For example, in the United States, federal mineral rights have been sold exclusively through first-price sealed-bid auctions, where the winner pays his bid, whereas timber rights have traditionally been sold through oral auctions. (The latter are, for our purposes, equivalent to second-price sealed-bid auctions, where the winner pays the highest losing bid.) Given the economic significance of these auctions, it is important to understand the relative performance of various auction formats.

Auctions with very different rules may yield similar outcomes. Consider the independent private-values setting with symmetric buyers, where valuations are independently and identically distributed. A large class of auctions generates the same expected revenue for the seller, despite the differences in rules. This "revenue equivalence" result relies on the insight that the rule for determining the winner, and the expected surplus that accrues to a buyer with the lowest possible valuation, completely determine the expected surplus to a given buyer. Total surplus is the same if the winner is the same. Since each buyer's expected surplus is also the same, the seller's expected revenue must be equal in the different auctions.

A consequence of revenue equivalence is that a seller should be indifferent among all auction formats within the relevant class. Yet sellers employ certain formats more frequently than others. In this paper, we show that the first-price sealed-bid auction yields higher expected revenue than the second-price sealed-bid auction if buyers face wealth constraints. Differences in the extent to which wealth constraints bind in the different formats generate the nonequivalence.

Many buyers face some form of wealth constraint when bidding. In the case of a consumption good, imperfect capital markets may constrain a buyer's ability to borrow against lifetime income (which may itself be below his subjective valuation of the object for sale). Similarly, the buyer could be a bureaucrat who internalizes the benefits from the acquisition, but not the costs, and who is therefore subject to tight budgetary control.

Anecdotal evidence of concern about the limited financial resources of small firms abounds in government auctions. For example, despite the presence of informational economies of scale, the U.S. government has limited the length and size of mineral leases.[1] In timber rights auctions, "set-aside sales" have been made available exclusively to small firms if such firms have not attained a specified market share in the prior 12 months (Bergsten et al. [1987]).

More recently, a proposal was made to require a substantial nonrefundable deposit to participate in the Federal Communication Commission's Personal Communications Service auction.[2] Requiring a deposit is an attempt to "pool" bidders' budget constraints by extracting revenue from all bidders, rather than from just the winner. Royalty payments, which are popular in mineral rights auctions, provide a method of spreading bidders' budget constraints across periods. Between 1953 and 1982, the revenue raised from royalty payments in Outer Continental Shelf (OCS) auctions amounted to $17.3 billion, or 41.9 percent of the revenue raised from up-front bids.[3]

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[1] The Mineral Leasing Act and the Outer Continental Shelf Land Act explicitly limit the size of leases, but allow consolidation of leases after bidding is complete. Leases are limited to five and ten years for producing and nonproducing tracts, respectively. See Bergsten et al. (1987).

[2] See Edmund L. Andrews, "U.S. Lays Out Rules for a Big Auction of Radio Airwaves," *New York Times*, September 24, 1993.

[3] Royalty payments do not solve the problem of budget constraints completely because an increased royalty rate lowers the incentive to develop and recover minerals.

The empirical work of Hendricks and Porter (1992) provides additional evidence of the importance of capital constraints. Since 1975, OCS regulations have permitted joint bidding by all but the eight largest firms. The authors study bidding behavior on OCS leases for the period 1954-1979. They examine the impact of joint bidding on bids and ex post profit rates. Their findings concerning the low profitability of joint ventures involving a large firm and small fringe firms are of particular interest. Formation of these joint ventures apparently leads to more competitive bidding. The authors suggest that joint ventures are "motivated primarily by capital constraints" (ibid, p. 510). McDonald (1979, pp. 106-07) reaches a similar conclusion.

We examine buyers who face an exposure limit that fixes their maximum feasible bid. This limit, referred to as the buyer's "wealth," is considered in two settings. The first corresponds to situations where heterogeneity of wealth is large compared to heterogeneity of valuations. In particular, we suppose that the value of the object, in the absence of wealth constraints, is v for all buyers. Wealth differs across buyers and is private information.[4] First- and second-price auctions each yield revenue of v in the absence of wealth constraints. If the wealth constraint binds, however, expected revenue differs.

The basic argument for nonequivalence can be developed along the following lines. Suppose that a buyer wins the object with probability X, that the expected payment is T, conditional upon winning, and that the maximum realized payment is m(T).[5] In the standard first-price auction, m(T) = T, since the winner pays his bid. In the standard second-price auction, m(T) is again the bid, but here it exceeds T.

4 An alternate interpretation is that this is a pure common-values case where buyers have identical information concerning the common value. Because no transmission of information concerning the common value takes place here, the "linkage" of bids described by Milgrom and Weber (1982) is not present.

5 There is a one-to-one correspondence between the maximum payment (the bid) and the expected payment in both auctions, so m(T) is well defined.

In equilibrium, a buyer with wealth w will select the feasible (X,T) pair that maximizes his expected surplus, (v-T)X, subject to $m(T) \leq w$. The corresponding Lagrangean is

$$L(X,T,\lambda;w) \equiv (v-T)X + \lambda[w-m(T)].$$

Let $(X^*(w),T^*(w),\lambda^*(w))$ denote the optimal values, and let $U^*(w) \equiv (v-T^*(w))X^*(w)$ be the maximized expected surplus. The Envelope Theorem implies

$$U^{*\prime}(w) = \partial L/\partial w = \lambda^*(w),$$

and integrating yields

$$(1) \qquad U^*(w) = U^*(\underline{w}) + \int_{\underline{w}}^{w} \lambda^*(u)du.$$

We immediately see that the expected surplus depends on the surplus in the benchmark case, where $w = \underline{w}$, and on how tightly the constraint binds. Therefore, the property that it depends only on the allocation rule and the expected surplus in the benchmark case does not hold. In other words, two auctions that always give the object to the buyer with the highest wealth, and that give zero expected surplus to a buyer with the lowest possible wealth, need not generate the same expected revenue.

The budget constraint binds differentially across auctions, which yields different expected surplus to the bidders as well as different expected revenues. For example, if v is very large, all buyers bid their wealth in both auctions. The expected surplus for a given buyer is lower in the first-price auction, since the winning bidder pays his bid. In the second-price auction, the winner's price is determined by the second-highest bid, and it is lower with probability one. Total surplus is the same in the two auctions, presuming that the reserve price (minimum bid) is the same, so expected revenue is higher in the first-price auction. In cases where buyers may or may not be constrained, we show that low-wealth buyers receive higher expected surplus in the second-price auction, all else equal, for the reasons just given. The same revenue ranking holds.

The second setting that we study corresponds to the opposite situation, where heterogeneity of valuations is large compared to heterogeneity of wealth. In particular, we suppose that wealth is equal for all buyers. Buyers have different valuations, however, and this is private information. Since this is an independent private-values model, revenue equivalence holds if wealth exceeds the highest possible valuation. Buyers with independent private values shade their bids below their valuations in a first-price auction, in the absence of budget constraints. A consequence is that, roughly speaking, budget constraints bind less frequently in a first-price auction. (The complete analysis accounts for possible changes in the equilibrium bidding strategies as well.) This again makes the seller's expected revenue lower in the second-price auction.

Although revenue nonequivalence has been noted in other contexts, few papers have examined the impact of budget constraints.[6] One exception is Pitchik and Schotter (1988), who consider the case of two buyers bidding for two goods in a complete-information sequential auction. In a second-price sequential auction, there is an incentive to bid relatively more aggressively in the initial auction. Since the losing bid determines the price paid in each auction, bidding aggressively in the first auction can enable a buyer to deplete her opponent's wealth, thereby making him a weaker competitor in the second auction. This feature leads to nonequivalence, but the revenue ranking is opposite to that found here.

Section 1 characterizes the equilibria of second-price auctions with heterogeneous wealth, followed by first-price auctions. We then give revenue comparisons, which are made by ranking buyers' expected surplus for each possible wealth. The first-price auction generates higher expected revenue either if no reserve prices are employed or if optimal reserve prices are employed. Section 2 repeats the

[6] For a comprehensive review of the literature, see McAfee and McMillan (1987). Other sources of revenue nonequivalence include buyer risk aversion and affiliation of valuations.

analysis for heterogeneous valuations, with the same qualitative results. The comparisons are made here by looking at the expected price paid to the seller for each possible highest valuation. Section 3 considers buyers with heterogeneous valuations and wealth.

## 1. Equilibria and Revenue Comparisons with Heterogeneous Wealth

There are N ex ante identical buyers who each value one unit of the good at v, so if a buyer's wealth exceeds v, his reservation price is v. Buyer i has wealth $w_i \in [\underline{w}, \overline{w}]$, which is private information. Wealth is independently and identically distributed, with cumulative distribution function $F(\bullet)$ and strictly positive density $f(\bullet)$. Buyers are risk-neutral. The seller has one indivisible unit of the good to sell, which she values at zero. We look for Nash equilibria throughout.

One case does not require analysis. If $v \le \underline{w}$, then all buyers are unconstrained. Standard Bertrand competition ensures that at least two buyers will bid v in either auction format, so the seller's revenue is v. Therefore, only the case of $v > \underline{w}$ requires analysis. A reserve price below $\underline{w}$ has no effect, while a reserve price strictly above v or $\overline{w}$ generates no revenue, so we need only consider reserve prices $r \in [\underline{w}, \min\{v,\overline{w}\}]$.

We note first that neither the first-price nor the second-price auction maximizes expected revenue if $v > \underline{w}$. Suppose that a buyer with wealth w has the option of receiving the object with ex ante probability X. He will not pay more than $\min\{Xv,w\}$ for this gamble. Summing over bidders, the seller's expected revenue cannot exceed $\min\{v,\Sigma w_i\}$, whatever mechanism she uses. We now show that this level of revenue can be attained, which means that we have found an optimal sales mechanism.

Consider a sales mechanism in which a buyer with wealth w has a probability $w/\Sigma w_i$ of receiving the good, and must pay a transfer equal to $vw/\max\{v,\Sigma w_i\} \le w$. If total wealth is below v, then all wealth is extracted. If total wealth exceeds v, then the seller's revenue is v. Overall, the mechanism generates revenue equal to $\min\{v,\Sigma w_i\}$. It can be implemented by a lottery in which v tickets are offered for sale at $1 apiece, and each ticket gives a 1/v chance of winning. There is random rationing of tickets in case of oversubscription. If each buyer j demands $d_j$ tickets, then the expected surplus to buyer i is

$$vd_i/[\min\{v,\Sigma d_j\}] - d_i[v/\min\{v,\Sigma d_j\}] = 0,$$

so it is weakly optimal to demand w tickets.

The first- and second-price auctions do not extract wealth from more than one buyer, so they cannot be optimal mechanisms. For legal reasons, however, lotteries are not a practical alternative for private sellers. Most states prohibit gambling, except for racetrack betting, state-run lotteries, and charity fund-raisers. It is partly for this reason that we focus on the more common auction formats. (While governments have used lotteries to allocate assets, there has been a movement away from them, even though concerns have been expressed that some bidders are budget constrained.[7]

*A. Second-Price Auctions*

In a second-price auction, buyers submit bids simultaneously. The high bidder wins (if the bid is at least equal to the reserve price) and pays the larger of the second-highest bid and the reserve price. Ties are broken randomly, here and elsewhere.

It is a dominant strategy for buyer i to bid $\min\{v,w_i\}$. If $v > w_i$, then the constraint binds, and it is dominant to bid one's wealth. (We can avoid the possibility that a buyer bids more than his wealth, and wins as a consequence, by imposing a small penalty on anyone who reneges on a bid.) If $v \leq w_i$, then the budget constraint does not bind, so it is a dominant strategy to bid v. As noted above, we need only analyze the case of $v > \underline{w}$, where the constraint may or may not bind.

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[7] See Edmund L. Andrews, "Airwaves Auction Bill Advances," *New York Times*, May 12, 1993.

Suppose that there is a reserve price $r \in [\underline{w}, \min\{v, \overline{w}\}]$. We first calculate the expected surplus that accrues to buyers. Consider a buyer with wealth w, $r \leq w \leq v$. Since all other buyers bid the smaller of v and their wealth, he will win if all other buyers have lower wealth. The expected price paid, conditional on winning, is

$$E[\max\{w_{(2)}, r\} | w_{(1)} = w] = [rF(r)^{N-1} + \int_r^w u(N-1)F(u)^{N-2} f(u)du] / [F(w)]^{N-1},$$

where $w_{(1)}$ and $w_{(2)}$ denote the first and second order statistics of wealth (i.e., the highest and second-highest wealths), respectively. Conditional on the highest wealth being equal to w, there is probability $[F(r)/F(w)]^{N-1}$ that all other buyers have wealth below r, in which case the high bidder pays r. The first term on the right-hand side gives the expected revenue generated by this event. Since $w_{(2)}$ is the first order statistic of N-1 random variables that are all below w, the second term gives the component of expected revenue generated by the event $w_{(2)} \geq r$. Integrating by parts, the expected price is

$$E[\max\{w_{(2)}, r\} | w_{(1)} = w] = w - [\int_r^w F(u)^{N-1} du] / [F(w)]^{N-1},$$

given $r \leq w \leq v$.

A buyer with wealth $w < v$ wins with probability $F(w)^{N-1}$, so his expected surplus is

$$U^*(w) \equiv (v - w)[F(w)]^{N-1} + \int_r^w F(u)^{N-1} du. \tag{2}$$

Recall from (1) that the multiplier on wealth is equal to $U^{*\prime}(w)$. Thus, the value of $1 of additional wealth to a buyer with $w < v$ is

$$U^{*\prime}(w) = (v-w)[(N-1)F(w)^{N-2} f(w)].$$

The price paid does not change in those cases where the buyer would have won anyway. The gain comes from the surplus (v-w) that accrues in those cases where the buyer would not have won previously. (The price paid in these latter cases is approximately w.)

A buyer with wealth $w \geq v$ wins if all other buyers have wealth strictly below v, and he may also win if other buyers have wealth above v, since buyers with wealth above v bid v. Therefore, he receives nonzero surplus if and only if the second order statistic of wealth is below v. It follows that a buyer with wealth $w \geq v$ has expected surplus

$$U^*(w) = U^*(v) = \int_r^v F(u)^{N-1} du.$$

If $w \geq v$, there is clearly no gain from additional wealth.

With a reserve price equal to r, the object sells with probability $1-F(r)^N$. Expected revenue is the difference between total surplus and total (ex ante) expected surplus for the buyers:

$$SER^S \equiv [1-F(r)^N]v - N\int_r^{\overline{w}} U^*(w)f(w)dw.$$

If the reserve is r = v, then no surplus accrues to the buyers, and $SER^S = [1-F(v)^N]v$.

B. *First-Price Auctions*

In a first-price sealed-bid auction, buyers submit bids simultaneously, and the high bidder wins and pays his bid. Once again, we need only analyze the case of $v > \underline{w}$. There is not a dominant strategy in this auction, so we must characterize the equilibrium payoffs.

Suppose that there is a reserve price $r \in [\underline{w}, \min\{v, \overline{w}\}]$. The expected surplus from submitting a bid $b \geq r$, if all other buyers bid their wealth, is

$$H(b) \equiv (v-b)F(b)^{N-1}.$$

Now define

$$U^*(w) \equiv \max_{b \in [r,w]} H(b).$$

$U^*(w)$ gives the highest expected surplus that a buyer with wealth w could receive if all other buyers bid their wealth. More important, it equals the equilibrium expected surplus that accrues to a buyer with wealth $w \geq r$. (We leave the dependence of $U^*(\bullet)$ on r implicit.)

Lemma 1. Suppose that $v > \underline{w}$. If there is a reserve price $r \in [\underline{w}, \min\{v, \overline{w}\}]$, then a buyer with wealth $w \geq r$ receives expected surplus of $U^*(w)$ in equilibrium.

Proof: Let $U(\bullet)$ denote the expected surplus in a candidate equilibrium. A bid b wins with probability $F(b)^{N-1}$ or more, since bids cannot exceed wealth. Therefore, the bid gives an expected surplus of at least $H(b) = (v-b)F(b)^{N-1}$. It follows that $U(w) \geq U^*(w)$ for all w. Now suppose that there exists a wealth w′ such that $U(w') > U^*(w')$. We show that this provides a contradiction.

Let $z \leq w'$ denote the smallest wealth for which the equilibrium expected surplus equals $U(w')$. If $z = r$, then $b(z) = r = z$. If $z > r$ but $b(z) < z$, then buyers with wealth $w \in [b(z),z)$ would be strictly better off bidding b(z), since $U(w) < U(z)$ for $w < z$. Therefore, $b(z) = z$. It likewise follows that $b(w) \geq z$ for all $w > z$.

The analysis above provides the following inequalities:

$$U(z) = U(w') > U^*(w') \geq H(z).$$

The first holds by definition, the second by assumption, and the third by definition. Since $U(z) > H(z) = (v-z)F(z)^{N-1}$ and $b(z) = z$, a buyer with wealth z must win with probability greater than $F(z)^{N-1}$. This requires that the first order statistic of the other N-1 bids have a mass point at z. But then an individual buyer could get a discrete increase in expected surplus by increasing his bid infinitesimally above z. We conclude that $U(w) = U^*(w)$ for all w. Q.E.D.

We can now provide explicit equilibrium bids. To simplify the exposition, we first impose a regularity condition on the distribution function:

(R1) $$w + \frac{F(w)}{(N-1)f(w)} \text{ is strictly increasing.}$$

(This condition is clearly weaker than the standard regularity condition in mechanism-design problems.) Condition (R1) ensures that there exists a critical wealth w* such that buyers with wealth below $w^*$ will bid their wealth in equilibrium, while those with wealth above $w^*$ will be indifferent among a range of bids. We can implicitly define $w^*$:

$$v = w^* + \frac{F(w^*)}{(N-1)f(w^*)}.$$

(If there is no solution, set $w^* = \overline{w}$.) Clearly, $\underline{w} < w^* < v$, since $F(v) > F(\underline{w}) = 0$.

By (R1) and the definition of $w^*$, $H(\bullet)$ is strictly increasing for $w < w^*$ and is strictly decreasing for $w > w^*$. Thus, $U^*(w) = H(w)$ for $w \leq w^*$, while $U^*(w) = U^*(w^*) = H(w^*)$ for $w > w^*$. An immediate consequence is that it is not optimal for all buyers with $w > w^*$ to bid their wealth, since they would be better off individually if they bid $w^*$ instead.

It is equilibrium behavior for all buyers to use the increasing bid function

(3) $$b^*(w) \equiv v - U^*(w)/F(w)^{N-1}.$$

For $w \leq w^*$, $U^*(w) = (v-w)F(w)^{N-1}$, so

$$b^*(w) = w.$$

For $w > w^*$, $U^*(w) = U^*(w^*)$, so

$$b^*(w) = v - (v-w^*)[F(w^*)/F(w)]^{N-1} < w.$$

To see that these are equilibrium bids, note first that a buyer who bids $b^*(w)$ wins with probability $F(w)^{N-1}$ if all other buyers use this bidding function. Expected surplus is therefore

$$[v-b^*(w)]F(w)^{N-1} = U^*(w).$$

Since $U^*(w)$ is strictly increasing for $w < w^*$ and is constant thereafter, buyers with wealth $w \leq w^*$ must bid their wealth, whereas buyers with $w > w^*$ are indifferent among all bids between $w^*$ and $\min\{w, b^*(\overline{w})\}$. Higher bids give a strictly lower expected surplus, so $b^*(\bullet)$ gives equilibrium bids. Moreover, we have found the unique symmetric equilibrium in which bids are an increasing function of wealth.

There exist other equilibria in which bids are not both symmetric and strictly increasing in wealth for w > w*. These equilibria yield the same expected surplus, however, and the same expected revenue.

We conclude the analysis by characterizing the seller's expected revenue. The impact of reserve prices is somewhat unusual here. Setting a reserve price r < w* excludes buyers with wealth below r, with no countervailing benefit, since buyers with wealth below w* bid their wealth. Therefore, the seller will not select a binding reserve price below w*. If v is sufficiently large that $w^* = \overline{w}$, for example, then all buyers bid their wealth, and the seller will not employ a binding reserve price. In the absence of a binding reserve price, the object sells with probability one, and the expected revenue can be written as

$$SER^f \equiv v - N\int_{\underline{w}}^{\overline{w}} U^*(w)f(w)dw.$$

It may be optimal to set a reserve price above w* if v is small. If $r \in (w^*,\min\{v,\overline{w}\}]$, a buyer with wealth $w \geq r$ has an expected surplus of $(v-r)F(r)^{N-1}$, so expected revenue is

$$SER^f \equiv [1-F(r)^N]v - N[1-F(r)](v-r)F(r)^{N-1}.$$

In particular, if r = v, then $SER^f \equiv [1-F(v)^N]v$. Figure 1 graphs the bids for the case without a binding reserve (i.e., $r = \underline{w}$).

We can now provide some additional comparisons that lead to the revenue ranking. Consider a buyer with wealth w satisfying r < w < w*. The buyer wins with probability $F(w)^{N-1}$, and he receives expected surplus

(4) $$U^*(w) = (v-w)F(w)^{N-1},$$

which is below the corresponding value in the second-price auction. The value of $1 of additional wealth is

$$U^{*\prime}(w) = (v-w)[(N-1)F(w)^{N-2}f(w)] - F(w)^{N-1}.$$

The first term reflects the increased probability of receiving the net surplus (v-w), while the second reflects the fact that the bid has increased for those cases in which the buyer would have won anyway (i.e., without the additional wealth). This second term is not present in the second-price auction. If $w > w^*$, the value of additional wealth is zero in the first-price auction. In terms of (1), therefore, the multiplier on the wealth constraint is lower in the first-price auction than in the second-price for all wealth levels.

If (R1) is not satisfied, there may be multiple bidding regimes, with buyers bidding their wealth on disconnected intervals. The basic intuition does not change, however. In particular, the highest level of wealth that induces a buyer to bid his wealth is still below v, and expected surplus does not increase with wealth thereafter.

C. *Revenue Comparison*

A buyer's expected surplus is at least as high in the second-price auction, for each possible wealth, given the same reserve price. Suppose that v exceeds $\underline{w}$. If wealth is below $w^*$, then the buyer bids his wealth in both auctions. Since the probability of winning is the same, but the expected price is lower in the second-price auction, the expected surplus is higher in the second-price auction. For wealth beyond $w^*$, expected surplus does not increase in the first-price auction. Since total surplus is the same in the two auctions, given the same reserve price, the revenue ranking follows.

Proposition 1. The first-price auction has a higher optimal expected revenue than the second-price. Expected revenue is strictly higher in the first-price auction if the optimal reserve in the second-price is not equal to v.

Proof: If $\underline{w} \geq v$, then all buyers are unconstrained, and revenue is equal to v in both auctions. Now suppose that $\underline{w} < v$, and that the reserve price $r \in [\underline{w}, \min\{v, \overline{w}\}]$ is used in both auctions. There are three cases to examine.

First, take $r < w^*$. In both auctions, a buyer with wealth r receives an expected surplus of $(v-r)F(r)^{N-1}$. Since $w^* < v$, direct comparison of (2) and (4) indicates that the expected surplus is strictly higher in the second-price auction for all $w \in (r, w^*]$. Moreover, expected surplus is constant in the first-price auction for all $w \geq w^*$. Since expected surplus is weakly increasing in wealth in the second-price auction, expected surplus in that auction is strictly higher for all $w > r$.

Second, let r satisfy $w^* \leq r < v$. Once again, in both auctions, a buyer with wealth r has an expected surplus of $(v-r)F(r)^{N-1}$. For $w > r$, expected surplus is constant in the first-price auction, but it is weakly increasing in wealth in the second-price.

Third, take $r = v$. Expected surplus is zero for all buyers in both auctions.

A reserve price r generates a total surplus of $v[1-F(r)^N]$ in both auction formats. The expected surplus ranking implies that the seller's expected revenue is weakly higher in the first-price auction. Now suppose, in particular, that the optimal reserve price in the second-price auction is not equal to v. If the same reserve price is employed in the first-price auction, then the analysis shows that the first-price auction yields a strictly higher seller's expected revenue.

The comparisons above assume that the same reserve price was used in the two formats. Clearly, the optimal reserve in the first-price auction may differ from the optimal reserve in the second-price, which only strengthens the result. Q.E.D.

We have shown that the first-price auction dominates the second-price in a setting where budget constraints are important and there is heterogeneity of wealth. The result holds with optimal reserve prices or no reserve prices, and the difference in revenue can be relatively large. Suppose, for example, that there are $N = 2$ bidders, with wealth uniformly distributed on $[0,1]$, and $v \geq 2$. Buyers bid their wealth in both auction formats so that the expected revenue in the first-price auction is $SER^f = 2/3$, whereas $SER^s = 1/3$. As v drops to 1, $SER^f$ falls, while $SER^s$ is unchanged initially. As v drops further, both terms fall until they each equal zero when v equals zero.

## 2. Equilibria and Revenue Comparisons with Heterogeneous Valuations

Each buyer has wealth equal to w. Buyer i's valuation of one unit of the good, in the absence of wealth constraints, is $v_i \in [\underline{v}, \overline{v}]$. Valuations are private information and are independently and identically distributed, with the cumulative distribution function $G(\bullet)$ and strictly positive density $g(\bullet)$. Buyers are risk-neutral. The seller has one indivisible unit of the good to sell, which she values at zero.

All buyers are unconstrained if $w \geq \overline{v}$, in which case the model collapses to a standard independent private-values model. Conversely, if $w \leq \underline{v}$, all buyers are constrained and find it optimal to bid their wealth. We therefore need only consider $w \in (\underline{v}, \overline{v})$. Moreover, we need only consider reserve prices satisfying $r \in [\underline{v}, w]$.

Neither auction maximizes expected revenue, in general. For instance, consider $w \leq \underline{v}/N$. It is optimal for the seller to allocate the object to each buyer with probability $1/N$, for all realizations of $\{v_i\}$, and to extract w from each buyer. Since the auctions cannot extract revenue from more than one buyer, they cannot be optimal sales mechanisms. As we noted earlier, lotteries are not a practical alternative for most private sellers.

### *A. Second-Price Auctions*

Once again, there is a dominant strategy for buyers in the second-price auction. Buyer i will bid $\min(v_i, w)$. As noted above, if $w \leq \underline{v}$, it is optimal for all buyers to bid w, while if $w \geq \overline{v}$, wealth does not bind. The rest of this section focuses on $w \in (\underline{v}, \overline{v})$.

Suppose that there is a reserve price $r \in [\underline{v}, w]$. Let $v_{(1)}$ and $v_{(2)}$ denote the highest and second-highest valuations, respectively. If $v_{(1)} < r$, then revenue is zero. Now suppose that $v_{(1)} = v$, where $r \leq v \leq w$. Bids are equal to valuations in this range, so the high bidder pays the second-highest valuation *if* it exceeds the reserve price. Otherwise, he pays the reserve. The expected price paid to the seller is therefore

(5) $$E[\max\{v_{(2)}, r\} | v_{(1)} = v] = [rG(r)^{N-1} + \int_r^v u(N-1)G(u)^{N-2}g(u)du]/[G(v)]^{N-1}$$
$$= v - [\int_r^v G(u)^{N-1}du]/[G(v)]^{N-1}.$$

Now suppose that $r \leq w < v$. If $v_{(2)} > r$, then the high bidder pays $\min\{v_{(2)},w\}$. If not, then he pays r. The expected price paid to the seller is now

(6) $$E[\max\{\min\{v_{(2)}, w\}, r\} | v_{(1)} = v] = [rG(r)^{N-1} + \int_r^w u(N-1)G(u)^{N-2}g(u)du]/[G(v)]^{N-1}$$
$$+ [\int_w^v w(N-1)G(u)^{N-2}g(u)du]/[G(v)]^{N-1}$$
$$< v - [\int_r^v G(u)^{N-1}du]/[G(v)]^{N-1}.$$

The inequality in (6) holds because the seller receives $w < v_{(2)}$ if the second order statistic exceeds w. Note also that the left-hand side of (6) is not the expected price paid by a buyer with valuation v, conditional on winning. If $v_{(2)} \geq w$, the price paid is w. Since ties are broken randomly, however, the high-valuation buyer does not necessarily win. As will be seen, this method of calculating expected revenue facilitates comparison with the first-price auction.

Given a reserve price $r \in [\underline{v},w]$, (5) and (6) imply that expected revenue can be written as

(7) $$SER^S \equiv \int_r^{\overline{v}} E[\max\{\min\{v_{(2)}, w\}, r\} | v_{(1)} = v]dG_{(1)}(v),$$

where $G_{(1)}(v) \equiv G(v)^N$ is the distribution of the first order statistic $v_{(1)}$. In particular, if $r = w$, then $SER^S = [1-G(w)^N]w$.

### B. *First-Price Auctions*

The wealth constraint does not bind if $w \geq \overline{v}$. Conversely, if $w \leq \underline{v}$, all buyers find it optimal to bid their wealth. We now consider the intermediate case where $w \in (\underline{w},\overline{w})$, and buyers may or may not be constrained. Once again, there is not a dominant strategy in the first-price auction, so we must characterize the equilibrium payoffs.

A buyer with valuation v has equilibrium expected surplus of the form

$$U^*(v) \equiv \max_b (v-b)p(b),$$

where p(b) denotes the probability of winning the auction with a bid of b. The Envelope Theorem implies that

$$U^{*\prime}(v) = p(B^*(v)),$$

where $B^*(\bullet)$ denotes the equilibrium bid function. Integrating implies

(8) $$U^*(v) = U^*(\underline{v}) + \int_{\underline{v}}^{v} p(B^*(u))^{N-1} du.$$

We now show that buyers employ a cutoff rule in determining their bids.

Lemma 2. Suppose that $w \in (\underline{v}, \overline{v})$ and that the reserve price is $r \in [\underline{v}, w)$. In equilibrium, there is a valuation, $v^* > w$, such that buyers with $v_i \in [\underline{v}, v^*)$ bid strictly below w, while those with $v_i \in (v^*, \overline{v}]$ bid w.

Proof: Feasibility of bids requires that each buyer's bid not exceed his wealth. Let $v^* \leq \overline{v}$ denote the infimum of the valuations for which the equilibrium bid is w. If all buyers with valuations $v < \overline{v}$ bid strictly below w, then $v^* = \overline{v}$, and the proof is complete. Now suppose that $v^* < \overline{v}$. Standard incentive-compatibility arguments show that the probability of winning is weakly increasing in an individual buyer's valuation. All buyers with $v > v^*$ must therefore bid w.

It is not optimal to bid more than one's valuation, so $v^* \geq w$. If $v^* = w$, then a buyer with valuation $v^*$ would receive zero expected surplus by bidding w. However, since $w > \underline{v}$, that buyer could get a strictly positive expected surplus by bidding below w, because he has a strictly positive probability of winning. Buyers with valuations infinitesimally above $v^*$ would also have an incentive to bid strictly below w, since $U^*(\bullet)$ is continuous, contradicting the definition of $v^*$. It follows that $v^* > w$. Q.E.D.

We determine the bids for buyers with valuations $v \leq v^*$ through their expected surplus. (If $v^* < \overline{v}$, a buyer with valuation $v^*$ is indifferent between bidding w and bidding strictly below. We assume that such a buyer bids below w.) If $v \leq v^*$, the probability of winning is the probability that all other buyers have lower valuations. (As noted above, the probability of winning is weakly increasing in v. If bids are constant, but below w, over an interval of valuations, then individual buyers would have an incentive to raise their bids infinitesimally.) Given a reserve price $r \in [\underline{v}, w)$, (8) indicates that

$$U^*(v) = \int_r^v G(u)^{N-1} du. \tag{9}$$

It follows that a buyer with valuation $v \leq v^*$ bids

$$B(v) = v - \left[\int_r^v G(u)^{N-1} du\right] / [G(v)]^{N-1}. \tag{10}$$

A buyer with valuation v* is indifferent between bidding B(v*) and w. Suppose that all other buyers bid w if and only if their valuation exceeds v*. A bid of w wins with probability 1/(n+1) if there are n other buyers with valuations above v*. It also wins if all other bids are below w. Straightforward calculations show that a buyer with valuation v*, who bids w, has an expected surplus of

$$\frac{(v^* - w)[1 - G(v^*)^N]}{N[1 - G(v^*)]}.$$

Equating this expected surplus to U*(v*) from (9) implicitly defines v*.

The seller's expected revenue is

$$SER^f \equiv \int_r^{v^*} B(v) dG_{(1)}(v) + \int_{v^*}^{\overline{v}} w dG_{(1)}(v). \tag{11}$$

If r = w, then $SER^f = [1-G(w)^N]w$.

## C. *Revenue Comparison*

The first-price auction dominates if no reserve prices are employed or if optimal reserves are employed. The proof compares the (expected) price paid for all possible realizations of the first order statistic of valuations, given a common reserve price. For each such valuation, the winning bid in the first-price auction is weakly higher than the expected price in the second-price auction. We graph the equilibrium bids in figure 2 for the case without a binding reserve price.

Proposition 2. The first-price auction has a higher optimal expected revenue. It has a strictly higher expected revenue if $w \in (\underline{v}, \overline{v})$ and the optimal reserve price in the second-price auction is not equal to w.

Proof: The wealth constraint is not binding if $w \geq \overline{v}$. The results of Myerson (1981) and Riley and Samuelson (1981) imply that the optimal reserve price is the same in the two auctions, as is the seller's expected revenue. The constraint binds for all buyers if $w \leq \underline{v}$, so revenue is equal to w in both auctions.

Now suppose that $w \in (\underline{v}, \overline{v})$ and that the reserve price is $r \in [\underline{v}, w]$ in both auctions. First consider $r < w$. The proof consists of looking at three possible ranges for the highest valuation. If $v_{(1)} \leq w$, (5) and (10) show that the expected price in the second-price auction is equal to the price in the first-price auction. If $w < v_{(1)} \leq v^*$, (6) and (10) show that the expected price in the second-price auction is strictly below the actual price in the first-price auction. Finally, if $v^* < v_{(1)}$, (6) shows that the expected price is strictly below w in the second-price auction, whereas the price is equal to w in the first-price auction. Now consider $r = w$. Expected revenue is equal to $[1-G(w)^N]w$ in both auction formats.

We conclude that the seller's expected revenue is weakly higher in the first-price auction. If $w \in (\underline{v}, \overline{v})$ and the optimal reserve price in the second-price auction is not equal to w, then the first-price auction generates a strictly higher expected revenue. Q.E.D.

## 3. Heterogeneous Wealth and Valuations

In the general case, buyers can differ in both wealth and valuation. Unfortunately, the equilibrium of the first-price auction cannot be characterized completely enough to make general revenue comparisons. There are regions over which comparisons can be made, however, and the first-price auction again dominates in those regions. We sketch the arguments below.

Suppose that valuations and wealth are distributed independently, with distribution functions G(•) and F(•), respectively. In the second-price auction, it is a dominant strategy to bid $\min\{v_i, w_i\}$. In the first-price auction, for a given wealth w, there is a critical valuation v*(w) such that the equilibrium bid is below w for v > v*(w), and is equal to w for v > v*(w). The various regions are graphed in figure 3, where $\underline{v} = \underline{w}$ and $\overline{v} = \overline{w}$. (We can set density equal to zero in the appropriate regions if the limits are not equal.) In the second-price auction, buyers bid their wealth if they are above the 45° line. In the first-price auction, they bid their wealth only if they are above a wealth-valuation locus that is itself above the 45° line.

Clearly, if all buyers are below the 45° line (with ex ante probability one), then revenue equivalence holds, since all buyers are unconstrained. If all buyers are above the wealth-valuation locus, then they all bid their wealth in both auctions. The first-price auction dominates, as was shown in section 1. Now suppose that all buyers are between the wealth-valuation locus and the 45° line, with probability one. In the first-price auction, the winning bid is the expectation of the second-highest valuation, given the highest valuation. In the second-price auction, the actual price paid is the second-highest wealth. Since valuations exceed wealth for all buyers, the first-price auction again dominates.

The probability of winning differs across auctions for a buyer with a given valuation-wealth pair, even with the same reserve price. Therefore, rankings for general distributions are not possible using the techniques of sections 1 and 2. Calculation of the wealth-constraint locus requires solving differential equations. Even the simplest case (N = 2 and uniform distributions) requires numerical solutions.

## 4. Concluding Remarks

In this paper, we have demonstrated that, in two cases, first-price sealed-bid auctions yield higher expected revenue than second-price sealed-bid auctions when buyers face wealth constraints: 1) heterogeneous wealth, which is the limiting case for settings where variation in wealth is greater than variation in valuations, and 2) heterogeneous valuations, which is the limiting case for settings where variation in valuations is greater than variation in wealth. We should therefore see first-price sealed-bid auctions, rather than second-price sealed-bid or oral ascending auctions, in settings where wealth constraints are present, all else equal. This finding is consistent with the government's predominant use of first-price sealed-bid auctions.[8]

A natural question concerns the robustness of the results to the availability of credit. Suppose that buyers have future income against which they can borrow. The case that we have considered corresponds to a situation where buyers can borrow only at a very high interest rate. At lower rates of interest, buyers at certain wealth levels will borrow. As long as buyers at some wealth levels find it optimal not to borrow, however, the first-price auction will still dominate the second-price. Once the borrowing rate is sufficiently low that buyers at all wealth levels borrow, revenue equivalence reappears.

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8 An interesting exception to this rule occurs with the sale of timber rights. The Federal Bureau of Land Management, which operates the auctions, experimented with first-price sealed-bid auctions and found that average winning bids were higher. It reverted to using oral ascending auctions, however, because of a strong preference on the part of the industry (Mead et al. [1981]).

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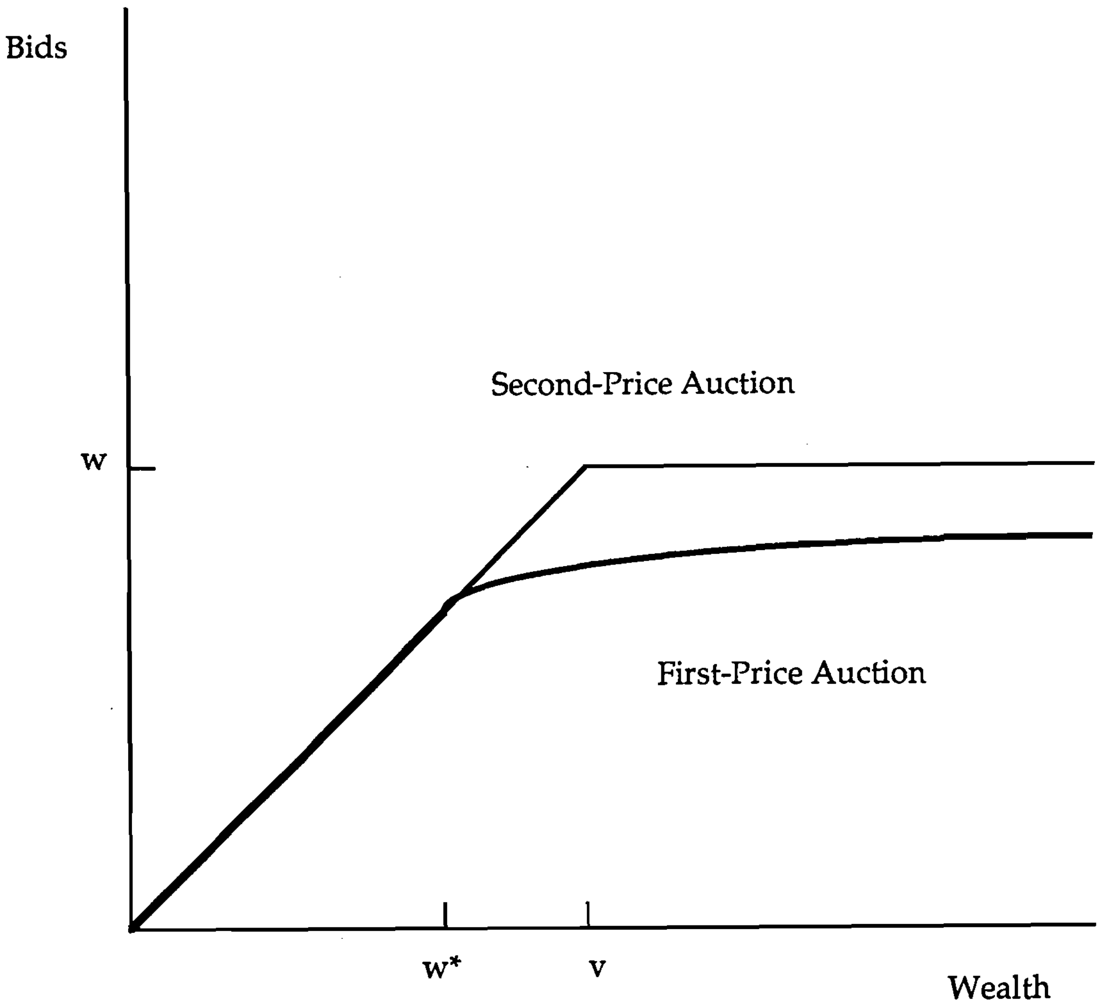
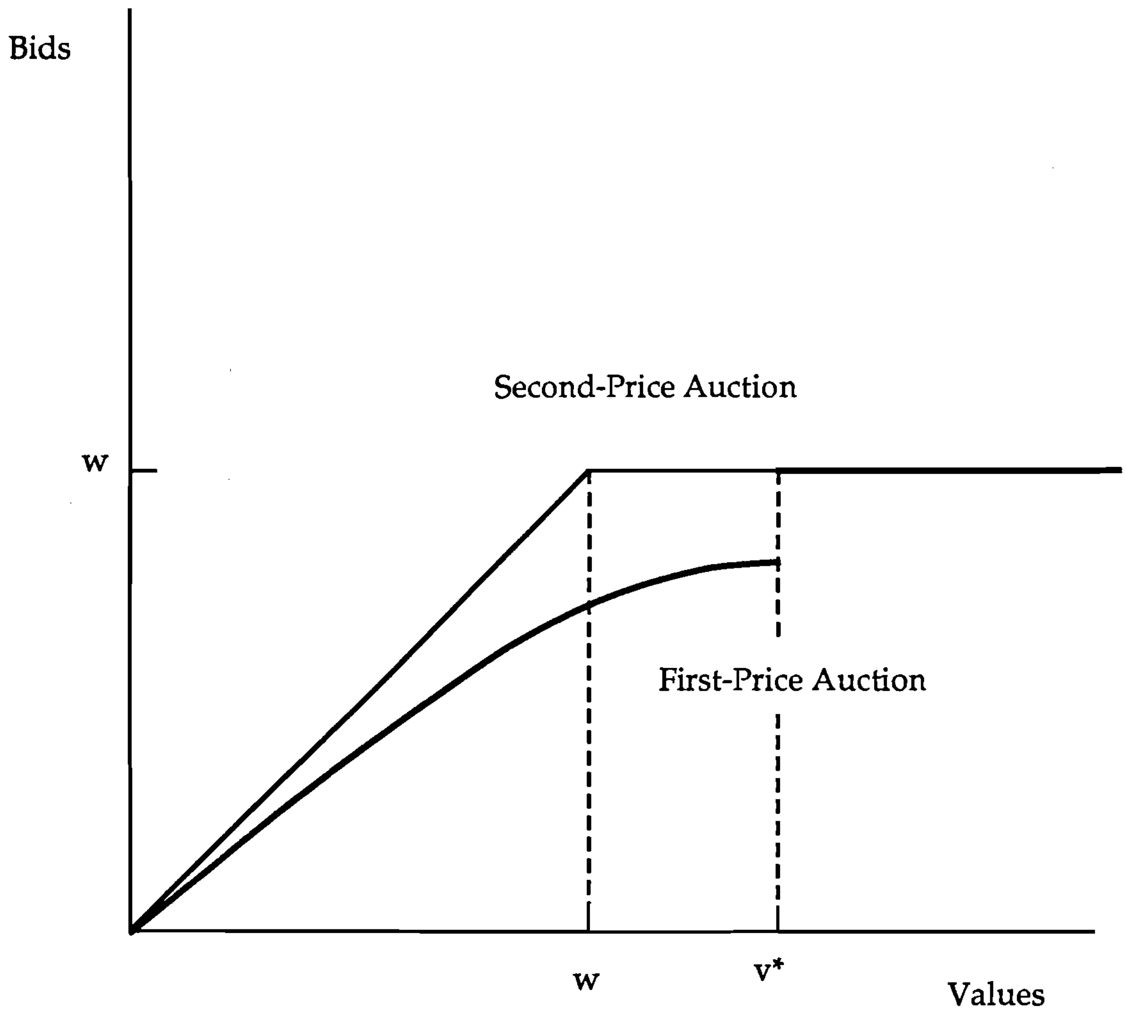
**Figure 1.**



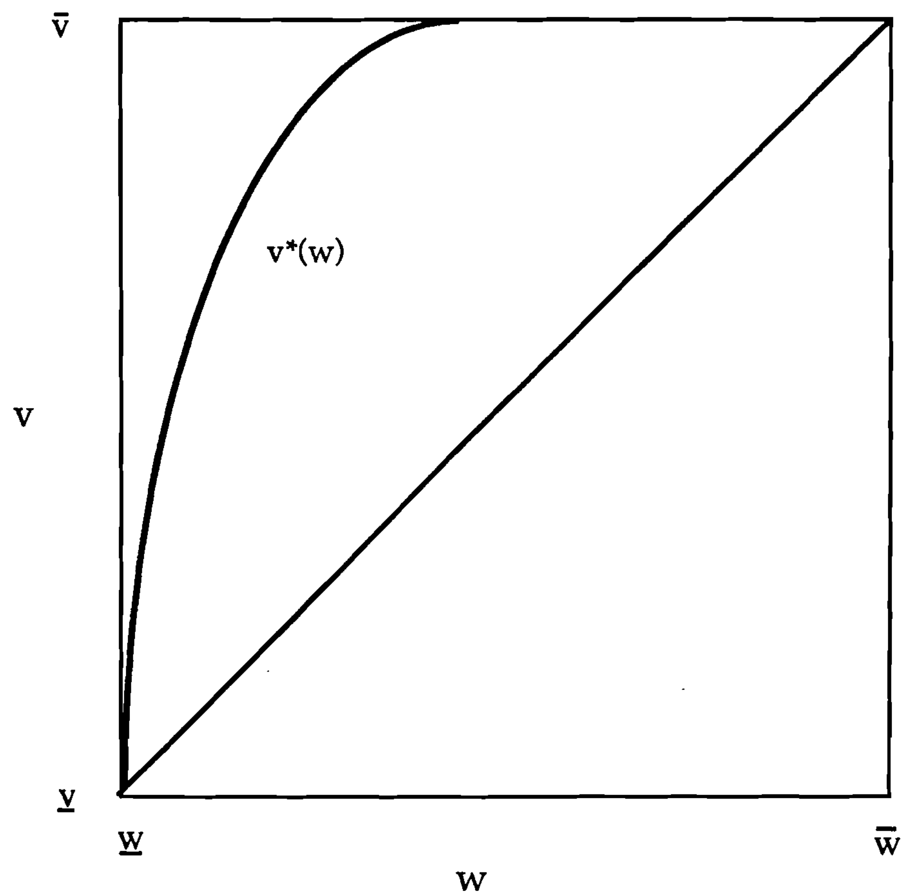

Source: Authors' calculations.

**Figure 2.**




Source: Authors' calculations.

**Figure 3.**




Source: Authors' calculations.