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by Ben Craig and Christopher J. Waller

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Ben Craig is at the Federal Reserve Bank of Cleveland. Christopher J. Waller is at the University of Kentucky. The authors thank Gabriel Camera for his comments on their work.

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Ben Craig
Research Department
Federal Reserve Bank of Cleveland,
OH 44101-1387
Ben.R.Craig@CLEV.frb.org

Christopher J. Waller
Department of Economics
University of Kentucky
Lexington, KY 40506-0034
cjwall@pop.uky.edu

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Abstract

We analyze a dual currency search model in which agents are allowed to hold multiple units of both currencies. Hence, agents hold portfolios of currency. We study equilibria in which the two currencies are identical and equilibria in which the two currencies differ according to the magnitude of the 'inflation tax' risk associated with each currency. The inflation tax is modeled by having government agents randomly confiscate the two currencies at different rates. We are able to obtain analytical results in a very special case but in general we must rely on numerical methods to solve for the steady-state distributions of currency portfolios, prices and value functions. We find that when one of the currencies has the right amount of 'risk', equilibria exist in which the safe currency trades for multiple units of the risky currency (pure currency exchange). As a result, the steady state has a distribution of *nominal* exchange rates. The mean and variance of the nominal exchange rate distribution is based on the fundamentals of the model such as the risk of confiscation, risk preferences, matching probabilities and relative money supplies. The mean and variance of this distribution typically change in predictable ways when the fundamentals change. While the ability to trade currencies improves average welfare, in general, the benefits of currency exchange are small.

We would like to thank Gabriel Camera for his comments on our work.

1. Introduction

The search theoretic model of money has become the dominant framework for studying monetary theory in the last few years. Until recently, a major drawback of the search theoretic framework has been the underlying assumption that agents can only hold one unit of currency at a time. This inventory restriction on money is imposed for analytical tractability. More recent work by Molico (1996), Green and Zhou (1996), Camera and Corbae (1998) and Taber and Wallace (1998) has relaxed the inventory assumption and these authors have studied monetary equilibrium when agents are allowed to hold multiple units of currency. These models have been used to study equilibrium price distributions, divisibility, and the welfare effects of changing the money stock.

All of the models listed above study economies in which only one currency circulates. In many countries around the world, two or more currencies act as a media of exchange and governments in these countries often try and restrict the use of foreign currency in order to maintain the value of the domestic currency. There are many papers in the search literature that look at dual currency issues but all of them rely on the one-unit inventory restriction on money holdings.[1] Unfortunately, the inventory constraint prevents us from addressing many interesting issues that arise in dual currency economies such as currency exchange, portfolio diversification and the equilibrium determination of *nominal* exchange rates. For example, a common situation in developing or transitional economies is that local residents begin to use a foreign currency in addition to the domestic currency to diversify against inflation risk associated with the domestic currency. The inflation risk creates incentives for local residents not only to acquire the foreign currency via sales of goods but also to engage in currency exchange in order to diversify their currency portfolios. As a result, currency exchange can be welfare improving. However, in a one unit of money model, currency exchange will never occur so it is impossible to study how changes in inflation risk affects portfolio diversification and nominal exchange rates.

In this paper we construct a one country, dual currency search model in which agents can hold multiple units of currency subject to an inventory restriction. In order to make the two currencies different, we introduce an 'inflation tax' that differs across currencies. The inflation tax is introduced as in Li (1995) by having government agents in the model randomly confiscate one of the currencies. As a result, we have a 'safe' currency that is never confiscated and a 'risky' currency that is confiscated whenever a private trader runs into a government agent.[2] By setting the inflation tax to zero for each currency, we can also study 'symmetric' equilibria in which the currencies are identical. We allow three types of trades to occur in the economy -- money for goods, money for goods and money, and money for money -- and study equilibria in which all or only a subset of these types of trades occurs in equilibrium.

We are able to obtain analytical results for a very special case of the model that mimics the equilibrium studied by Camera and Corbae (1998) in which agents only trade one unit of currency at a time. In this special case, agents can hold up to a total of 2 units of currency and the inflation tax on each currency is zero, hence they are identical. We find that that there many steady-state distributions of portfolios that can support a given set of equilibrium value functions. However, the marginal distribution of agents across portfolio size (0, 1, 2) is censored geometric as in Camera and Corbae and in Green and Zhou (1996).

When we allow agents to hold more than 2 units of currency or when we introduce the inflation tax, numerical methods are required to study the model. Our basic findings from these exercises are as follows. First, we typically are able to find equilibria when the inventory restraint is under 20 units although for large sections of the parameter space convergence to a steady state does not occur. For values over 20, our algorithm usually does not converge. Second, we are able to find equilibria for all of the trading patterns described above: money for goods, currency i for goods and some of currency j, and pure currency trades. Interestingly, we are able to find equilibria in which one of the currencies is more valuable in equilibrium even though they are identical (zero inflation tax). This reflects the 'self-fulfilling belief' nature of fiat currency equilibria -- if traders believe one currency is more valuable, then it will be more valuable in equilibrium.

Third, when we impose an inflation tax on the domestic currency, we can generate equilibria in which the safe foreign currency trades for multiple units of the risky domestic currency. Thus, we are able to obtain *endogenously* determined *nominal* exchange rates, something, which has not been done to date in the search literature.[3] The nominal exchange rate

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[1] See Craig and Waller (1999) for a brief survey of the search literature on dual currency economies.
[2] Curtis and Waller (1999) use a similar set-up to study currency restrictions that make one of the currencies 'illegal' and thus subject to confiscation.
[3] Zhou (1997) has a model in which currency exchange occurs but the nominal exchange is exogenous determined as 1 for 1. Head and Shi (1996) look at exchange rates in a search framework but under assumptions that eliminate all distribution issues.

reflects the inflation risk premium associated with domestic currency. When currency trades occur we have a pure financial 'market' -- even without a single coincidence of wants for goods, traders may still find it optimal to trade financial instruments. Due to the bilateral nature of trades in search models, within this financial market we observe an entire distribution of nominal exchange rates, not just one. Thus, one of the contributions of this paper is to use search theoretic models to examine issues in the realm of international finance. For example, simple comparative static exercises show higher inflation risk leads to a nominal depreciation of the domestic currency relative to the foreign currency, lowers its purchasing power over goods and leads to 'dollarization' of the economy.

In Section 2 we describe the economic structure of the model and the bargaining environment. Section 3 contains the definition of a stationary equilibrium that we employ. In Section 4 we present a special case of the model and discuss our analytical results. Section 5 contains a description of our numerical procedures. In Section 6 we present the results of our numerical analysis. Finally, Section 7 summarizes our findings and suggestions for future research.

2 The Search Environment

Our model is a standard random matching monetary model with divisible, non-storable goods in which goods prices are determined via bilateral bargaining. There is a continuum of agents uniformly distributed on the unit circle. Agents specialize in the production and consumption of goods. An agent located at point i on the unit circle is assumed to consume goods in the interval $i + x$ and produce goods in the interval $i - x$. For $x < 1/2$, double coincidences of wants cannot occur and thus no barter trades arise. This restriction on x greatly simplifies the model and will be maintained throughout the paper. Hence a trading equilibrium requires the existence of a medium of exchange. Given the matching process described below, x is the probability that a buyer meets a seller who produces a good in the buyer's desired consumption interval.

2.1 Preferences and Costs of Production

Agents are assumed to receive utility $u(q)$ from consumption of q units of their desired good and incur a utility loss of $c(q)$ from producing q units of their good. Both $u(q)$ and $c(q)$ are continuous, twice differentiable with $u' > 0$, $c' > 0$, $u'' \leq 0$, and $c'' \geq 0$ with at least one of the inequalities holding with strict inequality. Also assume that $u(0) = c(0) = 0$ with $u'(0) > c'(0)$ and there is a positive value of q, $\bar{q}$, such that $u(\bar{q}) = c(\bar{q})$. For the remainder of the paper we will assume that the cost function is linear and given by $c(q) = q$.

2.2 Media of Exchange and Currency Portfolios

In our economy, two fiat currencies can circulate as media of exchange. Let currency 1 denote the foreign currency (dollars) and currency 2 denote the domestic currency (rubles). One or both of the currencies are allowed to circulate in trade. Following Camera and Corbae, we assume that agents are able to hold up to N total units of currency at zero storage cost. These N units can be held in any combination of domestic and foreign currency and the support of N is given by the set $\boldsymbol{N}=\{0,1,2,\ldots,N\}$. Consequently, agents are able to hold portfolios of currencies whose support is the simplex $S^1 = \{n^i_{jt} \in \boldsymbol{N}: n^i_{1t} + n^i_{2t} \leq N\}$ where n^i_{jt} denotes the units of currency j = 1,2 held by individual i at time t. An individual's portfolio at time t is thus an ordered pair (n^i_{1t}, n^i_{2t}) on S^1.

2.3 Aggregate Money Stocks

Let $m_t(n_1,n_2)$ denote the proportion of the population holding currency portfolio (n_1,n_2) at time t. The per-capita foreign money stock is then given by

$$(1) \qquad M_1 = \sum_{n_2=0}^{N} \sum_{n_1=0}^{N-n_2} n_1 m_t(n_1, n_2)$$

while the per-capita domestic money stock is given by

$$(2) \qquad M_2 = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} n_2 m_t(n_1, n_2) \quad .$$

In a stationary steady state, $m_t(n_1,n_2) = m(n_1,n_2)$ for all t, n_1 and n_2.

2.4 Government Behavior

We distinguish the two currencies by their respective inflation tax. The inflation tax is modeled as in Li (1995). We assume that a subset of the agents in the economy are classified as government agents. The proportion of government agents in the economy is constant and given by ρ. Government agents consume but do not produce goods. Their main purpose is to confiscate and issue units of the domestic currency. Upon meeting a private agent, a government agent confiscates all or part of their domestic currency holdings with probability $0 \leq \mu \leq 1$. The foreign currency is not confiscated, hence its ‘inflation tax’ is zero. We assume that the government agents destroy confiscated currency. Let $\tau(n^I_2)$ denote the units of domestic currency confiscated by the government agent when he meets an agent holding n^I_2 units of the domestic currency where $0 \leq \tau(n^I_2) \leq n^I_2$.. We also interpret $\mu = 0$ as corresponding to the case where inflation tax on the domestic currency is zero. In order to have a stationary equilibrium with a positive stock of domestic currency in circulation, we need an inflow of the domestic currency to offset the outflow of currency arising through confiscation. When a government agent meets a $m(0,0)$ seller, with probability $0 \leq \eta \leq 1$, he issues $0 \leq \lambda \leq N$ units of domestic currency in return for goods.[4]

2.5 Matching Process

Agents meet at random according to a Poisson process with arrival rate α. With probability x, a single coincidence of wants occurs and one of the agents becomes a seller and the other a buyer. Trade requires that the buyer hold at least 1 unit of currency. Both the buyer and the seller may hold both currencies. When a single coincidence of wants occurs there are two possible types of trades: 1) The buyer gives money (foreign, domestic, or some of both) to the seller for some amount of goods, and 2) the buyer gives the seller some currency for the good and the seller 'makes change' by also giving the buyer some of the opposite currency in addition to the goods. For example, a buyer could give the seller 3 units of foreign currency for some units of the good and 1 unit of the domestic currency.

When a single coincidence of wants does not occur, traders may still face gains from trade via a pure financial transaction. Since the domestic currency is 'risky', traders may decide to diversify their portfolios by trading currencies. For example, a trader with many dollars and no rubles may meet a trader with many rubles and no dollars. By swapping dollars for rubles, the ruble trader gets rid of some of the risky currency and acquires the safe currency. If the dollar trader gets enough rubles per dollar, he will be willing to take on a greater risk position in order

[4] We could have modeled the government agents as using confiscated currency to buy goods rather than destroying it. However, this would have required us to solve for the distribution of portfolio holdings held by government agents in steady state. Therefore, modeling the government’s behavior as we have greatly simplifies the numerical routines.

to increase his total currency holdings. The amount that they trade will determine the nominal exchange rate. In general, the nominal exchange rate that occurs will be a function of the composition of the traders' current portfolios and the underlying trading values of the current portfolios. As a result, the nominal exchange rate occurring in these financial trades will vary across matches.

2.6 Bargaining

When a single coincidence of wants occurs, we assume that the buyer makes a take-it-or-leave-it offer to the seller. As a result, the buyer will offer a trade of goods for currency such that he extracts the entire trading surplus from the seller. The seller is indifferent between accepting and rejecting this offer and thus accepts the offer.[5] The buyer's offer, d, is a pair of foreign and domestic currency transfers $d = (d_1,d_2)$ in return for goods. If $d_1 > 0$ and $d_2 = 0$, the buyer offers to pay with the foreign currency while the reverse is true if $d_1 = 0$ and $d_2 > 0$. If both are greater than zero, than the buyer offers to pay the seller with a mixed bundle of foreign and domestic currency in return for goods. These are 'money for goods' trades. If $d_1 > 0, d_2 < 0$, the buyer offers d_1 units of foreign currency in return for goods and d_2 units of domestic currency. If the inequalities are reversed, the buyer offers domestic currency units in return for goods and units of the foreign currencies. Thus, $d_1 > (<) 0, d_2 < (>) 0$, will be referred to as 'making change' trades. Finally, we assume that the government agent who meets a $m(0,0)$ seller also makes a take-it-or-leave-it offer to the seller in return for λ units of the domestic currency.

When a single coincidence of wants does not occur but a currency swap is beneficial to both parties, we assume that each trader has a probability of 1/2 of making a take-it-or-leave-it offer to the other trader, which is either accepted or rejected. The first mover offers to trade y_1 units of currency 1 for y_2 units of currency 2. If $y_1 > 0, y_2 < 0$ and vice versa. The trader making the offer tries to extract as much surplus as possible from the other trader. However, unlike goods trades, the discreteness of the currency unit means the seller may end up with some of the surplus from trade. In the case where the seller is indifferent between accepting and rejecting the offer, we assume the offer is accepted. Thus, let y_i^1 denote the units of currency i given up when an

[5] Since q is continuous, the buyer can offer to take an infinitesimally smaller amount of q to induce the seller to accept the offer.

individual is the first mover and y_i^2 denote the units of currency i received when an individual is the second mover.

2.7 Value Functions

In a stationary steady-state, the returns to search for an agent with money holdings *(n_1,n_2)* will be given by

$$rV(n_1,n_2) = \alpha x \sum_{n_1^s \in \Omega} \sum_{n_2^s \in \Omega} [u(q(n_1, n_2, n_1^s, n_2^s))$$
$$+V(n_1 - d_1(n_1, n_2, n_1^s, n_2^s), n_2 - d_2(n_1, n_2, n_1^s, n_2^s)) - V(n_1, n_2)] m(n_1^s, n_2^s)$$
$$+\alpha x \sum_{n_1^b \in \Psi} \sum_{n_2^b \in \Psi} [-c(q(n_1^b, n_2^b, n_1, n_2))$$
$$+V(n_1 + d_1(n_1^b, n_2^b, n_1, n_2), n_2 + d_2(n_1^b, n_2^b, n_1, n_2)) - V(n_1, n_2)] m(n_1^b, n_2^b) \tag{3}$$
$$+(\alpha/2)(1-2x) \sum_{n_1^k \in K} \sum_{n_2^k \in K} \{[V(n_1 - y_1^1(n_1, n_2, n_1^k, n_2^k), n_2 + y_2^1(n_1, n_2, n_1^k, n_2^k)) - V(n_1, n_2)]$$
$$+[V(n_1 + y_1^2(n_1, n_2, n_1^k, n_2^k), n_2 - y_2^2(n_1, n_2, n_1^k, n_2^k)) - V(n_1, n_2)]\} m(n_1^k, n_2^k)$$
$$-\rho\mu[V(n_1,n_2) - V(n_1, n_2 - \tau(n_2))]$$

where Ω denotes the set of feasible sellers, ψ denotes the set of feasible buyers and K denotes the set of feasible currency traders. The first summation term of (3) is the expected payoff from being a buyer and offering the currency bundle d = $(d_1(n_1, n_2, n_1^s, n_2^s), d_2(n_1, n_2, n_1^s, n_2^s))$ for the quantity $q(n_1, n_2, n_1^s, n_2^s)$ from a seller with portfolio (n_1^s, n_2^s). The second summation term is the expected payoff from being a seller and producing $q(n_1^b, n_2^b, n_1, n_2)$ for a buyer who pays offers them a currency bundle d = $(d_1(n_1^b, n_2^b, n_1, n_2), d_2(n_1^b, n_2^b, n_1, n_2))$. The third double summation term captures the return from currency exchange with another trader holding portfolio (n_1^k, n_2^k) . The last term is the expected loss of running into a government agent and having $\tau(n_1)$ units of the domestic currency confiscated.

2.8 Bargaining

When a potential trade for goods occurs, the buyer makes a take-it-or-leave-it offer to the seller. This offer is a triplet (q,d_1,d_2) that satisfies

$$(4)\quad \begin{aligned}&\max_{q,d_1,d_2}[u(q(n_1,n_2,n_1^s,n_2^s))+V(n_1-d_1(n_1,n_2,n_1^s,n_2^s),n_2-d_2(n_1,n_2,n_1^s,n_2^s))-V(n_1,n_2)]\\ &s.t.\quad V(n_1+d_1(n_1,n_2,n_1^s,n_2^s),n_2+d_2(n_1,n_2,n_1^s,n_2^s))-V(n_1,n_2)\geq c(q(n_1,n_2,n_1^s,n_2^s))\ .\end{aligned}$$

and the constraint that d_1 and d_2 are feasible transfers of currency given the buyer's and seller's portfolios. When a trade actually occurs, the buyer's offer extracts the full surplus from the seller such that the constraint in (4) is satisfied with equality. This is possible because q is a continuous variable. As pointed out by Camera and Corbae, in general, there will be matches with a single coincidence of wants but no offer can be made which satisfies (4). Since the value functions are concave in money holdings, a currency 'rich' seller may not be willing to give up enough of the good for another unit of currency from a currency 'poor' buyer. The high price of the good makes the buyer willing to wait for a better deal than to trade now. Since the seller receives zero surplus when a goods trade occurs, the second double summation term in (3) will be zero.

Since agents without any currency units can only be sellers and sellers receive zero net surplus from trade, the returns to search for an agent without any currency units is

$$(5)\quad rV(0,0)=0\,.$$

When potential currency swaps exist, a coin flip determines who is the first mover and the first mover makes a take-it-or-leave-it offer to the other trader. The first mover chooses a currency swap $y = (y_1^1, y_2^1)$ to

$$(6)\quad \begin{aligned}&\max_{y_1^1,y_2^1}[V(n_1-y_1^1(n_1,n_2,n_1^k,n_2^k),n_2+y_2^1(n_1,n_2,n_1^s,n_2^s))-V(n_1,n_2)]\\ &s.t.\quad V(n_1^k+y_1(n_1,n_2,n_1^k,n_2^k),n_2-y_2(n_1,n_2,n_1^s,n_2^s))-V(n_1,n_2)\geq 0\ .\end{aligned}$$

subject to the constraint that the proposed portfolio changes are feasible. Unlike goods trades, the discreteness of the currency units will make it difficult for the first mover to extract the entire surplus from the second mover. Hence, in general each side will receive some surplus from currency exchange.

2.9 The Distribution of Portfolios

Let $F_t(n_1,n_2)$ denote the probability at time t that an individual agent has a portfolio that is smaller than or equal to (n_1,n_2). Thus, $F_t(n_1,n_2)$ is given by

$$(7)\qquad F_t(n_1,n_2)=\sum_{a=0}^{n_1}\sum_{b=0}^{n_2} m_t(a,b)$$

A stationary distribution of portfolios has $F_t(n_1,n_2) = F(n_1,n_2)$ for all t, n_1 and n_2.

At each point in time there are flows of agents into and out of each portfolio state. In steady state, the flows of agents out of a particular portfolio state must be matched by an inflow of agents into that portfolio state. Writing down these flow equations for each possible portfolio state is complicated. Nevertheless, it is relatively easy to describe what happens regarding matches at time t.

With regards to the inflows and outflows of the domestic currency, the government meets an agent at state (n_1,n_2) with probability ρ and confiscates their domestic currency holdings with probability μ. This reduces the proportion of people in that state by a factor of $\rho\mu$. Of course the proportion of the population, $m(n_1,0)$ increases by $\sum_{N_1} m(n_1,n_2)\mu\rho$. At the same time, those traders at $m(0,0)$ get the opportunity to receive a transfer from the government. Thus, if the transfer is one unit of domestic currency, the proportion of people at $m(0,1)$ will increase by the amount $\mu\eta m(0,0)$. In steady state, this outflow of the domestic currency must be equal to the inflow of domestic currency so $\mu\sum_{N_1} m(n_1,n_2)\tau(n_2) = \eta\lambda m(0,0)$. Note that when $\lambda = 1$ and $\tau(n_2) = n_2$, the proportion of agents at $m(0,0)$ is given by $m(0,0) = (\mu/\eta)\sum_{n_2} n_2 m(n_1,n_2) = (\mu/\eta)M_2$. For given values of μ and η, M_2 will be endogenous since $m(0,0)$ is endogenous. Thus, if we set μ and η, M_1 can be chosen ex ante but M_2 cannot.[6]

[6] In other words, adding this flow condition creates an extra equation in the model, hence an additional endogenous variable is needed to ensure that a solution exists. Thus, M_2 is the new endogenous variable.

Agents who are not matched with a government agent 'go to market'. Some of these matches generate a single coincidence of wants and thus a potential trade of goods for currency. Each trade of money for goods involves a flow into a new currency holding state and a flow out of the old currency state for both seller and buyer. For a transaction that involves the buyer paying $d_1(n_1, n_2, n_1^s, n_2^s)$ and $d_2(n_1, n_2, n_1^s, n_2^s)$, to use the notation above, the proportions $m(n_1 - d_1, n_2 - d_2)$ and $m(n_1^s + d_1, n_2^s + d_2)$ both increase by the amount $m(n_1, n_2)m(n_1^s, n_2^s)$, because of the new currency holdings. The old currency-holding proportions, $m(n_1, n_2)$ and $m(n_1^s, n_2^s)$, are decreased by the same amount. Matches that without a single coincidence of wants do not cause a change in portfolio positions. However, when currency trades are possible, these same matches can produce changes in portfolio positions and so the flow equations for each portfolio position must account for these pure currency trades. A steady-state equilibrium is achieved when the flows out of a given currency state are equal to the flows into it.

3 Equilibrium

We can define a stationary equilibrium as a set of functions $V^*(n_1,n_2)$, $q(n_1, n_2, n_1^s, n_2^s)$, $d_1(n_1, n_2, n_1^s, n_2^s)$, $d_2(n_1, n_2, n_1^s, n_2^s)$, $y_1^1(n_1, n_2, n_1^s, n_2^s)$, $y_2^1(n_1, n_2, n_1^s, n_2^s)$, $y_1^2(n_1, n_2, n_1^s, n_2^s)$, $y_2^2(n_1, n_2, n_1^s, n_2^s)$, $F(n_1,n_2)$ such that (1)-(7) are satisfied.

4 A Special Case: Camera-Corbae with Two Currencies

In general there are three classes of monetary equilibria that can occur in our dual currency model:

1. Equilibria in which trades only involve money for goods.
2. Equilibria that involve type 1 trades plus trades involving currency i for goods plus some amount of currency j, j≠ i.
3. Equilibria that involve type 2 trades plus pure currency trades of currency i for currency j.

We cannot solve this model analytically for the general case, hence it is difficult to study the steady-state distribution of money holdings. However, there is a special trading equilibrium that we can solve analytically in the two-currency model -- the Camera and Corbae (1998) equilibrium. Camera and Corbae study transaction patterns in a model where agents can hold up to *N* units of a single currency. They then look at a particular equilibrium in which agents only trade one unit of currency for goods, regardless of the quantity of money held by the buyers and sellers. They then show that the steady-state distribution of money holdings in this case is a censored-geometric distribution. Finally, they determine the parameter values needed to ensure that this trading strategy is optimal.

In this section we study a special case of the Camera-Corbae equilibrium in our two-currency model. In particular we concentrate on equilibria where:

i. agents can hold up to $N = 2$ units of currency (total).
ii. agents only trade one unit of currency for goods, regardless of: a) the currency used and b) the portfolio holdings of the buyers and sellers. This rules out the 'making change' equilibria.

The assumption that $N = 2$ implies that there are 6 portfolio states: *m(0,0), m(1,0), m(0,1), m(2,0), m(0,2) and m(1,1).*

Within this class of equilibria, we examine 'symmetric' equilibria. By symmetry we mean

1. currencies are identical (there is no confiscation or inflows of the domestic currency and the per-capita holdings of the foreign and domestic are the same).
2. equilibrium price distributions are symmetric, i.e., one unit of the domestic currency buys the same amount of goods from a given seller as a unit of the foreign currency.
3. equilibrium value functions are identical for portfolios with the same total amount of money holdings (i.e., $V(n_1,n_2) = V(n_2,n_1)$ for all $n_2 + n_1 = \iota$, $\iota = 0,1,2$).

These latter conditions do not restrict the steady-state distribution of portfolios to be symmetric. In general, there are three types of steady-state distributions:

a) Super symmetric probability distributions
   - The portfolio distribution of money holdings is symmetric (i.e., $m(n_1,n_2) = m(n_2,n_1)$).
   - Since agents only trade one unit of currency, super symmetry means that the *m(1,1)* buyer's choice of whether to give up the foreign or domestic currency is symmetric across sellers and ***does not*** depend on the seller's portfolio state.
b) Weak symmetric probability distributions
   - The portfolio distribution of money holdings is symmetric (i.e., $m(n_1,n_2) = m(n_2,n_1)$).
   - Since agents only trade one unit of currency, symmetry requires that the *m(1,1)* buyer's choice of whether to give up the foreign or domestic currency is symmetric across sellers and ***does*** depend on the seller's portfolio state.

c) Asymmetric probability distributions.

- The portfolio distribution of money holdings is not symmetric (i.e., $m(n_1,n_2) \neq m(n_2,n_1)$).
- The *m(1,1)* buyer's choice of whether to give up the foreign or domestic currency is ***not*** symmetric across sellers.

It turns out that there is one super symmetric equilibrium but many weak symmetric equilibria. Super symmetry imposes the condition that the *m(1,1)* buyer use a mixed strategy such that he gives up the domestic currency with probability 1/2 and the foreign currency with probability 1/2 when meeting any seller.[7]

The plethora of weak symmetric equilibria occurs because of an indeterminacy associated with the buying strategies used by a buyer with a (1,1) portfolio. Under symmetry, when buying from a seller with either a (1,0) or (0,1) portfolio, the buyer is indifferent as to which currency he gives up since they have equal value and buy the same quantity of goods. Thus, the buyer resorts to using a mixed strategy in determining which currency to give up. Symmetry restricts the choice of mixed strategy in that the buyer must treat identical sellers symmetrically. However, it does not mean that the buyer treats *all* sellers identically. In fact, there are an infinite number of mixed strategies, conditioned on the currency portfolio of the seller, which generate a symmetric steady-state portfolio distribution. For example, the strategy "give up the currency opposite the seller's currency holding with probability $p \in [0,1]$" is symmetric *conditional* on the seller's portfolio state; a seller with a unit of the domestic (foreign) currency will receive a unit of the foreign (domestic) currency with probability *p*. There is nothing in the model to pin down the solution for *p* under weak symmetry. Yet for every value of *p*, there is a different symmetric distribution of portfolio holdings and value functions.[8]

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[7] In actuality, the mixed strategy used by a m(1,1) buyer when meeting a m(0,0) seller turns out to be irrelevant for both the steady state distribution and the value functions. Its only important that the mixed strategy used when meeting a m(1,0) and m(0,1) seller equal 1/2.

[8] Indeterminacy also arises in Camera and Corbae and in Taber and Wallace (1998) but in a different fashion. In one-currency models, there is a possibility that a buyer is indifferent between giving up d units of a currency or d+1 units of the currency in exchange for goods. When this situation arises agents must resort to mixed strategies to determine which offer to make to the seller. In our model, this problem also arises but the addition of a second currency creates another source of indifference for the buyer which is the choice of giving up d units of the domestic currency or d units of the foreign currency.

With regards to the asymmetric probability distribution, there are mixed strategies the *m(1,1)* buyer can use that treat sellers the same but not symmetrically. For example, a *m(1,1)* buyer can adopt the strategy of always giving up the domestic currency first regardless of the seller's portfolio state.[9] However, this strategy does not treat sellers symmetrically, since it moves the sellers to portfolio states that are not symmetric; the *m(1,0)* seller moves to *m(2,0)* while the *m(0,1)* seller moves to *m(1,1)*.

What is interesting about our solutions for the steady state distribution is that when measured by *portfolio* size, our solution for the steady-state distribution is equivalent to that obtained by Camera-Corbae and Green and Zhou (1996).[10] In other words, the proportion of agents holding two units of currency, *regardless of the nationality of the currency*, is equal to the proportion of agents holding two units of a single currency in Camera and Corbae for $N=2$. Furthermore, this is true for the super symmetric, weak symmetric and asymmetric equilibria. In short, it is not affected by the mixed strategy used by the (1,1) buyer. Thus, define

$\mu_2 = m(2,0) + m(0,2) + m(1,1)$

$\mu_1 = m(1,0) + m(0,1)$

$1 = m(0,0) + \mu_1 + \mu_2$

then it can be shown that the steady-state distribution across portfolio size is given by

$$\mu_n = m_0\left(\frac{1-m_0}{1-\mu_2}\right)^n \quad \text{for n = 0,1,2.}$$

We suspect that this result holds for all N but have not tried to prove it. Hence, we conjecture that under symmetry, the steady-state distribution of currency holdings across portfolio size will be censored geometric. If this conjecture is true, then what remains to be determined is the marginal distribution of portfolios conditioned on portfolio size for $N > 2$.

Determining a steady-state equilibrium for this version of the model requires determining the steady state distribution of money holdings (portfolios) under the assumption that only one unit of currency is exchanged. We then solve for the value functions under symmetry. Given the solutions for the steady-state distribution and the value functions, we then need to determine the

[9] These are sometimes referred to as 'non-discriminatory' mixed strategies.
[10] In order to make the numerical values the same, one must adjust the per capita money stock in Camera and Corbae, in order to have equivalent *aggregate* per capita money stocks across the two models.

conditions under which it is optimal for buyers to follow the strategy of only giving up one unit of currency per trade.

In the appendix, we determine the conditions under which the super-symmetric equilibrium exists and obtain analytical expressions for the steady-state distribution. We do so under that specification that $u(q) = q^\sigma$ with $0 < \sigma < 1$.[11] The equilibrium solutions are:

$$\begin{aligned} &V(0,0) = V_0 = 0 \\ &V(1,0) = V(0,1) = V_1 = \{[A/(1-\mu_2)][m_0 + \mu_1 A^\sigma]\}^{1/(1-\sigma)} \\ &V(2,0) = V(0,2) = V(1,1) = (1+A)V_1 \\ &\quad \text{where } A = \alpha x(1-\mu_2)/[r + \alpha x(1-\mu_2)] < 1 \end{aligned} \tag{8}$$

$$[(1+A)^\sigma - 1]^{1/(1-\sigma)} \leq V_1 \leq A^{\sigma/(1-\sigma)} \tag{9}$$

$$\begin{aligned} &q_0^1 = V_1 \\ &q_1^2 = AV_1 \end{aligned} \tag{10}$$

$$M = M_1 = M_2 = m_1 + 2m_2 + m_{11} \tag{11}$$

$$1 = m_{00} + 2m_1 + 2m_2 + m_{11} \tag{12}$$

$$\begin{aligned} &m_{11} = 2m_2 \\ &m_2 = m_1^2 / m_0 \\ &m_{00} = 1 - M - m_1 \\ &m_1 = \left[\frac{1}{6}\right]\{-1 + [1 + 12M(1-M)]^{1/2}\} \end{aligned} \tag{13}$$

where $m(0,0) = m_{00}$, $m(1,0) = m(0,1) = m_1$, $m(2,0) = m(0,2) = m_2$ and $m(1,1) = m_{11}$. Equation (8) contains the equilibrium values of the value functions, (9) is the buyer-incentive compatibility constraint that must be satisfied to ensure that trading one unit of currency is optimal for all buyers, (10) gives the equilibrium quantities of goods given up by sellers with no currency and one unit of currency respectively under buyer-take-all bargaining, (11) is the money supply equation for each currency under symmetry, (12) is the adding up constraint under symmetry and (13) contains the solutions for distribution of money holdings.

As in Camera and Corbae, the value functions are concave and linearly dependent. Furthermore, the concavity of the value functions means that the marginal value of acquiring an additional unit of either currency is diminishing. Hence, as shown in (10), a seller with no

[11] Given the assumption of linear costs, this utility specification implies that the equilibrium quantities must lie in the unit interval, $q_i \in [0,1]$ in order to have $u(q) - c(q) \geq 0$.

currency will give up more units of the good for a unit of currency than will a seller with one unit of currency. Thus, (10) is the price distribution case of symmetric value functions. Finally, the incentive compatibility constraint is more likely to hold the greater the concavity of $u(q)$ $(\sigma \rightarrow 0)$ and the smaller are the search frictions associated with trading ($\alpha x/r \rightarrow \infty$).[12] The greater the concavity of $u(q)$ the smaller is the intertemporal elasticity of substitution, and the less likely buyers will be to postpone trades to the future. This basically means 'poor' buyers will not wait to find a better price tomorrow when they meet a 'rich' seller today. On the other hand, if search frictions are low, the likelihood of meeting a desired seller again tomorrow is reasonably high. This prevents 'rich' buyers from giving up more than one unit of currency today.

Under the assumption of super symmetry, the marginal distribution across the portfolios with 2 units of currency can be determined. Using the first expression in (13) we find that $m_2 = (1/4)\mu_2$ and $m_{11} = (1/2)\mu_2$. This essentially means that there are as many 'undiversified' 2 unit portfolios as 'diversified' 2 unit portfolios ($m_{11} = m_{20} + m_{02}$).

In the case of weak symmetry, equations (8)-(12) still hold but (13) does not. In this case, the four portfolio states, m_0, m_1, m_2, m_{11}, can be solved for using (11), (12) and the following two equations obtained from the steady-state flow conditions on m_2 and m_{11}:

$$\text{(14)} \quad \begin{aligned} \dot{m}_2 &= p m_1 m_{11} - m_2(m_1 + m_0) + m_1^2 = 0 \\ \dot{m}_{11} &= m_2 m_1 - p m_1 m_{11} - (m_0 m_{11}/2) + m_1^2 = 0 \end{aligned}$$

where p is the probability a (1,1) buyer gives up the 'left' currency to a seller at (1,0) and the 'right' currency to a (0,1) seller. When $p = 1/2$, the (1,1) buyer treats all sellers the same. When p differs from 1/2, the buyer's mixed strategy is conditioned on the seller's portfolio state. So for each value of p, a symmetric portfolio distribution can exist. Figure 1 shows a symmetric equilibrium with $p = 0$ for the parameter values $\sigma = .5$,
$\alpha x/r = 5$ and $M = 2/3$. The probability distribution is saddle-shaped. Given these parameters, we get the following proportions (with rounding) for $p = 0, .5, 1$:

[12] In fact, the solutions are exactly the same as in CC except that they use $u(q) = \sigma^{-1} q^{\sigma}$ rather than $u(q)=q^{\sigma}$. For this utility specification, the intertemporal elasticity is $1/(1-\sigma)$.

	p = 0	p = .5	p = 1
m_0	0.18085	0.18085	0.18085
m_1	0.15247	0.15247	0.15247
m_2	0.06974	0.12854	0.15925
m_{11}	0.374697	0.25709	0.19568

The values of m_0 and m_1 are unchanged across the values of p, hence μ_1 and μ_2 are constant. The only change is the marginal distribution across the portfolios with 2 currency units. Since the shares across portfolio sizes are constant for all values of p, the value functions will be constant since (8) shows that the solution for V_1 only depends on the portfolio shares m_0 and μ_1 which are constant given our results above.

In addition to symmetric distributions, we have asymmetric portfolio distributions even though the value functions are symmetric. Asymmetry in the distribution arises when the mixed strategy used by the (1,1) buyer is not symmetric across seller's portfolio states. For example, the (1,1) buyer may always give up the domestic currency first or vice versa. Figure 2 shows such a distribution

Finding that the shares of agents at each portfolio size is constant regardless of the mixed strategy used by the (1,1) buyer is important because it implies that the value functions are unaffected by the choice of mixed strategy. Hence, for parameter values such that (9) holds, there are multiple probability distributions, both symmetric and asymmetric, that can support the equilibrium value functions given in (8). In a sense this is not surprising. For example, if currency one was Federal Reserve Bank of Cleveland dollars and currency 2 was Federal Reserve Bank of Atlanta dollars, then we would expect that there are many equilibrium distributions of these two types of dollars that support the same equilibrium value functions associated with holding those dollars. The implication of this multiplicity of distributions is that when we let $N > 2$ and make the currencies identical, there are many steady-state portfolio distributions that can support the same set of symmetric value functions.

5 Numerical Methods

Numerical solution of the system of equations follows a classic fixed-point procedure. With assumed initial value functions, V_0, currency exchange functions, d_{10} and d_{20}, and probability distributions, M_0, we recursively calculate the economy where:

$$d_{1,t+1}\ d_{2,t+1} = \text{argMax(for the buyer)}\{\text{Feasible Bargain Conditions}(V_t,M_t)\}$$

$$M_{t+1} = \text{Implied new Probability Distribution}(d_{1,t+1}\ d_{2,t+1},M_t)$$

$$V_{t+1} = \text{Value from the Search Conditions}(d_{1,t+1}\ d_{2,t+1},M_{t+1})$$

and where convergence is achieved if the maximum difference between $d_{1,t+1}$ $d_{2,t+1}$, M_{t+1}, V_{t+1}, and $d_{1,t}$ $d_{2,t}$, M_t, V_t is under a specified tolerance. (In general the results did not differ for a wide range of tolerances.) The search conditions were rewritten in the numerical work to ensure a contraction mapping was as likely as possible. Under most initial conditions that we tried, the routine converged. In addition, we took the maximum operation globally for the bargaining conditions over all feasible bargains, which was allowed by the discrete number of currency units. Thus, we did not need to strongly specify local conditions in order to achieve an optimum.

6 Numerical Results

6.2 No Inflation Tax

6.1.1 *Symmetric Value Functions* with $M_1 = M_2$

In this section we explore the behavior of equilibria when $N > 2$. Given that we have some analytical results for symmetric equilibria in the $N = 2$ case, we will use these results as a guide and initially study symmetric equilibria when $N > 2$. In these simulations, there is no inflation tax on either currency and we only consider trades in which currency trades for goods. In these equilibria, the value functions should be symmetric since the currencies are identical.

Our analytical results were based on the 1 unit of currency per trade equilibrium first studied by Camera and Corbae. Their basic finding was that the 1 unit currency exchange equilibrium would most likely satisfy the buyers' incentive compatibility constraint if the intertemporal elasticity of utility was sufficiently low and search frictions were sufficiently low. Figure 3a shows a symmetric equilibrium in which only one unit of currency trades for $N = 10$, $\sigma = .15$, $\alpha x/r = 5$ and $M = 3.33$ and $\mu = \eta = 0$. The value functions are symmetric and, similar to the 2 unit of currency example, the equilibrium distribution is 'saddle shaped'. However, the probability distribution is not symmetric due to the fact that it is almost impossible for the numerical routine to find a symmetric distribution when the inventory constraint gets large.[13] As a result the distribution in Figure 3a is skewed to the left much like it is in Figure 2.

Table 1 contains summary statistics for this economy:

Table 1

Economic Data for Symmetric, No Inflation Tax Economy:

Expected ruble holdings:	3.33
Expected dollar holdings:	3.33
Expected quantity produced per match:	0.067
Expected dollar price per unit of output:	6.34
Expected ruble price per unit of output:	11.73
Implied real exchange rate from goods trades: (Rubles per dollar)	1.74
Percentage of matches producing goods trade:	35.3%
Percentage of goods trades involving:	
Dollars only:	61.3%
Rubles only:	38.7%

Given the equilibrium distribution, dollars are more frequently used than rubles and it appears that dollars are more valuable than rubles. However, this is simply a reflection of the asymmetric distribution of traders. Furthermore, there is a concentration of 'rich' ruble traders and 'poor' dollar traders. Since 'rich' ruble sellers give up relatively less for an additional unit of currency, on average the ruble has a higher monetary price per unit while the 'poor' dollar sellers give up relatively more for another unit of currency, hence the dollar price is much lower. However, a dollar buys the same quantity of goods as a ruble for any matched pair of traders.

[13] Our numerical routine could only find symmetric distributions when N=2

Figure 3b and 3c show the same parameterizations except that we increase the degree of intertemporal substitution by having $\sigma = .5$ and $\sigma = .75$. Increasing the rate of intertemporal substitution makes buyers less willing to wait for goods tomorrow, hence they are more likely to trade more than one unit of currency today when they meet a seller of their desired good. The equilibria depicted in Figure 3b and 3c have agents trading up to 3 units of currency per trade. However, the value functions are still symmetric.

However, we know from the 2-unit example that there are multiple distributions that support the same set of symmetric value functions. Although the numerical routine always finds the same distribution, we can 'trick' it by starting with a positive inflation tax on the domestic currency and letting it approach zero rather than starting at zero. Figure 3d shows an equilibrium in which $\mu = \eta = 0.0000001$. The value functions are identical in Figure 3d as in Figure 3a but the portfolio distribution has essentially flipped. This is what you would expect -- with identical currencies, you should be able to re-label the portfolio states and the distribution should simply flip as a result. This is essentially what we find from Figures 3a and 3d.

In Figure 4 we show outcomes for portfolio size N = 15. The value function maintains its symmetry and the distribution begins to take on a dome-like shape with most of the mass in the middle portfolios. However, the lesson of the N=2 analytical model is that under symmetry of the currencies there are many probability distributions that can support the value functions and that we have found only one of them in Figures 3a-3d and Figure 4.

6.1.2 *Symmetric Value Functions with* $M_1 \neq M_2$

In the section above, not only were the currencies identical but the per capita money stocks were the same for both currencies. Figure 5 shows the equilibrium associated with the parameter values in Figure 3a but now the per capita money stocks are different. In figure 5 the per capita value of dollars has been increased to 5.5 and the per capita stock of rubles has been lowered to 2.25. The value functions are still symmetric but, not surprisingly, since there are relatively more dollars in the economy there has been a dramatic shift of the probability distribution towards portfolios consisting of dollars.

6.1.3 *'Making Change' with* $M_1 = M_2$

Another type of equilibrium that can occur in dual currency models is that one currency may trade for a quantity of goods and some amount of the other currency. We call these 'making change' equilibria. What is interesting about these types of equilibria is that they can arise even though the currencies are identical in the sense that the inflation tax is zero for both currencies. This is a true 'belief' equilibrium -- if people believe that one currency is more valuable, they will act that way in trade and it will become more valuable in equilibrium. However, if this type of equilibrium exists then there must be two of them since you can arbitrarily reverse the labels on the currencies and get the mirror image of any equilibrium you find. This case is comparable to the two currency, one unit of money inventory restriction model in Aiyagari, Wallace and Wright (1996).

Figure 6 shows this equilibrium under the same parameter values as Figure 3a. In this case, the domestic currency (rubles) is viewed as being more valuable in equilibrium than the foreign currency (dollars) and so the value functions are asymmetric. The distribution becomes 'igloo' shaped. In this equilibrium, there is a wide range of quantities that are exchanged for the same currency exchange. Table 2 shows the transaction patterns for a subset of matches that involve making change:

Table 2

Buyer		**Seller**		**Currency Exchange**		
R	**$**	**R**	**$**	**R**	**$**	**Quantity of Goods**
0	3	1	0	-1	3	0.315205
0	3	1	1	-1	3	0.244833
0	3	1	2	-1	3	0.192668
0	3	1	3	-1	3	0.148377
0	3	1	4	-1	3	0.111199
0	3	1	5	-1	3	0.080804
0	3	1	6	-1	3	0.055992
0	3	1	7	-1	3	0.035733

In all of these trades, the seller gives up one ruble and some of the good for three dollars. It is clear that the 'richer' is the seller, the less output he is willing to give for the same exchange of currency. This just reflects the diminishing marginal value of additional units of currency. One ruble and goods for three dollars is not the only exchange of currency that occurs in the making change equilibrium, there are many types of trades involving the making of change. Table 3 gives the reverse portfolio case for the buyer:

<u>Table 3</u>

Buyer		Seller		Currency Exchange		
R	**$**	**R**	**$**	**R**	**$**	**Quantity of Goods**
3	0	1	1	1	-1	0.289271
3	0	1	2	1	-1	0.235117
3	0	1	3	1	-1	0.194010
3	0	1	4	1	-1	0.159778
3	0	1	5	1	-1	0.131915
3	0	1	6	1	-1	0.109233
3	0	1	7	1	-1	0.090737
3	0	1	8	2	-4	0.010707
3	0	1	9	2	-4	0.010906

In these subsets of trades, the seller has relatively more dollars and the buyer has many rubles but no dollars. Hence the seller gives up dollars and goods in return for rubles. In this specific example, no pure currency exchanges occur.

6.2 A Positive Inflation Tax

In the earlier exercises, the only way we allowed the two currencies to differ was by having different per capita money stocks. However, the currencies themselves were fundamentally identical. In this section we want to see how making the two currencies fundamentally different affects the equilibrium behavior of our dual currency economy.

6.2.3 Changing the Rate of Confiscation (No Making Change)

A common feature of developing and transitional economies is that the domestic currency is subject to a substantial amount of inflation risk. Inflation risk arises because of variability in the policymaker's use of money as a source of seigniorage. Consequently, agents in these economies resort to the use of a second currency that has a much lower inflation risk. Since one of the currencies is safe and the other is risky, there is an incentive for currency exchange in order to diversify currency portfolios. In this section, we explore how varying the degree of inflation risk affects the steady-state equilibrium.

In our model, the government has two methods for altering the stock of domestic currency in the economy. It can alter its confiscation policy or its injection policy. With regards to confiscation, it can increase the probability of confiscation and the amount of domestic currency confiscated. Similarly, it can alter its probability of injecting currency (via trades with *m(0,0)* sellers) and the amount transferred. In this section, we examine how changing the rate of confiscation affects the equilibrium.

We introduce confiscation and injection starting from the same parameterization as in Figure 3a. Initially we set the probabilities of confiscation and injection to be very small, $\mu = 0.0025$ $\eta = 0.001$. While the probability of confiscation is low, we assume that the government agent confiscates all of an agent's domestic currency $\tau(n_2) = n_2$. Furthermore, we assume the injection is only one unit of currency small $\lambda = 1$.

Table 4 contains summary statistics for this economy:

Table 4

Economic Data for Small Inflation Tax Economy:

Expected ruble holdings:	1.92
Expected dollar holdings:	3.33
Expected quantity produced per match:	0.0993
Expected dollar price per unit of output:	6.86
Expected ruble price per unit of output:	5.02
Implied real exchange rate from goods trades: (Rubles per dollar)	0.732
Percentage of matches producing goods trade:	36.8%
Percentage of goods trades involving:	
Dollars only:	26.0%
Rubles only:	74.0%

Comparing these data to that in Table 1, we see that introducing a small inflation tax has two very important effects. First, because of the severity of the inflation tax (complete confiscation of ruble holdings) the number of 'rich' ruble sellers declines significantly and the number of sellers with no money at all increases, means that rubles will now buy more goods on average. Hence, the purchasing power of a ruble actually increases relative to the dollar. The second major change from Table 1 is that the percentage of trades involving rubles as completely reversed -- agents are trying to get rid of rubles to avoid confiscation.

Figure 7 shows the value functions and probability distributions for this economy. Compared to the Figure 3a, which has the same parameters but no confiscation, the value functions for both currencies have *increased* in magnitude but the value of the risky domestic currency is marginally lower than the safe foreign currency at all portfolio sizes. For example, V(10,0) = 2.858 while V(0,10) = 2.793. Even though the probability of confiscation is small and the value functions similar, because of the severity of the confiscation policy, the distribution is dramatically shifted towards agents holding portfolios containing only the foreign currency or small amounts of the domestic currency.

We believe the intuition for higher value of both currencies is similar to the risk results found by Curtis and Waller (1999). Introducing confiscation makes the domestic currency more risky, hence the value of the foreign currency rises as traders substitute out of rubles and into dollars. This is a classic case of currency substitution. Since the domestic currency is risky, sellers have to be compensated to accept it in trade. Compensation can take one of two forms. Sellers will accept the risky currency if they believe they will get more goods for it when they become ruble buyers. In a steady-state equilibrium, in order for today's sellers to get more when they become buyers they must give up more for the currency today. As a result, the quantity of goods given up for the risky currency rises in equilibrium and thus so does the value of the currency. The alternative form of compensation to the seller is to give up less today in return for a unit of the risky currency. In equilibrium, the quantity traded per ruble then falls, as does the value functions associated with ruble portfolios. In Figure 7, the first effect is at work.

6.2.2 *Nominal Exchange Rates*

In the example above, despite the fact that the domestic currency is risky relative to the foreign currency, it is not risky enough to induce pure currency exchanges. For example, a 1-for-1 swap is not attractive to the dollar holder and a 10-for-9 swap (the largest possible) is too expensive for a ruble holder. However, if we made the degree of relative risk aversion very large, $1 - \sigma \approx 1$, we could induce currency exchange even at this low probability of confiscation. Furthermore, if we increased the inventory constraint, for the same parameter values we can obtain currency exchange since large quantities of currency can be traded to obtain exchange rates very close to one. To show this, we increase the inventory constraint to N = 15.

In Figures 8a-8c we simulated an economy with N = 15, for three different rates of confiscation: $\mu = 0.00125, 0.0025, \text{and } 0.005$. The value functions shift down slightly along the ruble axis showing the rubles are less valuable due to the risk of confiscation. The probability distributions shift towards dollar-weighted portfolios since more and more rubles are being drained from the economy. What is interesting about the change in the probability distribution is that we can see 'dollarization' of the economy occurring -- more and more agents in the economy are holding dollar-weighted currency portfolios. Dollarization does not mean the domestic currency is driven out completely, but rather means that it becomes a minor part of agents' portfolios. This is what we see in Figures 8a-8c.

With sufficient risk aversion or sufficiently large portfolio constraints, we now get currency exchange occurring. However, due to the decentralized nature of the economy we do not typically observe just one exchange rate but an entire distribution of nominal exchange rates.[14] Table 5 shows the nominal exchange rate distributions for these three economies and Figure 8d shows these exchange rates graphically.

Table 5

Economy 1: $\mu = 0.00125$

Nominal Exchange Rate	Proportion of CurrencyTrades
1.077	1.00

Economy 2: $\mu = 0.0025$

Nominal Exchange Rate	Proportion of Currency Trades
1.0909	0.000112
1.1000	0.202889
1.1111	0.491631
1.1250	0.284835
1.1429	0.019895

[14] This distribution is a cross-section of exchange rates, not a time series. Hence, discussion of the variance of the nominal exchange rate in this model differs from the traditional discussion of exchange rate variance that one would find in the international finance literature. Our nominal exchange rate distributions are more related to the dispersion of exchange rates observed in kiosks across a city at a point in time.

1.1667	0.000589
1.2000	0.000049

Mean exchange rate =1.1135

Standard Deviation = 0.009833

Economy 3: $\mu = 0.005$

Nominal Exchange Rate	Proportion of CurrrencyTrades
1.1250	0.724278
1.1428	0.192227
1.6667	0.080782
1.2000	0.001919
1.2500	0.000798
1.2857	0.000000
1.3333	0.000000

Mean exchange rate =1.1320

Standard Deviation = 0.013237

In the first economy there was only one observed exchange rate since the probability of confiscation and injection are so low. As the rate of confiscation increases, the domestic currency becomes riskier, hence there is a greater incentive to trade the risky currency. As a result, there are more currency trades occurring and over a wider range of nominal exchange rates. In economies 2 and 3 there are 7 observed exchange rates, although the bulk of currency trades are occur at only three exchange rates. The increase is risk associated with a higher rate of confiscation shows forces the risk premium on the domestic currency to rise and this appears as a depreciation of the domestic currency against the foreign currency. It is also interesting to note that as the level of the 'inflation tax' increases, the variance of the nominal exchange rate distribution also increases. However, despite the fact that the variance of the nominal exchange rate is greater in economy three, 72% of currency trades take place at the mode exchange rate while in economy 2 only 49% of trades occur at the mode exchange rate. Thus, while the variance increases, there nevertheless appears to be a tendency for the currency market to move towards a single exchange rate.

Although we found that the variance of the nominal exchange rate distribution increases as the risk of confiscation increases (starting from a low rate of confiscation), this is not a general result. At some point, the variance begins to decline. The reason is that as the rate of confiscation gets too large, the domestic currency gets too risky and agents willingness to accept it decreases. Associated with that is a decrease in the range of nominal exchange rates observed and consequently a decrease in the variance of observed exchange rates. Of course, at a sufficiently high rate of confiscation, currency exchange stops completely and the variance trivially goes to zero.

When the probability of meeting a seller in a single-coincidence match is set at .45, the unconditional probability of a no-coincidence match is .10. So less than 10% of all matches involve potential currency trades.[15] However, for the three economies simulated in Figures 8a-8c, the percentage of no coincidence matches that lead to currency trades is 0.119 E -18, only 0.23 E -11 and .258 E -9 respectively. In short, these economies have 'thin' financial markets since there is little trading volume. In general, we found through experimentation and intuition that the trading volume increased when: the discount rate was decreased (currency trades today were more highly valued), the probability of meeting a seller decreased, (the relative risk of meeting a government agent before traded increased), the foreign money stock is relatively low, the rate of injection is high and the rate of confiscation is low. The highest trading volume we found was 0.3098 E-02 or 31% of all no-coincidencematches. This occurred for the parameter values $N=10$, $M_1=1$, $M_2=3.33$, $\alpha x=.2$, $r=.05$, $\eta=.75$, $\mu=.01$.

6.2.3 Changing the Amount of Foreign Currency (No Making Change)

In the simulations shown in Figures 8a-8c, in equilibrium, the per capita holdings of the foreign currency are greater than the per capita holdings of the domestic currency. This would be a case of extreme dollarization of the economy. However, in many countries in which two currencies serve as a medium of exchange, agents still hold a larger portion of their wealth in the domestic currency even though it is risky. In order to study this case, we repeated the simulations corresponding to Figures 8a-8c but now we significantly reduce the equilibrium per capita holdings of the foreign currency from $M_1 = 5$ to $M_1 = 3$ in order to see how the equilibrium exchange rate distribution is affected. Figure 8e replicates the nominal exchange rate distribution in Figure 8d but with a smaller amount of dollars per capita in the economy. Table 6 compares the means and variances. As the supply of dollars in the economy decreases, its value increases relative to the risky domestic currency and the domestic currency depreciates in value. This is show by a shift to the right of all three distributions in figure 8e relative to their positions in Figure 8d. While this is not a surprising result, what is surprising is that the variance of all three distributions in Figure 8e increases compared to the ones in Figure 8.

Table 6

	$M_1 = 5$	$M_1 = 3$
Economy 1: $\mu = 0.00125$		
Mean exchange rate	1.077	1.083310141
Standard Deviation	0	0.000384877
Economy 2: $\mu = 0.0025$		
Mean exchange rate	1.1135	1.130727134
Standard Deviation	0.009833	0.015634396
Economy 3: $\mu = 0.005$		
Mean exchange rate	1.1320	1.15643755
Standard Deviation	0.013237	0.02075545

We suspect that this result is due to the fact that with fewer dollars in the economy and more traders holding portfolios with large amounts of the risky domestic currency. Thus, holders of foreign currency can extract a larger amount of domestic currency for each dollar traded from these 'rich' domestic currency traders. This cannot be done with 'poor' domestic currency holders since they have smaller margins to extract. As a result, the range of possible exchange rates increases. Furthermore, because these matches with 'rich' domestic currency holders occur with a greater frequency than before, there is more mass at this part of the probability distribution as well. Both effects lead to an increase in the variance of the distribution.

7. Conclusions

In this paper we have contributed to the growing literature involving the distribution of money and prices in search theoretic model. We have shown that monetary equilibria exist in which both currencies circulate as media of exchange and have shown that multiple equilibria exist even when the two currencies are indistinguishable. New aspects of our work that have not been studied in previous work are the issue of portfolio diversification, the endogenous determination of nominal exchange rates and the role of risk aversion in portfolio choice. Our comparative static analysis revealed sensible predictions regarding how the value of currencies

[15] For this economy, 41.4% of matches involved goods trades. This is less than 45% because potential sellers for whom the inventory constraint is binding cannot sell their goods for currency and because potential buyers without

change after various changes in policy regimes. They also revealed some surprising results regarding the behavior of the nominal exchange rate distributions that need to be explored further.

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any currency cannot buy.

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Appendix

Steady state flows:

There are 6 flows conditions, 2 money supply equations and 1 adding up constraint for determining the 6 steady state portfolio proportions. Thus, 3 of the flow equations are redundant. This implies we only need to consider 3 of the flow conditions to determine the equilibrium steady state. The other 3 flow conditions can be shown to be linear combinations of the other 3. Without loss of generality, we use the 3 flow conditions associated with the 2 unit of currency portfolios. We can then solve for the proportions of agents at the 1 unit portfolio state and the 0 unit portfolio state using the money supply equations and the adding up constraint. Under the assumption that only one unit of currency is traded we have the following 6 equations for determining the steady state distribution of portfolios:

(A1) $\dot{m}_{20} = \alpha x(m_{10}m_{11}p_{10}^{10} - m_{20}m_{00} - m_{20}m_{01} + m_{10}^{2})$

(A2) $\dot{m}_{02} = \alpha x(m_{01}m_{11}p_{01}^{01} - m_{02}m_{00} - m_{02}m_{10} + m_{01}^{2})$

(A3) $\dot{m}_{11} = \alpha x(m_{20}m_{01} + m_{02}m_{10} + 2m_{10}m_{01} - m_{10}m_{11}p_{10}^{10} - m_{01}m_{11}p_{01}^{01} - m_{00}m_{11})$

(A4) $M_1 = m_{10} + 2m_{20} + m_{11}$

(A5) $M_2 = m_{01} + 2m_{02} + m_{11}$

(A6) $1 = m_{00} + m_{10} + m_{01} + m_{20} + m_{02} + m_{11}$

where $m(0,0) = m_0$, $m(1,0) = m(0,1) = m_1$, $m(2,0) = m(0,2) = m_2$, $m(1,1) = m_{11}$, p_{10}^{10} is the probability a (1,1) buyer gives up the 'left' (foreign) currency to a seller at (1,0) and p_{01}^{01} is the probability a (1,1) buyer gives up the 'right' (domestic) currency to a seller at (0,1). Consequently, $p_{10}^{10} + p_{10}^{01} = 1$ and $p_{01}^{10} + p_{01}^{01} = 1$. In steady state, (A1)-(A3) equal zero.

Censored Geometric Marginal Distribution:

Let

$\mu_2 = m_{20} + m_{02} + m_{11}$

$\mu_1 = m_{10} + m_{01}$

$1 = m_{00} + \mu_1 + \mu_2$

The sum of (A1)-(A3) gives the net flow into μ_2 and this sum equals zero in steady state, yielding

(A7) $\mu_2 = \mu_1^2 / m_{00}$.

Substitute (A7) into (A6) and solve for μ_1 as

(A8) $\mu_1 = m_{00}(1 - m_{00})/(1 - \mu_2)$.

Substituting back into (A7) yields

(A9) $\mu_1 = m_{00}(1 - m_{00})^2 / (1 - \mu_2)^2$.

(A8) and (A9) show that the marginal distribution of portfolios conditioned on portfolio size is censored geometric.

Symmetric steady-state distributions:
Under symmetry, $m_{10} = m_{01} = m_1$ and $m_{20} = m_{02} = m_2$. Imposing these conditions on (A1)-(A6) yields:

(A10) $\dot{m}_{20} = m_1 m_{11} p_{10}^{10} - m_2 m_{00} - m_2 m_1 + m_1^2 = 0$

(A11) $\dot{m}_{02} = m_1 m_{11} p_{01}^{01} - m_2 m_{00} - m_2 m_1 + m_1^2 = 0$

(A12) $\dot{m}_{11} = 2m_2 m_1 + 2m_1^2 - m_1 m_{11} p_{10}^{10} - m_1 m_{11} p_{01}^{01} - m_{00} m_{11} = 0$

(A13) $M_1 = m_1 + 2m_2 + m_{11}$

(A14) $M_2 = m_1 + 2m_2 + m_{11}$

(A15) $1 = m_{00} + 2m_1 + 2m_2 + m_{11}$

Symmetry requires $M_1 = M_2 = M$ and (A10) = (A11) which imposes $p_{10}^{10} = p_{01}^{01}$. This condition implies that a (1,1) buyer treat sellers symmetrically conditional on their portfolio state but it does not require $p_{10}^{10} = p_{01}^{01} = 1/2$. It merely requires that a (1,1) buyer give up the 'left' currency to a seller holding the 'left' currency with the same probability he gives up the 'right' currency to a seller holding the 'right' currency. We define a super symmetric equilibrium to correspond to the case where $p_{10}^{10} = p_{01}^{01} = 1/2$ and we define a weak symmetric equilibrium to correspond to the case where $p_{10}^{10} = p_{01}^{01} = p \neq 1/2$. Under either definition of symmetry, (A11) and (A12) are used to solve for the steady state values of m_2 and m_{11}:

(A16) $\dot{m}_2 = m_1 m_{11} p - m_2 m_{00} - m_2 m_1 + m_1^2 = 0$

(A17) $\dot{m}_{11} = 2m_2 m_1 + 2m_1^2 - 2m_1 m_{11} p - m_{00} m_{11} = 0$.

It is easy to see that when $p = 1/2$, $m_{11} = 2m_2$ satisfies both equations. Using this expression in (A13) yields $m_2 = (1/4)(M - m_1)$. (A15) shows that $m_{00} = 1 - M - m_1$ which requires $M < 1$ to ensure a non-negative value of m_{00} for positive values of m_1. Using these three expressions in (A16) produces a quadratic equation in m_1. Non-negativity of m_1 requires the positive discriminant as the solution. The solution for m_1 is given in equation (13) in the text.

Symmetric Value Functions
When the currencies are identical, it is reasonable to conjecture that the value functions will be symmetric across portfolio size. Under symmetry, V(2,0) = V(0,2) = V(1,1) = $V_2 = V_{11}$ and V(1,0) = V(0,1) = V_1. As a result of symmetry, the inventory constraint, buyer-take-all and the conjecture that only one unit of currency trades at a time, we only have two equilibrium quantities to determine; the quantity given up by a (0,0) to move to V_1 and the quantity a (1,0) or (0,1) seller will give up to move to V_2 or V_{11}. Thus, buyer-take-all implies

(A18) $V_1 - V_{00} = q_0^1$

(A19) $V_2 - V_1 = V_{11} - V_1 = q_1^2$.

When the currencies are identical, currency trades do not occur. Hence, under the conjecture that only one unit of currency trades per transaction, the returns to search show that the values functions can be written as;

(A20) $V_{00} = 0$

(A21) $V_1 = \dfrac{\alpha x}{(r + \alpha x(1-\mu_2))}[m_{00}u(q_0^1) + \mu_1 u(q_1^2)]$

(A22) $$V_2 = V_{11} = \frac{\alpha x}{(r + \alpha x(1-\mu_2))}[m_{00}u(q_0^1) + \mu_1 u(q_1^2)] + \frac{\alpha x(1-\mu_2)}{(r + \alpha x(1-\mu_2))}V_1 \quad . \\ = (1+A)V_1$$

Using (A21) and (A22) in (A18)-(A19) yields equation (10) in the text. Furthermore, substituting, $V_1 = q_0^1$ and $V_2 - V_1 = AV_1 = q_1^2$ into (A21) and using $u(q) = q^\sigma$ yields equation (8) in the text.

Incentive Compatibility

Given the solutions in (A20)-(A22), we need to verify when it is individually rational for a buyer to trade and, conditional on choosing to trade, they only trade one unit of currency per transaction. Buyers will choose to trade when

(A23) $u(q_1^2) = u(AV_1) \geq V_1$

(A24) $u(q_0^1) = u(V_1) \geq V_1$

(A25) $u(q_1^2) = u(AV)_1 \geq V_2 - V_1 = AV_1$

(A26) $u(q_0^1) = u(V_1) \geq V_2 - V_1 = AV_1$.

Since $0 \leq A \leq 1$, if (A23) holds, (A24)-(A26) holds. Thus from (A23) we obtain an upper bound on V_1 that ensures all buyers will trade

(A27) $A^{\sigma/91-\sigma)} \geq V_1$.

Since $0 \leq A \leq 1$, from (A27), in equilibrium $0 \leq V_1 \leq 1$. Finally, we need to determine, under the conjectured equilibrium, whether or not it is rational for an individual buyer with 2 units of currency to spend only 1 unit of currency rather than both units when meeting a seller with no currency. This condition imposes the constraint that

(A28) $u(q_0^1) - (V_2 - V_1) \geq u(q_0^2) - (V_2 - V_0)$

where q_0^2 is the quantity a seller with no currency would give up to acquire two units of currency. Under buyer-take-all bargaining, the best the buyer could do is offer $q_0^2 = V_2 - V_0 = (1+A)V_1$ to the seller. Thus we have

(A29) $u(V_1) \geq u((1+A)V_1) - V_1$.

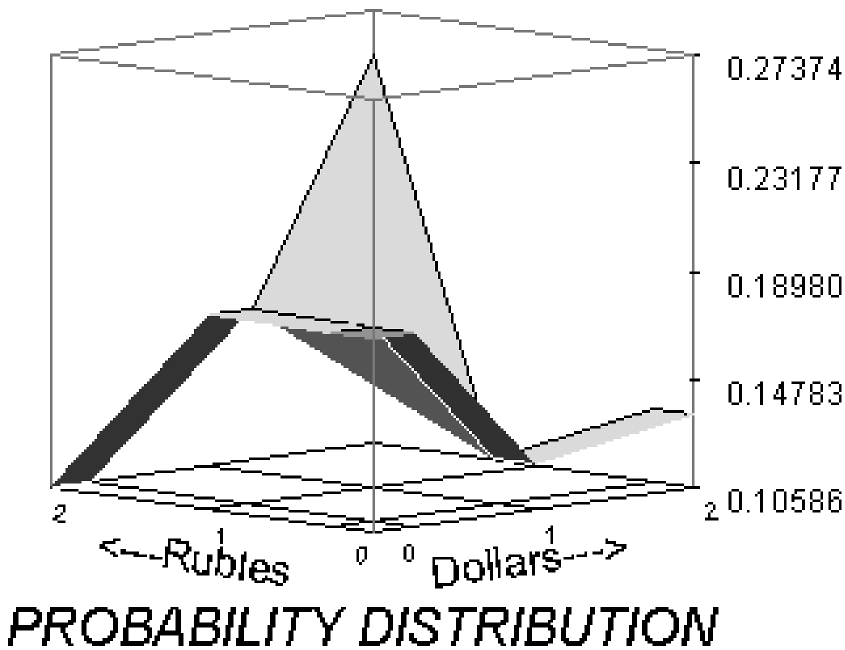
Substituting in the utility function and rearranging generates

(A30) $V_1 \geq [(1+A)^{\sigma} - 1]^{1/(1-\sigma)}$

which places a lower bound on V_1.

Figure 1

Figure 2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0.23177
0.18980
0.14783
0.10586
2
1
0
<---Rubles
0
1
2
Dollars--->
PROBABILITY DISTRIBUTION

Figure 3a

N =10, Symmetric Currencies M_1=M_2=3.3 αx/r = 4.5, σ = .15 μ=η=0

Figure 3b

N =10, Symmetric Currencies M_1=M_2 =3.3 αx/r = 4.5, σ = .5 μ=η=0

Figure 3c

N =10, Symmetric Currencies $M_1=M_2=3.3$ $\alpha x/r = 4.5$, $\sigma = .75$ $\mu=\eta=0$

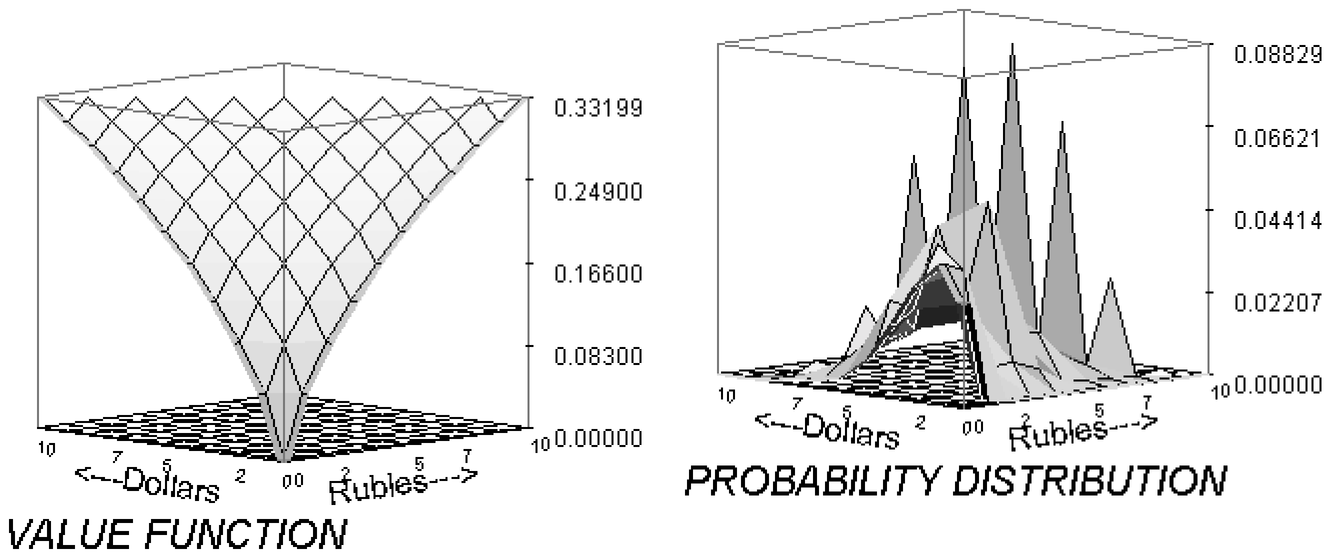


Figure 3d

N =10, Symmetric Currencies $M_1=M_2=3.3$ $\alpha x/r = 5$, $\sigma = .15$ $\mu = \eta = 0.0000001$

Figure 3d

N =10, Symmetric Currencies $M_1=M_2=3.3$ $\alpha x/r = 4.5$, $\sigma = .15$ $\mu=\eta=0.0000001$

Figure 4

N =15, Symmetric Currencies M_1=M_2=5 $\alpha x/r$ = 4.5, σ = .15 μ=η=0

Figure 5

N =10, Symmetric Currencies M_1= 5.5, M_2= 2.25, $\alpha x/r$ = 4.5, σ = .15, μ=η=0

Figure 6

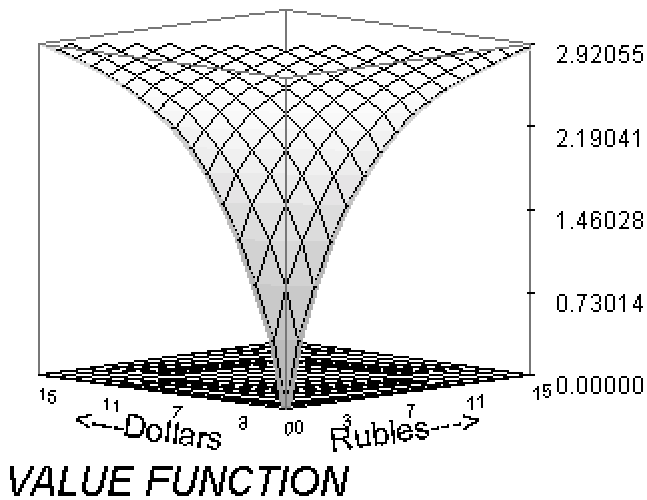
Making Change, $N = 10$, Symmetric Currencies $M_1 = M_2 = 3.3$ $\alpha x/r = 4.5$, $\sigma = .15$ $\mu = \eta = 0$

Figure 7

$N = 10$, $M_1 = 3.3$, $M_2 = 1.9$, $\alpha x/r = 4.5$, $\sigma = .15$ $\mu = 0.0025$, $\eta = 0.01$

Figure 8a

N =15, M_1= 5, M_2 =2.36, $\alpha x/r$ = 4.5, σ = .15 μ=0.00125, η=0.01



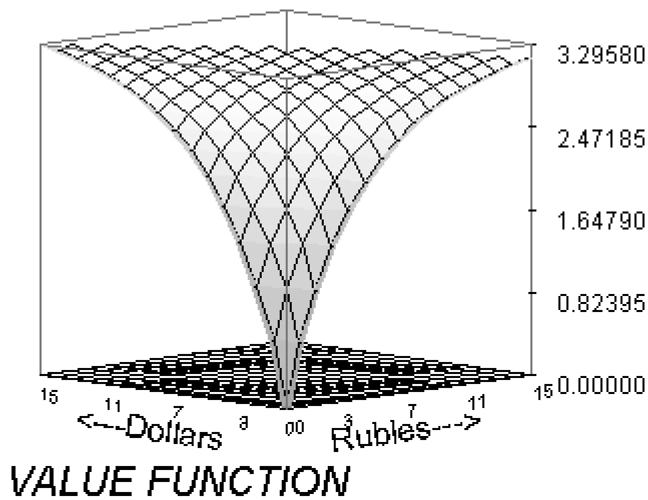
Note: Axis reversed on the right probability distribution for graphical clarity.

Figure 8b

$N = 15$, $M_1 = 5$, $M_2 = 2.36$, $\alpha x/r = 4.5$, $\sigma = .15$ $\mu = 0.0025$, $\eta = 0.01$

Figure 8c

N =15, M_1= 5, M_2 = 1.15, αx/r = 4.5, σ = .15 μ=0.005, η=0.01



Note: Axis reversed on the right probability distribution for graphical clarity.

Figure 8d

Distribution of Exchange Rates

N =15, M_1 = 5, $\alpha x/r$ = 4.5, σ = .15, η=0.01

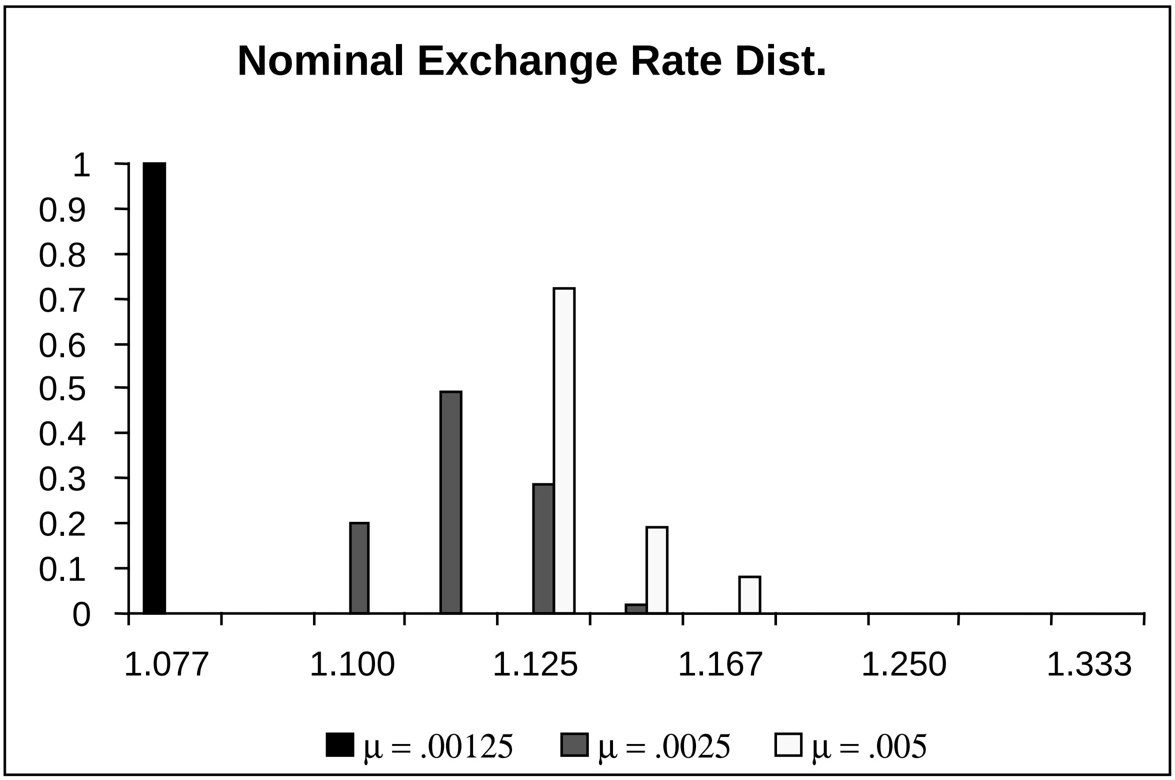
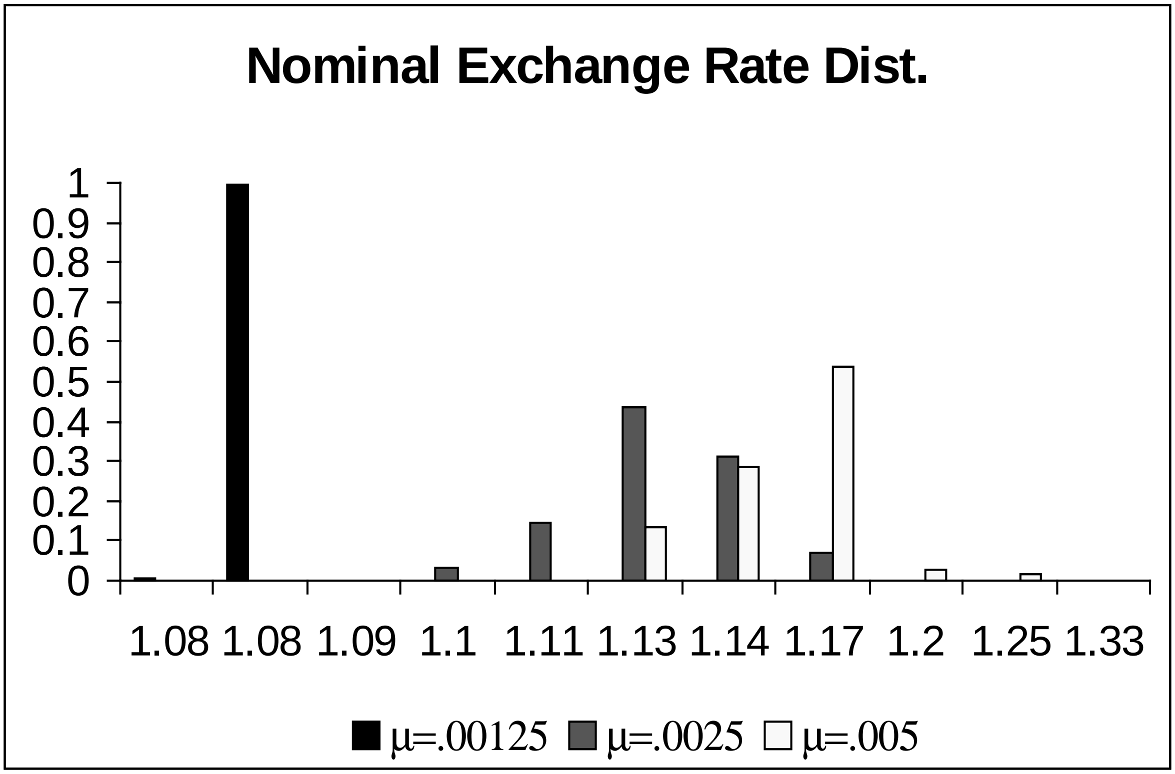


Figure 8e

Distribution of Exchange Rates

N =15, M_1 = 3, $\alpha x/r$ = 4.5, σ = .15, η=0.01



Abstract

We analyze a dual currency search model in which agents are allowed to hold multiple units of both currencies. Hence, agents hold portfolios of currency. We study equilibria in which the two currencies are identical and equilibria in which the two currencies differ according to the magnitude of the 'inflation tax' risk associated with each currency. The inflation tax is modeled by having government agents randomly confiscate the two currencies at different rates. We are able to obtain analytical results in a very special case but in general we must rely on numerical methods to solve for the steady-state distributions of currency portfolios, prices and value functions. We find that when one of the currencies has the right amount of 'risk', equilibria exist in which the safe currency trades for multiple units of the risky currency (pure currency exchange). As a result, the steady state has a distribution of *nominal* exchange rates. The mean and variance of the nominal exchange rate distribution is based on the fundamentals of the model such as the risk of confiscation, risk preferences, matching probabilities and relative money supplies. The mean and variance of this distribution typically change in predictable ways when the fundamentals change. While the ability to trade currencies improves average welfare, in general, the benefits of currency exchange are small.