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# EMPIRICAL COMPARISONS OF SEASONAL ARIMA AND ARIMA COMPONENT (STRUCTURAL) TIME SERIES MODELS

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## Abstract

This paper compares seasonal ARIMA models as presented in Box and Jenkins (1970) with ARIMA component (structural) models as presented in Harvey (1989). Both models are augmented as appropriate with the same regression variables to account for calendar effects, level shifts, and additive outliers. The models are compared on a set of 40 Census Bureau monthly time series in regard to fit using AIC and related statistics. Bell and Pugh (1990) made similar comparisons of ARIMA models with the basic structural model (BSM). This paper extends their work by also considering ARIMA component models with trigonometric seasonal components. For the 40 time series considered, AIC and the other model comparison statistics express a strong overall preference for the ARIMA models over the ARIMA component models.

Key Words: time series, seasonality, AIC, seasonal adjustment.

## 1. Introduction

Two popular models for seasonal time series are multiplicative seasonal ARIMA (autoregressive–integrated–moving average) models (Box and Jenkins 1970) and ARIMA component (structural) models (Harvey 1989). Despite the rising popularity of ARIMA component models in the time series literature of recent years, empirical studies comparing these models with seasonal ARIMA models have been relatively rare. One exception is Bell and Pugh (1990), hereafter BP, which compared seasonal ARIMA models and the basic structural model (BSM) of Harvey and Todd (1983) on a set of 45 time series. The present paper extends the work of BP to include empirical comparisons of seasonal ARIMA models with ARIMA component models that use the trigonometric seasonal models of Harvey (1989).

Section 2 briefly reviews some relevant literature. The specific models to be compared are then presented in section 3. Section 4 discusses the data used and section 5 presents the empirical comparisons of model fit. Section 6 discusses the results.

## 2. Previous Work

BP compared seasonal ARIMA models with the BSM on a set of 44 monthly time series from the Census Bureau and one from the Bureau of Labor Statistics. AIC (Akaike 1973) was used to compare model fit, and showed strong differences in favor of the ARIMA models. Essentially similar results were obtained when the individually selected ARIMA models were replaced by the single "airline model" of Box and Jenkins (1970). For a few time series BP also examined use of ARIMA models versus the BSM for signal extraction in seasonal adjustment and in repeated survey estimation. For the few series considered the signal extraction point estimates were very similar under the two alternative models, but signal extraction variances differed.

Findley (1990) followed up BP's study on 40 of their time series, using a new graphical diagnostic and a robust test statistic to compare the ARIMA models and the BSM. These approaches to model comparison make very mild assumptions – e.g., the "correct" model is not assumed to be included in either model class being compared, and invertibility is not assumed for the time series models. For this reason, it is not unusual for these procedures to be inconclusive, which was the case for about half the 40 series. For the remaining series, however, the procedures expressed a general preference for the ARIMA models, confirming BP's results.

BP give a few additional references that make some comparisons (either theoretical or empirical for a small set of time series) between ARIMA and ARIMA component models. Some additional comparisons are given by Harvey (1989) and by Garcia–Ferrer and del Hoyo (1992). Also, a recent paper by Bruce and Jurke (1992) applies non–Gaussian versions of the ARIMA component models (the BSM and trigonometric seasonal models) to 29 Census Bureau time series fit by the MING (mixture–based non–Gaussian) program. Bruce and Jurke provide AIC comparisons among the various component models that will be mentioned in section 5. They also fit ARIMA models to the series, but these were used only for forecast extension in seasonal adjustment by the X–11 method, as the main focus of their paper was to compare these seasonal adjustments with those from the MING program.

## 3. ARIMA and ARIMA Component Models

Let $Y_t$ for $t=1,\ldots,n$ be observations on a time series, which in this paper will always be the logarithms of an original time series. Let

$$Y_t = \underset{\sim}{x}_t' \underset{\sim}{\beta} + Z_t \tag{3.1}$$

where $\underset{\sim}{x}_t' \underset{\sim}{\beta}$ is a linear regression mean function and $Z_t$ is the (zero mean) stochastic part of

$Y_t$. The regression variables used here will be to account for, as appropriate, trading–day and Easter holiday variation (Bell and Hillmer 1983), as well as any detected additive outliers (see Chang, Tiao, and Chen 1988) and level shifts (see Bell 1983). For each time series considered here the same set of regression variables $\underset{\sim}{x}_t$ will be used for all the different models for $Z_t$ to be compared.

The ARIMA models to be used for $Z_t$ can all be written in the form

$$\phi(B)(1 - B)(1 - B^{12})Z_t = \theta(B)(1 - \theta_{12}B^{12})a_t \tag{3.2}$$

where B is the backshift operator ($BZ_t = Z_{t-1}$), $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^P$ and $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ are nonseasonal AR and MA operators, and $a_t$ is white noise (i.i.d. $N(0, \sigma_a^2)$). Using the same seasonal part of the model (the $1 - B^{12}$ and $1 - \theta_{12}B^{12}$) for all series may seem restrictive, but, in fact, other seasonal forms are rarely chosen for the type of series used here. The exception that sometimes occurs is the use of fixed seasonal effects (seasonal dummy regression variables in $\underset{\sim}{x}_t$), but this is equivalent to the model (3.2) with $\theta_{12} = 1$ (Bell 1987). Notice also that the differencing in (3.2), $(1 - B)(1 - B^{12})$, will be the same for all series. Again this can be questioned, but it is the most common choice of differencing employed in practice, and $(1 - B)(1 - B^{12})$ is the differencing operator implied by use of the ARIMA component models to be considered for $Z_t$. Using the same differencing operator for all models being compared on a given series is necessary for the model comparison statistics used here (AIC, bias–corrected AIC, Hannan–Quinn, and BIC) to be valid.

The particular seasonal ARIMA model known as the "airline model" (Box and Jenkins 1970, sec. 9.2) is

$$(1 - B)(1 - B^{12})\, Z_t \;=\; (1 - \theta_1 B)(1 - \theta_{12}B^{12})\, a_t. \tag{3.3}$$

This model is very commonly used and so will be considered here as a default choice of an ARIMA model, i.e., the ARIMA model that might be used if only one ARIMA model were to be used for all the time series being modelled.

The ARIMA component models considered are discussed by Harvey (1989). They begin with a seasonal + trend + irregular decomposition for $Z_t$:

$$Z_t = S_t + T_t + I_t \, . \tag{3.4}$$

The models for the trend and irregular components are always the following:

$$(1 - B)^2 T_t = (1 - \eta B) b_t \qquad b_t \sim \text{i.i.d. } N(0, \sigma_b^2) \tag{3.5}$$

$$I_t \sim \text{i.i.d. } N(0, \sigma_I^2) \, . \tag{3.6}$$

Harvey (1989) actually uses an alternative parameterization to (3.5) that is equivalent except that it implies $\eta \geq 0$. For the series modelled here $\eta$ is always estimated to be positive (often equal or close to 1), so the restriction is always satisfied. If $\eta$ were ever estimated to be negative for a series, this might lead one to question the trend model as formulated by Harvey (1989) for that series.

The three different component models to be used here differ only in terms of their models for the seasonal component $S_t$. These models for $S_t$, and the associated models for $Z_t$ obtained by combining them with (3.4) – (3.6), will be referred to as the BSM (after Harvey and Todd 1983), and the TRIG–1 and TRIG–6 models (a name given by Bruce and Jurke (1992) to the non–Gaussian versions). The three models for $S_t$ can be written in ARIMA component form as follows:

BSM: $(1+B+\dots+B^{11})S_t = \varepsilon_t \qquad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma_\varepsilon^2)$ (3.7)

TRIG–6: $S_t = \sum_{j=1}^{6} S_{jt}, \quad \delta_j(B)S_{jt} = (1-\alpha_j B)\,\varepsilon_{jt}, \quad \varepsilon_{jt} \sim \text{i.i.d. } N(0, \sigma_j^2)$ (3.8)

TRIG–1: TRIG–6 with the restriction $v_1 = \cdots = v_6$ (discussed below). (3.9)

Details of the TRIG–6 and TRIG–1 models are given in Table 1. The "differencing" operators $\delta_j(B)$ each correspond to a factor of $1 + B + \cdots + B^{11}$ at a different seasonal frequency $\lambda_j = 2\pi j/12$ ($j = 1, \dots, 6$). Thus, $\prod_{j=1}^{6} \delta_j(B) = 1 + B + \cdots + B^{11}$. Notice also that $1 - B^{12} = (1 - B)(1 + B + \cdots + B^{11})$, so that the differencing operator for $Z_t$ implied by the BSM, TRIG–1, and TRIG–6 models is, from (3.5) and (3.7) – (3.9), $(1 - B)^2(1 + B + \cdots + B^{11}) = (1 - B)(1 - B^{12})$. In (3.9) the $v_j$ denote the innovation variances in Harvey's formulation of the model for the $S_{jt}$. These determine the innovation variances $\sigma_j^2 = \mathrm{Var}(\varepsilon_{jt})$ in the ARIMA representation (3.8) through the relations $2v_j = (1 + \alpha_j^2)\sigma_j^2$. From the $\alpha_j$ values in Table 1.a, the TRIG–1 restriction $v_1 = \cdots = v_6$ restricts the $\sigma_j^2$ as follows: $\sigma_j^2 = 1.5\sigma_6^2$ for $j = 1, 5$; $\sigma_j^2 = (1 + \sqrt{3}/2)\sigma_6^2 \approx 1.866\sigma_6^2$ for $j = 2, 4$; and $\sigma_3^2 = 2\sigma_6^2$.

Harvey's (1989, pp. 42–43) formulation of the models for the TRIG–6 seasonal components $S_{jt}$ uses a bivariate model for $[S_{jt}, S_{jt}^*]'$, where $S_{jt}^*$, "appears as a matter of construction," and so is not actually needed in the $S_t$ model. Starting from this bivariate model, it is then straightforward to derive the ARIMA representations of the univariate models for the $S_{jt}$ as given in Table 1.a, as well as the variance relation $2v_j = (1 + \alpha_j^2)\sigma_j^2$. The ARIMA representation for the TRIG–1 model for $S_t$ is obtained by applying $U(B) = 1 + B + \cdots + B^{11}$ to the TRIG–6 equation (3.8), giving

$U(B)S_t = \sum_1^6 [U(B)/\delta_j(B)](1 - \alpha_j B)\, \varepsilon_{jt}$. The right hand side of this equation is the sum of 6 independent MA(10) processes, which is itself then an MA(10) process. Using the TRIG–1 restriction on the $\sigma_j^2$, the autocovariances through lag 10 of $U(B)S_t$ can be obtained up to a constant of proportionality. The resulting autocovariance generating function for $U(B)S_t$ can then be factored to give the MA(10) representation given in Table 1.b. The unknown constant of proportionality is absorbed into the innovation variance $\sigma_c^2$, which is the one seasonal parameter to be estimated in the TRIG–1 model.

Hannan (1970, p. 174) suggests the following model for a stochastic seasonal component:

$$\begin{aligned} S_t &= \sum_{j=1}^{6} [\gamma_{jt}\cos(\lambda_j t) + \zeta_{jt}\sin(\lambda_j t)] \\ (1 - B)\, \gamma_{jt} &= \xi_{jt} \qquad \xi_{jt} \text{ - i.i.d. } N(0, v_j) \qquad j = 1, \dots, 6 \\ (1 - B)\, \zeta_{jt} &= \xi_{jt}^* \qquad \xi_{jt}^* \text{ - i.i.d. } N(0, v_j) \qquad j = 1, \dots, 5 \end{aligned} \tag{3.10}$$

where the $\xi_{jt}$ and $\xi_{kt}^*$ series are independent of each other for all j, k = 1, ..., 6. Notice that $\sin(\lambda_6 t) = 0$ for all t, so this term drops out of $S_t$. Using standard trigonometric identities, it can be shown (after tedious manipulation) that the models for $\gamma_{jt}\cos(\lambda_j t) + \zeta_{jt}\sin(\lambda_j t)$ implied by (3.10) are equivalent to those for the $S_{jt}$ in (3.8), up to multiplication of the vector of innovations $[\xi_{jt}, \xi_{jt}^*]'$ by an orthogonal 2×2 matrix. Therefore, Hannan's seasonal model and Harvey's TRIG–6 seasonal model are the same in the Gaussian case considered here, and they are also the same when the innovations follow a mixture–of–normals distribution as in Bruce and Jurke (1992).

Haywood and Wilson (1992) suggest a generalization of Harvey's formulation of the TRIG–6 model. Without going into details, their generalization still implies that $S_{1t}, \dots, S_{5t}$ follow the nonstationary ARMA(2,1) models given in (3.8). The effect of the generalization is to turn $\alpha_1, \dots, \alpha_5$ into parameters to be estimated. (This includes $\alpha_3$, which is constrained to 0 in Harvey's formulation. Also, note that the model for $S_{6t}$ is unchanged.) For the 40 time series used in this paper, however, this more general model was found to fit uniformly worse (as measured by AIC) than the standard TRIG–6 model. For this reason, the Haywood–Wilson model will not be considered further here.

The TRIG–6 model can also be written in an alternative "canonical" form, along the lines that BP developed a canonical version of the BSM. Details are available from the author on request. This canonical form provides an equivalent model for the observed series $Y_t$ (at least for the Gaussian case), while transferring as much white noise as possible from the individual seasonal components $S_{jt}$ to the irregular $I_t$. This is in the spirit of the approach to seasonal decomposition developed by Burman (1980) and Hillmer and Tiao (1982) for ARIMA models. A canonical form of the TRIG–1 model can also be developed from the TRIG–6 model under the restriction $v_1 = \cdots = v_6$. BP found the canonical BSM to be nearly identical to the original BSM, but such is not the case for the TRIG–6 and TRIG–1 models. Thus, use of the canonical form has potential consequences for seasonal adjustment, although for the one example I have examined so far, seasonal adjustment results from the original and canonical TRIG–6 models have been essentially the same.

BP review some alternative ARIMA component models that have been proposed. One such model involves augmenting the BSM with a fourth component to account for possible cyclical behavior (Harvey 1985). BP encountered severe numerical problems in trying to fit such models, and so did not report results for them. Since that writing I have been successful in fitting these models with the fourth component following an AR(2) model. The augmenting of the BSM with this fourth component did nothing to improve its performance relative to the ARIMA models for the series considered in BP. In fact, for about 3/4 of the series the innovation variance of the fourth component was estimated to be essentially zero, indicating that the fourth component was not really present anyway. (Harvey (1985) also found little evidence of cyclical components in the 5 U.S. postwar economic time series he analyzed, though he found more evidence of cycles in the prewar data.) Given these results, models with a fourth cycle component will not be considered in this paper.

## 4. Data

The ARIMA and ARIMA component models will be compared using a set of 40 monthly time series that are broadly representative of the time series that are seasonally adjusted by the Census Bureau. 27 of these series were also used in BP, although the time frames of the actual data used here are different. The time frames used here are those of Kramer, Bell, and Koreisha (1993), and yield $n = 200$ observations for each series.

Space constraints prevent my giving further information on the data here. Details on the data series are available on request.

## 5. Empirical Results

The ARIMA and the ARIMA component models (BSM, TRIG–1, and TRIG–6) were fit to the 40 time series by maximum likelihood using the REGCMPNT computer program developed by the time series staff of the Census Bureau. The likelihood function is defined as the joint density of the differenced data: $(1 - B)(1 - B^{12})Y_t$ for $t = 14, \ldots, n$. REGCMPNT evaluates the likelihood using the Kalman filter initialized as described in Bell and Hillmer (1991). The fit of the various models will be compared here in terms of various commonly used model selection criteria. These are AIC (Akaike 1973); $AIC_c$, a bias corrected version of AIC (Hurvich and Tsai 1989); HQ (Hannan and Quinn 1979); and BIC (Schwarz 1978). If $\hat{L}$ denotes the value of the log–likelihood evaluated at the maximum likelihood parameter estimates for a given model, and m is the number of parameters in the model, then the criteria are defined as follows, using $n - 13$ $(= 187)$ as the number of observations of the differenced data.

$$\mathrm{AIC} = -2\hat{\mathrm{L}} + 2\mathrm{m} \tag{5.1}$$

$$\mathrm{AIC}_c = -2\hat{\mathrm{L}} + 2\mathrm{m}/[1 - (\mathrm{m}+1)/(\mathrm{n}-13)] \tag{5.2}$$

$$\mathrm{HQ} = -2\hat{\mathrm{L}} + 2\mathrm{m}\,[\log(\log(\mathrm{n}-13))] \tag{5.3}$$

$$\mathrm{BIC} = -2\hat{\mathrm{L}} + [\log(\mathrm{n}-13)]\,\mathrm{m}\,. \tag{5.4}$$

When comparing models for a given time series using any of the criteria (5.1) – (5.4), the model giving the smallest value of the criterion being used is the preferred model.

The criteria (5.1) – (5.4) differ in regard to the "penalty terms" added to $-2\hat{L}$. Since the BSM and TRIG–1 models have the same number of parameters (4 plus the number of regression parameters), and the ARIMA models chosen generally have a similar number of parameters (e.g., the airline model contributes 3 parameters in addition to the regression parameters), the use of different criteria matters most in comparisons involving the TRIG–6 model (which has 9 parameters in addition to the regression parameters). Therefore, BP's comparisons of ARIMA models with the BSM would have come out essentially the same if one of the other three model selection criteria had been used.

The particular ARIMA models used here are those I selected for a study comparing alternative approaches to ARIMA model selection (Kramer, Bell, and Koreisha 1993). These selections were based on examination of sample autocorrelations and partial autocorrelations of $(1 - B)(1 - B^{12})[Y_t - \sum_i{}' \tilde{\beta}_i x_{it}]$, where $\sum_i{}' \tilde{\beta}_i x_{it}$ includes trading–day and Easter holiday regression variables when appropriate, but not outlier terms, which are not available at the model identification stage. The $\tilde{\beta}_i$ are obtained by ordinary least squares regression of $(1 - B)(1 - B^{12})Y_t$ on the $(1 - B)(1 - B^{12})x_{it}$. The other judgemental

ARIMA model selections reported in Kramer, Bell, and Koreisha (1993) produced, on average, comparable AIC values. Therefore, the empirical results presented here might be regarded as representative in regard to the performance of ARIMA models.

Table 2.a presents a summary of the AIC differences over the 40 time series for each pair of models considered. The first 3 lines of the table show the results comparing the ARIMA and ARIMA component models. Positive values favor the ARIMA models; negative values favor the ARIMA component models. AIC expresses a strong preference overall for the ARIMA models: only a few of the AIC differences in these comparisons are negative, and those that are negative are almost all small to moderate in magnitude ($\leq 8$), while most of the AIC differences are positive, and many of these positive differences are large ($> 8$).

The middle 3 lines of Table 2.a present comparisons of the ARIMA component models with each other. The median AIC differences are small for the two comparisons involving the BSM, and there is a median AIC difference of 3.2 slightly favoring the TRIG–1 over the TRIG–6 model. However, the presence of negative values large in magnitude in all of these comparisons shows that sometimes the BSM fits quite poorly in comparison with the TRIG–1 and TRIG–6 models, as can the TRIG–1 in comparison to the TRIG–6. Thus, the TRIG–1 and TRIG–6 models appear definitely favored over the BSM, while the choice between the TRIG–1 and TRIG–6 models is less clear. In general, having the 6 different seasonal variance parameters of the TRIG–6 model sometimes is unnecessary, but occasionally this achieves a major improvement in model fit over the TRIG–1 model (as measured by AIC).

Bruce and Jurke (1992) report AIC's for the non–Gaussian versions of the BSM, TRIG–1, and TRIG–6 models as fitted by the MING program to 29 of the 40 series considered here. Their results would not be expected to be the same as those reported here because of the non–Gaussian fitting, and because Bruce and Jurke used different time frames of the series. Still, the AIC relationships for their non–Gaussian BSM, TRIG–1, and TRIG–6 models are similar to those found here for most of the series they consider, although there are some notable exceptions.

The final four lines in Table 2.a compare the fit of the four models already considered (ARIMA, BSM, TRIG–1, and TRIG–6) with the airline model (3.3). As in BP, these comparisons are presented to check the possibility of selection bias favoring the ARIMA model. (The particular ARIMA models chosen were not chosen to minimize AIC, but the usual identification procedures could have a similar effect.) The first of these lines compares the airline model with the selected ARIMA models. The median AIC difference is 0; in fact, 14 of the 40 selected ARIMA models were the airline model. Aside from this, the results are as would be expected: almost all the nonzero AIC differences are negative, indicating that selection of a particular ARIMA model for each series improved model fit as measured by AIC. There are a few exceptions where the airline model comes out slightly favored, but what is more important is the occurrence of some negative AIC differences that are large in magnitude, indicating that for a few series the airline model provided poor fits. Two of these series are responsible for the large–in–magnitude negative AIC differences in the final three lines of Table 2.a, indicating a strong preference for the ARIMA component models over the airline model for these two series. Apart from these two series, the results are not very different from the results of the ARIMA and ARIMA component model comparisons. On balance, AIC expresses a strong preference for the airline model over the ARIMA component models.

As expected, the effects of using one of the model comparison statistics other than AIC is very slight for comparisons that do not involve the TRIG–6 model. The effect of using the bias corrected AIC ($AIC_C$) is to make the TRIG–6 model look slightly worse relative to the other models. The effect of using HQ or BIC, however, is to make the TRIG–6 model look dramatically worse relative to the other models. This is seen by comparing the results in Tables 2.a and 2.b that summarize AIC and HQ differences involving the TRIG–6 model. The effects in going to the BIC are even more dramatic. This effect is, of course, due to the penalty terms. The increases in the HQ and BIC penalty terms for the TRIG–6 model versus the BSM or TRIG–1 models are:

$$\text{HQ}: \text{ penalty increase} = 2\{\log[\log(187)]\}(9-4) = 16.5$$

$$\text{BIC}: \text{ penalty increase} = [\log(187)](9-4) = 26.2\ .$$

## 6. Discussion

The results presented here provide further evidence to that in BP and Findley (1990) of the superiority of ARIMA models to the BSM. Actually, for the 27 of the 40 series used here that were used in BP, albeit over different time frames, the ARIMA–BSM comparisons reported here are best interpreted as confirming the findings of BP with a more controlled study of almost the same data. What is both new and important in this paper is that the results also show marked superiority of the ARIMA models (and the airline model) over the TRIG–1 and TRIG–6 models for the series considered. Thus, in general, it appears that all the ARIMA component (structural) models tend to provide a poor fit to the sort of time series seasonally adjusted by the Census Bureau.

BP noted that the BSM is much more difficult to fit than ARIMA models. In performing this study I have further found the TRIG–6 model to be even more difficult to fit than the BSM. The TRIG–1 model is also more difficult to fit than the BSM, though not so difficult to fit as the TRIG–6 model. One problem that arises in fitting all the ARIMA component models occurs when the maximum likelihood estimate (MLE) of $\eta$ in (3.5) is 1, something that occurs fairly frequently. Another problem that surfaces with the TRIG–6 model occurs when the MLE of one or more of the 6 seasonal variance parameters is essentially 0. This is also not an infrequent occurrence. The need to estimate parameters at such boundary values poses rather difficult numerical problems for nonlinear optimization algorithms. The same sort of difficulty arises with the ARIMA model (3.2) when $\hat{\theta}_{12} = 1$. This occurs occasionally, but much less frequently than the analogous problems for the ARIMA component models. It also appears that, for all the ARIMA component models, the likelihood is rather flat in certain directions in the parameter space. Given this, I find the oft–claimed advantages of simplicity and interpretability for ARIMA component models difficult to accept.

## REFERENCES

Akaike, H. (1973) "Information Theory and an Extension of the Likelihood Principle," in the 2nd International Symposium on Information Theory, eds. B. N. Petrov and F. Czaki, Budapest: Akademia Kiado, 267–287.

Bell, W. R. (1983), "A Computer Program for Detecting Outliers in Time Series," Proceedings of the American Statistical Association, Business and Economic Statistics Section, 634–639.

_______ (1987) "A Note on Overdifferencing and the Equivalence of Seasonal Time Series Models With Monthly Means and Models With (0,1,1)12 Seasonal Parts When $\Theta = 1$," Journal of Business and Economic Statistics, 5, 383–387.

Bell, W. R. and Hillmer, S. C. (1983) "Modeling Time Series with Calendar Variation," Journal of the American Statistical Association, 78, 526–534.

_______ (1991) "Initializing the Kalman Filter for Nonstationary Time Series Models," Journal of Time Series Analysis, 12, 283–300.

Bell, W. R. and M. G. Pugh (1990) "Alternative Approaches to the Analysis of Time Series Components," Analysis of Data in Time, Proceedings of the 1989 International Sypmposium, A.C. Singh and P. Whitridge eds., Statistics Canada, 105–116.

Box, G.E.P. and Jenkins, G. M. (1970) Time Series Analysis: Forecasting and Control, San Francisco: Holden Day.

Bruce, A. G. and Jurke, S. R. (1992) "Non–Gaussian Seasonal Adjustment: X–12 ARIMA Versus Robust Structural Models," Research Report No. 92/14, Statistical Research Division, Bureau of the Census, Washington, D.C.

Burman, J. P. (1980) "Seasonal Adjustment by Signal Extraction," Journal of the Royal Statistical Society, A, 143, 321–337.

Chang, I., Tiao, G. C., and Chen, C. (1988) "Estimation of Time Series Parameters in the Presence of Outliers," Technometrics, 30, 193–204.

Findley, D. F. (1990) "Making Difficult Model Comparisons," Research Report No. 90/11, Statistical Research Division, Bureau of the Census, Washington, D.C.

Garcia–Ferrer, A. and del Hoyo, J. (1992) "On Trend Extraction Models: Interpretation, Empirical Evidence, and Forecasting Performance" (with discussion), Journal of Forecasting, 11, 645–674.

Hannan, E. J. (1970) Multiple Time Series, New York: John Wiley.

Hannan, E. J. and Quinn, B. G. (1979) "The Determination of the Order of an Autoregression," Journal of the Royal Statistical Society, B, 41, 190–195.

Harvey, A. C. (1985), "Trends and Cycles in Macroeconomic Time Series," Journal of Business and Economic Statistics, 3, 216–227.

______ (1989) Forecasting, Structural Time Series Models and the Kalman Filter, Cambridge, U. K.: Cambridge University Press.

Harvey, A. C. and Todd, P. H. J. (1983), "Forecasting Economic Time Series With Structural and Box–Jenkins Models: A Case Study," (with discussion), Journal of Business and Economic Statistics, 1, 299–315.

Haywood, J. and Wilson, G. Tunnicliffe (1992) "On the Choice of State Space Representation for Trigonometric Components of Seasonal Time Series," unpublished manuscript, Mathematics department, Lancaster University, U.K.

Hillmer, S. C., and Tiao, G. C. (1982) "An ARIMA–Model–Based Approach to Seasonal Adjustment," Journal of the American Statistical Association, 77, 63–70.

Hurvich, C. M. and Tsai, C. (1989) "Regression and Time Series Model Selection in Small Samples," Biometrika, 76, 297–307.

Kramer, M., Bell, W. R., and Koreisha, S. (1993) "A Comparison of Automatic and Judgemental ARIMA Model Selection," paper presented at the 1993 meeting of the American Statistical Association, San Francisco, CA.

Schwarz, G. (1978) "Estimating the Dimension of a Model," Annals of Statistics, 6, 461–464.

**Table 1**

**a. ARIMA Representations for the Individual TRIG–6 Seasonal Components**

$$\text{TRIG–6 Seasonal Model: } S_t = \sum_{1}^{6} S_{jt}$$

$$[1 - 2\cos(\lambda_j)B + B^2]\, S_{jt} = (1 - \alpha_j B)\, \varepsilon_{jt} \qquad j = 1, \ldots, 5 \quad (\lambda_j = 2\pi j/12)$$

$$(1 + B)\, S_{6t} = \varepsilon_{6t} \qquad \varepsilon_{jt} \sim \text{i.i.d. } N(0, \sigma_j^2) \qquad j = 1, \ldots, 6$$

| j | $\lambda_j$ | "Differencing" Operator | MA Operator |
|---|---|---|---|
| 1 | $\pi/6$ | $1 - \sqrt{3}B + B^2$ | $1 - (\sqrt{3}/3)B$ |
| 2 | $\pi/3$ | $1 - B + B^2$ | $1 - (2-\sqrt{3})B$ |
| 3 | $\pi/2$ | $1 + B^2$ | 1 |
| 4 | $2\pi/3$ | $1 + B + B^2$ | $1 + (2-\sqrt{3})B$ |
| 5 | $5\pi/6$ | $1 + \sqrt{3}B + B^2$ | $1 + (\sqrt{3}/3)B$ |
| 6 | $\pi$ | $1 + B$ | 1 |

**b. ARIMA Representation of the TRIG–1 Seasonal Component**

$$(1 + B + \cdots + B^{11})\, S_t = (1 - \alpha_1 B - \cdots - \alpha_{10} B^{10})\, c_t \qquad c_t \sim \text{i.i.d. } N(0, \sigma_c^2)$$

$\alpha_1 = -.737378$ $\quad \alpha_2 = -.627978$ $\quad \alpha_3 = -.430368$ $\quad \alpha_4 = -.360770$

$\alpha_5 = -.219736$ $\quad \alpha_6 = -.180929$ $\quad \alpha_7 = -.088488$ $\quad \alpha_8 = -.071423$

$\alpha_9 = -.020306$ $\quad \alpha_{10} = -.016083$

## Table 2

### a. AIC Comparisons

| | (−∞,−30) | [−30,−16) | [−16,−8) | [−8,−2) | [−2,2] | (2,8] | (8,16] | (16,30] | (30,∞) |
|---|---|---|---|---|---|---|---|---|---|
| BSM–ARIMA | 0 | 0 | 0 | 1 | 9 | 12 | 7 | 9 | 2 |
| TRIG1–ARIMA | 0 | 0 | 0 | 1 | 12 | 14 | 8 | 4 | 1 |
| TRIG6–ARIMA | 0 | 0 | 2 | 1 | 2 | 13 | 20 | 2 | 0 |
| TRIG1–BSM | 0 | 3 | 5 | 7 | 23 | 2 | 0 | 0 | 0 |
| TRIG6–BSM | 3 | 2 | 5 | 8 | 4 | 12 | 6 | 0 | 0 |
| TRIG6–TRIG1 | 0 | 2 | 4 | 6 | 6 | 19 | 3 | 0 | 0 |
| ARIMA–Airline | 1 | 3 | 2 | 7 | 24* | 3 | 0 | 0 | 0 |
| BSM–Airline | 1 | 1 | 1 | 0 | 10 | 11 | 8 | 6 | 2 |
| TRIG1–Airline | 1 | 1 | 1 | 0 | 14 | 15 | 5 | 2 | 1 |
| TRIG6–Airline | 1 | 2 | 1 | 3 | 2 | 13 | 17 | 1 | 0 |

### b. Hannan–Quinn (HQ) Comparisons

| | (−∞,−30) | [−30,−16) | [−16,−8) | [−8,−2) | [−2,2] | (2,8] | (8,16] | (16,30] | (30,∞) |
|---|---|---|---|---|---|---|---|---|---|
| BSM–ARIMA | 0 | 0 | 0 | 2 | 4 | 15 | 8 | 9 | 2 |
| TRIG1–ARIMA | 0 | 0 | 0 | 1 | 5 | 19 | 10 | 4 | 1 |
| TRIG6–ARIMA | 0 | 0 | 0 | 0 | 3 | 2 | 15 | 19 | 1 |
| TRIG1–BSM | 0 | 3 | 5 | 7 | 23 | 2 | 0 | 0 | 0 |
| TRIG6–BSM | 1 | 4 | 1 | 4 | 5 | 7 | 18 | 0 | 0 |
| TRIG6–TRIG1 | 0 | 1 | 1 | 3 | 4 | 8 | 23 | 0 | 0 |
| ARIMA–Airline | 1 | 2 | 1 | 11 | 21* | 4 | 0 | 0 | 0 |
| BSM–Airline | 1 | 1 | 0 | 1 | 1 | 19 | 8 | 7 | 2 |
| TRIG1–Airline | 1 | 1 | 0 | 1 | 1 | 28 | 4 | 3 | 1 |
| TRIG6–Airline | 0 | 1 | 1 | 1 | 2 | 2 | 15 | 17 | 1 |

* 14 of the AIC and HQ differences are exactly 0 because 14 of the selected ARIMA models are the airline model.