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Real Indeterminacy in Monetary Models with Nominal Interest Rate Distortions

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This paper demonstrates that in a standard monetary model with a cash-in-advance constraint on consumption there exists real indeterminacy whenever the nominal interest rate moves too closely with the real rate. A particular example of such a policy is an inflation rate target. This is not a knife-edge result. The conclusion is robust to a wide range of calibrations, and a monetary environment that allows for endogenous velocity.

JEL Codes: E4, E5
Key Words: Interest rates, monetary policy, central banking

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Introduction

There is a long literature on real indeterminacy in monetary models with infinitely-lived agents. Brock (1975) and Obstfeld and Rogoff (1983) examine a money-in-the-utility-function (MIUF) model in which the central bank follows a k% money growth rule and demonstrate the existence of a large class of *non-stationary* monetary equilibria. However, there are few examples of similar models in which there are multiple *stationary* equilibria.[1] The key feature of these equilibria is that they are subject to stationary sunspot fluctuations. More recently, following the empirical work of Taylor (1993), there has been considerable interest in examining whether stationary sunspot fluctuations can arise with interest rate operating procedures in which the nominal rate responds to movements in inflation with elasticity τ. Influential papers include Benhabib, Schmitt-Grohe and Uribe (1998), Clarida, Gali, and Gertler (1997), and Schmitt-Grohe and Uribe (2000).

A review of these papers leaves one with a bewildering set of conditions necessary for equilibrium determinacy. The purpose of this paper is to provide a set of benchmark results in a fairly general flexible price model.[2] In contrast to the previous work, we analyze a model with a standard production economy that includes capital accumulation. The paper demonstrates that the inclusion of investment and capital accumulation tightens up the determinacy regions so that they are essentially independent of labor supply and money demand elasticities. A second extension of this paper is to consider a much more general transactions structure. Following Feenstra (1986), we utilize a general transactions cost function which is isomorphic to a MIUF economy with an arbitrary cross-partial of utility with respect to consumption and real balances, U_{cm}. This

is particularly interesting as Benhabib, Schmitt-Grohe and Uribe (1998) examine a flexible price continuous time MIUF endowment economy and show that the sign of the cross-partial U_{cm} is critical to whether or not there is indeterminacy. They conclude that "active monetary policies" ($\tau > 1$) are indeterminate (determinate) if $U_{cm} < (>) 0$.[3] The current paper demonstrates that in a production economy with constant returns to scale the determinacy regions are independent of the sign of U_{cm}.[4]

The key assumption that generates indeterminacy is that some goods require cash (eg., consumption purchases) while others do not (eg., leisure, investment purchases). This cash vs. non-cash distinction produces a wedge of inefficiency in the economy, a wedge that depends on the nominal rate of interest. If the nominal rate were pegged ($\tau = 0$), then the wedge would be constant and there would be real determinacy in this flexible price economy.[5] More generally, there would be real determinacy for any exogenous interest rate path. However, under a Taylor rule in which the targeted rate moves with inflation, this distortionary nominal rate becomes endogenous, so that real indeterminacy may arise. In the case of a Taylor rule responding to forecasted inflation, we need $\tau < 1$ to ensure determinacy. In the case of a current-based Taylor rule, only $\tau = 0$ ensures determinacy. However, if the central bank looks backward, and only responds to past inflation, then $\tau > 1$ is needed for real determinacy. It is important to note that in this case of a backward rule the current nominal rate is predetermined and does not respond to current conditions. Aggressive responses to inflation are in order only if the Taylor rule is backward looking.

Because the results depend crucially on the nominal interest rate acting like a consumption tax we consider another monetary environment where both consumption

and investment purchases are subject to a strict cash-in-advance (CIA) constraint. In this case the nominal interest rate is equivalent to an income tax. We show that the range in which the economy is uniquely determined is dramatically expanded in this case. The question of whether consumption or total output (consumption plus investment) is the more appropriate scale variable for money demand regressions is clearly a much more important question than previously thought.

The paper proceeds as follows. The next section presents the basic results. Section two provides some sensitivity analysis. Section three concludes.

1. A Flexible Price Monetary Model with Production

The economy consists of numerous identical and infinitely-lived households with preferences over consumption given by

$$\sum_{t=0}^{\infty} \beta^t U(c_t, 1-L_t),$$

where β is the discount rate, c_t is consumption, and $1-L_t$ is leisure. Utility is given by:

$$U(c,1-L) \equiv (c^{1-\sigma}-1)/(1-\sigma) - L^{1+\varepsilon}/(1+\varepsilon),$$

where $1/\varepsilon$ is the Frisch labor supply elasticity. Since we are interested in issues of equilibrium determinacy, we restrict the analysis to a deterministic economy without loss of generality. As is well known, if the deterministic model is subject to real indeterminacy then it is possible to construct sunspot equilibria within the model.

At the beginning of the period, the household has nominal financial wealth denoted by W_t consisting of fiat money and one-period risk-free bonds. The household first travels to the financial market where it receives a lump sum monetary transfer from the monetary authority (X_t), and can buy/sell bonds (B_t) that earn a gross nominal return of

R_t. Any remaining cash (M_t) is used to aid in contemporaneous transactions. Hence we have

$$W_t + X_t = M_t + B_t.$$

The cash balances M_t enter into the following generalized cash-in-advance constraint:

$$P_t c_t \leq M_t v_t.$$

Below we limit the discussion to equilibria in which this constraint binds (equivalently, equilibria in which the gross nominal rate exceeds unity). The household chooses the fraction of its consumption purchases that it will finance with cash. With fraction $1/v_t \leq 1$ being purchased with cash, the remainder $(1-1/v_t)$ is purchased with non-cash means (eg, credit, check, etc.). These other means entail a real resource cost of $c_t h(v_t)$, where h is increasing. Following Marshall (1992) we set $h(v) \equiv (a/\gamma)v^{\gamma} - (a/\gamma)$ where a and γ are positive constants, and $h(1) = 0$.[6] The household's choice of velocity (v_t) implies the following demand for money:

$$M_t / P_t = c_t \, [a/(R_t - 1)]^{[1/(\gamma+1)]}.$$

This demand function has a unit transactions velocity, and a constant interest elasticity of $[1/(\gamma+1)]$. As γ goes to infinity, the model collapses into a rigid cash-in-advance model with $v_t = 1$.

The household produces output for sale to other households using a CRS Cobb-Douglas production function $f(K_t, L_t)$ with a capital share of α and a labor share of $(1-\alpha)$. K_t denotes the household's accumulated capital stock as of the start of time-t. The household's intertemporal constraint is given by

$$W_{t+1} = M_t + R_tB_t - P_tc_t\,[1+ h(v_t)] + P_tf(K_t,L_t) - P_tI_t,$$

where $I_t = K_{t+1} - (1-\delta)K_t$.

Household optimization is defined by the binding cash constraint and the following Euler equations:

$$U_2(t)/U_1(t) = f_L(t)/(\lambda_tR_t) \tag{1}$$

$$U_1(t)/(\lambda_tR_t) = [\beta U_1(t+1)/(\lambda_{t+1}R_{t+1})][f_K(t+1) + (1-\delta)] \tag{2}$$

$$U_1(t)/(P_t\lambda_t) = R_t\beta U_1(t+1)/(P_{t+1}\lambda_{t+1}) \tag{3}$$

$$v_th'(v_t) = (R_t - 1)/v_t \tag{4}$$

$$\lambda_t \equiv \{1 + h(v_t) + [(R_t - 1)/v_t]\}/R_t. \tag{5}$$

The aggregate resource constraint is given by

$$c_t\,[1 + h(v_t)] + I_t = f(K_t,L_t). \tag{6}$$

The variable $\lambda_t \leq 1$ represents the beginning-of-period, private cost of consumption: the unit good cost is augmented by the resource cost of the fraction of goods purchased with

credit and the foregone interest cost of the fraction of goods purchased with cash. The sum of these costs is then discounted to express it in beginning-of-period terms. Note that the social cost of consumption does not include the foregone interest so that the resource constraint (6) does not include the nominal rate.

We close the model by assuming that the central bank follows a Taylor-type rule where the nominal rate varies with inflation in the following way:

$$R_t = R_{ss}\left(\frac{\pi_{t+j}}{\pi_{ss}}\right)^{\tau}, \textit{where } \tau \geq 0, R_{ss} = \frac{\pi_{ss}}{\beta}$$

or log-linearizing (with tildes denoting log deviations from steady-state)

$$\widetilde{R}_t = \tau\widetilde{\pi}_{t+j}$$

where $\pi_{t+j} \equiv P_{t+j}/P_{t+j-1}$ is the price movement between t+j and t+j-1, and ss denotes steady-state. Taylor (1993) originally posited a current rule (j = 0), but there is substantial empirical and anecdotal evidence that central banks are forward-looking and base current policy on forecasts of inflation (j = 1). For completeness we also consider a backward-looking rule (j = -1).

Note that under any interest rate policy the central bank allows the money supply to move endogenously to hit the target. As demonstrated by Carlstrom and Fuerst (2000b), it is very unlikely for real indeterminacy to arise in this model when the central bank follows an exogenous money growth process. Hence, the real indeterminacy that arises under Taylor rules arises because of the endogenous movement in money supply growth rates. That is, when there is real indeterminacy under an interest rate policy it is because there is more than one money growth rate consistent with the given interest rate policy.

There are actually two types of indeterminacy that may arise. First, there is *nominal indeterminacy*—are the initial values of the price level and all other nominal variables pinned down? In our notation this corresponds to the question of whether $\pi_t \equiv P_t/P_{t-1}$ is determined (where t is the initial time period). This nominal or price level determinacy is a typical occurrence under many interest rate operating procedures, the most celebrated example being an interest rate peg. This nominal indeterminacy is of no consequence in and of itself, but is important only if it leads to real indeterminacy.

By *real indeterminacy*, we mean a situation in which the behavior of one or more real variables is not pinned down by the model. This possibility is of great importance as it immediately implies the existence of sunspot equilibria which, in the present environment, are necessarily welfare reducing. In the flexible price model of this paper real indeterminacy manifests itself as an indeterminacy in the nominal interest rate path. Since this rate distorts real behavior, this is a form of real indeterminacy.

We now state our principle results:

Proposition 1: Assume that $\varepsilon = 0$ (linear labor). If the central bank follows a forward-looking Taylor rule, then a necessary and sufficient condition for real determinacy is $\tau < 1$. The economy, however, is always subject to nominal indeterminacy.

Proof: See the appendix.

Proposition 2: Assume that $\varepsilon = 0$ (linear labor). If the central bank follows a current-looking Taylor rule then there is real and nominal indeterminacy for all values of $\tau \neq 0$.

Proof: See the appendix.

Proposition 3: Assume that $\varepsilon = 0$ (linear labor). If the central bank follows a backward-looking Taylor rule, then a necessary and sufficient condition for real determinacy is $\tau = 0$

or $\tau > 1$. In the case of $\tau = 0$ there is nominal indeterminacy, but there is nominal determinacy if $\tau > 1$.

Proof: See the appendix.

Although the exact analytical results of Propositions 1-3 disappear as ε increases and labor supply becomes less elastic, the quantitative impact is extremely small. Later we provide some sensitivity analysis.

Notice that the determinacy results are entirely independent of money demand. The propositions imposed no restrictions on the size of the interest elasticity, $1/(\gamma+1)$. Following Feenstra (1986) we can gain some insight into this robustness by considering the corresponding MIUF economy. We can rewrite the above problem in terms of the indirect utility function $W(e_t,m_t) - V(L_t)$, where $e_t = c_t\,[1 + h(v_t)]$ (e = consumption expenditures inclusive of transactions costs). This implies $W_e = U'(c_t)\partial g/\partial e = U'(c_t)/\lambda_t R_t$ (where $c = g(e,m)$). In a production economy, the labor market and capital accumulation margins restrict the behavior of W_e. For example, with linear production and linear leisure, W_e is constant and independent of real money balances. This will be true, and thus the exact nature of the results will occur, whenever the assumption of linear leisure is coupled with a CRS production function. Differing money demand elasticities simply correspond to differing supporting money supply processes.[7]

To develop intuition about why indeterminacy may arise it is helpful to consider the extreme version of the model in which γ goes to infinity and the model collapses to the familiar rigid cash-in-advance model as in Carlstrom and Fuerst (1995). In this case, $\lambda_t = 1$, and (1) and (3) become

$$U_2(t)/U_1(t) = f_L(t)/R_t \tag{7}$$

$$U_1(t)/R_t = [\beta U_1(t+1)/R_{t+1}][f_K(t+1) + (1-\delta)]. \tag{8}$$

Equation (7) embodies the cash vs. non-cash distortion—the income from selling leisure is not available for consumption until next period. The nominal rate acts as a distortion, driving a wedge between current labor effort and current consumption. Similarly, equation (8) is the standard optimal growth condition distorted by the nominal rate of interest. The nominal rate can thus be interpreted as a consumption tax of (1+tax)= R_t. A high nominal rate is equivalent to a high consumption tax, and a low nominal rate is equivalent to a low consumption tax.

As noted by Carlstrom and Fuerst (1995), if the nominal interest rate is constant, then equation (8) collapses into the optimal growth Euler equation. Hence, in a model with inelastic labor, an interest rate peg is the optimal monetary policy and real behavior will be determinate although there is nominal indeterminacy.[8] But if the nominal rate (consumption tax) is endogenous, as occurs under a Taylor rule, then real indeterminacy may arise.

Under a forward-looking Taylor rule, increases in expected inflation increase the nominal rate but depending on the elasticity τ these increases may or may not increase the real rate. The Taylor rule implies

$$\tilde{R}_t - \tilde{\pi}_{t+1} = (\tau - 1)\tilde{\pi}_{t+1} = \frac{(\tau - 1)}{\tau}\tilde{R}_t .$$

For aggressive policies ($\tau > 1$), nominal rate increases are associated with increases in the real rate of interest. Thus, we have an implicit consumption tax (the nominal rate) correlated tightly with expected consumption growth (the real rate). The self-fulfilling

circle goes something like this. A sunspot-driven increase in expected inflation increases today's nominal interest rate and thus increases the opportunity cost of holding money. This lowers current consumption, which in turn increases the real rate of interest. With $\tau > 1$, the nominal rate (consumption tax) rises with this real rate movement which completes the circle. The initial increase in expected inflation is therefore rational.[9] This circle is broken in the case of $\tau < 1$.

Note that in the case of a forward rule π_t is free for all values of τ. This nominal indeterminacy also arises in the case of a current based rule, but now it is more problematic. Since π_t is free, a current-based policy implies that R_t is also free. Since R_t acts like a tax on consumption, real behavior is not pinned down. The nominal indeterminacy from before is now real. The reason for this indeterminacy is because the policy rule is responding to current inflation, which is not pinned down for standard nominal indeterminacy reasons. Because of the policy rule this nominal indeterminacy becomes real. More generally the potential for indeterminacy in both the current- and forward-looking rules arises whenever policy responds to endogenous variables that are not predetermined. This is why an interest rate peg ($\tau = 0$) has real determinacy.

This discussion suggests that the central bank should look backwards so that it responds only to predetermined variables. Remarkably, by looking backwards the conditions for determinacy are (almost) entirely flipped on their head from when the Taylor Rule is forward-looking. If the monetary authority responds aggressively to past inflation ($\tau > 1$) initial inflation and hence R_{t+1} is pinned down (the policy rule implies that R_t is predetermined by last period's inflation rate). Hence, there is real and also

nominal determinacy if and only if $\tau > 1$. (An interest rate peg of $\tau = 0$ yields real but not nominal determinacy.)

The intuition why an aggressive backward-looking policy can eliminate nominal and real indeterminacy is as follows. Suppose there is an increase in the current price level π_t of 1%. This implies that *next period's* nominal rate must rise by τ%. This increase in the future nominal rate (consumption tax) leads to an increase in current consumption. This implies that the real rate falls. The policy rule implies that the current nominal rate does not respond to π_t. Hence, the decline in the real rate must lead to an increase in π_{t+1} that is greater than the initial increase in π_t. This behavior is explosive, and thus eliminates this as a possible equilibrium path.

The idea that responding to a nominal variable can pin down prices is not new. This result is a general equilibrium generalization of McCallum's (1981) earlier result. He argued that because an interest rate peg suffered from nominal indeterminacy the monetary authority needed a nominal anchor, which could be accomplished by responding to a nominal variable. This analysis confirms this but shows that merely responding to a nominal variable, like past inflation, is not enough. The monetary authority has to aggressively respond to past inflation to ensure both real and nominal determinacy.

2. Sensitivity Analysis.

Because of the irrelevance of the money demand elasticity we impose the convenient rigid CIA constraint (γ equal to infinity) for the remainder of the paper. However, we reiterate that none of the results are affected in a quantitative sense by this assumption.

a. General Labor Supply Elasticity.

The key simplifying assumption in Propositions 1-3 is that $\varepsilon = 0$, so that labor supply is infinitely elastic (recall that labor supply is $1/\varepsilon$). Hence, the first issue we consider is the sensitivity of these results to a less elastic labor supply ($\varepsilon > 0$). This is particularly interesting as Schmitt-Grohe and Uribe's (2000) results were sensitive to the assumed labor supply elasticity. [10]

Before proceeding, we can replicate Schmitt-Grohe and Uribe's (2000) results and demonstrate the importance of labor supply elasticity by dropping capital from the model, assuming that the production function is linear in labor, and imposing $\sigma = 1$ so that consumption preferences are logarithmic. Under these assumptions, the labor market equation (1), the Fisher equation (3), and the resource constraint c = L, can be combined to yield the following equation

$$\frac{-\widetilde{R}_{t+1}}{\varepsilon+1} + \frac{\widetilde{R}_t}{\varepsilon+1} = \widetilde{R}_t - \widetilde{\pi}_{t+1},$$

where $\lambda_t = 1$ because of the rigid CIA constraint.

Under a current-looking rule this difference equation becomes

$$\widetilde{\pi}_{t+1} = \left(\frac{\tau\varepsilon}{\varepsilon+1-\tau}\right)\widetilde{\pi}_t .$$

We have real and hence nominal determinacy if and only if this mapping is explosive. We have two cases. If $\varepsilon < 1$, then there is determinacy if and only if $1 < \tau < (1+\varepsilon)/(1-\varepsilon)$ (this is Schmitt-Grohe and Uribe's Proposition 3 with $1/\varepsilon$ equal to their θR). If $\varepsilon > 1$, then there is determinacy as long as $\tau > 1$ (this is their Proposition 4). As in the model with capital if we assume linear labor ($\varepsilon = 0$) there is always indeterminacy as $\pi_t \equiv P_t/P_{t-1}$

and hence R_t is free. But surely an infinite labor supply elasticity is not compelling and as we increase ε to, say, 2 (a labor supply elasticity of $1/\varepsilon = 0.5$), then the determinacy region apparently becomes quite large with any $\tau > 1$ ensuring determinacy.

However, the inclusion of capital and the added investment region almost completely eliminates this determinacy region. Determinacy is more likely as labor supply becomes less elastic. Suppose we drive ε to infinity so that we stack the deck in favor of indeterminacy. Even in this case, the determinacy region is incredibly small. For example, if $\alpha = 0.34$, $\beta = 0.99$, $\delta = 0.02$, $\sigma = 1$ the range of determinacy is $1 < \tau < 1.0014$. As ε falls below infinity, this region shrinks even more.

What about a forward-looking rule? Proceeding as before with a labor-only economy the relevant difference equation is

$$\tilde{\pi}_{t+2} = \left[1 + \frac{(1-\tau)(\varepsilon+1)}{\tau}\right]\tilde{\pi}_{t+1}.$$

This mapping is explosive and we have determinacy if $\tau < 1$, or if $\varepsilon > 1$ and $\tau > (1+\varepsilon)/(\varepsilon-1)$. This latter region is operative for plausible labor supply elasticities. However, once again the addition of capital restores the earlier result that there is determinacy if and only if $\tau < 1$. For example, if $\alpha = 0.34$, $\beta = 0.99$, $\delta = 0.02$, $\sigma = 1$ then the range of determinacy remains at $\tau < 1$.

Finally, in the case of a backward-looking rule we have

$$\tilde{\pi}_{t+1} = \left(\frac{\tau}{1+\varepsilon}\right)\tilde{\pi}_t + \left(\frac{\tau\varepsilon}{1+\varepsilon}\right)\tilde{\pi}_{t-1}$$

We have two cases. If $\varepsilon > 1$, then there is determinacy if and only if $1 < \tau < (1+\varepsilon)/(\varepsilon-1)$. If $\varepsilon < 1$, then there is determinacy as long as $\tau > 1$. As before, the addition of capital

eliminates the former region, and we are left with $\tau > 1$ being necessary and sufficient for determinacy.

The thrust of these results is that in a model with capital, real indeterminacy under a Taylor rule becomes much more likely (compared to a labor-only economy). Recall that in this monetary model the nominal interest rate manifests itself as an endogenous consumption tax. Real indeterminacy under such a tax is much more likely when we add an investment margin to the analysis as the household can more readily shift consumption across time.

b. Investment Subject to CIA.

A key feature of the previous analysis is that only consumption spending is subject to a transactions constraint. Given the relative sizes of consumption and investment purchases this assumption is entirely reasonable as one would expect that the advantage of alternative transactions arrangements would be increasing in the size of the transaction. However as a test of robustness we consider the opposite extreme and assume that both consumption and investment are subject to a rigid CIA constraint. In this case the investment margin (2) becomes

$$U_c(t) = \beta U_c(t+1)\left[\frac{f_k(t+1)}{R_{t+1}} + (1-\delta)\right] \tag{9}$$

In this case, the nominal interest rate distorts all factor payments as it enters symmetrically on the productivity of labor and the return to capital. Hence, the nominal interest rate now acts as an income tax.

The principle result of this section is that when both consumption and investment are treated symmetrically, the likelihood of real indeterminacy shrinks dramatically. In

the earlier sections consumption and investment were treated asymmetrically so that an increase in the nominal rate had a powerful depressing effect on consumption via a direct effect on the current cost of conducting transactions, and an indirect effect of increasing investment spending as the household postponed consumption until transactions costs fell. But in a model in which both consumption and investment are both subject to a common transactions constraint, the latter effect is reversed so that a higher nominal interest rate path also discourages investment spending. This effect acts as an automatic stabilizer and makes real indeterminacy much less likely. In particular we have the following:

Proposition 4: Assume that $\varepsilon = 0$ (linear labor). Suppose that consumption and investment are subject to a strict CIA constraint and that monetary policy is given by a forward-looking rule. A sufficient condition for real determinacy is $\tau < 1$. If $(1-\alpha-\beta(1-\delta)(1+\alpha)) > 0$, $\tau < 1$ is necessary and sufficient for determinacy. If $(1-\alpha-\beta(1-\delta)(1+\alpha)) < 0$, the determinate regions are $\tau < 1$ and

$$\tau > \tau^* = 1 + \frac{2(1-\beta(1-\delta))}{(\alpha+\beta(1-\delta)(1+\alpha)-1)}.$$

The economy, however, is always subject to nominal indeterminacy.

Proof: See the appendix.

Remark: For all plausible parameter values, the condition $(1-\alpha-\beta(1-\delta)(1+\alpha)) < 0$ will be satisfied so that the model is determinate for all τ outside of $[1, \tau^*]$. This range is quite small. With $\alpha = 1/3$, $\beta = .99$, and $\delta = .02$, the model is determinate unless τ is between 1 and 1.0475. Recall that in the model in which only consumption is subject to

CIA there is determinacy if and only if $\tau < 1$. Hence, the determinacy region is much larger in the model in which investment is also subject to CIA.

Proposition 5: Assume that $\varepsilon = 0$ (linear labor). Suppose that consumption and investment are subject to a strict CIA constraint and that monetary policy is given by a current-looking rule. An interest rate peg of $\tau = 0$ exhibits real determinacy but nominal indeterminacy. For nonzero τ, a necessary condition for real and nominal determinacy is $\tau > 1$. This is both necessary and sufficient if

$$\left(1-\alpha-\beta(1-\delta)(1+\alpha)\right)\tau-(1-\alpha)\left(1-\beta(1-\delta)\right)-2\alpha<0 .$$

Proof: See the appendix.

Remark: As before, the condition in the proposition is satisfied for all plausible parameter values. Hence, there is determinacy for all $\tau > 1$. Recall that in the model in which only consumption is subject to CIA, there is indeterminacy for all values of τ. Once again, imposing the CIA constraint on investment significantly expands the region of determinacy.

Proposition 6: Assume that $\varepsilon = 0$ (linear labor). Suppose that consumption and investment are subject to a strict CIA constraint and that monetary policy is given by a backward-looking rule. An interest rate peg of $\tau = 0$ exhibits real determinacy but nominal indeterminacy. For nonzero τ, a necessary condition for real and nominal determinacy is $\tau > 1$. If $\left(1-\alpha-\beta(1-\delta)(1+\alpha)\right)<0$, then $\tau > \tau^* > 1$ is necessary and sufficient for real and nominal determinacy where τ^* is given by

$$\tau^* = 1+\frac{2\left(1-\beta(1-\delta)\right)}{\left(\alpha+\beta(1-\delta)(1+\alpha)-1\right)} .$$

Proof: See the appendix.

Remark: For plausible parameter values the condition $(1-\alpha-\beta(1-\delta)(1+\alpha))<0$ is surely satisfied and there is determinacy as long as $\tau > 1.0475$. Thus the general result that responding aggressively to past inflation guarantees determinacy continues to hold with both consumption and investment subject to a CIA constraint.

Table I summarizes the bounds for real determinacy for all three monetary policy rules when only consumption is in the CIA constraint and when both consumption and investment are in the constraint. While the bounds for uniqueness were clearly expanded for both a forward and a current-looking rule, the bounds for determinacy are actually slightly narrower for a backward-looking rule. This does not contradict our earlier result that the region in which the economy is indeterminate shrinks with investment in the CIA constraint. The reason is because unlike propositions 1-3 where the economy was necessarily indeterminate when there was not real determinacy, in propositions 4-6 the economy is frequently overdetermined when it is not uniquely determined. See the proofs in the appendix for details.

3. Conclusion

This paper has utilized a standard business cycle model to explore the issue of equilibrium determinacy when the central bank conducts policy according to a Taylor rule. In a model in which consumption but not investment purchases are subject to a transactions constraint, there is real indeterminacy whenever the central bank responds too aggressively to current or future inflation. Under a forward-looking Taylor rule we need $\tau < 1$ for determinacy, while there is indeterminacy under a current Taylor rule for all $\tau > 0$. To ensure real and nominal determinacy, the central bank should respond

aggressively ($\tau > 1$) to lagged inflation rates. By responding aggressively, the central bank can pin down prices and achieve nominal determinacy but avoids real determinacy by making the current nominal rate predetermined. The empirical robustness of this result warrants restatement. The endogenous velocity model of Section 1 is remarkably general, and the results are quantitatively unaffected by the interest elasticity of money demand, the share of consumption goods purchased with cash (steady-state velocity), the sign of U_{cm} in the isomorphic MIUF economy, or the labor supply elasticity. This is not a knife-edge result.

However, these results are affected by the assumption on which transactions are subject to a finance constraint. In an environment with a strict CIA constraint on both consumption and investment purchases, there is determinacy under a forward-looking rule for essentially all values of τ, while essentially any $\tau > 1$ ensures determinacy under both a current-looking and backward-looking rule. We thus conclude that the investment margin has an enormous influence on the determinacy regions. Future work should not only include explicit investment spending and capital accumulation, but also carefully examine the assumption on which purchases are subject to finance constraints. The question of whether consumption or total output (consumption plus investment) is the more appropriate scale variable for money demand regressions is clearly a much more important question than previously thought.

Another important question is how all these results are affected by the presence of nominal rigidities. Not surprisingly the form this rigidity takes is critical. For example, its easy to see that in a strict limited participation environment such as Fuerst (1992) (see

eg., Carlstrom and Fuerst (1999)) that determinacy becomes extraordinarily difficult whenever the central bank operates off of interest rate rules.

Recently there have been papers that have attempted to answer this question in sticky price models. For example, Clarida, Gali and Gertler (1998) and Kerr and King (1996) analyze reduced-form sticky price models and conclude that $\tau > 1$ is needed for determinacy. There are two fundamental differences between these papers and the current work.

First, these sticky price models use a MIUF model with "cash-when-I'm-done" timing. As demonstrated by Carlstrom and Fuerst (2000a), if one adjusts the money demand formulation to the more appropriate timing convention, then forward-looking rules are always subject to real indeterminacy.

The second principle difference between the analyses is that these sticky price models ignore the role of investment spending as they are labor-only production economies. The current paper makes clear that ignoring this margin is problematic. Hence, an important research question is the bounds for determinacy in a sticky price model that explicitly includes investment spending.

Table I

Bounds for Real Determinacy in Various Monetary Models

	Forward Rule	**Current Rule**	**Backward Rule**
Consumption s.t. CIA Constraint— see Propositions 1-3	$\tau < 1$	$\tau = 0$	$\tau > 1$ and $\tau = 0$
Cons. & Investment s.t. CIA Constraint-- see Propositions 4-6	$\tau < 1$ & $\tau > 1.0475$*	$\tau > 1$* & $\tau = 0$	$\tau > 1.0475$* & $\tau = 0$

*Denotes that the conditions depend on assumed numerical values. See the text for details

Appendix

Proposition 1: Assume that ε = 0 (linear labor). If the central bank follows a forward-looking Taylor rule, then a necessary and sufficient condition for real determinacy is τ < 1. The economy, however, is always subject to nominal indeterminacy.

Proof: Given the assumption that utility is linear in leisure (ε = 0) the first order conditions (1) and (2) are

$$\left(\frac{U_1(t)}{\lambda_t R_t}\right)=\frac{x_t^{\alpha}}{(1-\alpha)}, where\ x_t \equiv \frac{L_t}{K_t} \tag{A1}$$

$$\left(\frac{U_1(t)}{\lambda_t R_t}\right)=\beta\left(\frac{U_1(t+1)}{\lambda_{t+1} R_{t+1}}\left\{\alpha x_{t+1}^{1-\alpha}+(1-\delta)\right\}\right)$$

Plugging (A1) and equation (A1) scrolled forward one period into (A2) yields:

$$x_t^{\alpha}=E_t\left(\alpha\beta x_{t+1}+\beta(1-\delta)x_{t+1}^{\alpha}\right)) \tag{A3}$$

Substituting (A1) into Fisher equation (3) yields

$$x_t^{\alpha}=\beta\, x_{t+1}^{\alpha}\frac{R_{t+1}}{\pi_{t+1}} \tag{A4}$$

The remaining condition is the resource constraint $K_{t+1} = H(x_t,c_t,K_t)$. Since capital does not enter into either (A3) or (A4) the resource constraint separates out. The resource constraint provides one eigenvalue $e_1=\frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta}>1$. There is only one predetermined variable. Therefore for the economy to be determinate one of the two eigenvalues of the remaining log-linearized system (A3 and A4) need to lie outside the unit circle.

$$\begin{bmatrix} 1-\beta(1-\delta)(1-\alpha) & 0 \\ \alpha & \tau \end{bmatrix}\begin{bmatrix} \tilde{x}_{t+1} \\ \tilde{\pi}_{t+2} \end{bmatrix}=\begin{bmatrix} \alpha & 0 \\ \alpha & 1 \end{bmatrix}\begin{bmatrix} \tilde{x}_t \\ \tilde{\pi}_{t+1} \end{bmatrix}$$

The remaining two eigenvalues are the roots of the following quadratic equation:

$$e^2+\left(\frac{1}{\tau}-\frac{\alpha}{1-\beta(1-\alpha)(1-\delta)}\right)e-\frac{\alpha}{1-\beta(1-\alpha)(1-\delta)}=0$$

The three eigenvalues of the system are thus

$$e_1=\frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta}>1,\ e_2=\frac{\alpha}{1-\beta(1-\alpha)(1-\delta)}<1,\ e_3=\frac{1}{\tau}.$$

Since there is only one predetermined variable, for the economy to be determinate two eigenvalues need to lie outside the unit circle. e_3 is outside the unit circle if $\tau < 1$. In any event, the initial π_t is free so that there always is nominal indeterminacy. QED

Proposition 2: Assume that $\varepsilon = 0$ (linear labor). If the central bank follows a current-looking Taylor rule then there is real and nominal indeterminacy for all values of $\tau \neq 0$.

Proof: In the case of $\tau = 0$, (A4) backs out the path of π_{t+1} implied by the dynamics of (A3) and the resource constraint. This latter system has eigenvalues

$$e_1=\frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta}>1,\ e_2=\frac{\alpha}{1-\beta(1-\alpha)(1-\delta)}<1,$$

so that we have real determinacy because we have one predetermined variable (the capital stock). There is nominal indeterminacy because π_t is free. For $\tau > 0$, we follow the proof in Proposition 1 and are once again left with the first order conditions (A3) and (A4) and the eigenvalue $e_1 > 1$ given by the resource constraint. Once again there is only one predetermined variable. Therefore for the economy to be determinate one of the two eigenvalues of the following log-linearized two dimensional system (A3 and A4) need to lie outside the unit circle.

$$\begin{bmatrix}1-\beta(1-\delta)(1-\alpha) & 0\\ \alpha & \tau-1\end{bmatrix}\begin{bmatrix}\tilde{x}_{t+1}\\ \tilde{\pi}_{t+1}\end{bmatrix}=\begin{bmatrix}\alpha & 0\\ \alpha & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_t\\ \tilde{\pi}_t\end{bmatrix}$$

The remaining two eigenvalues are the roots of the following quadratic equation:

$$e^2 - \left(\frac{\alpha}{1-\beta(1-\alpha)(1-\delta)} \right) e = 0$$

The three eigenvalues of the system are thus

$$e_1 = \frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta} > 1,\ e_2 = \frac{\alpha}{1-\beta(1-\alpha)(1-\delta)} < 1,\ e_3 = 0.$$

The economy is therefore always indeterminate. This also implies nominal indeterminacy as π_t is free. QED

Proposition 3: Assume that $\varepsilon = 0$ (linear labor). If the central bank follows a backward-looking Taylor rule, then a necessary and sufficient condition for real determinacy is $\tau = 0$ or $\tau > 1$. In the case of $\tau = 0$ there is nominal indeterminacy, but there is nominal determinacy if $\tau > 1$.

Proof: For the case of $\tau = 0$, the proof is symmetric to Proposition 2. For $\tau > 0$, we proceed as in Proposition 2, and have the two variable system

$$\begin{bmatrix} 1-\beta(1-\delta)(1-\alpha) & 0 \\ \alpha & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_{t+1} \\ \tilde{\pi}_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \alpha & -\tau \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ \tilde{\pi}_t \end{bmatrix}$$

Including the resource constraint we have the following eigenvalues

$$e_1 = \frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta} > 1,\ e_2 = \frac{\alpha}{1-\beta(1-\alpha)(1-\delta)} < 1,\ e_3 = \tau.$$

Hence, there is real determinacy if and only if $\tau > 1$. Note that since the system begins with π_t, we also have nominal determinacy in the case of $\tau > 1$. QED

Proposition 4: Assume that $\varepsilon = 0$ (linear labor). Suppose that consumption and investment are subject to a strict CIA constraint and that monetary policy is given by a forward-looking rule. A sufficient condition for real determinacy is $\tau < 1$. If

$(1-\alpha-\beta(1-\delta)(1+\alpha))>0$, τ < 1 is necessary and sufficient for determinacy. If $(1-\alpha-\beta(1-\delta)(1+\alpha))<0$, the determinate regions are τ < 1 and

$$\tau > \tau^* = 1+\frac{2(1-\beta(1-\delta))}{(\alpha+\beta(1-\delta)(1+\alpha)-1)}.$$

The economy, however, is always subject to nominal indeterminacy.

Proof: Given the assumption that utility is linear in leisure (ε = 0) the first order condition (1) can be written as

$$\left(\frac{U_1(t)}{R_t}\right)=\frac{x_t^{\alpha}}{(1-\alpha)}, where\ x_t \equiv \frac{L_t}{K_t} \qquad \text{(A5)}$$

Substituting (A5) and equation (A5) scrolled forward one period into (9) yields the rewritten capital accumulation equation:

$$x_t^{\alpha} R_t = \left(\alpha\beta\ x_{t+1} + \beta(1-\delta)x_{t+1}^{\alpha} R_{t+1}\right) \qquad \text{(A6)}$$

Substituting the labor equation into the Fisher equation yields:

$$x_t^{\alpha} = \beta x_{t+1}^{\alpha}\frac{R_{t+1}}{\pi_{t+1}}$$

Capital once again separates out. Log-linearizing and substituting in the forward-looking Taylor rule yields

$$\begin{bmatrix} 1-\beta(1-\delta)(1-\alpha) & \tau\beta(1-\delta) \\ \alpha & \tau \end{bmatrix}\begin{bmatrix} \tilde{x}_{t+1} \\ \tilde{\pi}_{t+2} \end{bmatrix}=\begin{bmatrix} \alpha & \tau \\ \alpha & 1 \end{bmatrix}\begin{bmatrix} \tilde{x}_t \\ \tilde{\pi}_{t+1} \end{bmatrix}$$

Since capital does not enter into either (A5) or (A6) the resource constraint separates out providing the following eigenvalue

$$e_1=\frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta}>1.$$

Solving the above system of equations the other two eigenvalues are given by the solution to the following equation:

$$J(e) = \tau(1-\beta(1-\delta))e^2 + [\alpha\beta\tau(1-\delta) - (1-\beta(1-\alpha)(1-\delta))]e + \alpha(1-\tau) = 0$$

For the economy to be determinate one of these two eigenvalues need to lie outside the unit circle. After some algebra we have

$$J(-1) = (1-\beta(1-\delta))(1-\alpha) + 2\alpha + \tau[\,1-\alpha-\beta(1-\delta)(1+\alpha)]$$

$$J(0) = \alpha(1-\tau)$$

$$J(1) = (1-\beta(1-\delta))(1-\alpha)(\tau-1)$$

When $\tau < 1$ J(-1) > 0, J(0) > 0, and J(1) < 0 and therefore only one root lies within the unit circle and the economy is determinate. Hence, $\tau < 1$ is sufficient for determinacy. If $\tau > 1$, then J(0) < 0 and J(1) > 0 so that there is one root in (0,1). For the other root to be outside the unit circle, the necessary and sufficient condition is J(-1) < 0. If $(1-\alpha-\beta(1-\delta)(1+\alpha)) > 0$, this is never satisfied and the system is underdetermined. If $(1-\alpha-\beta(1-\delta)(1+\alpha)) < 0$, and $\tau > \tau^*$ then J(-1) < 0, and again the economy is determinate, while if $1 < \tau < \tau^*$ J(-1) > 0 and the system is underdetermined. In any event, the initial π_t is free so that there always is nominal indeterminacy. QED

Proposition 5: Assume that $\varepsilon = 0$ (linear labor). Suppose that consumption and investment are subject to a strict CIA constraint and that monetary policy is given by a current-looking rule. An interest rate peg of $\tau = 0$ exhibits real determinacy but nominal indeterminacy. For nonzero τ, a necessary condition for real and nominal determinacy is $\tau > 1$. This is both necessary and sufficient if

$$(1-\alpha-\beta(1-\delta)(1+\alpha))\tau - (1-\alpha)(1-\beta(1-\delta)) - 2\alpha < 0\,.$$

Proof: Following the proof for a forward rule we obtain

$$\begin{bmatrix} 1-\beta(1-\delta)(1-\alpha) & \tau\beta(1-\delta) \\ \alpha & \tau-1 \end{bmatrix}\begin{bmatrix} \tilde{x}_{t+1} \\ \tilde{\pi}_{t+1} \end{bmatrix}=\begin{bmatrix} \alpha & \tau \\ \alpha & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_t \\ \tilde{\pi}_t \end{bmatrix}$$

Since capital does not enter into either (A5) or (A6) the resource constraint separates out providing the following eigenvalue

$$e_1=\frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta}>1.$$

Solving the above system of equations the other two eigenvalues are given by the solution to the following equation:

$$[(\tau-1)(1-\beta(1-\delta))-\alpha\beta(1-\delta)]e^2+[\alpha(1+\beta\tau(1-\delta))]e-\alpha\tau=0$$

For the economy to be determinate one of these two eigenvalues need to lie outside the unit circle. After some algebra we have

$$J(-1)=(1-\alpha-\beta(1-\delta)(1+\alpha))\tau-(1-\alpha)(1-\beta(1-\delta))-2\alpha$$

$$J(0)=-\alpha\tau$$

$$J(1)=(1-\beta(1-\delta))(1-\alpha)(\tau-1)$$

Note that J(0) < 0 and J'(0) > 0 for all τ. If $\tau < 1$, J is concave and J(1) and J(-1) are both negative implying that both roots are either outside or inside the unit circle. In the first case the system is overdetermined, while in the latter it is underdetermined. Hence, $\tau > 1$ is necessary for determinacy in which case J(1) > 0. If J(-1) < 0, $\tau > 1$ is necessary and sufficient for determinacy. QED

Proposition 6: Assume that $\varepsilon = 0$ (linear labor). Suppose that consumption and investment are subject to a strict CIA constraint and that monetary policy is given by a backward-looking rule. An interest rate peg of $\tau = 0$ exhibits real determinacy but nominal indeterminacy. For nonzero τ, a necessary condition for real and nominal determinacy is $\tau > 1$. If $(1-\alpha-\beta(1-\delta)(1+\alpha))<0$, then $\tau > \tau^* > 1$ is necessary and

sufficient for real and nominal determinacy where τ^* is given by

$$\tau^* = 1 + \frac{2(1-\beta(1-\delta))}{(\alpha+\beta(1-\delta)(1+\alpha)-1)}.$$

Proof: Proceeding as in the previous two propositions we have

$$\begin{pmatrix} 1-\beta(1-\delta)(1-\alpha) & 0 & \tau\beta(1-\delta) \\ \alpha & 0 & \tau \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_{t+1} \\ \tilde{\pi}_{t+1} \\ \tilde{\pi}_t \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \tau \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_t \\ \tilde{\pi}_t \\ \tilde{\pi}_{t-1} \end{pmatrix}$$

As before, the resource constraint separates out and provides one explosive eigenvalue, $e_1 = \frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta} > 1$.

and one predetermined variable, K_t. Solving the above system of equations yields the following cubic characteristic equation:

$$J(e) = [1-\beta(1-\delta)(1-\alpha)]e^3 - \{\alpha+\tau[1-\beta(1-\delta)]\}e^2 - \alpha\tau\beta(1-\delta)e + \alpha\tau$$

Note that J(0) > 0 and J'(0) < 0. Since the cubic coefficient is positive, we know that there is one negative root, and two positive roots. For determinacy, we need one root outside of the unit circle (we have two predetermined variables, K_t and π_{t-1}). Algebra implies

$$J(1) = (1-\beta(1-\delta))(1-\alpha)(1-\tau)$$

$$J(-1) = (1-\tau)[1-\alpha-\beta(1-\delta)(1+\alpha)] - 2[1-\beta(1-\delta)]$$

If $\tau < 1$ J(-1) < 0 so that there is one root in (-1,0). We also have J(1) > 0, so we either have two roots in (0,1) or two roots outside (0,1). In the former case we have indeterminacy, while in the latter the system is overdetermined. Hence, for determinacy we need $\tau > 1$. If $\tau > 1$ we have J(1) < 0 and one root in (0,1). We now must consider J(-1). If $(1-\alpha-\beta(1-\delta)(1+\alpha)) > 0$, then J(-1) < 0 for all $\tau > 1$, and the system is overdetermined. If $(1-\alpha-\beta(1-\delta)(1+\alpha)) < 0$, then for $\tau > \tau^*$, J(-1) > 0 and we have determinacy (where τ^* satisfies J(-1) = 0).

QED

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Footnotes:

[1]A few significant counterexamples include Matsuyama (1990,1991) and Woodford (1994) who demonstrate the possibility of endogenous fluctuations in monetary economies operating under k% money growth rules. Woodford also considers an interest rate peg and demonstrates that such a policy typically ensures equilibrium determinacy.

[2] The paper closest in spirit to the current analysis is Schmitt-Grohe and Uribe (2000). They consider a flexible-price environment in which the fiscal authority follows a balanced-budget requirement and the central bank follows a Taylor-type interest rate operating procedure. One result of Schmitt-Grohe and Uribe (2000) is that a balanced budget policy is Ricardian. Hence although we do not impose a balanced-budget constraint below, our results on determinacy under interest rate rules are directly comparable to theirs. Schmitt-Grohe and Uribe consider both a current Taylor rule in which the central bank responds to movements in the current inflation rate, and a forward-looking Taylor rule in which the central bank responds to expected inflation. In the case of the current rule, they report that there is real determinacy for a range of τ between one and some upper bound, while the forward-looking rule is determinate only if $\tau < 1$. Their results, however, were for a labor-only cash-in-advance economy and were sensitive to the assumed labor supply elasticity.

[3] Calvo (1979) demonstrates the possibility of multiple stationary equilibria in a continuous-time, money-in-the-production-function (MIPF) model under several different monetary policies (including an inflation peg). He concludes that an inflation peg is indeterminate. Since a MIPF environment is a MIUF endowment economy with $U_{cm} < 0$, Calvo's results are simply a special case of Benhabib et al. (1998).

[4] Along with the explicit production technology, a second important difference between this paper and Benhabib et al. (1998) is that the discrete time counterpart to their continuous time model is a "cash when I'm done model". That is, money "at the end of the period" is used to facilitate trade, while this paper uses money "at the beginning of the period." Surprisingly this last difference is quite important. Carlstrom and Fuerst (2000a) discuss the effect that these timing differences have on indeterminacy.

[5] The standard nominal indeterminacy would arise but this is of no consequence for real behavior.

[6] Using M1, Marshall (1992) estimates a = .00923, $\gamma = 1.79$.

[7] Straightforward calculations imply that the sign of W_{em} is given by the sign of $(1+\gamma)-\sigma$. As γ becomes large and we move towards a CIA economy and W_{em} necessarily becomes positive. Since we are not restricting γ, we are also not restricting W_{em}.

[8] Carlstrom and Fuerst (1995) note that while *real* behavior is unique under an interest rate peg there is *nominal* indeterminacy since the money growth process is only unique up to the addition of a mean-zero iid term. In a limited participation model this nominal indeterminacy spills over into real indeterminacy. This occurs because in a limited participation model equation (2) holds only in an expected value sense. Real behavior will still be uniquely determined, however, for a given money growth process.

[9] Woodford (1994) shows that a k% money growth rule can produce stationary sunspot equilibria in a Lucas-Stokey (1987) cash-in-advance model only if the constant of relative risk aversion (σ) is large enough (see his Proposition 6). With a k% money growth rule and a rigid cash-in-advance constraint the nominal rate tracks the real rate with an elasticity of $(\sigma-1)/\sigma$, where σ is the constant degree of relative risk aversion. Hence, for σ large enough ($\sigma > 2$) this elasticity is positive and we have sunspots. Woodford's (1994) example is thus a special case of our much more general result.

[10] Schmitt-Grohe and Uribe (2000) impose restrictions on the steady-state leisure-consumption ratio. For the preference specification they use this ratio is proportional to the steady-state labor supply elasticity.