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Fairness in Apportionment

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> To apportion is to distribute by right measure, to set off in just parts, to assign in due and proper proportion.
>
> Daniel Webster, 1832

1. The Constitutional background

Apportionment is concerned with the allocation of Congressional seats within the U.S. House of Representatives. It therefore deals with the very substance of political power. Over the course of U.S. history, as waves of immigration and migration have shifted the demographic balance among different sections of the country, the accompanying changes in political power have played out through the apportionment formula. Indeed, the history of apportionment provides a remarkable window on the larger political and demographic currents sweeping over our society. In this paper I shall give a thumbnail sketch of this fascinating history, focusing particularly on the perceived fairness of different approaches, which is the common thread that runs through almost all debates on the subject. A more complete account may be found in *Fair Representation: Meeting the Ideal of One Man, One Vote*, 2nd edition (Balinski and Young 2002).

Language governing apportionment may be found in Article I, Section 2 of the U.S. Constitution:

> Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons. The actual Enumeration shall be made within three years after the first meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such manner as they shall by Law direct. The Number of Representatives shall not exceed one for every thirty thousand, but each State shall have at least one Representative.

Here the term "apportioned" means that representation should be made *proportional* to the number of free persons in each state, subject to the modifications spelled out in the subsequent clauses of the article. While proportionality to numbers may seem reasonable or even obvious from our vantage point, it was by no means obvious to some of the delegates to the First Constitutional Convention. Indeed the issue of what constitutes the proper basis of apportionment was the subject of much discussion. One could easily argue that the number of voters in a state, or the number of voting age, or the number of citizens of voting age would be as reasonable as counting the total number of inhabitants. Some of the delegates went so far as to maintain that the most reasonable basis is the economic contribution of each state to the federal coffers, that is, the amount of its taxes. Implicit in this view is the idea that property rather than persons is the natural object of representation. While this position did not ultimately prevail, it was not entirely jettisoned either. It lingers in the infamous three-fifths rule, which gave representation to slaves (a particular form of property). It is also implicit in the language linking representation and taxation: instead of making representation proportional to economic product, the framers insisted that direct federal taxation be proportional to representation. (For this reason, early attempts by Congress to levy an income tax were declared unconstitutional; it required the passage of the sixteenth amendment to remedy the situation.)

An important adjunct to the requirement of apportionment "according to their respective Numbers" is that an enumeration be carried out every ten years to keep the Numbers current. This represented a striking departure from practices in other representative democracies of the day, notably Great Britain, where the allocation of seats in Parliament was largely a matter of historical precedent and often had little to do with the actual population of the various voting districts. While the Constitution is clear enough that representation should be proportional to the number of inhabitants, it does not spell out exactly how the apportionment should be calculated. This raises a difficulty, for in practice exact proportionality will almost always be unachievable due to the rounding problem. Inevitably, some states get more than their fair share and others less. The question then becomes: what is meant by "equity as near as may be" when perfect equity cannot be achieved? The history of apportionment in the United States can be viewed as a two hundred year debate on how to resolve this question. As we shall see, it has involved some of the country's greatest statesmen, as well as leading mathematicians, statisticians, and political analysts. Often the fiercest debates have been over the disposition of a single seat. Even one seat can tip the balance of political power, which makes the argument over principle all the more telling.

2. Formal statement of the apportionment problem.

An *apportionment problem* involves a group of states with given populations, say $p_1, p_2, \ldots, p_n$, and a whole number a of seats to be distributed among them. An *apportionment* allots a whole number of seats a_i to each state i, the sum of the allotments being *a*. An *apportionment method* is a criterion that determines one or more apportionments for every possible combination of populations and number of seats. The ideal is for each state to receive its exact proportional share of seats, known as its *quota,* subject to each state receiving at least one representative.

QUOTA. The *quota* of a state is the fraction that the state's population represents of the total population, multiplied by the total number of seats.

In almost every real case, quotas are whole numbers. The problem, therefore, boils down to finding solutions in whole numbers that are as *nearly proportional* to the populations as possible. While this may seem easy at first, over two hundred years of U.S. history show that this is far from being the case.

3. The 1791 contest between Jefferson and Hamilton

To see why, let us consider the first apportionment bill, which was passed by Congress in 1791 and sent to President George Washington for his signature.

State	Apportionment Population	Quota	First Bill (Hamilton)
Virginia	630,560	20.926	21
Massachusetts	475,327	15.774	16
Pennsylvania	432,879	14.366	15
North Carolina	353,523	11.732	12
New York	331,589	11.004	11
Maryland	278,514	9.243	9
Connecticut	236,841	7.860	8

South Carolina	206,236	6.844	7
New Jersey	179,570	5.959	6
New Hampshire	141,822	4.707	5
Vermont	85,533	2.839	3
Georgia	70,835	2.351	2
Kentucky	68,705	2.280	2
Rhode Island	68,446	2.271	2
Delaware	55,540	1.843	2
Total	3,615,920	120.000	120

Table 1. First apportionment bill: Hamilton's method

The proposed allocation of seats seems eminently reasonable: states with fractions larger than one-half are rounded up, while those with fractions smaller than one-half are rounded down. Of course, it is quite serendipitous that ordinary rounding would yield the required total number of seats, because typically there will either be too many or too few remainders that exceed one-half. A simple example is shown in Table 2. Since all remainders are less than .5, ordinary rounding will allot 20 seats instead of the required 21.

State	Population	Quota	Hamilton apportionment
A	7,270,000	14.24	14
B	1,230,000	2.41	3
C	2,220,000	4.35	4
Total	10,720,000	21.00	21

Table 2. A hypothetical example illustrating Hamilton's method

One way to deal with this problem was first suggested by Alexander Hamilton, then Secretary of the Treasury. In a letter to George Washington he proposed giving out one extra seat to the states with the largest fractions until the required number of seats is apportioned, irrespective of whether the resulting cutoff point equals one-half. In the above example, Hamilton's approach rounds state B up to three seats (because its fraction is largest), and the other two states are rounded down.

HAMILTON'S METHOD. *First give to each state the integer part of its quota, but not less than one seat. If any seats remain to be apportioned, give one each to the states with the highest fractional remainders.*

Meanwhile, Jefferson (Secretary of State, and Hamilton's arch-rival in Washington's Cabinet) sent another letter to Washington recommending against this scheme and its "difficult and inobvious doctrine of fractions." He alleged that the Constitution had left fractions "unprovided for" and they must therefore be neglected. Of course, virtually any real example will produce fractions, so it is not immediately obvious how to avoid representing them. Jefferson proposed the following ingenious device to deal with the situation: divide all the state populations by a common divisor and neglect the fractions; the divisor may then be *adjusted* so that this process yields the desired number of seats.

Table 3 shows the 1791 populations divided by the number 28,500. When the fractions are dropped, the outcome is the apportionment shown in the right-hand column, which sums to 120 seats.[1]

State	Apportionment Population	Quotient by 28,500	Jefferson

[1] Any divisor between 28,356 and 28,511 apportions 120 seats by this method, and the apportionments are all the same.

Virginia	630,560	22.125	22
Massachusetts	475,327	16.678	16
Pennsylvania	432,879	15.189	15
North Carolina	353,523	12.404	12
New York	331,589	11.635	11
Maryland	278,514	9.772	9
Connecticut	236,841	8.310	8
South Carolina	206,236	7.236	7
New Jersey	179,570	6.301	6
New Hampshire	141,822	4.976	4
Vermont	85,533	3.001	3
Georgia	70,835	2.485	2
Kentucky	68,705	2.411	2
Rhode Island	68,446	2.402	2
Delaware	55,540	1.949	2
Total	3,615,920	120.000	120

Table 3. Jefferson's method applied to the 1791 populations and 120 seats

The alert reader will notice that there is a special dividend from this proposal: Jefferson's (and Washington's) home state of Virginia gains one seat–indeed, it receives 22 seats even though its quota is only 20.986!

JEFFERSON'S METHOD. *Choose a common divisor and divide it into each of the state populations to obtain quotients. Give each state the whole number in its quotient, but not less than one seat. If the total number of seats allotted by this process is too large, increase* *the divisor; if the total is too small, decrease the divisor until a value is found that apportions the correct number of seats.*

Washington, after having "maturely considered" the input from various members of his cabinet, proceeded to veto the proposed apportionment, the first-ever exercise of the Presidential veto power. His stated objections to the plan were first, that it did not employ a common divisor (as proposed by Jefferson), and second, that it gave several states more than one representative per 30,000, contrary to the language of the Constitution. The legislation was therefore reconsidered, and after further wrangling Congress passed a new apportionment bill based on Jefferson's method, but with a common divisor of 33,000. This resulted in a House of 105 seatswith 19 seats for Virginia even though its quota of 105 seats was only 18.310.

This early tussle between Jefferson and Hamilton has a number of features that bob up repeatedly in later apportionment debates. First, the bone of contention often involves the disposition of just one or two seats-- a seemingly innocuous matter, but often of great political significance. Second, the deeper issue is usually about gaining a pivotal advantage for some political party or section of the country. (In 1791, the looming power struggle between the South and the North was uppermost in people's minds.) Third, the nature of the contest is often camouflaged beneath a debate over competing fairness principles that produce the desired political outcome as a byproduct.

4. The 1830s debate: Adams and Webster

All of these elements can be seen in the next major spat over apportionment, which broke out in the 1830s. In this instance, the deeper political issue was the rapidly waning power of New England, and the corresponding rise of the West and South. In 1791, for example, Massachusetts had been the second most populous state and New England had over 25 percent of the apportionment population. By the census of 1830, Massachusetts had slipped to eighth place and New England's share had dwindled to about 15 percent of the apportionment population. Meanwhile, the bias of Jefferson's method in favor of large states was becoming increasingly evident. Table 4 shows the apportionments to New York (by then the largest state) as compared to Delaware (one of the smallest) over the period 1791-1830.

Year:	1791	1800	1810	1820	1830	Total
New York's Quota	9.629	16.661	26.199	32.503	38.593	123.585
New York's Allotment	10	17	27	34	40	128
Delaware's Quota	1.613	1.782	1.952	1.685	1.517	8.549
Delaware's Allotment	1	1	2	1	1	6

Table 4. Jefferson allotments to New York and Delaware, 1791 - 1830

Notice that in every one of these apportionments, Delaware had a quota greater than 1.5, yet in four out of the five cases it received only one seat. Meanwhile, New York received 34 seats when its quota was 32.40, and 40 when its quota was 38.59.

The reason for this bias toward large states (which Jefferson doubtlessly recognized) is the following: by dropping the fractional part of each quotient, a large state gives up only a small part of its entitlement, whereas a small state may give up a major part. For example, if one state's quotient is 40.5 and another's is 1.5, then dropping the fraction--as Jefferson's method requires--results in a 1.2 percent loss for the large state but a 33 percent loss for the small one. In general, dropping the fractional part means that the per capita representation of a small state tends to be marked down by a larger percentage than the per capita representation of a large state. Hence, larger states are likely to be favored by Jefferson's method, and smaller states disfavored.

Bias, and ways of avoiding it, is probably the single most important issue that has arisen in debates over apportionment; we shall return to it later. In the meanwhile let us return to the contentious history of apportionment in the 1830s and 1840s. Prompted by a steady loss of political power and the blatant bias of Jefferson's method toward large states, New Englanders led the charge for apportionment reform. John Quincy Adams, who at that time was serving as a Representative from Massachusetts, wrote to his colleague in the Senate, Daniel Webster, advocating a new apportionment method that would benefit New England. Adams's proposal was both simple and cunning: follow the general outlines of Jefferson's method, but instead of dropping the fractional parts of quotients, round all of them up.

J. Q. ADAMS'S METHOD. *Choose a common divisor and divide it into each of the state populations to obtain quotients. Give each state its quotient rounded up to the next whole number. If the total number of seats allotted by this process is too large, increase the divisor; if the total is too small, decrease the divisor until a value is found that apportions the correct number of seats.*

This proposal cleverly turns the tables on Jefferson by giving a marked advantage to the small states (now in ample supply in New England). Webster judged it more prudent, however, to propose a remedy that was less obviously self-serving. This led him to formulate a compromise between Jefferson's and Adams's approaches.

WEBSTER'S METHOD. *Choose a common divisor and divide it into each of the state populations to obtain quotients. Round each quotient to the nearest whole number, but not less than one. If the total number of seats allotted by this process is too large, increase the divisor; if the total is too small, decrease the divisor until a value is found that apportions the correct number of seats.*

This proposal alleviates the bias toward the larger states inherent in Jefferson's method, but does not go too far by trying to shift the balance toward the other extreme.

Webster did not prevail in the 1830's apportionment debate. Discontent with Jefferson's method continued to grow, however, and following the 1840 census it was displaced by Webster's method. Unfortunately, the victory was short-lived: in 1850, a supposedly new method was proposed by Representative Samuel F. Vinton, which in fact was nothing but a thinly-disguised version of Hamilton's method. By a quirk of political fate, "Vinton's Method of 1850" was voted into law and remained on the books until the turn of the twentieth century.

5. A fatal flaw in Hamilton's method: the Alabama paradox.

During the latter part of the nineteenth century the number of members in both the House and Senate increased rapidly to accommodate new states and a growing population. From 1850 to 1900, the U.S. population rose more than three-fold and 22 states were added to the union. During this entire period Hamilton's method was the formula *de jure*, though not always *de facto*. Politics during the 1860s and 1870s were unusually fractious, and results were sometimes manipulated to satisfy the whims of various regional interests. In 1870, for example, the original apportionment was supplemented a few months later with a special hand-out of nine seats to certain Northern states. This had consequences beyond the skewing of representation in the House: in the 1876 Presidential election, Rutherford B. Hayes beat Samuel J. Tilden by a margin of one electoral vote. If Hamilton's (alias Vinton's) method had been used as prescribed, Tilden would have won instead.[12]

The steady enlargement of the House during this period led to the discovery of a curious, and ultimately fatal, flaw in Hamilton's method. Following the 1880 census, the chief clerk of the Census Office, C.W. Seaton, computed apportionments using Hamilton's method for all House sizes between 275 and 350 seats. In a letter to Congress he stated that: "While making these calculations I met with the so-called 'Alabama' paradox where Alabama was allotted 8 Representatives out of a total of 299, receiving but 7 when the total became 300. Such a result as this is to me conclusive proof that the process employed in obtaining it is defective."[2]

ALABAMA PARADOX. An apportionment method suffers from the *Alabama paradox* if there is a situation in which the total number of seats *increases*, all populations remain fixed, and a state's allotment *decreases*.

This peculiar phenomenon can be illustrated by the three-state example in Table 2. If there are 22 seats to be apportioned instead of 21, then the quotas are as follows: 14.92 for state A, 2.52 for state B, and 4.56 for state C. Therefore, the Hamilton apportionment is 15 for A, 2 for B, and 5 for C. Comparing this with the Hamilton solution for 21 seats, we see that B has lost a seat in going to a larger House! The reason is that the quota of B increases less rapidly in *absolute* terms than the quotas of A and C. B's remainder was largest when 21 seats were to be distributed; it is the smallest when 22 seats are available. Hence, it does not get an extra seat even though there are more seats to go around.

It is important to note that this paradox cannot occur under Webster's or Jefferson's methods, nor indeed under any method that employs a common divisor. The reason is quite simple: if the number of seats is increased by one, then the common divisor must be decreased enough to push up the quotient of some state to qualify for one more seat. Since this lower common divisor is applied to *all* of the states simultaneously, none of them can lose a seat in this process. Therefore, any such method avoids the Alabama paradox.

Congress was understandably disturbed by Seaton's findings, but no measures were taken to revise the apportionment statute for some years to come. Instead, the size of the House was enlarged after each census so that the effects of the Alabama paradox would not be felt. After the census of 1900, however, the difficulties with Hamilton's method became so acute that they could no longer be ignored. In 1901, for example, a report was submitted to the Congress giving Hamilton apportionments for all House sizes between 350 and 400. Maine kept bobbing up and down between three and four seats, as shown in Table 5. On being informed of this, Representative Littlefield of Maine vented his indignation against the chairman of the Select Committee on the Census: "In Maine comes and out Maine goes . . .God help the State of Maine when mathematics reach for her and undertake to strike her down." To which the chairman responded: "if Dame Rumor is to be credited, the seat of [Littlefield] is the one in danger . . . if the gentleman's statement be true that Maine is to be crippled by this loss, then I can see much force in the prayer he uttered here when he said "God help the State of Maine." [3]

House size	350-382	383-385	386	387-388	389-390	391-400
Maine's allotment	3	4	3	4	3	4

Table 5. Effect of Hamilton's method on Maine: 1900

After this episode, Hamilton's method was effectively abandoned though the statute itself remained on the books for another decade. The House was enlarged to 386 members, a number with the property that no state lost a seat. In 1911, the apportionment statute was changed and once again. Webster's method became the law of the land but not for long. That same year an entirely new proposal surfaced that has evolved into the method used today.

6. The method of Joseph Hill.

In 1911, Joseph A. Hill, chief statistician in the Census Bureau, wrote a letter to the chairman of the House Committee on the Census describing an entirely new approach to the apportionment question. Hill began with the proposition that the *per capita* representation in each of the states should be made as nearly uniform as possible. Here, "*per capita* representation" means the number of a state's representatives divided by the number of its inhabitants (a_i/p_i). The question is: What does it mean for these numbers to be "as nearly uniform as possible"?

Consider, for example, the Webster solution to the problem in Table 2. State A receives 15 seats and state B receives 2 seats. Thus, the per capita representation in A is 2.063 per million and in B it is 1.626 per million. Clearly, state B is less represented than A, and the absolute difference in representation is .437 per million. Now suppose that we take away one of A's seats and give it to B. B's per capita representation becomes 2.439 and A's becomes 1.926, a difference of .513. Accordingly, the transfer is *not justified* if we measure the discrepancy by looking at the absolute difference in per capita representation; but, said Hill, we should look at the *relative* difference rather than the absolute difference between the two numbers. In the first case, A has 26.9 percent more representation per capita than B; in the second case, B has 26.6 percent more representation per capita than A. Therefore, the transfer *is justified* because it lessens the *relative* discrepancy in per capita representation between the two states.

HILL'S METHOD. *Allocate the seats among the states so that no transfer of a seat between any two states reduces the percentage difference in per capita representation between them.*

It is by no means clear that such an allocation exists when there are more than two states. The reason is that, in transferring a seat from A to B in order to reduce the amount of inequality between those two states, there is no guarantee that the inequality between some other two states--say B and C--will not increase. The difficulty was resolved in 1921 by a professor of mathematics at Harvard University, Edward V. Huntington. He demonstrated that there always exists a solution satisfying Hill's criterion for all pairs of states simultaneously; moreover, there is a simple and elegant method for computing it. To describe Huntington's solution, we need to introduce the following concept:

GEOMETRIC MEAN. The *geometric mean* of two numbers is the square root of their product.

HUNTINGTON'S RULE FOR COMPUTING HILL'S METHOD. *Choose a common divisor and divide it into each of the state populations to obtain quotients. Round the quotient down if it is less than the geometric mean of the two nearest whole numbers, otherwise round it* *up. If the total number of seats allotted by this process is too large, increase the divisor; if the total is too small, decrease the divisor until a value is found that apportions the correct number of seats.*[4]

To illustrate how Huntington's procedure works in practice, consider Table 2 again. Let d = 500,000 be a trial common divisor. The quotients are then 14.540 for A, 2.460 for B, and 4.440 for C. The geometric mean of 14 and 15 is $\sqrt{14 \times 15}$ = 14.491, which is less than 14.540, hence A must be rounded up. Similarly, B must be rounded up because B's quotient exceeds $\sqrt{2 \times 3}$ = 2.449. Rounding both A and B upwards apportions too many seats, however, so the common divisor must be increased. If instead we consider d = 502,000, the quotients are 14.482 for A, 2.450 for B, and 4.422 for C. Thus, only B's quotient exceeds the geometric mean, and the Huntington apportionment is 14 for A, 3 for B, and 4 for C.

Since this method is based on a common divisor, it avoids the Alabama paradox as we noted earlier. Huntington christened his technique for computing a Hill apportionment the "method of equal proportions" and thereafter viewed the method as his own invention. He also devoted the next twenty years to the tireless promotion of it, both in congressional testimony and scientific journals. Moreover, his quest was ultimately successful: on November 15, 1941, President Franklin D. Roosevelt signed into law a bill establishing the Hill-Huntington formula as the statute method for apportioning the U.S. House of Representatives. It has been used ever since. Once again, Webster's method was displaced in favor of a supposed improvement.

Two events conspired to deliver Huntington this victory. One was political: in 1941 it happened that the methods of Webster and Hill differed in the assignment of exactly one seat. Hill's method gave one more to Arkansas and one less to Michigan than did Webster's method. In that era, Arkansas was a solidly Democratic state while Michigan had Republican leanings. In switching from Webster's method to Hill's, the Democrats saw a sure way to pick up one more seat. The vote reflected this: all House Democrats (except those from Michigan) voted for the change, while all Republicans voted against it.

The second reason for Huntington's triumph was that he had recruited several leading members of the mathematical community to his cause, including John von Neumann, Luther P. Eisenhart, and Marston Morse. They and others supported his method over Webster's in a series of reports to the National Academy of Sciences. Their argument was essentially the following: First, the only methods that should be considered are those that avoid the Alabama paradox. Second, of those methods that avoid the paradox, the method of equal proportions (Hill's method) is the only one that is "mathematically neutral" in its treatment of small and large states. But is this correct?

7. Bias

Let us say that an apportionment method is *unbiased* if every state can expect to receive its quota of seats on average over many apportionment situations. To examine whether Hill's method is unbiased in practice, let us evaluate its results had it been used in each of the twenty-two censuses from 1791 to 2000. First, eliminate from consideration all states whose quota is less than .5. These states are favored (by any method) because of the constitutional provision that every state must receive at least one seat. After these states have been removed, rank the states according to their populations. The largest one-third of the states will be called "large", the next one-third "medium," and the bottom one-third "small". (If the number of states is not divisible by three, put the extras into the "medium" category). Now compute the apportionment by Hill's method for each census year. Let a_S be the number of seats that the small states (as a group) would have received under Hill's method in any given census, and let q_S be their quota as a group. The percentage by which a_S exceeds q_S measures the extent to which Hill's method would have favored the small states in a given period.

It happens that the methods of Webster and Hill give the same result in the 2000 census, which is very slightly tilted in favor of the small states, but some favoritism is bound to occur in any given year (because of the rounding issue). What should be avoided is consistent favoritism toward some class of states over many years. Let us therefore examine the results if Hill's method had been used in all twenty-two censuses from 1791 to 2000. On average, Hill's method would have given the small states about 3.6 percent more seats than their quotas allow. Under Webster's method, though, small states would have received almost exactly their due (the discrepancy is less than 0.1 percent). These empirical results are backed up by theoretical analysis. We can estimate the theoretical probability that the small states will be favored in any given apportionment, and the extent of the favoritism. If the populations of the fifty states are distributed approximately as they are now, Hill's method would give the small states, as a group, about 3 percent more than their quotas. By contrast, Webster's method is almost exactly neutral.[5]

Given these results, it seems odd that several prestigious committees of the National Academy of Sciences reached the conclusion that Hill's method is even-handed in its treatment of small and large states. Obviously, they could not have looked at the data, and indeed they did not.[6] Rather, theyreasoned their way toward a solution. The argument was quite ingenious, and had originally been formulated by Huntington in his 1921 paper:

> Between any two states there will practically always be a certain inequality which gives one of the states a slight advantage over the other. A transfer of one representative from the more favored state to the less favored state will ordinarily reverse the sign of this inequality, so that the more favored state now becomes the less favored, and vice versa. Whether such a transfer should be made or not depends on whether the amount of inequality between the two states after the transfer is less or greater than it was before . . . The fundamental question therefore at once presents itself, as to how the 'amount of inequality' between two states is to be measured. [7]

Huntington examined each of the ways in which the difference in representation between two states can be written: the absolute difference in the number of persons per representative, the relative difference in the number of persons per representative, the absolute difference in the number of representatives per person, et cetera. Each such measure of inequality between two states can be translated into a method for apportioning seats between two states, namely, by distributing the seats so that the amount of inequality between every two states is minimized. In other words, transferring a seat from one state to another should not *increase* the degree of inequality between them.

It is not at all obvious that such a distribution exists. One of Huntington's major contributions was to show that only some ways of measuring inequality can be made to work; in fact, there are only *five* such methods. Interestingly, *every one* of them had turned up in historical debates on apportionment. Four of the five methods have already been discussed: Jefferson, Adams, Webster, and Hill. The fifth had been proposed in an 1832 letter to Daniel Webster by one of his former teachers at Dartmouth, Professor James Dean. Table 6 shows the five methods and criteria of pair-wise inequality that they minimize. Note that Webster's method minimizes the absolute difference between the number of representatives per person for every pair of states, whereas Dean's method minimizes the absolute difference between the average district sizes for every pair of states. Hill's method minimizes the relative difference in district sizes (as well as the relative difference in number of representatives per person) for every pair of states.

The mathematicians appointed to study the problem by the National Academy of Sciences rested their case for Hill's method on two main points. First, they noted that Hill's method minimizes both the relative difference between district sizes and the relative difference between number of representatives per person. Hence, one does not have to choose between these two criteria if one accepts the relative difference as the correct measure of difference. (It leaves open the question, however, of why the relative difference is to be preferred over the absolute difference.) Second, they pointed out that the five methods can be ranked according to their tendency to favor small versus large states. Moving down the list in Table 6, the methods increasingly favor the large states relative to the small states. Thus, Adams's is most favorable to the small, and Jefferson's is most favorable to the large. The method of Hill is mathematically neutral, said the NAS committee, because it is the "middle" of the five methods. It is certainly fortunate for this reasoning that the number of methods under consideration was odd!

Method	Inequality measure
Adams	$a_i - a_j (p_i/p_j)$
Dean	$p_j/a_j - p_i/a_i$
Hill	$(a_i/p_i)/(a_j/p_j) - 1$
Webster	$a_i/p_i - a_j/p_j$
Jefferson	$a_i(p_j/p_i) - a_j$.

Table 6. Inequality measures for the five classical divisor methods.

The flaws in this reasoning are all too obvious. It establishes that Hill's method favors the small states less than two methods (Dean and Adams) and favors them more than two other methods (Jefferson and Webster). It does not show that Hill's method is unbiased in any absolute sense. Indeed, as we have already seen, Hill's method is measurably biased toward small states.

8. The population paradox

The historical debates over apportionment have ultimately been driven by the extraordinarily dynamic character of the nation's growth in numbers. Indeed, the implications of rapid growth were foreseen by the architects of the Constitution, and lie behind the requirement of regular censuses and reapportionments. As we have already seen, the steady expansion in the number of representatives during the nineteenth century revealed a serious flaw in one method–Hamilton's–and led to its political demise. Since 1920, however, the House size has been fixed at 435 (except for a temporary expansion in the 1950s to accommodate the new states of Alaska and Hawaii). Thus, we might be tempted to argue that the Alabama paradox no longer has any force. Furthermore, it can be shown that Hamilton's method, like Webster's, is unbiased. Does this mean that Hamilton's method ought to be resuscitated?

Unfortunately, the method suffers from another fatal defect that can occur even when the House size is fixed. To illustrate, suppose that Hamilton's method had been in use during the 1960s, and suppose for purposes of exposition that a reapportionment had been conducted every year. During this period, North Dakota was the only state that was losing population; nevertheless, had there been an apportionment in 1960 and again in 1961, North Dakota would have *gained* a seat at the expense of states that were increasing in population.

The reason for this paradoxical result can be explained as follows: Under Hamilton's method, the states' fractional remainders determine their priority for getting an extra seat. Large states that are growing more slowly than the national average may see their remainders decrease quite sharply–even more sharply, in fact, than the remainders of small states that are not growing. To be specific, in 1960, North Dakota's quota was 1.541 and Pennsylvania's was 27.576. Under Hamilton's method, Pennsylvania would have received an extra seat but not North Dakota, because its remainder fell just below the cutoff point for receiving an extra seat. In 1961, their quotas would have been 1.519 and 27.350 respectively. Pennsylvania's quota decreased because it was growing more slowly than the national population, and the decrease is large because it is a large state. The upshot is that its remainder is now below that of North Dakota, which gains a seat at Pennsylvania's expense.

POPULATION PARADOX. A method exhibits the *population paradox* if a state that loses population gains a seat at the expense of a state that gains population.

We have just seen that Hamilton's method, even though it is unbiased, suffers from both the population paradox and the Alabama paradox. It would therefore seem unwise to reinstate it. The question, however, is whether there exist any methods that avoid the paradox? It turns out, in fact, that all of the classical divisor methods–Jefferson, Webster, Hill, Dean, and Adams--are immune to it. The results of U.S. history, as well as theoretical considerations, show that Webster's is the only one of these methods that is unbiased in its treatment of small and large states. In fact, it can be shown that Webster's method is essentially the only rule that is unbiased and avoids these two paradoxes.[8]

<u>9. Conclusion.</u>

Had the drafters of the Constitution prescribed a specific rule for apportioning representation among the states, much subsequent debate would have been spared. Perhaps they appreciated the subtlety of the problem and wanted to avoid a wrangle in the First Constitutional Convention that would have impeded progress in other matters. In any event, the historical debates over this gap in Article I show how difficult the problem is to resolve in practice. The debates also illustrate a larger point: disputes over the allocation of political power ultimately derive their force from appeals to principle. Put differently, even though the recurrent contests over apportionment might have taken the form of naked and unprincipled power struggles, history shows that they were usually clothed in the language of fairness and how to meet the ideal of one person, one vote. Although many may view this as mere hypocrisy, in my view it shows that appeals to principle represent powerful instruments of persuasion.

The principles that have been forged in the crucible of political debate are simple to state and appeal to our intuitions about fairness. First and foremost is even-handedness or lack of bias: all states, large or small, should get their fair share on average. Second, as the number of seats goes up (the pie grows) no one's share should go down. Third, as populations change, a growing state should not give up seats to a shrinking state. The implications are surprisingly strong: there is essentially only one method–Webster's–that satisfies all three of these principles. It is commonly used in other representative democracies (where it goes under the name St. Lagüe's method); in the United States it was the law of the land in the 1840s and again during the period 1910-1940 when a combination of political interests and scientific confusion led to its abandonment. The ideal of one person, one vote would be well served if Congress reinstated this simple and eminently sensible solution to the apportionment problem.

Notes

[1] The 1876 election is one of the few instances in U.S. history where the popular vote and the electoral vote gave contrary results. The most recent example is the 2000 election, in which George W. Bush received 271 electoral votes and Albert Gore 266 but Gore had over 500,000 more popular votes than did Bush. Unlike the 1876 case, the electoral college outcome would not have been reversed by using any of the six historical apportionment methods discussed in this paper.

[2] Congressional Record, 47th Congress, 1st session, 1881. Vol. 12, pp. 704-705.

[3] Congressional Record, 56th Congress, 2nd session, 1901, vol. 34 (House), pp. 591-93 and pp.729-30.

[4] This method automatically gives at least one seat to each state, however small. The reason is that the geometric mean of 0 and 1 is zero, so any state with quotient less than 1 gets rounded up.

[5] See Balinski and Young, 2002, Appendix A. The bias of Jefferson's and Adams's methods is even more pronounced.

[6] That task was undertaken by Huntington's rival, Walter F. Willcox, who was one of the country's foremost applied statisticians. As early as 1915 Willcox had enunciated his position in his presidential address to the American Economic Association: "The use of [Hill's method] has recently been advocated. To use it, however, would . . . result in defeating the main object of the Constitution, which is to hold the scales even between the small and the large states. For the use of [it] inevitably favors the small state . . . the method of major fractions [Webster's] is the correct and constitutional method of apportionment." (Willcox, 1916). Willcox reached this conclusion in the old-fashioned, and in those days rather tedious, way of computing solutions to many sample problems. Huntington and Willcox debated this point in the pages of *Science* for over a decade and a half.

[7] Huntington, Edward V. 1921. "The Mathematical Theory of the Apportionment of Representatives." *Proceedings of the National Academy of Sciences of the United States of America* 7: 123-127.

[8] See Balinski and Young , 2002, Theorem 5.3.

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