State-Dependent Pricing, Inflation, and Welfare in Search Economies.*

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Abstract

We investigate the welfare effects of inflation in economies with search frictions and menu costs. We first analyze an economy where there is no transaction demand for money balances: Money is a mere unit of account. We determine a condition under which strictly positive inflation is desirable. We relate this condition to a standard efficiency condition for search economies. Second, we consider a related economy in which there is a transaction role for money. In the absence of menu costs, the Friedman rule is optimal. In the presence of menu costs, the optimal inflation rate is negative for our numerical examples provided menu costs are small. A deviation from the Friedman rule can be optimal depending on the extent of the search externalities.

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1 Introduction

Lagos and Wright (2005) showed that models with explicit microfoundations for monetary exchange can be used for policy analysis and they can generate new useful predictions. While the Lagos-Wright model emphasizes search frictions to explain the usefulness of money, it omits nominal frictions that many think are important to capture the effects of monetary policy. For instance, Diamond (1993, p. 53) argued that "*some degree of price stickiness is a necessary part of a realistic transaction technology.*" This paper brings monetary theory one step closer to policy analysis by introducing menu costs into a continuous time version of the Lagos-Wright model.

Building on the literature pioneered by Kiyotaki and Wright (1991, 1993), we adopt a model where agents trade in bilateral meetings and where means of payment are needed to mitigate a double-coincidence-of-wants problem. We also endogenize the frequency of trades through a free-entry condition so that inflation affects both the quantities traded in individual matches (i.e., the intensive margin) and the number of matches (i.e., the extensive margin). The introduction of nominal rigidities is based on the model of Sheshinki and Weiss (1977). Sellers who incur a fixed cost to change their prices adjust them only infrequently by following an endogenous (S, s) rule.[1]

In order to disentangle different inefficiencies associated with inflation, and in order to contrast our analysis with previous search models with sticky prices (e.g., Benabou (1988) and Diamond (1993)), we first study a cashless economy where there is no transaction demand for money balances. We are able to derive analytically a condition under which a positive inflation rate is optimal. This condition states that inflation can be good for society when sellers have too much market power, or equivalently, when the congestion imposed by sellers in the goods market is too severe. In the presence of sticky prices, a deviation from price stability mitigates

[1] For recent empirical evidence suggesting that price adjustments are infrequent, see Levy, Bergen, Dutta, and Venable (1997) and Bils and Klenow (2004). The macroeconomic literature on state-dependent price adjustments was pioneered by Caplin and Spulber (1987) and Caplin and Leahy (1991), and in search-theoretic environments, by Benabou (1988, 1992), Diamond (1993), and Diamond and Felli (1992). Recent contributions include Dotsey, King, and Wolman (1999) and Golosov and Lucas (2003).

this inefficiency by reducing sellers' incentives to enter the market.

The introduction of a transaction role for money into the previous environment brings two new insights. First, the result obtained in cashless economies according to which deflation is never optimal does not survive the introduction of an inflation tax. In the absence of menu costs, the Friedman rule is optimal, and in the presence of menu costs, provided menu costs are small, the optimal inflation rate is negative. Second, the presence of nominal rigidities matters for the optimality of the Friedman rule. Depending on the extent of the search externalities and sellers' market power, it is sometimes optimal to keep inflation above the level prescribed by the Friedman rule.[2] Also, we illustrate how the presence of nominal frictions can enhance welfare and how it eliminates a real indeterminacy at the Friedman rule.

The paper is organized as follows. Section 2 describes the environment. Section 3 solves a model in which there is no money but prices are posted in terms of a unit of account. Section 4 investigates a monetary model with flexible prices, and then with sticky prices. All proofs of lemmas and propositions are relegated to the appendix.

2 The environment

Time is continuous and goes on forever. Some trades take place in a centralized market, and others in a decentralized market with bilateral random matching. There are two types of perishable goods, a *special good* and a *general good*. Whereas the general good is produced and traded in the centralized market, the special good is traded in the decentralized market.[3]

The economy is populated with a continuum of infinitely-lived agents divided into two categories, called *buyers* and *sellers* to reflect their trading behaviors in the decentralized market. The measure of buyers is normalized to one. The measure of sellers, denoted n, will be endogenous. Buyers differ from sellers both in the goods they produce and in their consumption

[2] These results are not inconsistent with findings in new Keynesian models based on monopolistic competition and time-dependent pricing. In those models, the Friedman rule is optimal if prices are flexible despite the presence of imperfect competition. With sticky prices, deflation is still optimal but the deflation rate can be lower than the one at the Friedman rule. For details and references, see Khan, King, and Wolman (2002).

[3] The assumption that trades take place in both centralized and decentralized markets was introduced by Lagos and Wright (2005). The environment described in this paper is closer to Rocheteau and Wright (2005).

preferences. Whereas both types of agents consume and produce the general good, buyers, unlike sellers, also want to consume the special good, and sellers, unlike buyers, produce the special good.

	Buyers	**Sellers**
General good	consume produce	consume produce
Special good	consume	produce

Agents' trading behaviors

The utility of consuming q units of the special good is $u'q$ with $u' > 0$.[4] The disutility of producing the special good is $c(q)$ with $c(0) = c'(0) = 0$, $c'(q) > 0$ and $c''(q) > 0$ for $q > 0$, and $c(q) = u'q$ for some $q > 0$. The instantaneous utility function of buyers and sellers in the centralized market is simply x, where x is the net consumption flow of general goods.[5] Given this specification, producing the general good for oneself is worthless. Buyers and sellers discount future utility at the same rate, $\rho > 0$.

Unmatched agents trade in the centralized market. They are thrown into a bilateral match, i.e., in the decentralized market, according to a stochastic Poisson process. When matched, agents do not have access to the centralized market.[6] Matched agents choose whether or not to trade, split apart immediately after the trade has occurred, and return to the centralized market. Since an agent does not have the ability to produce general goods while in the decentralized market, he can only transfer the special good he produces or the assets he holds at the time he is matched.[7]

We will consider two polar cases regarding agents' abilities to use credit. We will first

[4] The linearity of the utility function in terms of special goods, also used by Benabou (1988), will simplify greatly sellers' pricing strategy. We will argue, however, that our main results should be robust to alternative specifications.

[5] The linear specification for the utility functions for centralized market goods is a key assumption to obtain a tractable model in which the distribution of money holdings is easy to handle. See Lagos and Wright (2005).

[6] For a somewhat related formalization where centralized and decentralized markets open concurrently, see Williamson (2006).

[7] One could assume instead that even though general goods can be produced in bilateral matches, the seller does not wish to consume the general good produced by the buyer in the match.

consider an economy in which buyers are able to commit to repay their debts and therefore can use IOUs to trade in the decentralized market. We will call this economy a *cashless economy.*[8] We will then consider a *monetary economy* where buyers are unable to commit to repay their debt and need to use money in order to trade in the decentralized market.

The trading opportunities in the decentralized market are described by a standard random-matching technology. The instantaneous matching probability of a buyer is $\alpha(n)$, whereas the instantaneous matching probability of a seller is $\alpha(n)/n$. Furthermore, $\alpha' > 0$ and $\alpha'' < 0$, $\alpha(0) = 0$, $\alpha'(0) = \infty$, $\alpha'(\infty) = 0$ and $\lim_{n\to\infty} \alpha(n) = \infty$. We denote $\eta(n) = \alpha'(n)n/\alpha(n)$ the elasticity of the matching function. As we also want to endogenize n, we assume that sellers who participate in the decentralized market incur a flow cost, $k > 0$, to search for buyers and to advertise their products.[9]

There exists a good called money that is intrinsically useless but that serves as a unit of account. The monetary price of the general good is $w(t)$. It will be exogenous in the cashless economy, and will be determined by a market-clearing condition in the monetary economy. In the decentralized market, we adopt the following pricing protocol. Unmatched sellers post a monetary price. They can change their posted prices at any time at the cost γ in terms of utility. When a match occurs, the transaction price is chosen as follows. With probability $1-\theta$, every unit that is produced is sold at the seller's posted price. The quantity traded is the minimum of the buyer's demand and the seller's supply at this price. With probability $\theta \in (0,1)$, however, the buyer makes a take-it-or-leave-it offer. If this offer is rejected by the seller, no trade takes place. One can think of this pricing procedure as bargaining with nominal rigidities, or price posting with imperfect commitment.[10] This pricing captures the observation that transaction prices often differ from posted prices. There are two additional reasons to give

[8]Related cashless economies with state-dependent pricing are studied in Caplin and Spulber (1987), Benabou (1988, 1992), Diamond (1993), Golosov and Lucas (2003), among others.

[9]The assumption of free-entry is standard in the search literature to endogenize the number of trades. See, among others, Pissarides (2000), Diamond (1993) and Rocheteau and Wright (2005).

[10]For instance, a seller instructs a sales clerk to sell his output at the posted price. The commitment technology is imperfect in the sense that the sales clerk is not always in the shop, let's say, because it is too costly to have an employee full time. This is related to the assumption of costly price commitment introduced by Bester (1994).

buyers some market power by allowing them to make offers in some matches. First, without this assumption there would be no monetary equilibrium in the economy with flexible prices.[11] Second, this assumption will allow us to derive a simple condition on θ and $\eta(n)$ under which a deviation from price stability is optimal in the cashless economy.

3 Cashless economies

In this section, we describe an economy in which there are no monetary frictions and agents do not hold nominal assets. As emphasized earlier, this environment is closely related to the one in Diamond (1993) and it will provide a useful benchmark to compare against our monetary economy in Section 4.[12] Also, we will derive several analytical results that will prove useful to build our intuition on the effects of inflation in the presence of nominal and search frictions.

Buyers use credit arrangements to trade in the decentralized market. They commit to repay their debt in the general goods market straight after a trade has occurred.[13] Sellers post a monetary price at which they commit to sell their output. The monetary price of general goods, $w(t)$, is exogenous and is growing at rate $\pi \geq -\rho$. In the following, we will refer to the real price p as the nominal price posted by sellers divided by the price of general goods, w. Note that p decreases at rate π as long as the monetary price remains unchanged.

Consider a buyer. The Poisson arrival rate of a match in the decentralized market is $\alpha(n)$. With probability θ, the buyer makes a take-it-or-leave-it offer (q_b, d_b), where q_b is the quantity of the special good produced by the seller, and d_b is the quantity of general goods that the buyer

[11] In order to allow for the existence of a monetary equilibrium in an economy with price posting, one can introduce heterogeneity across buyers. See Curtis and Wright (2004) and Ennis (2004). One can interpret our assumption that buyers get the whole surplus of the match with probability θ as a reduced-form for this heterogeneity.

[12] Our model differs from Diamond (1993) in several dimensions. First, buyers can appropriate the surplus of a match in a fraction θ of the meetings, whereas in Diamond's model, θ is assumed to be 0. Second, the quantity traded in each match is endogenous, whereas in Diamond, it is set exogenously at 1. Third, buyers can trade repeatedly in the search market whereas buyers only trade once in Diamond's environment. As we will show, the last two assumptions are not crucial to Diamond's results while the first one – the fact that sellers have all the market power – is.

[13] Diamond (1993, p.56) assumes that the purchasing power held by customers while searching is earning the nominal interest rate which increases point for point with the inflation rate. He argues that "this assumption fits with payments by check or credit card rather than currency".

commits to deliver (the subscript b reflects the assumption that the offer is made by a buyer). With probability $1-\theta$, the buyer trades at the posted price: He consumes q_s in exchange for $d_s = pq_s$ units of general goods. The quantity q_s, which is the minimum of the buyer's demand and the seller's supply at the posted price, is a function of p. The value function of a buyer, W^b, satisfies the following flow Bellman equation

$$\rho W^b = \alpha(n)\left\{\theta\left[u(q_b) - d_b\right] + (1-\theta)\int \left[u' q_s(p) - pq_s(p)\right] dH(p)\right\}, \tag{1}$$

where $H(p)$ is the distribution of real prices across sellers.

Consider next a seller. As shown by Sheshinsky and Weiss (1977), in the presence of menu costs sellers change their price according to an (s, S) rule. If $\pi > 0$, the real price of the seller falls until it reaches the *trigger point* s and is then readjusted to the *target point* S. Conversely, if $\pi < 0$ the real price increases steadily from s to S. The length of the period of time between two price adjustments is denoted τ. Suppose the seller has not adjusted his price for a period of time of length $h \in (0, \tau)$. His posted price expressed in terms of the general good is $p(h) = Se^{-\pi h}$ if $\pi > 0$, and $p(h) = se^{-\pi h}$ if $\pi < 0$. The seller's expected utility, $W^s(h)$, obeys the following Bellman equation (see the Appendix)

$$\rho W^s(h) = -k + \frac{\alpha(n)}{n} G\left[p(h)\right] + \frac{\partial W^s(h)}{\partial h}, \tag{2}$$

where $G(p)$, the seller's expected trade surplus, satisfies

$$G(p) = (1-\theta)\left\{q_s(p)p - c\left[q_s(p)\right]\right\} + \theta\left\{d_b - c(q_b)\right\}. \tag{3}$$

Equation (2) has the following interpretation. The seller incurs the cost k to participate in the market. A match occurs with instantaneous probability $\alpha(n)/n$, in which case the seller's expected surplus is $G(p)$. The last term on the right-hand side of (2) reflects the fact that the seller's value function is not constant over the (S, s) cycle.

At the end of the (S, s) cycle, i.e., when $h = \tau$, the seller readjusts his price and starts a new cycle so that $W^s(\tau) = W^s(0) - \gamma$. Furthermore, the free entry of sellers implies $W^s(0) - \gamma = 0$, and $W^s(\tau) = 0$. This simply means that a seller who readjusts his price is in the same position as a new entrant. Using this terminal condition, equation (2) can be written in integral form as

$$W^s(h) = \int_h^\tau e^{-\rho(t-h)} \left[-k + \frac{\alpha(n)}{n} G\left[p(t)\right] \right] dt. \tag{4}$$

Finally, $W^s(0) = \gamma$ yields

$$\int_0^\tau e^{-\rho t} \left[-k + \frac{\alpha(n)}{n} G\left[p(t)\right] \right] dt = \gamma. \tag{5}$$

According to (5), the expected discounted utility of a seller over the (S, s) cycle has to be equal to the cost of setting a new price.

3.1 Equilibrium

To characterize equilibrium, we need to specify how terms of trade are formed in the decentralized market. Consider a match between a buyer and a seller whose posted price is p. As previously stated, the transaction price differs from the posted price with probability θ. In this case, the buyer makes a take-it-or-leave-it offer (q_b, d_b) in order to maximize his utility $u(q_b) - d_b$, subject to the sellers' participation constraint, $-c(q_b) + d_b \geq 0$. The solution is then $q_b = q^*$ and $d_b = c(q^*)$, where q^* solves $u' = c'(q^*)$.

With probability $1 - \theta$, agents trade at the posted price. However, the seller can choose not to serve all the buyer's demand at that price. The buyer's demand corresponds to the value of q that maximizes his surplus $u'q - pq$. It is unbounded if $u' > p$ and it is 0 if $u' < p$.[14] We call u' the buyer's reservation price. Also, sellers produce no more than the quantity q that maximizes $qp - c(q)$. Therefore, $q_s(p)$ is given by

$$q_s(p) = \begin{cases} c'^{-1}(p) & \text{if} \quad p \leq u' \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

It should be noticed here that the assumption of divisible output which will play an important role in the monetary economies is not crucial for the results of the cashless economies.

The optimal pricing policy of sellers is such that S and τ maximize sellers' expected utility as given by the left-hand side of (5). It is characterized in the following Lemma.

[14] In the knife-edge case, where $u' = p$, the buyer is indifferent between trading or not. To guarantee that the seller's pricing problem has a solution, we assume that the buyer's demand is at least equal to the quantity q^* that the seller is willing to produce at this price.

Lemma 1 *(i) If $\gamma = 0$ then sellers set a real price equal to u'. (ii) Assume $\gamma > 0$. The sellers' optimal pricing policy (S,τ) satisfies $S = u'$ and*

$$\alpha(n)G(u'e^{-|\pi|\tau}) = \begin{cases} nk, & \text{if} \quad \pi > 0 \\ n(\rho\gamma + k), & \text{if} \quad \pi < 0 \end{cases}, \tag{7}$$

and $\tau = \infty$ if $\pi = 0$.

For all inflation rates, sellers target the buyers' reservation price u'. They do not let their price go beyond this target. This result is intuitive since a seller's expected sales fall to 0 if his price is above u'. Obviously, this particular form for the (S, s) rule hinges crucially on the linearity of buyers' utility function.[15] In the presence of inflation, the opportunity cost of delaying the price adjustment is $\rho W^s(\tau) = 0$ (in flow terms), whereas the instantaneous benefit is $\frac{\alpha(n)}{n} G[p(\tau)] - k$. Therefore, the seller adjusts his price when his instantaneous utility falls to 0.[16] In the case of deflation, the seller readjusts his price when it reaches the buyer's reservation price, u', irrespective of his choice for s. The opportunity cost for the seller of delaying the price adjustment by setting a price smaller than s is equal to $\rho W^s(0) = \rho\gamma$. From (7) there is a symmetry between inflation and deflation only when $\rho \to 0$.

All through the paper, we focus on time-invariant cross-sectional distributions of real prices by assuming that real prices are log-uniformly distributed over $[s, S]$. As shown by Caplin and Spulberg (1987) and Benabou (1988), the log-uniform distribution is the only one that is time invariant and consistent with the (s, S) rule.[17] Equivalently, the length of the period of time during which a seller's price has been kept unchanged is uniformly distributed on $[0, \tau]$.

Definition 1 *An equilibrium is a pair (n, τ) that satisfies (5) and (7) if $\pi \neq 0$ or (5) and $\tau = \infty$ if $\pi = 0$.*

[15] For alternative specifications for the utility function, the (S, s) rule would be such that the upper-bound S overshoots the ideal price of the seller. See, for instance, Benabou (1992). We discuss the importance of this assumption at the end of the section.

[16] One may wonder why the menu cost γ does not appear in Equation (7) when $\pi > 0$. The reason is that the continuation value of a seller who readjusts his price is $-\gamma + W^s(0)$. From the free-entry condition, this term is 0. Note, however, that the menu cost γ appears in the free-entry condition.

[17] Under a steady inflation, a log-uniform distribution of prices is equivalent to a uniform staggered timing (Rotemberg, 1983).

We illustrate the determination of equilibrium for the case $\pi > 0$ in Figure 1. The free-entry curve corresponds to (5), whereas the pricing curve corresponds to (7). The pricing-curve slopes downward since sellers need to readjust their prices more frequently when the market is congested. The pricing-curve intersects the free-entry-curve when the latter reaches a maximum: The number of sellers is highest when the frequency of price adjustment is chosen optimally. As π increases, the free-entry-curve shifts downward (see dotted curve), n decreases, and $\pi\tau$ increases. Inflation drives sellers out of the market and raises (real) price dispersion.[18] Results are analogous when $\pi < 0$. An increase in deflation (a reduction in π) raises price dispersion and reduces the measures of sellers in the market. These results are summarized in the following proposition.

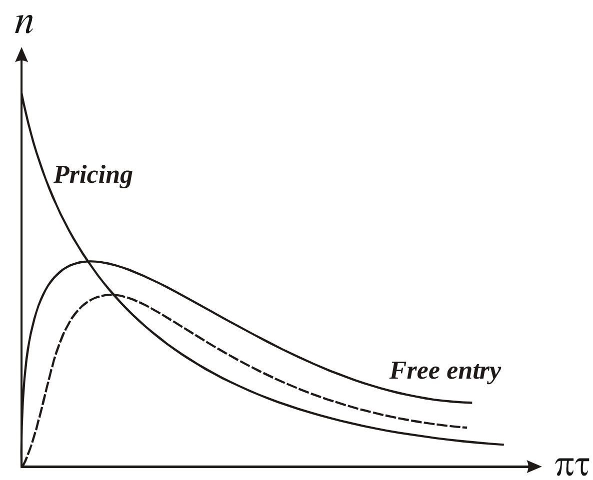

Figure 1: Equilibrium ($\pi > 0$).

Proposition 1 *If $\theta < 1$, equilibrium exists and is unique. If $\gamma = 0$, n is independent of π. If $\gamma > 0$ and $\pi > 0$ then $\partial\pi\tau/\partial\pi > 0$ and $\partial n/\partial\pi < 0$. If $\gamma > 0$ and $\pi < 0$ then $\partial\pi\tau/\partial\pi > 0$ and $\partial n/\partial\pi > 0$.*

[18] For a search-theoretic model with endogenous price dispersion and flexible prices, see Head and Kumar (2005) and Head, Kumar and Lapham (2004).

We measure society's welfare as the expected surplus of buyers per unit of time.

$$\mathcal{W}^b = \alpha(n) \left\{ \theta \left[u'q^* - c(q^*) \right] + (1-\theta) \int_0^{|\pi|\tau} u'q_s(u'e^{-h}) \left(1 - e^{-h}\right) \frac{dh}{|\pi|\,\tau} \right\}. \qquad (8)$$

This welfare measure is legitimate since the free-entry condition drives sellers' expected utility to 0. Denote n_0 the measure of sellers when $\pi = 0$.

Proposition 2 *If $\gamma = 0$, equilibrium is efficient iff $1 - \theta = \eta(n_0)$.*

The condition in Proposition 2 is similar to the one derived by Hosios (1990) for an efficient allocation in the presence of congestion externalities. It states that the measure of sellers is socially efficient if the fraction of the matches where sellers appropriate the whole surplus of a match coincides with sellers' contribution to the matching process as measured by the elasticity of the matching function. If $1 - \theta > \alpha'(n)n/\alpha(n)$, n is too high. Although it may not be well known, this is the main inefficiency in the Diamond (1993) economy.

Proposition 3 *Provided γ is sufficiently small, the optimal inflation rate is strictly positive if $\theta < [1 - \eta(n_0)] \,/\, [1 + \eta(n_0)]$.*

Proposition 3 indicates under which circumstances positive inflation is desirable. An increase in inflation has two opposite effects on buyers' welfare. It raises the ability of buyers to extract a higher surplus, but it also reduces the number of sellers and therefore the frequency of trades. If θ is low, the first effect dominates and price dispersion raises buyers' welfare. In Diamond's (1993) economy, $\theta = 0$ so the condition in Proposition 3 is satisfied.

According to Proposition 1, inflation and deflation have similar effects on price dispersion and the measure of sellers. It is therefore not obvious that optimal inflation is positive when $\theta < [1 - \eta(n_0)] \,/\, [1 + \eta(n_0)]$. The intuition for this result goes as follows.[19] If buyers could choose the inflation rate, they would face a trade-off between the larger share in the gains from trade that is associated with higher price dispersion and the lower frequency of trade that is associated with the smaller number of sellers in the market. For a given price dispersion $|\pi\tau|$,

[19] This result explains the numerical example provided by Diamond and Felli (1992).

inflation hurts sellers less than deflation does. Indeed, if $\pi > 0$, sellers set their prices to S and get high profits at the beginning of the (S, s) cycle, whereas if $\pi < 0$, they set their prices to s and get low profits first. Therefore, it is optimal to reduce sellers' market power by running a positive inflation instead of a deflation.

4 Monetary economies

We now introduce a transaction role for money by assuming that buyers are anonymous in the centralized market and cannot commit to repay their debts. In the absence of credit arrangements, trades in bilateral matches need to be quid pro quo, and this requires buyers hold money balances. The price of general goods in terms of money is now endogenous. We will determine the optimal monetary policy in the absence of menu costs, and we will investigate how the presence of nominal frictions affects policy.

The quantity of fiat money per buyer is $M(t) > 0$. The growth rate of the money supply is constant over time and equal to $\pi \geq -\rho$; that is, $\dot{M} = \pi M$. Money is injected (withdrawn if $\pi < 0$) by lump-sum transfers (taxes). For simplicity, transfers go only to buyers. We will restrict our attention to steady-state equilibria in which the real value of money M/w is constant over time, i.e., $\dot{w} = \pi w$.

Let $W^b(z)$ be the value of an unmatched buyer holding z units of real money (expressed in terms of the general good). The stochastic time for a buyer to find a seller, denoted T_b, is characterized by an exponential distribution with mean $1/\alpha$. The value function $W^b(z)$ satisfies

$$W^b(z_0) = \max_{\{x(t),z(t)\}} \mathbb{E}\left[\int_0^{T_b} e^{-\rho t} x(t)dt + e^{-\rho T_b} V^b\left[z(T_b)\right]\right] \tag{9}$$

$$\begin{aligned} \text{s.t.} \quad x + \dot{z} &= L - \pi z, && (10) \\ z(0) &= z_0, && (11) \end{aligned}$$

where $x(t)$ is the net consumption flow of general goods at time t, where $V^b(z)$ is the value function of a matched buyer holding z units of real money, and where the trajectory $\{x(t), z(t)\}$ is contingent on $t < T_b$.[20] The first term on the right-hand side of (9) is the utility of consumption minus the disutility of production over the time interval $[0, T_b]$. The second term is the present value of being matched at time T^b with $z(T_b)$ units of real money. Equation (10) is a budget identity. The term L on the right-hand side is a lump-sum transfer expressed in terms

[20] Implicitly, we allow for jumps in the state variables. For a presentation of optimal control problems with jumps in state variables, see Seierstad and Sydsaeter (1987, chapter 3).

of the general good, and the last term reflects the fact that real balances depreciate at rate π.[21] The initial condition for real balances is given by (11).

From the assumption that T_b is exponentially distributed, (9) can be rewritten

$$W^b(z_0) = \max_{\{x(t),z(t)\}} \int_0^\infty e^{-[\rho+\alpha(n)]t} \Big[x(t) + \alpha(n)V^b\left[z(t)\right]\Big]\, dt, \tag{12}$$

subject to (10) and (11).[22] Interestingly, (12) is analogous to a deterministic optimal control problem in which the effective discount rate is $\rho + \alpha(n)$ and the instantaneous utility is $x + \alpha(n)V^b(z)$.

Lemma 2 *Assume $V^b(z)$ is concave. Buyers adjust their real balances instantly to a $\hat{z}$ that satisfies*

$$V^b_z(z) = 1 + \frac{\rho + \pi}{\alpha(n)}. \tag{13}$$

The left-hand side of (13) is the benefit for the buyer of an additional unit of real balances, whereas the right-hand side is the marginal cost of real balances. This cost, measured in terms of the general good, is the sum of the forgone unit of the general good and the cost of holding real balances, as measured by the sum of the discount rate and the inflation rate, over a period of time of length $1/\alpha(n)$. Assuming $V^b(z)$ is strictly concave over a relevant range, the solution to (13) is unique and the steady-state distribution of real balances across buyers is degenerate at $z = \hat{z}$.[23] Given that the buyer adjusts his real balances to $\hat{z}$ instantly, we have $W^b(z) = -(\hat{z} - z) + W^b(\hat{z})$, and in particular, $W^b(z) = z + W^b(0)$.

The value function of a matched buyer satisfies

$$V^b(z) = \theta\Big\{u'q_b(z) + W^b\left[z - d_b(z)\right]\Big\}$$
$$+(1-\theta)\int \Big\{u'q_s(z,p) + W^b\left[z - pq_s(z,p)\right]\Big\}\, dH(p), \tag{14}$$

[21] Let m be the buyer's nominal balances. Then, $z = m/w$ and $\dot{z} = -(\dot{w}/w)(m/w) + \dot{m}/w$. To obtain (10), use the fact that $\dot{w}/w = \pi$ and $\dot{m}/w = -x + L$.

[22] See the Appendix for a derivation of (12).

[23] If $V^b(z)$ is linear over the relevant range for the choice of z, we restrict ourselves to symmetric equilibria. This will be the case for $\gamma > 0$ and π close to $-\rho$.

where q_s depends on both the buyer's real balances and the real price p posted by the seller. With probability θ, the buyer has the ability to make an offer (q_b, d_b). He enjoys the utility of consumption $u'q_b$ and becomes unmatched with $z - d_b$ units of real balances. With probability $1-\theta$, the buyer trades at the posted price p and consumes q_s units of goods for pq_s units of real balances, where q_s is determined as before. Using the linearity of $W^b(z)$, (14) can be rewritten

$$V^b(z) = \theta \left\{ u'q_b(z) - d_b(z) \right\}$$

$$+(1-\theta) \int \left(u' - p \right) q_s(z,p) dH(p) + z + W^b(0). \tag{15}$$

The seller's value function obeys the Bellman equation (2), where G is now a function of $F(z)$, the distribution of real balances across buyers,

$$G(p) = \int (1-\theta) \left\{ q_s(z,p)p - c\left[q_s(z,p) \right] \right\} + \theta \left\{ d_b(z) - c[q_b(z)] \right\} dF(z). \tag{16}$$

Let us turn to the determination of prices. In the fraction θ of the matches where the buyer has the ability to make a take-it-or-leave-it offer, he proposes (q_b, d_b) that satisfies

$$\max_{(q_b,d_b)} \left[u'q_b - d_b \right] \quad \text{s.t.} \quad -c(q_b) + d_b = 0 \quad \text{and} \quad d_b \leq z. \tag{17}$$

The main difference with respect to the previous section is that the transfer d_b is constrained by the buyer's (monetary) wealth. The solution to (17) is

$$q_b = \begin{cases} q^* & \text{if} \quad z \geq z^* \equiv c(q^*) \\ c^{-1}(z) & \text{otherwise} \end{cases} . \tag{18}$$

According to (18), if $z > z^*$, then $d_b \leq z$ is not binding and agents trade the efficient quantity q^*. If $z < z^*$, then the constraint binds and the buyer spends all his real balances to buy less than q^*. In the fraction $1-\theta$ of the matches where agents trade at the posted price, the buyer demands z/p if $u' \geq p$ and 0 otherwise, and sellers are willing to produce up to $\bar{q}$ such that $c'(\bar{q}) = p$. Therefore,

$$q_s(z,p) = \begin{cases} \min\left(c'^{-1}(p), \frac{z}{p} \right), & \forall p \leq u', \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

4.1 Flexible prices

The case where prices can be adjusted at no cost ($\gamma = 0$) will allow us to contrast the monetary economy with the economy described in the previous section.[24] In particular, we will show that even though inflation drives sellers out of the market, as in the cashless economy with sticky prices, this effect is not welfare-enhancing in the monetary economy with flexible prices.

When prices can be adjusted at no cost, the sellers' optimal pricing strategy is $p = u'$ for any distribution of real balances $F(z)$. The reasoning is similar to the one in Lemma 1. If buyers choose $p > u'$, they make no trade, and sellers have no incentive to choose a price lower than u' since for all $p \leq u'$ buyers spend all their real balances. This pricing strategy allows sellers to extract all the surplus of a match whenever agents trade at the posted price. From (15), the Bellman equation for the value function of a buyer can be simplified to

$$V^b(z) = \theta \left\{ u' q_b(z) - c\left[q_b(z)\right] \right\} + z + W^b(0).$$

Differentiate $V^b(z)$ to get

$$V_z^b(z) = \begin{cases} \theta \left\{ \frac{u'}{c'[q_b(z)]} - 1 \right\} + 1 & \text{if} \quad z < z^*, \\ 1 \quad \text{otherwise.} \end{cases} \tag{20}$$

Equation (20) can be interpreted as follows. Consider a match where the buyer makes a take-it-or-leave-it offer. If the buyer brings one additional unit of real balances, he increases his consumption by $1/c'(q_b)$, which is worth $u'/c'(q_b)$ in terms of utility. Consider next a match where the buyer trades at the posted price. The buyer can increase his consumption by $1/u'$, which is worth 1 in terms of utility. From (20), $V_{zz}^b < 0$ for all $z < z^*$. From Lemma 2, there is a unique solution $\hat{z} < z^*$ to (13), and the distribution of real balances across buyers is degenerate.[25] Furthermore, (13) and (20) yield:

$$\frac{\rho + \pi}{\alpha(n)\theta} = \frac{u'}{c'\left[q_b(z)\right]} - 1. \tag{21}$$

[24] This model is related to the search equilibrium described in Rocheteau and Wright (2005).

[25] To check that $z < z^*$, note first that $V_z^b(z)$ is strictly decreasing for all $z < z^*$. Furthermore, $\rho + \alpha(n) + \pi > \alpha(n) V_z^b(z)$ for all $z \geq z^*$ and for all $\pi > -\rho$.

Free entry of sellers implies $W^s(0) = 0$, which gives from (2)

$$\frac{\alpha(n)}{n}(1-\theta)\left\{u' q_s(z,u') - c\left[q_s(z,u')\right]\right\} = k. \tag{22}$$

Definition 2 *A monetary equilibrium with flexible prices is a list* (z, n) *that satisfies (21) and (22).*

Assume $\pi > -\rho$ (We will treat the case $\pi = -\rho$ separately). We cannot rule out multiplicity of steady-state equilibria. When such multiplicity arises, we focus on the equilibrium with the highest values for q and n as it converges to an equilibrium at the Friedman rule.

Proposition 4 *If* $\theta = 0$ *or* $\theta = 1$*, there is no monetary equilibrium. If* $\theta \in (0,1)$*, there exists* $\bar{\pi} > -\rho$ *such that a monetary equilibrium exists for all* $\pi \in (-\rho, \bar{\pi})$*. At the equilibrium with the highest* z*,* $\partial z/\partial\pi < 0$ *and* $\partial n/\partial\pi < 0$*.*

If $\theta = 0$, there is no demand for real balances and hence no monetary equilibrium. Similarly, if $\theta = 1$, there is no active equilibrium because sellers have no incentive to enter the market. So, for a monetary equilibrium, we need $\theta \in (0,1)$. In contrast to the model in the previous section, inflation is no longer neutral when $\gamma = 0$. An increase in π reduces the quantities traded in bilateral matches as well as the measure of sellers in the market. Therefore, one may conjecture that the result for cashless economies, that inflation is welfare improving when n is too high, carries over to monetary economies. The following proposition shows that this conjecture is wrong.

Proposition 5 *Welfare is decreasing with* π*, and the optimal monetary policy is the Friedman rule. Furthermore,* $\lim_{\pi\downarrow-\rho} q_b = q^*$ *and* $\lim_{\pi\downarrow-\rho} q_s < q^*$*.*

According to Proposition 5, the monetary equilibrium is inefficient since the quantities traded in matches where agents trade at the posted price are always inefficiently low, even at the limit when π approaches $-\rho$. This monopolistic competition inefficiency is related to the fact that sellers do not internalize the effects of their pricing decisions on buyers' real balances.[26] Even if the quantities traded in bilateral matches were efficient, the entry of sellers would generically be inefficient because of the presence of search externalities.

Let us turn to the case $\pi = -\rho$. Denote n^* the value of n that satisfies (22) when $q_s = q^*$.

Proposition 6 *If $\pi = -\rho$ then any (z, n) such that $z \in [c(q^*), u'q^*]$ and n satisfies (22) is an equilibrium. Equilibria are strictly positively Pareto-ranked according to z. There is an equilibrium that generates the first-best allocation iff $\theta = 1 - \eta(n^*)$.*

From the previous proposition, there is a real indeterminacy at the Friedman rule. There exists a continuum of equilibria and these equilibria are Pareto-ranked. Intuitively, when $\pi+\rho = 0$ there is no cost of holding real balances so that buyers are indifferent between any level of real balances above $c(q^*)$. However, an increase in real balances above $c(q^*)$ allows sellers to extract a higher surplus in matches where buyers trade at the posted price. If one selects the equilibrium by taking the limit $\pi \to -\rho$, this equilibrium corresponds to the one with the lowest welfare, i.e., the one with the lowest real balances.

Calibration We calibrate this model using the methodology in Lucas (2000). A unit of time corresponds to a year and $r = 0.03$. We define B as the aggregate output in the centralized market.[27] The functional forms for the disutility of production is $c(q) = q^{\delta+1}/(\delta+1)$. We adopt the normalization $u' = 1$. As in Lagos and Wright (2005), we choose the parameters (δ, B) to fit money demand in the model to the data. Money demand is defined as $L \equiv M/PY$. In the model, nominal output in the centralized market is pB, and nominal output in decentralized market is $\alpha(n)M$. Hence, $PY = pB+\alpha(n)M$ and $Y = B+\alpha(n)M/p$. In equilibrium, $M/P = z$, and so

$$L = \frac{M/P}{Y} = \frac{z}{B + \alpha(n)z},$$

[26] This inefficiency should be distinguished from the bargaining inefficiency based on the nonmonotonicity of the generalized Nash solution in Lagos and Wright (2005).

[27] It is indeterminate in the model because of the linearity of the utility function. The only requirement imposed by the model is $B \geq \alpha(n)z$. We introduce this parameter to pin down the relative size of the decentralized market and to make the calibration as close as possible to the one in Lagos and Wright (2005).

where the pair (z, n) solves (21) and (22). Hence L is a function of $i = r + \pi$. Following Lucas (2000), i is taken to be the commercial paper rate and let M be $M1$. The sample period is 1900-2000.

The matching technology takes the following functional form: $\alpha(n) = n^{\eta}$ where $\eta \in (0, 1)$. Since there are no study on the matching technology in goods market and the best fit for money demand is obtained for small values of η we pick $\eta = 0.1$. We choose k to generate the same frequency of trade as in Lagos and Wright (2005), that is $\alpha(n) = 0.5$, at $\pi = 2\%$.[28] Finally, the parameter θ is chosen so as to generate an average markup of 10% when $\pi = 2\%$, i.e.,

$$\frac{B}{B + \alpha(n)z} + \left[\frac{\alpha(n)z}{B + \alpha(n)z}\right]\left[\theta \frac{z}{q_b c'(q_b)} + (1 - \theta)\frac{z}{q_s c'(q_s)}\right] = 1.1.$$

The markup in the centralized market is 1. In the decentralized market, the markup is $z/q_b c'(q_b)$ if the buyer makes the offer and $z/q_s c'(q_s)$ otherwise. The parameter values for our benchmark example are recapitulated in Table 1.

Preferences	$u' = 1$ $\delta = 0.65$
Discount rate	$r = 0.03$
Matching	$\eta = 0.10$
Pricing	$\theta = 0.34$
Entry cost	$k = 76.16$
General good	$B = 0.79$

Table 1: Parameter values.

We measure welfare as buyer's expected utility in the decentralized market plus the net consumption of buyers and sellers in the centralized market (which is 0 by definition of the utility function). We omit sellers' expected utility in the decentralized market since it is 0 from the free-entry condition. Welfare is then

$$\mathcal{W}_{\pi}^{b} = \alpha(n_{\pi})\theta \left[u' q_{\pi}^{b} - c(q_{\pi}^{b})\right],$$

where (n_{π}, q_{π}^{b}) are the equilibrium values of n and q_b when the inflation rate is π. Suppose next that the inflation rate is set at $\pi = 0$ but total consumption is reduced by a factor $1 - \Delta$ (where

[28] As in Lagos and Wright (2005) the best fit for money demand is obtained for low values of buyers' frequency of trade.

Δ will be our measure of the welfare cost of inflation). Society's welfare is then

$$\mathcal{W}_0^b(\Delta) = \alpha(n_0)\theta \left[u' q_0^b (1-\Delta) - c(q_0^b) \right] - B\Delta,$$

where the last term is the reduction in consumption in the centralized market. The welfare cost of an inflation rate π is the fraction Δ of total consumption that agents would be willing to give up to be in a steady state with no inflation instead of a steady state with inflation π, i.e., Δ solves $\mathcal{W}_\pi^b = \mathcal{W}_0^b(\Delta)$.

π (%)	-3	0	5	10
z	0.61	0.40	0.22	0.13
$\alpha(n)$	0.52	0.51	0.48	0.46
Δ (%)	-0.43	0	1.17	2.26

Table 2: Flexible price economy

Table 2 reports the equilibrium values for real balances (z), the frequency of trade (α) and the welfare cost of inflation (Δ) at different inflation rates. As shown in Proposition 4 real balances and the frequency of trade fall with inflation. The welfare cost of inflation increases with π and the Friedman rule is the optimal monetary policy. The first-best allocation is such that $q_b = q_s = 1$ and $\alpha(n) = 0.43$. So, in equilibrium the measure of sellers is too high and the quantities traded in bilateral matches are too low. The welfare cost of 10 percent inflation is 2.26% of total consumption which is lower than the estimates in Lagos and Wright (2005) —but still higher than Lucas' (2000) estimates. The difference with Lagos and Wright (2005) arises from the fact that the participation of sellers is endogenous in our model. An increase of the inflation rate reduces the inefficiently large measure of sellers which mitigates the negative effect of inflation on real balances.

4.2 Sticky prices

Assume now that sellers must incur a fixed cost to change prices. As in the previous section, we focus on steady-state equilibria in which the distribution of (real) prices, $H(p)$, is time-invariant and price adjustments are uniformly staggered. The seller's value function obeys a flow Bellman equation analogous to condition (2), but where G is given by (16). The pricing policy of sellers is still given by Lemma 1.

We describe the buyer's choice of real balances in the case where the growth rate of the money supply is positive ($\pi > 0$). If a buyer meets a seller at random, the price posted by the seller is $p(h) = u'e^{-\pi h}$, where h is uniformly distributed over $[0, \tau]$. The quantity q_s satisfies

$$q_s(z, u'e^{-\pi h}) = \begin{cases} z/u'e^{-\pi h} & \text{if } h \leq \tilde{h}, \\ c'^{-1}\left(u'e^{-\pi h}\right) & \text{if } h > \tilde{h}, \end{cases}$$

where $\tilde{h}$ is the value of h such that $c'(q_s) = p(\tilde{h})$; i.e., $ze^{\pi\tilde{h}} = u'c'^{-1}\left(u'e^{-\pi\tilde{h}}\right)$. The seller serves all the buyer's demand if $p(h) \geq p(\tilde{h})$; otherwise, the buyer is rationed. From (15) and (19), the expected utility of a matched buyer can be rewritten as[29]

$$V^b(z) = \theta\left\{u'q_b(z) - c\left[q_b(z)\right]\right\} + (1-\theta)z\int_0^{\min\left[\tilde{h}(z),\tau\right]} \tau^{-1}\left(e^{\pi h} - 1\right)dh$$

$$+(1-\theta)u'\int_{\min\left[\tilde{h}(z),\tau\right]}^{\tau} \tau^{-1}c'^{-1}\left(u'e^{-\pi h}\right)\left(1 - e^{-\pi h}\right)dh + z + W^b(0). \tag{23}$$

Let us interpret the second term on the right-hand side of (23). If $h < \tilde{h}$, the seller satisfies all the buyer's demand and the buyer's surplus is equal to $(u' - p)q_s = z\left(\frac{u'-p}{p}\right) = z\left(e^{\pi h} - 1\right) > 0$. Therefore, the buyer can extract a positive surplus even when trading at the posted price. The third term on the right-hand side has a similar interpretation.

Differentiate $V^b(z)$ to get

$$V_z^b(z) = \theta\left[\frac{u'}{c'\left[q_b(z)\right]} - 1\right]^+ + (1-\theta)\int_0^{\min\left[\tilde{h}(z),\tau\right]} \tau^{-1}\left(e^{\pi h} - 1\right)dh + 1, \tag{24}$$

where $[x]^+ = \max(x, 0)$. If the buyer can make a take-it-or-leave-it-offer, one additional unit of real balances allows him to raise his utility by $\frac{u'}{c'[q_b(z)]}$. If the buyer trades at the posted price, $p(h) = u'e^{-\pi h}$, and assuming that the seller satisfies all his demand, an additional unit of real balances allows him to buy $e^{\pi h}/u'$ units of the special good which is worth $e^{\pi h}$ in terms of utility.

[29] To obtain (23), we use the fact that for all $h < \tilde{h}$, $(u' - p\phi)\, q_s = ze^{\pi h} - z$, whereas for all $h > \tilde{h}$, $(u' - p\phi)\, q_s = u'\left(1 - e^{-\pi h}\right)c'^{-1}(u'e^{-\pi h})$.

From (24), $V^b(z)$ is concave, and strictly concave for all $z < z^*$ and for all z such that $\tilde{h}(z) < \tau$. From (13) and (24),

$$\frac{\rho + \pi}{\alpha(n)} = \theta \left[\frac{u'}{c'(q_b)} - 1 \right]^+ + (1 - \theta) \int_0^{\min\left[\tilde{h}(z), \tau\right]} \tau^{-1} \left(e^{\pi h} - 1 \right) dh. \tag{25}$$

Inflation has two opposite effects on the value of money. It raises the opportunity cost of holding cash, the left-hand side of (25), and it transfers some market power to the buyer, the second term on right-hand side of (25). The second effect is absent from the model with flexible prices.

The buyer's choice of real balances in the case of deflation ($\pi < 0$) is derived using similar reasoning,

$$\frac{\rho + \pi}{\alpha(n)} = \theta \left[\frac{u'}{c'(q_b)} - 1 \right]^+ + (1 - \theta) \int_0^{\min\left[\tilde{h}(z), \tau\right]} \tau^{-1} \left[e^{-\pi l} - 1 \right] dl, \tag{26}$$

where $\tilde{h}(z)$ satisfies $ze^{-\pi\tilde{h}} = u'c'^{-1}(u'e^{\pi\tilde{h}})$.

Definition 3 *A steady-state monetary equilibrium with menu cost is a list* (z, n, τ) *that satisfies (5), (7) and, if* $\pi > 0$*, (25), or if* $\pi < 0$*, (26).*

At the Friedman rule, $\rho + \pi = 0$ and, from (26), $q_b = q^*$, and $\tilde{h}(z) = 0$. Therefore, the quantities produced and consumed in matches where agents trade at the posted price obey $c'(q_s) = u'e^{-|\pi|h}$ for all $h \in [0, \tau]$. In particular, $z = u'q^*$ and $q_s(0) = q^*$. The economy is then analogous to the cashless economy studied in the previous section.

4.3 Calibrated examples

The model with endogenous real balances and state-dependent pricing is hard to study analytically. Therefore, we conduct our analysis through numerical examples. We use the same parameter values as in Table 1. We choose the menu cost γ so that prices are adjusted once a year at $\pi = 2\%$ ($\gamma = 0.536$). In Figure 2 the panels on the left (right) plot the endogenous variables for negative (positive) inflation rates.

In presence of menu costs and positive inflation (top right panel of Figure 2), z is a decreasing function of π. As outlined in (25), inflation raises the cost of holding real balances, but it also allows buyers to extract a larger share of the gains from trade. For our parametrization, the

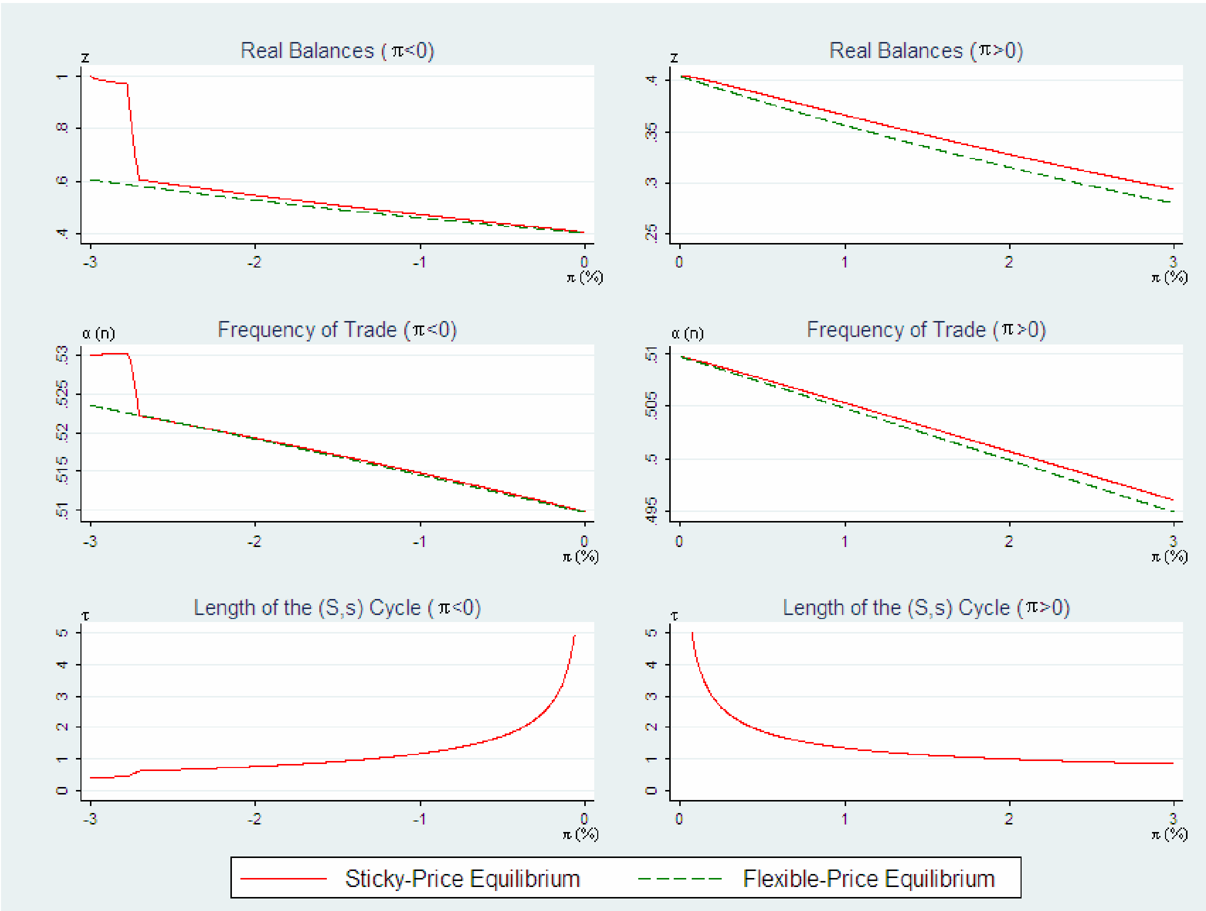

Figure 2: Equilibrium for the baseline calibration.

first effect dominates.[30] Real balances are higher in the presence of menu costs since the second effect of inflation is absent from the model with flexible prices.

Consider next the case of negative money growth rates (top left panel of Figure 2). An increase in the deflation rate reduces the cost of holding real balances and it raises buyers' average share in the gains from trade. As a consequence, z increases. The relationship between z and π exhibits an inflection point when z is in the neighborhood of z^*. This result can be explained as follows. At $z = z^*$, $\tau < \tilde{h}$ so that buyers are never rationed. When the inflation

[30]For some parameter values, the second effect dominates at low but positive inflation rates. In this case z is a hump-shaped function of π. See our examples with larger menu costs in Figure 5.

rate falls below the value that generates z^*, buyers increase their real balances until they get rationed in some matches: z increases to the value that satisfies $\tilde{h}(z) = \tau$. Also, the presence of nominal frictions eliminates the real indeterminacy at the Friedman rule: as π tends to $-\rho$, z approaches $u'q^*$.

Let us turn to the effects of inflation on the frequency of trades —See the two panels on the second row of Figure 2. Inflation reduces sellers' incentives to enter the market since they have to readjust prices more frequently. Hence, $\alpha(n)$ decreases with π. Deflation has two effects on the measure of sellers. On the one hand buyers carry more real balances which gives sellers higher incentives to enter the market. On the other hand a higher deflation rate implies that sellers need to readjust their prices more often. For low deflation rates, the first effect dominates while when π gets closer to the Friedman rule, the second effect dominates.

The effects of inflation on the length of the (S, s) cycle are in accordance with those obtained in cashless economies (See the bottom panels of Figure 2). As inflation increases, sellers need to readjust their prices more often (but price dispersion increases).

Figure 3 plots the welfare cost of inflation expressed as a fraction of total consumption (in the centralized and decentralized market). For positive money growth rates, the welfare cost of inflation is a U-shaped function of π. A small increase in inflation above price stability is welfare-improving because inflation raises buyers' gains from trade and therefore their incentives to invest in real balances. Note however that the welfare gains of positive inflation are tiny. A stronger effect occurs when one reduces π below 0. For our example, the welfare gain of reducing π from 0 to the Friedman rule is about 0.7% of total consumption. So, deflationary policies dominate inflationary ones. This result is robust across the various numerical examples we have considered provided menu costs are not (unreasonably) large.

To summarize, the presence of nominal rigidities can explain why a small positive inflation rate generates a higher welfare than price stability. However, when looking at the optimal monetary policy, deflation does better than inflation (subject to the caveat that menu costs are not too large). This contrasts with our result for cashless economies.

Figure 3: Cost of inflation under the baseline calibration

4.4 Optimality of the Friedman rule

Close to the Friedman rule, the economy behaves similarly to the cashless economy described in Section 3. The number of sellers decreases as π decreases (equivalently, the deflation rate increases) because of the negative effect of price dispersion on sellers' expected utility. If the number of sellers is inefficiently high at the Friedman rule —which is the case in our calibrated example— then the Friedman rule is optimal since an increase of the inflation rate would make the number of sellers even higher. The case where there are too many sellers because of congestion externalities corresponds to low values of θ relative to $(1-\eta)/(1+\eta)$ —See Proposition 3.

In Figure 4, we provide numerical examples where the Friedman rule is not optimal. Consider the following parameter values: $\eta = 0.5$, $k = 0.1$, $\delta = 0.65$, $\rho = 0.03$ and $\gamma = 0.001$. For $\theta = 0.45$ the number of sellers is too low at the Friedman rule. So a deviation from the Friedman

Figure 4: (Sub)optimality of the Friedman rule

rule can be optimal because an increase in inflation raises the entry of sellers. In contrast, when $\theta = 0.25$ the measure of sellers is too large at the Friedman rule so that increasing inflation would reduce welfare. For this parametrization we found that the threshold for θ above which the Friedman rule is no longer optimal is slightly above 1/3, the threshold for θ above which a deviation from price stability is suboptimal in cashless economies. So the extent of search externalities matters in cashless economies to explain the optimality of price stability while it matters in monetary economies to explain the optimality of the Friedman rule.

4.5 The effects of menu costs

Figure 5 illustrates how the size of menu costs affects real balances and the frequency of trade. The parameter values are the same as in Table 1 except for γ. According to the top panels of Figure 5 a larger menu cost raises real balances by increasing the buyer's (expected) share in the match surplus. Also, for positive inflation rates the curve for real balances is hump-shaped. A small increase in inflation can increase real balances. The medium-left panel of Figure 5 shows that the measure of dealers increases with inflation close to the Friedman rule and this effect is amplified for large menu costs. By deflating at a higher rate the monetary authority reduces sellers' market power (they have to readjust prices more often) and it drives sellers out. On the contrary, for positive inflation rates the frequency of trade increases with the size of the menu cost since buyers' real balances are higher. Finally, and not surprisingly, the bottom panels of Figure 5 reveals that sellers readjust prices less frequently when menu costs are larger.

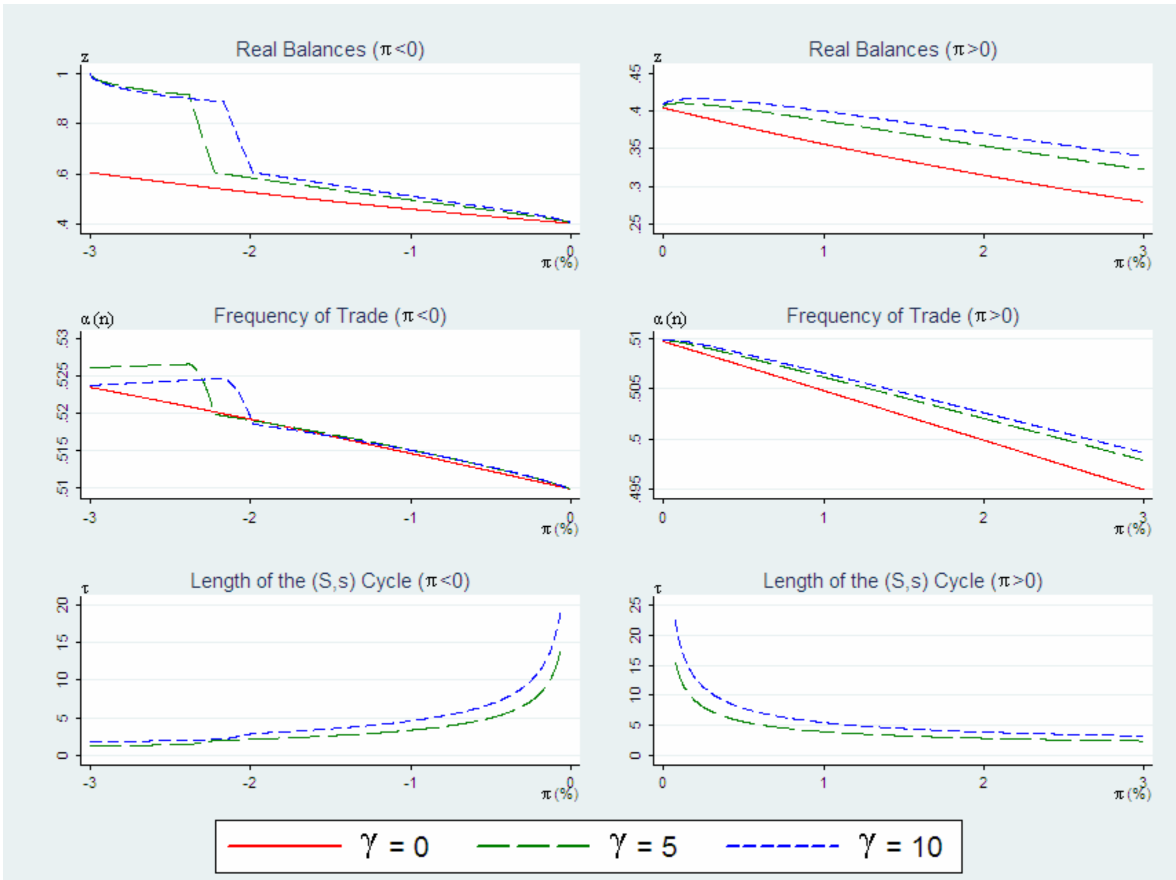

Figure 5: Equilibrium for $\gamma \in \{0, 5, 10\}$

Figure 6 plots buyers' welfare for different values of γ. (We do not plot Δ since the calibration is not quantitatively relevant for large values of γ.) Buyers are better-off when menu costs are larger. The reason for this result are twofold. Larger menu costs reduce the inefficiently high number of sellers (under our parametrization), and it also raises real balances. Welfare is maximized at the Friedman rule for all values of γ. (As argued earlier, this result holds because $\theta < (1-\eta)/(1+\eta)$.)

Figure 6: Buyers' welfare for $\gamma \in \{0, 5, 10\}$

Finally, we have been able to find (atypical) examples where the optimal inflation is positive. Such examples require relatively large menu costs. For instance, for the parameter values $\rho = 0.03$, $\delta = 1.5$, $\eta = 0.5$, $k = 0.1$, $\gamma = 0.05$ and $\theta = 0.29$ the optimal inflation is slightly less than 1%.

5 Discussion

We studied optimal monetary policy in environments with search and nominal frictions. We showed that search frictions generate a congestion externality in the goods market that can rationalize a role for positive inflation in cashless economies and a role for a deviation from the Friedman rule in monetary economies. In the presence of menu costs, inflation erodes sellers' market power by preventing them from maintaining a monopoly price. This effect can be beneficial to society when sellers do not internalize the congestion they impose on other sellers. In monetary economies, however, a monetary wedge associated with a positive nominal interest rate makes deflation optimal. A deviation from the Friedman rule is optimal when sellers have relatively low market power. Our results are summarized in Table 3.

From the standpoint of models of nominal price rigidity, our paper brings two new elements: a micro-founded demand for real balances and search externalities. Both elements are essential to understand the welfare effects of inflation. The first element yields a somewhat standard inflation tax effect. Inflation reduces real balances and therefore it lowers the quantities that agents trade in bilateral meetings. One novelty comes from the fact that our pricing mechanism tends to amplify the monetary wedge as in Lagos and Wright (2005). The second element, search externalities, arises naturally in random-matching environments when participation decisions are introduced. Since inflation act as a tax on participation in the market it can exacerbate or mitigate the inefficiencies associated with participation decisions. In the absence of search externalities the optimal inflation would always be 0 in our cashless economy. (And there would be no welfare effect of inflation in Diamond's (1993) model). In the monetary economy, the presence externalities matter to the extent that they can make the Friedman rule suboptimal. Without entry the Friedman rule would always optimal.

	Flexible prices	**Sticky prices**
No monetary wedge	π is neutral	$\pi^* = 0$ if θ high $\pi^* > 0$ if θ low
Monetary wedge	$\pi^* = -\rho$	$\pi^* > -\rho$ if θ high $\pi^* = -\rho$ if θ low

Table 3. Summary of the results

To conclude, we discuss the robustness of our results to alternative assumptions. The main result of Section 3, according to which the optimal inflation can be strictly positive in the presence of search frictions, was derived under a linear utility function. As a consequence, S is equal to the seller's ideal price, i.e., the one he would choose if $\gamma = 0$. For strictly concave utility functions, the upper-bound S can be larger than the seller's ideal price. The positive welfare effect of inflation in our model relies on the congestion externality that prevails in the goods market. Inflation in the presence of menu costs drives sellers out of the market, which raises welfare when the congestion is too severe. We conjecture that inflation will reduce sellers' expected profits by preventing them from setting a monopoly price irrespective of the specific form taken by the (S, s) rule.

In our description of the monetary economy, all trades in the decentralized market are conducted with money. Alternatively, one could assume that only a fraction of trades involve monetary exchange, while the remaining trades use credit (say, because agents have access to a record-keeping technology with some probability). As the fraction of monetary trades goes to 0, the economy approaches the cashless economy described in Section 3. We conjecture that the optimal inflation would depend on the extent of monetary exchange in the decentralized market and that it could be positive provided that the share of monetary trades is sufficiently small.

As emphasized above, our welfare results rely heavily on the search externalities associated with sellers' participation decisions. While we have assumed a free entry of sellers as in Diamond (1993) or Rocheteau and Wright (2005), one could consider alternative ways to capture participation decisions: Agents could choose to be buyers or sellers in the decentralized market (Shi, 1997) or buyers' and sellers' search intensities could be endogenous (Berentsen, Rocheteau, and Shi, 2006). As long as the Hosios (1990) condition for efficiency is violated and sellers have too much bargaining power, inflation in the presence of menu costs should raise welfare in cashless economies by raising the buyer's share of the gains from trade. In monetary economies, inflation has a direct negative effect on buyers' welfare by acting as a proportional tax on real balances. Its effect on society's welfare is not obvious. But, as illustrated by Craig and Rocheteau (2006), positive inflation can be optimal in such economies, even in the absence of nominal frictions.

Appendix

A1. Derivation of (2)

The seller's value function $W^s(h)$ obeys the following Bellman equation:

$$W^s(h) = \Pr[T_s \leq \tau - h]$$
$$\times \mathbb{E}\left[\int_0^{T_s} e^{-\rho t}(-k)\,dt + e^{-\rho T_s}\left[G\left[p(h+T_s)\right] + W^s(h+T_s)\right]\middle| T_s \leq \tau - h\right]$$
$$+ \Pr[T_s > \tau - h]\left\{\int_0^{\tau-h} e^{-\rho t}(-k)\,dt + e^{-\rho(\tau-h)}W^s(\tau)\right\}, \tag{27}$$

where τ is the length of the period of time between two price adjustments, which has the following interpretation. If $T_s \leq \tau - h$, then the seller meets a buyer before he readjusts his price, and his expected surplus is $G\left[p(h+T_s)\right]$, where $p(h+T_s)$ is the price posted by the seller. If $T_s > \tau - h$, then no trade occurs before the seller's real price hits the trigger point s.

The distribution for T_s conditional on $T_s \leq \tau - h$ is a truncated exponential distribution. Thus, $\Pr[T_s \leq \tau - h] = 1 - e^{-\frac{\alpha(n)}{n}(\tau-h)}$. Consequently, (27) can be rewritten

$$W^s(h) =$$
$$\int_0^{\tau-h} \frac{\alpha(n)}{n} e^{-\frac{\alpha(n)}{n}t}\left[\frac{(-k)\left(1-e^{-\rho t}\right)}{\rho} + e^{-\rho t}\left[G\left[p(h+t)\right] + W^s(h+t)\right]\right]dt$$
$$+ e^{-\frac{\alpha(n)}{n}(\tau-h)}\frac{(-k)}{\rho}\left(1-e^{-\rho(\tau-h)}\right) + e^{-\left(\rho+\frac{\alpha(n)}{n}\right)(\tau-h)}W^s(\tau). \tag{28}$$

Using the change of variable $u = h + t$, (28) yields

$$W^s(h) =$$
$$\int_h^{\tau} \frac{\alpha(n)}{n} e^{-\frac{\alpha(n)}{n}(u-h)}\left[\frac{(-k)\left(1-e^{-\rho(u-h)}\right)}{\rho} + e^{-\rho(u-h)}\left[G\left[p(u)\right] + W^s(u)\right]\right]du$$
$$+ e^{-\frac{\alpha(n)}{n}(\tau-h)}\frac{(-k)}{\rho}\left(1-e^{-\rho(\tau-h)}\right) + e^{-\left(\rho+\frac{\alpha(n)}{n}\right)(\tau-h)}W^s(\tau). \tag{29}$$

Differentiate (29) with respect to h to obtain

$$\frac{\partial W^s(h)}{\partial h} = -\frac{\alpha(n)}{n} G\left[p(h)\right] + k + \rho W^s(h).$$

A2. Proof of Lemma 1.

We prove Lemma 1 for the case where buyers hold real balances z and cannot use credit. The distribution of real balances across buyers is $F(z)$.

Case 1: $\gamma = 0$. The seller chooses p in order to maximize

$$G(p) \equiv (1-\theta) \int \max_{q \leq z/p} \left[pq - c\left(q\right)\right] \mathbf{1}_{\{p \leq u'\}} dF(z), \tag{30}$$

where $\mathbf{1}_{\{p \leq u'\}}$ is an indicator function that is equal to 1 if $p \leq u'$ and 0 otherwise. According to (30), in each match the seller chooses the quantity q to produce, subject to the constraint that q is not greater than the buyer's demand. If $p > u'$ the buyer's demand is 0, and if $p \leq u'$ then the buyer's demand is z/p. Denote $\xi(z)$ the Lagrange multiplier corresponding to $pq \leq z$. The seller's problem can be rewritten as

$$\max_p \int \max_q \left\{\left[pq - c\left(q\right)\right] \mathbf{1}_{\{p \leq u'\}} + \xi(z)\left[z - pq\right]\right\} dF(z). \tag{31}$$

Assume $p \leq u'$ so that $\mathbf{1}_{\{p \leq u'\}} = 1$. The first-order condition for $q(z)$ is

$$p\left[1 - \xi(z)\right] = c'\left[q(z)\right]. \tag{32}$$

Differentiate $G(p)$ and use (32) to obtain

$$G'(p) = (1-\theta) \int \frac{q(z)c'\left[q(z)\right]}{p} dF(z) > 0, \quad \forall p \leq u'.$$

Therefore, the optimal price is $p = u'$.

Case 2: $\gamma > 0$ and $\pi > 0$. First, we show $S = u'$ using a proof by contradiction. Assume that the optimal (S, s) rule is such that $S > u'$. Using the fact that $G(p) = 0$ for all $p > u'$, (4) yields

$$W^s(0) = -\int_0^{\ln(S/u')/\pi} e^{-\rho t} k dt + \int_{\ln(S/u')/\pi}^{\ln(S/s)/\pi} e^{-\rho t} \left[-k + \frac{\alpha(n)}{n} G(S e^{-\pi t})\right] dt. \tag{33}$$

The first term on the right-hand side corresponds to the interval of time during which the seller's price is above u'. Since this term is negative, we have

$$W^s(0) < \int_{\ln(S/u')/\pi}^{\ln(S/s)/\pi} e^{-\rho t} \left[-k + \frac{\alpha(n)}{n} G(Se^{-\pi t}) \right] dt,$$

which can be re-expressed as

$$W^s(0) < e^{-\rho \frac{\ln(S/u')}{\pi}} \int_{\ln(S/u')/\pi}^{\ln(S/s)/\pi} e^{-\rho\left[t - \frac{\ln(S/u')}{\pi}\right]} \left[-k + \frac{\alpha(n)}{n} G(Se^{-\pi t}) \right] dt.$$

Since $e^{-\rho \frac{\ln(S/u')}{\pi}} < 1$ we have

$$W^s(0) < \int_{\ln(S/u')/\pi}^{\ln(S/s)/\pi} e^{-\rho\left[t - \frac{\ln(S/u')}{\pi}\right]} \left[-k + \frac{\alpha(n)}{n} G(Se^{-\pi t}) \right] dt.$$

Adopt the change of variable $\tilde{t} = t - \frac{\ln(S/u')}{\pi}$ to rewrite the previous inequality as

$$W^s(0) < \int_0^{\ln(u'/s)/\pi} e^{-\rho \tilde{t}} \left[-k + \frac{\alpha(n)}{n} G(u'e^{-\pi \tilde{t}}) \right] d\tilde{t}.$$

Consequently, a profitable deviation is to set $S = u'$ while keeping s unchanged. Therefore, $S > u'$ is not optimal. Consider next a (S, s) rule such that $S < u'$. Since $G(Se^{-\pi t})$ is increasing in S, it can be checked from (4) that for given τ, $W^s(0)$ is a strictly increasing function of S for all $S \leq u'$. Consequently, $S < u'$ is not optimal. Second, to determine the optimal length of the (S, s) cycle, differentiate the right-hand side of (4) with respect to τ to obtain (7).

Case 3: $\gamma > 0$ and $\pi < 0$. The reasoning for $S = u'$ is similar to the one for the case $\pi > 0$. To determine the optimal τ, express $W^s(0)$ as

$$W^s(0) = \int_0^{\tau} e^{-\rho(\tau - t)} \left[-k + \frac{\alpha(n)}{n} G(u'e^{\pi t}) \right] dt. \tag{34}$$

Differentiate (34) with respect to τ and use the fact that $W^s(0) = \gamma$ to obtain (7).

A3. Proof of Proposition 1

We consider three cases.

Case 1: $\gamma = 0$. The seller sets $p = u'$ and $\tau = \infty$. From (5),

$$-k + \frac{\alpha(n)}{n} G(u') = 0, \tag{35}$$

where $G(u') = (1-\theta)\left[u'q^* - c(q^*)\right] > 0$. Since $\alpha(n)/n$ is continuous and strictly decreasing, $\lim_{n\to 0} \alpha(n)/n = \infty$ and $\lim_{n\to\infty} \alpha(n)/n = 0$, there exists a unique n that satisfies (35) and it is such that $\partial n/\partial \pi = 0$.

Case 2: $\gamma > 0$ and $\pi > 0$. Using $p(t) = u'e^{-\pi t}$, Equations (5) and (7) yield

$$\max_{\tau \geq 0} \int_0^\tau e^{-\rho t} \left[-k + \frac{\alpha(n)}{n} G(u'e^{-\pi t})\right] dt = \gamma, \tag{36}$$

where

$$G(p) = (1-\theta) \max_q [pq - c(q)] \quad \text{if } p \leq u'$$

and $G(p) = 0$ otherwise. Let n^f be the value of n that satisfies (35). The left-hand side of (36) tends to ∞ as n approaches 0, and it is equal to 0 for all $n \geq n^f$. Furthermore, for all $n \in \left(0, n^f\right)$, the left-hand side of (36) is strictly decreasing in n. Consequently, there exists a unique $n \in \left[0, n^f\right]$ that satisfies (36). Given n, price dispersion $\pi\tau$ is determined by (7). Totally differentiate (36) to obtain

$$\frac{\partial n}{\partial \pi} = \frac{-n \int_0^\tau e^{-\rho t} \left[t u' e^{-\pi t} G'(u'e^{-\pi t})\right] dt}{\left[1 - \eta(n)\right] \int_0^\tau e^{-\rho t} G(u'e^{-\pi t}) dt} < 0,$$

where $\eta(n) = -\alpha'(n)n/\alpha(n)$. From (7), $\partial \pi\tau/\partial \pi > 0$.

Case 3: $\gamma > 0$ and $\pi < 0$. Using the fact that $p(t) = u'e^{\pi(\tau - t)}$, (5) and (7) can be rewritten as

$$\max_{\tau \geq 0} \int_0^\tau e^{-\rho t} \left[-k + \frac{\alpha(n)}{n} G\left(u'e^{\pi(\tau-t)}\right)\right] dt = \gamma. \tag{37}$$

Following the reasoning of Case 2, it is easy to show that there exists a unique $n \in \left[0, n^f\right]$ that satisfies (37). Furthermore, $\partial n/\partial \pi > 0$ and $\partial \left|\pi\right| \tau/\partial \pi < 0$.

A4. Proof of Proposition 2.

In the case $\gamma = 0$, $\mathcal{W} = \mathcal{W}^b = \alpha(n)[u(q) - c(q)] - kn$. The efficient value for n satisfies $\alpha'(n)\left[u'q^* - c(q^*)\right] = k$, whereas the equilibrium value for n satisfies $\alpha(n)(1-\theta)\left[u'q^* - c(q^*)\right] = nk$. The two coincide iff $\alpha'(n)n/\alpha(n) = 1 - \theta$.

A5. Proof of Proposition 3

We first show that an increase in inflation above price stability is optimal when $\theta < [1-\eta(n_0)]\,/\,[1+\eta(n_0)]$. The welfare metric is

$$\mathcal{W}^b = \alpha(n)\left\{\theta\left[u'q^* - c(q^*)\right] + (1-\theta)\int_0^1 u'q_s(l\pi\tau) - u'e^{-l\pi\tau}q_s(l\pi\tau)dl\right\}.$$

Differentiate and take the limit as $\pi\tau \to 0$ to obtain

$$\frac{d\mathcal{W}^b}{d\pi\tau} = \alpha'(n_0)\theta\left[u'q^* - c(q^*)\right]\frac{dn}{d\pi\tau} + \alpha(n_0)(1-\theta)\frac{u'q^*}{2}, \tag{38}$$

where the relationship between n and $\pi\tau$ is given by (7), i.e.,

$$\alpha(n)(1-\theta)\max_q[u'e^{-\pi\tau}q - c(q)] = nk. \tag{39}$$

From (39),

$$\frac{dn}{d\pi\tau} = \frac{-n_0 u'q^*}{[u'q^* - c(q^*)]\{1-\eta(n_0)\}}. \tag{40}$$

Substitute $dn/d\pi\tau$ by its expression given by (40) into (38) to get

$$\frac{d\mathcal{W}^b}{d\pi\tau} = \alpha(n)u'q^*\left\{\frac{(1-\theta)}{2} - \frac{\theta\eta(n_0)}{[1-\eta(n_0)]}\right\}.$$

It can be checked that $d\mathcal{W}^b/d\pi\tau > 0$ if $\theta < [1-\eta(n_0)]\,/\,[1+\eta(n_0)]$. As shown in Proposition 1 price dispersion $\pi\tau$ increases with π. Furthermore, $\lim_{\pi\downarrow 0}\pi\tau$ can be made arbitrarily close to 0 by choosing γ sufficiently small. Therefore, an increase in inflation above $\pi = 0$ is welfare-improving provided γ is sufficiently small.

In order to show that the optimal inflation rate is not negative, we use a proof by contradiction. Assume the optimal inflation rate is $\pi^* < 0$. Let $\tilde{\pi}$ be the positive inflation rate such that price dispersion $|\pi|\,\tau$ at $\pi = \tilde{\pi}$ is equal to price dispersion at $\pi = \pi^*$. From (7) the measure of sellers satisfies $n(\tilde{\pi}) > n(\pi^*)$. Therefore, from (8), $\mathcal{W}^b(\tilde{\pi}) > \mathcal{W}^b(\pi^*)$. A contradiction.

A6. Derivation of (12)

Since T_b is exponentially distributed, the maximand on the right-hand side of (9) can be rewritten as

$$RHS = \int_0^\infty\left[\int_0^t e^{-\rho s}x(s)ds + e^{-\rho t}V^b\left[z(t)\right]\right]\alpha(n)e^{-\alpha(n)t}dt. \tag{41}$$

Equation (41) yields

$$RHS = \int_0^\infty \int_0^\infty e^{-\rho s} x(s)\alpha(n) e^{-\alpha(n)t} \mathbf{1}_{\{s \leq t\}} ds dt$$
$$+ \int_0^\infty V^b [z(t)] \alpha(n) e^{-[\rho+\alpha(n)]t} dt. \tag{42}$$

Interchange the order of integration in the repeated integral to get

$$RHS = \int_0^\infty e^{-\rho s} x(s) \int_s^\infty \alpha(n) e^{-[\alpha(n)]t} dt ds$$
$$+ \int_0^\infty \alpha(n) V^b [z(t)] e^{-[\rho+\alpha(n)]t} dt. \tag{43}$$

Finally, substitute $e^{-[\alpha(n)]s}$ for $\int_s^\infty \alpha(n) e^{-[\alpha(n)]t} dt$ into (43) to obtain

$$\begin{aligned} RHS &= \int_0^\infty x(s) e^{-[\alpha(n)+\rho]s} ds + \int_0^\infty \alpha(n) V^b [z(t)] e^{-[\rho+\alpha(n)]t} dt, \\ &= \int_0^\infty e^{-[\rho+\alpha(n)]t} \left\{ x(t) + \alpha(n) V^b [z(t)] \right\} dt. \end{aligned}$$

A7. Proof of Lemma 2

Let λ be the current value costate variable associated with z. The current value Hamiltonian is

$$\mathcal{H}(x, z, \lambda) = x + \alpha(n) V^b (z) + \lambda (-x + L - \pi z). \tag{44}$$

Assuming an interior solution for z, the necessary conditions from Pontryagin's maximum principle are

$$\lambda = 1, \qquad \forall t \tag{45}$$
$$\rho\lambda = \alpha(n) \left[V_z^b (z) - \lambda \right] - \pi\lambda + \dot{\lambda}, \tag{46}$$

where V_z^b is the derivative of the value function $V^b(z)$. We add the following transversality condition from the Mangasarian sufficiency theorem

$$\lim_{t \to \infty} e^{-[\rho+\alpha(n)]t} \lambda(t) z(t) = 0. \tag{47}$$

To get (13), combine (45) and (46). Assuming $V^b(z)$ is concave, the Hamiltonian $\mathcal{H}(x, z, \lambda)$ is jointly concave in (x, z). Since the transversality condition is satisfied for the solution given by (45) and (46), it is a maximum. Furthermore, (10) implies $\dot{\lambda} = 0$ and z satisfies (13). The uniqueness of the solution to (13) follows from the strict concavity of $V^b(z)$. Note that the state variable z jumps to the solution of (13) instantly.

A8. Proof of Proposition 4

Part 1. There is no monetary equilibrium if $\theta = 0$ or $\theta = 1$. From (21), if $\theta = 0$, then $q = 0$ or $n = \infty$. From (22), if $n = \infty$, then $q = 0$. As a consequence, $q = 0$. From (22), if $\theta = 1$, then $n = 0$, which from (21) implies $q = 0$. In both cases, money is not valued in exchange.

Part 2. If $\theta \in (0, 1)$, there exists $\bar{\pi} > -\rho$ such that a monetary equilibrium exists for all $\pi < \bar{\pi}$. At the equilibrium with the highest z, $\partial z/\partial \pi < 0$ and $\partial n/\partial \pi < 0$. Equilibrium condition (21) can be reexpressed as

$$q = q(n; \pi) \equiv c'^{-1}\left[\frac{u'\alpha(n)\theta}{\rho + \pi + \alpha(n)\theta}\right], \tag{48}$$

with $q(0; \pi) = 0$ and $q(\infty; \pi) = q^*$. Furthermore, $q(n; \pi)$ is strictly increasing in n and strictly decreasing in π. Using (48), equilibrium condition (22) can be reformulated as $\Gamma(n; \pi) = 0$ with

$$\Gamma(n; \pi) \equiv (1 - \theta)\frac{\alpha(n)}{n}\left\{c\left[q(n; \pi)\right] - c\left(\frac{c\left[q(n; \pi)\right]}{u'}\right)\right\} - k. \tag{49}$$

We first show that under the Friedman rule ($\pi = -\rho$) a monetary equilibrium always exists. From (48), $q(n; -\rho) = q^*$ for all $n > 0$. Therefore, given that $c(q^*) < u'q^*$, $\Gamma(0; -\rho) = \infty$ and $\Gamma(\infty; -\rho) = -k$. Consequently, if $\pi = -\rho$ there exists a $n > 0$ that satisfies $\Gamma(n; \pi) = 0$.

Consider next $\pi > -\rho$. For all $\pi > -\rho$, $\Gamma(\infty; \pi) = -k$. For all $n > 0$, $\Gamma(n; \pi)$ is continuous and decreasing in π. Using the continuity of $\Gamma(n; \pi)$ one can deduce that there is a threshold $\bar{\pi} > -\rho$ such that for all $\pi \in (-\rho, \bar{\pi})$ there exists $n > 0$ such that $\Gamma(n; \pi) = 0$.

Finally, let us show that $\partial z/\partial \pi < 0$ and $\partial n/\partial \pi < 0$ at the equilibrium with the highest z. Equations (21) and (22) give two positive relationships between z and n. Furthermore, at the equilibrium with the highest z the curve (21) cuts the curve (22) by below in the space (z, n). An increase in π moves the curve (21) upward leading to a decrease of both z and n.

A9. Proof of Proposition 5

The two measures of welfare $\mathcal{W}$ and $\mathcal{W}^b$ are the same in the flexible price economy,

$$\mathcal{W} = \alpha(n)\left\{\theta\left[u'q_b - c(q_b)\right] + (1-\theta)\left[u'q_s - c(q_s)\right]\right\} - nk. \tag{50}$$

From (22) and (50), social welfare in equilibrium reduces to $\mathcal{W} = \alpha(n)\,\theta\,[u'q - c(q)]$. Given that an increase in π reduces both q and n, the optimal monetary policy is the Friedman rule.

Let us turn to the second part of the proposition. Since $z = c(q_b)$ and $c'(q^*) = p = u'$, $q^s = \min[q^*, c(q_b)/u']$. From (18), for all $z \leq z^*$, $q_b \leq q^*$ which implies $c(q_b) \leq c(q^*) < u'q^*$. Therefore, $q_s = c(q_b)/u' < q^*$.

A10. Proof of Proposition 6

From (18), $q_b(z) = q^*$ for all $z \in [c(q^*), u'q^*]$. Consequently, from (21), any $z \in [c(q^*), u'q^*]$ corresponds to an equilibrium when $\pi = -\rho$. Seller's welfare is 0 and buyer's welfare is $\mathcal{W}^b = \alpha[n(z)]\{u'q^* - c(q^*)\}$ where $n(z)$ is the value of n that satisfies (22). Since $n(z)$ is strictly increasing in z, buyer's welfare is strictly increasing in z so that equilibria with higher values for z Pareto-dominate equilibria with lower values for z. The first-best allocation is such that $q_b = q_s = q^*$ and n satisfies $\alpha'(n)[u'q^* - c(q^*)] = k$. From (18) and (19) this requires $z = u'q^*$, and from (22),

$$1 - \theta = \frac{\alpha'(n^*)n^*}{\alpha(n^*)}.$$

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