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Recent monetary policy experience suggests a simple test of models of monetary non-neutrality. Suppose the central bank pegs the nominal interest rate below steady state for a reasonably short period of time. Familiar intuition suggests that this should be inflationary. But a monetary model should be rejected if a reasonably short nominal rate peg results in an unreasonably large inflation response. We pursue this simple test in three variants of the familiar dynamic new Keynesian (DNK) model. All of these models fail this test. Further some variants of the model produce inflation reversals where an interest rate peg leads to sharp deflations.

Keywords: fixed interest rates, new Keynesian model, zero lower bound.

JEL Classication: E32.

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1 Introduction

This paper examines a very simple question: what is the behavior of inflation if a central bank credibly announces a fixed interest rate for an extended period of time? To be precise, we examine the perfect foresight path of sticky price models in which the nominal interest rate is pegged for a finite period of time. It is of course well known that if the peg is perpetual, then there is local equilibrium indeterminacy and thus sunspot equilibria. But if the peg is finite, and the subsequent monetary policy is consistent with a unique equilibrium, then this subsequent behavior provides the needed terminal condition that implies determinacy along the entire path.

Our analysis is motivated by the use of such monetary policies in the aftermath of the 2008 financial crisis. But we do not view this paper as an investigation of policy during the financial crisis. Nor do we see this as a data-matching exercise. Instead we see a transient interest rate peg as an experiment for testing the reasonableness of monetary models. For example, suppose we want to consider the reasonableness of a particular model of monetary non-neutrality. If a reasonably short fixed-interest rate policy leads to an unreasonably large behavior for inflation and output within that model, then this model should be rejected. The modifiers "reasonable" and "unreasonable" are open for discussion, but the basic logic of the experiment is not.

Our experiments consider a fully anticipated and unconditional lowering of the monetary policy rate for a finite number of periods. We find that the workhorse Calvo (1983) models of time dependent pricing currently used for monetary policy analysis deliver unreasonably large responses of inflation and output in response to such a policy. Furthermore, if there are endogenous state variables, these models suggest that the initial responses can become arbitrarily large as the duration of the fixed rate regime approaches some critical value and then switch sign and become arbitrarily negative as this critical value is exceeded slightly. For empirically realistic models such as the Smets and Wouters (2007) model, the critical duration for which these asymptotes occur is around eight quarters - well within the duration of the low interest rate environment in the U.S. following the financial crisis.

We believe these results call into question the usefulness of these models - at least for studying monetary policy at the zero lower bound or when interest rate are fixed for other reasons. We conjecture that state dependent pricing models as in Dotsey, King, and Wolman (1999) or Golosov and Lucas (2007) may not suffer from these deficiencies and may thus be particularly appropriate when considering fixed interest rate environments. These unreasonably large responses of inflation and output are unlikely to occur with state dependent pricing, because the incentive for firms to change their price instead of changing quantities would seem large in the experiments we consider.

Several other authors have examined the issues of this type. This previous work includes Blake (2011), Laseen and Svensson (2011) and Galí (2011). Laseen and Svensson (2011) provide a convenient mechanism for solving monetary models in which nominal rates are set exogenously for a finite period of time. Relatedly, Braun and Korber (2011) and Christiano, Eichenbaum, and Rebelo (2011) have examined the size of the fiscal multiplier when the nominal rate is pegged for a finite time period.

Section 2 examines the issue in the simplest new Keynesian model. Section 3 adds inertia in the Phillips curve and Taylor rule and provides conditions for the existence of asymptotes. Section 4 shows that our results also obtain in the Smets and Wouters (2007) model at their estimated parameter values. Section 5 concludes. The appendix contains proofs, an analysis of a stochastic exit from a fixed interest rate regime, and a stacked time approach to the issue.

2 The DNK model with no endogenous states

The canonical DNK model is given by the following:

$$\pi_t = \kappa y_t + \beta \pi_{t+1} \tag{1}$$

$$y_t = y_{t+1} - \frac{1}{\sigma}\left(i_t - \pi_{t+1}\right) \tag{2}$$

where π_t, y_t, and i_t, denote inflation, the output gap, and the nominal rate, respectively, all measured as deviations from the steady-state. We dispense with expectations notation as we are focusing on the perfect foresight path. To close the model, we assume the central bank announces an interest rate rule given by the following:

$$i_t = \begin{cases} i^* & t = 1,2,...,\mathrm{T} \\ \phi_\pi \pi_t + \phi_y y_t & t = \mathrm{T}+1,\mathrm{T}+2,\ ... \end{cases} \tag{3}$$

Under standard assumptions on ϕ_y and ϕ_{pi} there is a unique equilibrium for $t > T$. Since there are no state variables nor exogenous shocks, the unique equilibrium for $t > T$ is given by $\pi_t = y_t = 0$. For the first T periods, the constant interest rate suggests that there is equilibrium indeterminacy. But since this policy regime is finite, the subsequent uniqueness for $t > T$ serves as a terminal condition and thus ensures uniqueness of equilibrium along the entire path. During the constant interest rate policy, the inflation dynamics are governed by

$$\pi_t = -\frac{\kappa}{\sigma} i^* + \left(\frac{\kappa}{\sigma} + 1 + \beta\right) \pi_{t+1} - \beta \pi_{t+2} \tag{4}$$

with the two terminal conditions:

$$\pi_T = -\frac{\kappa}{\sigma} i^* \tag{5}$$

$$\pi_{T-1} = -\left(\frac{\kappa}{\sigma} + 2 + \beta\right) \frac{\kappa}{\sigma} i^* \tag{6}$$

It is convenient to invert this system, running time backwards from the end of the extended period. That is, let z_s denote the value of inflation s-periods before time T: $z_s \equiv \pi_{T+1-s}$. The difference equation (5) can now be written as:

$$z_s = \frac{\kappa}{\sigma} i^* + \left(\frac{\kappa}{\sigma} + 1 + \beta\right) z_{s-1} - \beta z_{s-2} \tag{7}$$

with the two *initial* conditions:

$$z_1 = \pi_T = -\frac{\kappa}{\sigma} i^* \tag{8}$$

$$z_2 = \pi_{T-1} = -\left(\frac{\kappa}{\sigma} + 2 + \beta\right) i^* \tag{9}$$

The two initial conditions (8)-(9) imply that there is a unique solution to the difference equation (7) The solution has a simple form:

Proposition 1. *The inflation rate during the extended period is given by*

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 e_2^s \quad \text{for } s = 0, 1, ..., T \tag{10}$$

for $s = 1, 2, .., T$*, where* $e_1 > 1, e_2 < 1$*, and* m_1 *and* m_2 *come from the two restrictions* (8)-(9)*. Proof: see appendix A.*

Several remarks are in order. First, the solution to the difference equation is independent of T. That is, m_1 and m_2 are independent of T. This arises because there are only restrictions on inflation at the end of the extended period, and not at the beginning of the extended period. Second, the inflation path during the extended period is entirely independent of the Taylor rule that commences at the end of the extended period. But this subsequent policy is necessary as it provides a terminal condition to the indeterminate difference equation. These first two remarks will not be true if there are endogenous or exogenous state variables. Third, the inflation behavior given by (10) is explosive since $e_1 > 1$. That is, if we run time backwards from the end of the extended period, the initial inflation rate explodes exponentially in T:

$$\pi_1 \equiv z_T = i^* + m_1 e_1^T + m_2 e_2^T \tag{11}$$

This unstable root is a manifestation of the familiar local indeterminacy induced by an interest rate peg. This explosive behavior is already evident in (5) - (6):

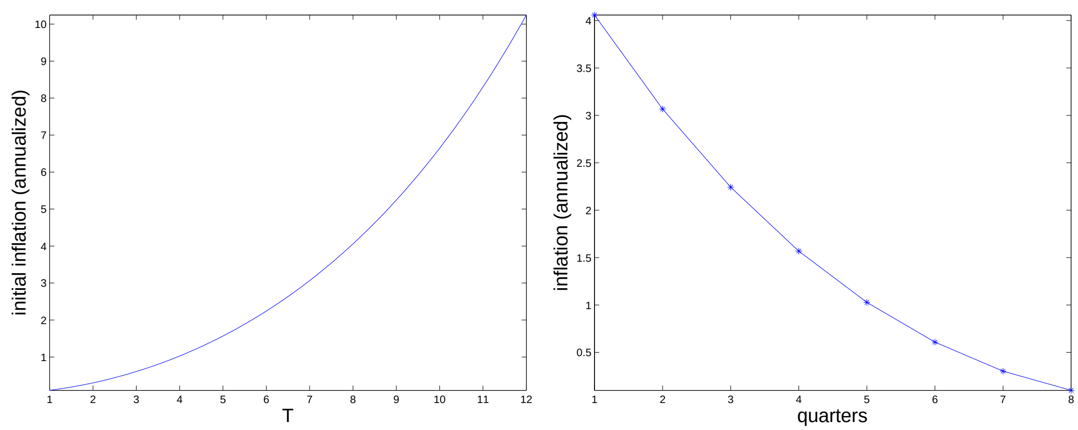
$$\pi_{T-1} = -\left(\frac{\kappa}{\sigma} + 2 + \beta\right) i^* = \left(\frac{\kappa}{\sigma} + 2 + \beta\right) \pi_T > \pi_T \tag{12}$$

The unstable eigenvalue (e_1) during the extended period has a fairly simple form if we assume $\beta = 1$. It is given by:

$$e_1 = x + \sqrt{x^2 - 1} \quad \text{where: } x = 1 + \frac{\kappa}{2\sigma} \tag{13}$$

The key variable determining inflation during the extended period is $\frac{\kappa}{2\sigma}$. A smaller value for σ magnifies the inflation response, because it makes spending more interest rate sensitive. A larger value for κ increases the inflation response, because with fixed nominal rates, the more flexible prices are (the higher κ) the more the real rate interest rate falls, which stimulates demand and puts even further upward pressure on prices.

Figure 1 provides a quantitative example of these effects. The calibration is standard: $\kappa = 0.025$, $\sigma = 1$, $\beta = .99$. As noted, the policy rule subsequent to the extended period is irrelevant (as long as it produces determinacy). We assume a steady state annualized nominal rate of 4%, and consider a policy of pegging the nominal rate at zero for T periods.

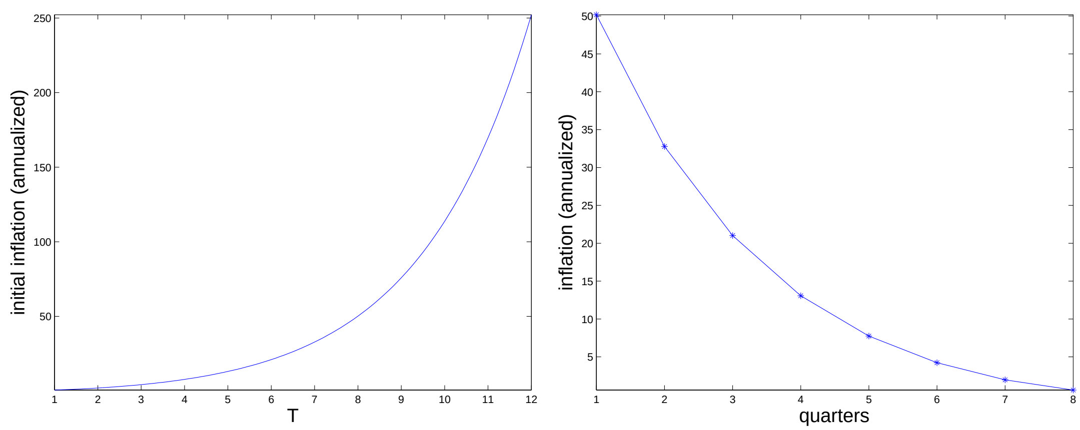


(a) Initial inflation as a function of T

(b) Time path of inflation for $T = 8$

Figure 1: Inflation dynamics in new Keynesian model with fixed interest rates, $\sigma = 1$

The magnitude of the inflation response depends crucially on the ratio $\frac{\kappa}{\sigma}$ and our baseline calibration assumes conservative values for this. Thus, the response of inflation shown in Figure 1 may not seem unreasonable. Woodford (2003) argues that in a model without capital one should use a lower value for σ as that makes spending more interest rate sensitive, partly compensating for the absence of investment which is more interest rate sensitive.



(a) Initial inflation as a function of T (b) Time path of inflation for $T = 8$

Figure 2: Inflation dynamics in new Keynesian model with fixed interest rates, $\sigma = 0.16$

For the dynamics of inflation, only the ratio κ/σ matters. So a graph similar to 2 would obtain for changes in κ. But the output dynamics do depend on whether σ or κ are varied. Figure 2 shows that the initial inflation response can become extremely large when $\sigma = 0.16$ as in the Woodford calibration. This suggests that for reasonable calibrations of the simple DNK model, a reasonably short interest rate peg delivers an unreasonable level of inflation.

3 Adding state variables to the model

Many have argued that the simple DNK model is a poor description of macro behavior because of its inability to generate inflation persistence, see Fuhrer and Moore (1995). One manifestation of this lack of inertia is the explosive behavior of the initial inflation rate in Figure 2. If inflation in the initial period is linked to the inflation rate in the previous period (the period before the announcement of the extended period), then the initial inflation behavior will likely be dampened. Hence, in this section we explore the effect of adding inflation inertia to the model as in Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007). The Phillips curve is now given by:

$$\pi_t = \frac{\kappa}{1+\beta\lambda} y_t + \frac{\beta}{1+\beta\lambda}\pi_{t+1} + \frac{\lambda}{1+\beta\lambda}\pi_{t-1} \tag{14}$$

where $0 \leq \lambda \leq 1$ is the degree of indexation. Similarly, the policy rule subsequent to the extended period is characterized by inertia $0 \leq \rho < 1$, so that for $t > T$ we have

$$i_t = \rho i_{t-1} + (1-\rho)\left(\phi_\pi \pi_t + \phi_y y_t\right) \tag{15}$$

Again, under standard assumptions on the policy rule, there is a unique equilibrium for $t > T$. We denote these decision rules for $t > T$ by:

$$\pi_t = b_1 \pi_{t-1} + b_2 i_{t-1} \tag{16}$$

$$i_t = d_1 \pi_{t-1} + d_2 i_{t-1} \tag{17}$$

During the extended period of time, lagged inflation is the only state variable. The difference equation governing dynamics is given by:

$$\begin{aligned}\pi_t =& \frac{-\kappa i^*}{\sigma[1+\lambda(1+\beta)]} + \frac{\lambda}{[1+\lambda(1+\beta)]}\pi_{t-1} + \frac{\sigma[1+\lambda(1+\beta)]+\kappa}{\sigma[1+\lambda(1+\beta)]}\pi_{t+1} \\ &- \frac{\beta}{1+\lambda(1+\beta)}\pi_{t+2}\end{aligned} \tag{18}$$

We will find it convenient to write this as:

$$\pi_t = c + \gamma_0 \pi_{t-1} + \gamma_1 \pi_{t+1} + \gamma_2 \pi_{t+2} \tag{19}$$

The state variables imply that we have one initial and two terminal conditions:

$$\begin{aligned}&\pi_0, \\ \pi_T &= c + \gamma_0 \pi_{T-1} + \gamma_1(b_1 \pi_T + b_2 i^*) + \gamma_2[b_1(b_1 \pi_T + b_2 i^*) + b_2(d_1 \pi_T + d_2 i^*)], \\ \pi_{T-1} &= c + \gamma_0 \pi_{T-2} + \gamma_1 \pi_T + \gamma_2(b_1 \pi_T + b_2 i^*)\end{aligned}$$

As before, the system can be expressed in z form:

$$z_s = c + \gamma_0 z_{s+1} + \gamma_1 z_{s-1} + \gamma_2 z_{s-2} \tag{20}$$

With two initial and one terminal condition in z-form:

$$z_{T+1} = \pi_0, \tag{21}$$

$$z_1 = c + \gamma_0 z_2 + \gamma_1(b_1 z_1 + b_2 i^*) + \gamma_2[b_1(b_1 z_1 + b_2 i^*) + b_2(d_1 z_1 + d_2 i^*)], \tag{22}$$

$$z_2 = c + \gamma_0 z_3 + \gamma_1 z_1 + \gamma_2(b_1 z_1 + b_2 i^*) \tag{23}$$

The solution to this system is given by the following.

Proposition 2. *The system in (20) has one eigenvalue e_1 inside the unit circle and two eigenvalues (e_2 and e_3) outside the unit circle. The inflation rate during the extended period is then given by*

1. *If e_2 and e_3 are real:*

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 e_2^s + m_3 e_3^s \tag{24}$$

2. *If e_2 and e_3 are complex, $(e_2, e_3) = a \pm bi$:*

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 r^s \cos(\theta s) + m_3 r^s \sin(\theta s) \tag{25}$$

   where $r = \sqrt{a^2 + b^2} > 1$, $\theta = \tan^{-1}(\frac{b}{a})$, and m_1, m_2, and m_3, come from the three restrictions (21) - (23). These constants do depend upon T because of the condition (21).

Proof: See appendix A.

Proposition 2 demonstrates that once again we have explosive inflation behavior as we increase the length of the extended period. An additional implication of Proposition 2 is that we can get inflation reversals. Suppose that we are in a parameter region with complex roots. The initial inflation behavior is then given by

$$\pi_1 \equiv z_T = i^* + m_1 e_1^T + m_2 r^T \cos(\theta T) + m_3 r^T \sin(\theta T) \tag{26}$$

Since $r > 1$, the initial inflation rate is explosive in T. But the sign of this effect will switch between positive and negative. This switch is marked by an asymptote, where initial inflation goes from arbitrarily positive to arbitrarily negative as we increase T. Strictly speaking there are no asymptotes when T is restricted to take on integer values, but sign reversals will still occur. Figure 3 plots inital inflation against the duration of the fixed rate regime and shows these reversals. The calibration is given by: $\kappa = 0.025$, $\sigma = 1$, $\beta = 0.99$, $\lambda = 1$, $\rho = 0.80$, $\phi_\pi = 1.5$, $\phi_y = 0.5$. The plot somewhat masks the asymptote that first arises between T = 6 and T = 7 (and periodically thereafter). But note that as we approach the asymptote, initial inflation goes from unboundedly positive to unboundedly negative.

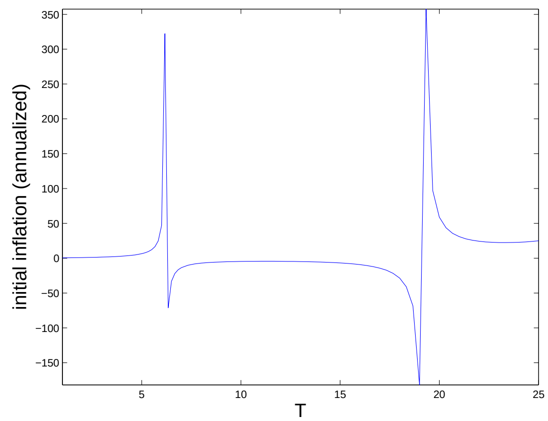
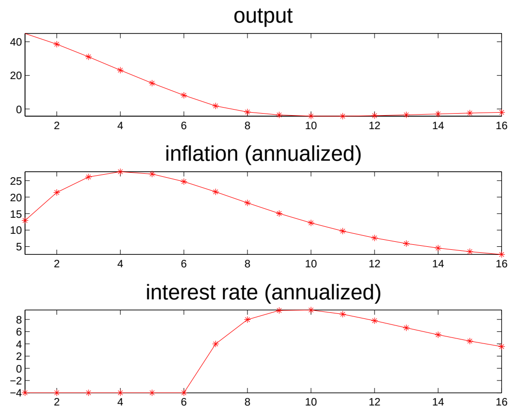
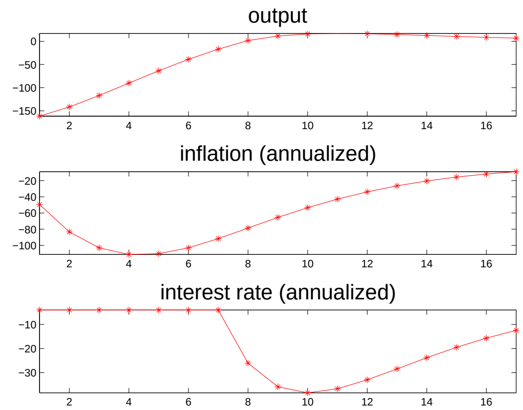


Figure 3: Initial inflation as a function of the duration T

Figure 4 shows dynamics of the model for two different values of T on either side of the asympote. The figures plot the inflation behavior for $T = 6$, and 7 periods. Because these are trigonometric functions, we of course have further sign reversals as we increase T. Along with these peculiar reversals, we also have enormous inflation rates from very modest extended periods. This is in contrast to the model without state variables as it arises even with $\sigma = 1$. Hence, adding state variables has two peculiar implications: (i) the existence of enormous inflation rates for modest extended periods, and (ii) sign reversals so that a low nominal rate peg for 6 periods can have the opposite effect of a low nominal rate for 7 periods.



(a) Dynamics for $T = 6$



(b) Dynamics for $T = 7$

Figure 4: Dynamics in new Keynesian model with inflation inertia

The nominal interest rate in Figure 4(b) violates the zero lower bound after the exit from the fixed rate regime and is thus not an equilibrium. We show these dynamics nevertheless to emphasize the possibility of sudden reversals mentioned before. Taking the lower bound into account, we find numerically that the economy remains in the extended period until period $T = 12$. Keeping the interest rate low for a few extra quarters does not eliminate the perverse dynamics in the fixed rate regime where output and inflation still collapse (inflation troughs at -13 %, and output falls by -15%). This suggests that according to the model, it is simply not possible for the central bank to keep the rate pegged at zero for 7, 8, 9, 10, or 11, quarters and then revert to a Taylor rule.

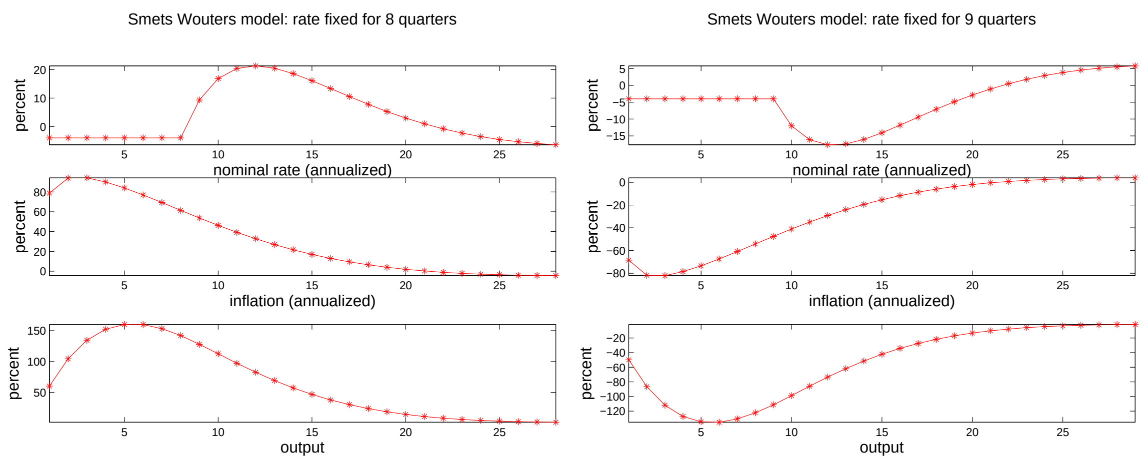
A necessary condition for this perverse sign-switch behavior is the existence of complex roots to the characteristic equation. If we assume $\beta = 1$, the condition for the existence of complex roots has a particularly simple form. There are complex roots if and only if $\Delta < 0$, with

$$\Delta \equiv -4\frac{\kappa}{\sigma}\lambda^3 + \left[12\frac{\kappa}{\sigma} - 8\left(\frac{\kappa}{\sigma}\right)^2\right]\lambda^2 - 4\left[\left(\frac{\kappa}{\sigma}\right)^3 + 5\left(\frac{\kappa}{\sigma}\right)^2 + 3\left(\frac{\kappa}{\sigma}\right)\right]\lambda + \left(\frac{\kappa}{\sigma}\right)^2 + 4\left(\frac{\kappa}{\sigma}\right) \tag{27}$$

This relationship is a convex and decreasing mapping between $\frac{\kappa}{\sigma}$ and λ. For example, for $\kappa = 0.025$, there are complex roots if and only if $\lambda > 0.57$. For $= \frac{\kappa}{\sigma} = 0.25$, there are complex roots if $\lambda > 0.29$. The existence of complex roots implies that there is an asymptote at some value of T. Because the size of the complex part is increasing as Δ becomes more negative, the periodicity of the asymptotes is decreasing in $\frac{\kappa}{\sigma}$ and λ. Note from (27), that interest rate inertia has no effect on the existence of complex roots (and thus asymptotes).

4 The Smets-Wouters Model

In this section we investigate whether the perverse effects outlined in Section 3 continue to arise in a more complete model economy. A classic reference here is Smets and Wouters (2007) small scale estimated DSGE model that includes habit persistence in consumption, capital accumulation, nominal wage indexation, and nominal price indexation. We will use their benchmark model and estimated parameter values to investigate the existence of perverse behavior in a more realistic model of the economy. The reader is referred to the paper by Smets and Wouters (2007) for the derivation of the model and the estimated parameter values.



(a) Model dynamics for $T = 8$

(b) Model dynamics for $T = 9$

Figure 5: Dynamics in Smets Wouters model with fixed interest rates

Remarkably, we again observe sign switches in the behavior for inflation and output for reasonably short durations of the fixed rate regime. Apparently, the additional frictions and features introduced in the empirically relevant model leave our main conclusion unchanged. We note again that dynamics in Figure 5(b) are not an equilibrium, because they violate the zero lower bound on the nominal interest rate. Imposing the lower bound, the interest rate would have to remain fixed until period $T = 13$. Nevertheless, in that equilibrium, annualized inflation falls by almost 20 percent on impact. So the qualitative features of a reversal of initial inflation dynamics still remain when taking the zero lower bound into account.

5 Conclusion

This paper began by proposing the following diagnostic test of a monetary model: reject a model if a reasonably short extended period of time generates an unreasonably large inflation effect. We have pursued this test in a variety of familiar DNK models. In all cases the test suggests that these models should be rejected.

We acknowledge that this analysis considers only local linear approximations to the nonlinear model. Furthermore, the variables move far away from their expansion points. Whether results similar to those obtained from the linear approximation can also be obtained when studying the nonlinear model globally is an open question that we leave for future research. See Braun and Korber (2011) for a global analysis of the dynamics in the new Keynesian model at low interest rates.

This is not an econometric test. Nor is this a statement about other possible shocks hitting the system. This is instead a question of prima facie sensibility. Our results suggest that these models fail this simple test. Therefore, these models or the linear approximation technique commonly used to analyze them at the zero lower bound are not fit for this purpose.

A standard rejoinder to this conclusion is that the DNK model with time-dependent pricing would break down into state-dependent pricing along such inflationary paths. This is surely correct. But this confirms the notion that the standard DNK model is inappropriate for situations in which the interest rate is fixed because of the zero lower bound or for other reasons.

Appendix

A Proofs

Proposition 1. *The inflation rate during the extended period is given by*

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 e_2^s \quad \textit{for } s = 0, 1, ..., T$$

for $s = 1, 2, ..., T$, *where* $e_1 > 1, e_2 < 1$, *and* m_1 *and* m_2 *come from the two restrictions* (8)-(9).

Proof. The difference equation is given by

$$z_s = -\frac{\kappa}{\sigma} i^* + \left(\frac{\kappa}{\sigma} + 1 + \beta \right) z_{s-1} - \beta z_{s-2} \equiv c + \gamma_1 z_{s-1} + \gamma_2 z_{s-2}$$

with the two initial conditions:

$$\begin{aligned} z_1 &= c \\ z_2 &= c + \gamma_1 c \end{aligned}$$

The particular solution is given by: $m = c + \gamma_1 m + \gamma_2 m$. Solving this we have $m = i^*$. The characteristic equation of the homogeneous system is given by

$$q^2 - \gamma_1 q - \gamma_2 = 0$$

or

$$h(q) \equiv q^2 - \left(\frac{\kappa}{\sigma} + 1 + \beta \right) q + \beta$$

Since h is convex, $h(0) > 0$, and $h(1) < 0$, the difference equation has two real eigenvalues denoted by $e_1 > 1$ and $e_2 < 1$. The general solution is thus given by $z_s = m + m_1 e_1^s + m_2 e_2^s$. □

Proposition 2. *The system in* (20) *has one eigenvalue* e_1 *inside the unit circle and two eigenvalues (*e_2 *and* e_3*) outside the unit circle. The inflation rate during the extended period is then given by*

1. *If* e_2 *and* e_3 *are real:*

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 e_2^s + m_3 e_3^s$$

2. *If* e_2 *and* e_3 *are complex,* $(e_2, e_3) = a \pm bi$*:*

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 r^s \cos(\theta s) + m_3 r^s \sin(\theta s)$$

where $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}(\frac{b}{a})$, *and* m_1, m_2, *and* m_3, *come from the three restrictions* (21) - (23)*. These constants do depend upon* T *because of the condition*(23)*.*

Proof. The difference equation(20) is given by

$$z_s = c + \gamma_0 z_{s+1} + \gamma_1 z_{s-1} + \gamma_2 z_{s-2}$$

The particular solution to this system is given by the m that satisfies

$$m = c + \gamma_0 m + \gamma_1 m + \gamma_2 m$$

or

$$m = \frac{c}{1 - \gamma_0 - \gamma_1 - \gamma_2} = i^*$$

The characteristic equation of the homogeneous system is

$$g(q) \equiv \frac{\lambda}{[1 + \lambda (1 + \beta)]} q^3 - q^2 + \left[\frac{\sigma (1 + \beta\lambda + \beta) + \kappa}{\sigma [1 + \lambda (1 + \beta)]} \right] q - \frac{\beta}{[1 + \lambda (1 + \beta)]}$$

We can express this more conveniently as

$$g(q) \equiv \lambda q^3 - [1 + \lambda (1 + \beta)] q^2 + \left[(1 + \beta\lambda + \beta) + \frac{\kappa}{\sigma} \right] q - \beta$$

The product of the three roots is is $\frac{\beta}{\lambda} > \beta$. We also have $g(0) < 0$, $g(\beta) > 0$, so there is a real root in $(0, \beta)$. But since the product of the roots exceeds β, the other two roots must be outside the unit circle. The solution to the difference equation is then given by the sum of the particular solution and the homogeneous solution. □

B Stacked time approach

A complementary look at the issue of reversal can be obtained via a stacked time solution. During the fixed interest rate period, the dynamic system can be expressed in terms of leads, lags and contemporaneous values of the vector $Z_t \equiv [\pi_t, y_t]'$ as below

$$BZ_{t+1} + CZ_t + DZ_{t-1} = V$$

Here, B, C and D are coefficient matrices of appropriate dimension and V is a vector of constants. After the fixed interest rate period ($t > T$) and under perfect foresight, the solution is of the form $Z_t = AZ_{t-1}$. This decision rule is then used to substitute out expectations of future values in the ultimate period of the fixed interest rate regime.

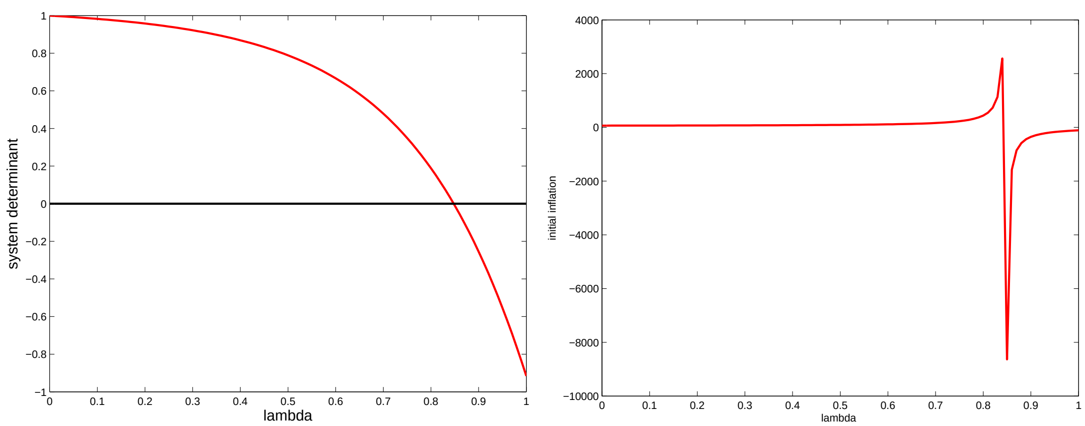
We can define a stacked vector $Z \equiv [Z_1', Z_2', ..., Z_{T-1}', Z_T']'$ and express the entire system of equilibrium conditions during the T periods in a stacked system of equations

$$JZ = M$$

Here, the matrix J is given by

$$J = \begin{bmatrix} C & B & 0 & \cdots & 0 & 0 & 0 \\ D & C & B & .. & 0 & 0 & 0 \\ \vdots & & & .. & & & \\ 0 & 0 & 0 & .. & D & C & B \\ & & & .. & 0 & D & C+BA \end{bmatrix}$$

and the matrix M collects constants. The solution can then be obtained via simple matrix inversion, i.e. $Z = J^{-1}M$. Clearly, the system determinant, $\det(J)$, is crucial for the solution of the endogenous variables. Figure 6 below plots the determinant of this system and the initial inflation response when the interest rate is fixed for 8 quarters as a function of the inertia in price setting λ.



(a) System determinant as a function of λ (b) Initial inflation as a function of λ

Figure 6: Sensitivity with respect to λ for given $T = 8$

Clearly, as λ exceeds a critical value in the neighborhood of the value 0.85, the determinant approaches zero and switches sign. This implies that the dynamics of the model also asymptotes and switches sign at that value. These results are reminiscent of the bifurcation analysis in the new Keynesian model analyzed by Barnett and Duzhak (2010).

C Stochastic exit

This section considers a stochastic exit from the fixed rate regime. This is motivated by the the influential papers of Eggertsson and Woodford (2003) and Christiano, Eichenbaum, and Rebelo (2011) that assume Markov process for the shocks that drive the economy to the zero lower bound. We assume that as long as no exit has occurred previously, there is a probability p that the interest rate remains pegged, and probability $(1-p)$ that an exit to an active interest rate regime occurs. Exit is an absorbing state.

In this model with no endogenous states, when the exit occurs $\pi_t = y_t = 0$. Hence, the equilibrium is given by two numbers: the levels of inflation and gap during the extended period. The model is given by equations(1) and (2) from section 2.

The solution for inflation is given by:

$$\pi_t = -\frac{\frac{\kappa}{\sigma} i^*}{(1-p)(1-\beta p) - p\frac{\kappa}{\sigma}} \tag{28}$$

The mean duration of the peg is given by $T = 1/(1-p)$, or $p = (T-1)/T$, so that we can write (28) as

$$\pi_t = -\frac{\frac{\kappa}{\sigma} i^* T^2}{\beta + T(1-\beta) + T(T-1)\frac{\kappa}{\sigma}} \tag{29}$$

A simple case is to set $\beta = 1$ so that we have

$$\pi_t = \frac{-\frac{\kappa}{\sigma} i^* T^2}{1 - T(T-1)\frac{\kappa}{\sigma}} \tag{30}$$

The normal case is where the denominator is positive, or

$$T_{crit} = \left(1 + \sqrt{1 + 4\frac{\sigma}{\kappa}}\right)/2 \tag{31}$$

There is an asymptote and then sign switch at T_{crit}. If we use $\kappa = .025$, and $\sigma = 1$, we have that $T_{crit} = 6.84$. Thus, asymptotes and sign switches are possible even without endogenous states in the simple DNK model as long as the exit is stochastic.

But with a stochastic exit, there is also the possibility of equilibrium indeterminacy whenever the expected duration of the fixed rate regime is too large. It is thus of interest to see whether the sign switch occurs in the determinacy region of the parameter space. Combining the Phillips curve with the IS curve, we have:

$$\beta p^2 \pi_{t+2} - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] p\pi_{t+1} + \pi_t + \frac{\kappa}{\sigma} i^* = 0 \tag{32}$$

To compute the determinacy region, we postulate a candidate equilibrium with unknown coefficients a,b, and c:

$$\pi_t = ca^t + b \tag{33}$$

Put into (32):

$$\beta p^2(ca^{t+2} + b) - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] p \left(ca^{t+1} + b\right) + ca^t + b + \frac{\kappa}{\sigma} i^* = 0 \tag{34}$$

The value of b is the particular solution and is given by the solution to:

$$\beta p^2 b - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] pb + b + \frac{\kappa}{\sigma} i^* = 0 \tag{35}$$

But this is just the solution given by (3).

$$b = \frac{\frac{\kappa}{\sigma} i^*}{p\frac{\kappa}{\sigma} - (1-p)(1-\beta p)} \tag{36}$$

The value of a is given by the root of the characteristic equation:

$$h(a) \equiv \beta p^2 a^2 - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] pa + 1 \tag{37}$$

This equilibrium is stationary if a is in the unit circle. Note that h is quadratic, convex, $h(0) > 1$, $h'(0) < 0$. The key issue then is the value of $h(1)$. The value of a is in the unit circle if $h(1) < 0$.

$$h(1) = \beta p^2 - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] p + 1 < 0 \tag{38}$$

But this is equivalent to the issue of whether there are reversals. That is, if $h(1) < 0$, then $a < 1$ and there are equilibrium of the form (33). But c is free, there are an infinite number of equilibria. Hence, the boundary of the determinacy region is also the boundary of the region for which reversals occur. Thus for parameters in the determinacy region, the stochastic exit model agrees with the deterministic exit model that reversals are not possible without endogenous states.

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