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ABSTRACT

We investigate frequency domain properties of seasonal adjustment filters of the model-based signal extraction method of SEATS, using corresponding properties of X-11/12-ARIMA filters as a convenient reference. One-sided and symmetric filters determined by Box and Jenkins' airline models for time series of lengths 49 and 109 are considered. Our analyses are preceded by a discussion of three approaches to filter design: the Digital Signal Processing approach, the Optimal Filtering approach motivating SEATS, and a synthesis that we find appropriate for SEATS. Our most important conclusion is that the frequency domain diagnostics of SEATS are inadequate. In particular, the gain function of the infinite symmetric model-based filter provided by SEATS can fail to show significant features of the finite filter gain function, and it provides virtually no insight into the properties of the one-sided concurrent filter, whose gain and phase delay offer more relevant information for most users of seasonally adjusted data.

Key Words and Phrases: Gain function, Phase delay function, Turning point, Seasonal adjustment diagnostics, Model-based seasonal adjustment.

1. INTRODUCTION

Seasonal adjustment is a signal extraction procedure in which seasonal movements constitute the "noise" that must be suppressed to better reveal the "signal" of interest. In the simplest situation (often achieved by taking logs) the seasonal movement S_t of the data point Z_t in month t is independent of the level and other nonseasonal components of the series, and Z_t is decomposed as

$$Z_t = S_t + N_t, \tag{1.1}$$

where N_t denotes the nonseasonal component.

Under (1.1), for a given span of data Z_t, $1 \le t \le T$ without detectable outlier, trading day and moving holiday effects, the most widely used seasonal adjustment methods estimate N_t, $1 \le t \le T$ by applying linear filters to the available data. That is, the estimates $\hat{N}_t$ have formulas of the form

$$\hat{N}_t = \sum_{s=-(T-t)}^{t-1} c_{s,t}^T Z_{t-s}, 1 \le t \le T, \tag{1.2}$$

with series-length and t-dependent filter coefficients $c_{s,t}^{T}$ which sum to one and are determined by the methods and software options chosen. In principle, this makes it possible to use standard filter diagnostics to analyze properties of such seasonal adjustment methods, at least for the times t of greatest interest, e.g. $t=T$. However, to achieve conceptual and computational simplifications, the versions of these diagnostics provided by software or presented in the published literature are those for large, even infinite, T. Also, most often they are for the symmetric filter that produces the adjustment of the midpoint of series.

This article presents frequency domain filter diagnostics for series of small and moderate lengths, $T=49$ and $T=109$, for the filters that produce the concurrent (initial) seasonal adjustment ($t = T$), and the symmetric (midpoint) adjustments (at t = 25 and 55, respectively). The filters examined are those used (for additive decompositions) by two seasonal adjustment programs of current interest, namely TRAMO/SEATS (Gómez and Maravall 1994) and X-12-ARIMA[1] (Findley, Monsell, Bell, Otto and Chen 1998), in the situation in which the series to be adjusted is modeled with the airline model of Box and Jenkins (1976) with any of several typical pairs of coefficient values. The diagnostics we investigate are the squared gain functions of both the concurrent and symmetric filters and also the phase delay function of the concurrent adjustment filter.

As mentioned, the filters that apply to the midpoint of an observation interval of odd length $T=2t+1$,

$$\hat{N}_{t+1}=\sum_{s=-t}^{t} c_{s,t+1}^{T} Z_{t+1-s} \tag{1.3}$$

are symmetric, i.e.

$$c_{s,t+1}^{T}=c_{-s,t+1}^{T}, \quad s=1,2,\ldots,t \ . \tag{1.4}$$

This is true both for X-12-ARIMA with forecast and backcast extensions of equal length and for ARIMA-model-based seasonal adjustments whose calculations, like those of SEATS, implement Assumption A of Bell (1984) and the assumptions of Kohn and Ansley (1986) for the initial values suppressed by differencing operations. In contrast to (1.3)-(1.4), the concurrent adjustment $\hat{N}_T$ is obtained from a one-sided filter,

$$\hat{N}_{T}=\sum_{s=0}^{T-1} c_{s,T}^{T} Z_{T-s} \ . \tag{1.5}$$

As the first-published adjustment for month T, this is the adjustment that receives the most attention, although the later adjustments for this month, obtained with more data from a longer and more symmetric filter, can be expected to have higher quality. Thus it is especially important to consider properties of the concurrent filters (1.5). However, for model-based approaches like that of SEATS, simple general formulas for frequency response and gain functions of signal extraction filters in terms of coefficients of models for the components S_t and N_t of Z_t are available only for the symmetric seasonal adjustment filters for bi-infinite data, Z_t,

[1] We used X-12-ARIMA for our analysis, but only considered filters also available in X-11-ARIMA (Dagum, 1980) with forecast and backcast extension, so we shall use the hybrid name X-11/12-ARIMA to indicate the dual source of the filters. The SEATS' filters considered would be common to any software implementing the method of Hillmer and Tiao (1982) and Burman (1980) with the now customary initial value treatment. In describing these procedures as linear, we are ignoring nonlinearities introduced by the use of estimated model coefficients and by any pretreatments of the data to remove calendar effects, extreme values, level-shifts, etc.

$-\infty < t < \infty$ (see (3.2) below). SEATS provides graphs of these gains, and in several articles, e.g. Maravall (1999) and Gómez and Maravall (2000), frequency domain properties of these (bi)infinite model-based symmetric filters have been compared to those of a certain symmetric X-11 filter (without forecast or backcast extension) which is the default filter for series of length $T \geq 169$. However, in the published literature, there are no frequency domain analyses of SEATS symmetric filters for finite T or of the concurrent seasonal adjustment filters (1.5) of SEATS for infinite or finite T, a gap filled by this article.

In Section 2, we review definitions and some basic properties of gain and phase functions. In Section 3 we discuss the possibly conflicting perspectives of Digital Signal Processing (Subsection 3.1) and Optimal Filtering (Subsection 3.2) concerning what properties are desirable for these functions. The first perspective is the more natural of the two for X-11/12-ARIMA, whereas SEATS' method is motivated by the second. However, as we explain (Subsection 3.3), a balance between the two perspectives is what is appropriate for seasonal adjustment methods like SEATS because of ambiguities concerning model determination and concerning what properties of seasonal adjustment filters are desirable. Subsection 3.4 discusses problems of interpretation arising from nonstationarity of the data.

Frequency domain properties of a range of seasonal adjustment filters are presented in Section 4, where, due to limitations of space and the lack of existing published studies of SEATS' filters for finite length time series, the analysis (in Subsection 4.2) is mainly focused on SEATS' filters, with *default* filters of X-11/12-ARIMA providing relatively constant reference graphs for comparing features of SEATS' filters of different models. (We first show, in Subsection 4.1, that *non-default* filters of X-11/12-ARIMA can be used to obtain squared gain and phase delay functions that closely resemble those of SEATS for airline models with parameter values from a somewhat more restricted but typical range.) In Subsection 4.2.4, to provide some perspective on the differences between gain and phase delay plots of finite and infinite filters, the plots for one model's infinite symmetric and concurrent filters are provided. These are enough to show that it is impossible to infer certain important spectral properties of the finite filters from spectral properties of the infinite filters. Finally, in Section 5, to underscore the importance of the availability of these diagnostics, we provide an example of two models that provide comparable and acceptable fits to a Census Bureau time series but whose model-based seasonal adjustment filters are shown by the diagnostics to have quite different properties. An Appendix presents formulas for solutions of the airline model equation for four extreme pairs of parameter values. These formulas suggest and illuminate properties of SEATS filters of some models considered.

Analyses like those we present could be performed for other model-based signal extraction methods, e.g. those of BAYSEA (Akaike and Ishiguro 1980, based on Akaike 1980), DECOMP (Kitagawa 1985, based on Kitagawa 1981) and STAMP (Koopman, Harvey, Doornik and Shepherd 1995, based on Harvey 1989).

2. GAIN, PHASE AND PHASE DELAY FUNCTIONS

For a linear filter for monthly data Z_t with output Y_t,

$$Y_t = \sum_j c_j Z_{t-j}, \tag{2.1}$$

the *frequency response function* $C(l)$ of the filter is the possibly complex-valued function defined by $C(l)=\sum_j c_j e^{-i\frac{2p}{12}jl}, -6 \le l \le 6$ when l is in units of cycles per year. For the filters we consider, $C(l)$ is continuous. Its amplitude function,

$$G(l)=\left|\sum_j c_j e^{-i\frac{2p}{12}jl}\right|, \tag{2.2}$$

is the *gain function* of the filter. A function $f(l)$ defined whenever $C(l)\neq 0$ (but possibly undefined when $C(l)=0$) that satisfies

$$C(l)=\pm G(l)e^{i\frac{2p}{12}f(l)} \tag{2.3}$$

is a *phase function* of the filter. The simplest possibility is

$$f_a(l)=\frac{12}{2p}\arctan\left(\mathrm{Im}\ C(l)/\mathrm{Re}\ C(l)\right).$$

For standard seasonal adjustment filters, $C(0)=\sum_j c_j=1$ so $f_a(0)=0$, and $C(l)=0$ only at $l=\pm1,\ldots,\pm6$. The gain is symmetric, $G(-l)=G(l)$, and, where defined, the phase function $f_a(l)$ is antisymmetric, $f_a(-l)=-f_a(l)$. Thus only frequencies $0\le l\le 6$ need be considered.

Remark 2.1. At any l_0 with $C(l_0)=0$ around which one of the functions $\mathrm{Re}C(l)$ and $\mathrm{Im}C(l)$ changes sign and the other does not, $f_a(l)$ will have a jump of ±6. Elsewhere $f_a(l)$ will be continuous because of the continuity of $C(l)$. Since modifying a phase function by adding values that are multiples of ±6 preserves the property (2.3), it follows that a phase function $f(l)$ exists that is continuous wherever $C(l)\neq 0$. For the symmetric filters of our examples below, $C(l)$ is always nonnegative, so $f_a(l)$ is always zero for these filters.

2.1 The Squared Gain and Phase Delay Functions

Recall that the squared gain function $G(l)^2$ has the important property that if Z_t is a stationary time series with spectral density $f_Z(l)$, then the spectral density $f_Y(l)$ of the filter output series Y_t in (2.1) is given by

$$f_Y(l)=G(l)^2 f_Z(l). \tag{2.4}$$

Thus this function quantifies the extent to which the filter increases, decreases, or leaves unchanged the contribution to variance of each frequency component of the input series. An approach to an analogue of (2.4) for nonstationary ARIMA processes is discussed in Remark 3.1.

The phase function is most easily interpreted for the backshift operator (filter), $BZ_t=Z_{t-1}$ and its powers $B^kZ_t=Z_{t-k}$ for $k=0,\pm1,\pm2\ldots$. Since the frequency response function of B^k is $e^{-i\frac{2p}{12}kl}$, we can take its phase function to be the continuous function $f_k(l)=-kl$. Note that if

$l \neq 0$, then $-f_k(l)/l = k$, the lag or lead induced by B^k. Generalizing, for any seasonal adjustment filter with frequency response $C(l)$ and continuous phase function $f(l)$ and every $l \neq 0$ for which $C(l) \neq 0$, the value of the function

$$t(l) = -\frac{f(l)}{l} \ , \tag{2.5}$$

is interpreted as the *delay* (or the *advance* if its sign is negative) induced by the filter on the frequency component of Z_t with frequency l (see Remark 2.1 below). $t(l)$ is called the *phase delay function*; see Rabiner and Gold (1977, p. 80). We only consider phase delays for $0 < l < 1$, i.e. for periods greater than a year, because the delays of interest are those associated with detecting turning points or business cycle movements, whose dominant components would normally have periods greater than three years, i.e. frequencies in $0 \leq l \leq 1/3$. (A changing trend whose direction is reversing sharply at a turning point could have significant components at somewhat higher frequencies.) Findley (2000) describes an experiment with a polygonal turning point that shows how phase properties of trend filters can be predictive of their observed delay properties. (Delay properties are the most useful information conveyed by the phase function. We focus on the phase delay function instead, because it reveals delay properties more directly.)

Remark 2.2. There are simple interpretations of gain, phase and phase delay in the weakly stationary case. In that case the time series Z_t has a spectral representation $Z_t = \int_{-6}^{6} e^{i\frac{2p}{12}lt} z(dl)$ and the filter output has the representation

$$\sum_j c_j Z_{t-j} = \int_{-6}^{6} \sum_j c_j e^{i\frac{2p}{12}l(t-j)} z(dl) = \int_{-6}^{6} e^{i\frac{2p}{12}lt} C(l) z(dl)$$
$$= \int_{-6}^{6} \pm G(l) e^{i\frac{2p}{12}\{lt + f(l)\}} z(dl) = \int_{-6}^{6} \pm G(l) e^{i\frac{2p}{12}l\{t - t(l)\}} z(dl).$$

Thus, over subintervals $[l_0, l_1]$ where $t(l)$ and $G(l)$ change little, say $t(l) \doteq \bar{t}$ and $G(l) \doteq \bar{G}$, the filter, in essence, delays the component $\int_{l_0}^{l_1} e^{i\frac{2p}{12}lt} z(dl)$ of Z_t by $\bar{t}$ and changes its amplitude by the factor $\bar{G}$. Where $G(l)$ is effectively zero, the behavior of $t(l)$ immaterial.

Frequency domain properties of X-11-ARIMA seasonal adjustment filters have been considered extensively. Dagum[2] (1983) provided plots of squared gain and phase functions for default concurrent filters of X-11-ARIMA for series of length $T = 84$ associated with various choices of time series models used to provide twelve forecasts of the series. Her models include several Box-Jenkins "airline", i.e. ARIMA(011)$(011)_{12}$ models. Dagum, Chhab and Chiu (1996) presented spectral analyses of maximum-length concurrent seasonal adjustment, trend and irregular filters from all combinations of 9, 13, and 23-term Henderson trend filters and 3x3, 3x5 and 3x9 seasonal filters with forecast extrapolation from airline models as well as without forecast extrapolation. For the case of no forecast extrapolation, Bell and Monsell (1992) showed squared gain function graphs for all of the full-length symmetric X-11 filters (e.g. length 169 for the default adjustment filters). Huot, Chiu and Higginson (1986) analyzed revisions (defined as the mean square difference between the frequency response functions of the central and

[2] The squared gain is referred to as the transfer function in Dagum (1983) and as the power transfer function in Bloomfield (2000).

concurrent filters over all frequencies) from X-11-ARIMA. The only report (unpublished) on short filters known to us, Cholette (1979), provided plots of the gain functions of X-11 filters that estimate "central" and concurrent seasonal factors of series of lengths 36, 48, 60 and 84, together with comments about their phases.

Because the model-based signal extraction perspective on frequency domain properties of filters differs from that implicit in these studies of X-11 or X-11-ARIMA filters, we precede our presentation of gain and phase delay function plots with a discussion of relevant perspectives.

3. ALTERNATIVE PERSPECTIVES ON SQUARED GAINS AND PHASE DELAYS

It will be convenient to start with the Wiener-Kolmogoroff formula for filtering bi-infinite data Z_t, $-\infty < t < \infty$. If Z_t is a (zero-mean) covariance stationary time series having a decomposition (1.1) into uncorrelated components S_t and N_t with spectral densities $f_S(l)$ and $f_N(l)$ respectively, the spectral density of Z_t has the decomposition

$$f_Z(l) = f_S(l) + f_N(l) . \tag{3.1}$$

Assuming $f_Z(l) > 0$ for all l, the estimator of N_t of the form $\hat{N}_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$ that minimizes mean squared error $E\left(\hat{N}_t - N_t\right)^2$ has the frequency response (and gain) function given by the Wiener-Kolmogoroff formula

$$\frac{f_N(l)}{f_Z(l)} = \frac{f_N(l)}{f_S(l) + f_N(l)} \tag{3.2}$$

(see, for example, Chapter 5 of Whittle 1963 or Section 27 of Yaglom 1962).

In the situation in which the components occupy non-overlapping frequency bands, meaning $f_S(l) = 0$ whenever $f_N(l) > 0$ and vice versa, then the gain function $G(l) = f_N(l)/f_Z(l)$ and its square take on only the values one and zero,

$$G(l) = \begin{cases} 1 & , f_N(l) > 0 \\ 0 & , f_S(l) > 0 \end{cases} , \tag{3.3}$$

and, by (2.4), the spectral density $f_{\hat{N}}(l)$ of the estimate $\hat{N}_t$ satisfies $f_{\hat{N}}(l) = f_N(l)$, i.e. the filtering leads to perfect recovery of the spectral amplitudes of N_t.

3.1 The Digital Signal Processing Perspective

We first discuss the Digital Signal Processing (DSP) filtering perspective (see Rabiner and Gold 1975 or Bloomfield 2000), which is the easiest to apply to visual inspection of squared gain and phase plots. Without explicit regard to the assumptions and mean square optimality considerations that led to (3.3), the conventional DSP perspective takes (3.3) as the idealized goal of filtering: complete elimination of the variance components of the noise with no change to those of the signal. Because the frequency response function of a filter of finite length L is zero at no more than $L-1$ frequencies, ideal filters whose gain functions are actually zero (or one) over some interval (but not everywhere) must have infinitely many non-zero coefficients and so are not implementable. The shorter the time series, the larger are the deviations from the ideal that must be expected. In practice, what is desired of a signal extraction filter is that its gain function be close to zero on λ-intervals where the noise has strength and close to one where the signal has significant frequency content. (There is no standard prescription for frequency intervals where both signal and noise have strong components.) After application of a filter with such a gain function, the noise will be largely suppressed and the amplitudes of the frequency components of the variance of the signal little changed. Phase considerations that may compromise achieving these gain function goals are sometimes also included in the filter design/choice if, say, large or erratic phases or phase delays are undesirable for a certain range of l.

From this perspective, seasonal adjustment filters should have gain functions that are zero at the seasonal frequencies $l = 1,2,3,4,5,6$ and remain small in narrow intervals surrounding each of these frequencies. The more the seasonal pattern is changing over time, the wider these surrounding intervals should be (see Bloomfield 2000, pp. 81-82). Elsewhere, and especially in an interval around zero associated with trend and cyclical movement, say $0 \le l < 1/3$, the gain function should ideally be close to one except when this property compromises other desired properties of the adjustment too much.

When the gain function is less than one in an interval away from seasonal frequencies, movements in the nonseasonal component that have frequency content in this interval are being suppressed. When the gain function is greater than one, nonseasonal frequency components are being magnified. From the simplest DSP perspective, all such suppressions and magnifications are unattractive. In practice, many users of seasonally adjusted data would worry about suppression or magnification of data components in the trend and business cycle range of frequencies, but a substantial number would also prefer that higher frequency components be suppressed in order to obtain a smoother adjusted series. (Figures presented in Sections 4 and 5 and in Findley, Martin and Wills (2002) show that one of the costs of additional smoothing is additional phase-delay.)

By contrast, when the “optimal” perspective considered next is appropriate, some gain function movements above or below one at any nonseasonal frequencies can be desirable.

3.2. The Optimal Filtering Perspective for ARIMA Models

In the optimal filtering framework for finite length series presented by Anderson and Moore (1975), the goal is to obtain linear estimates of unobserved components of the time series that are optimal in the sense that they minimize mean squared error, conditional on assumed time series models for the components. In the special case of the optimal signal extraction approach to seasonal adjustment of Hillmer and Tiao (1982) used by SEATS, the models for the components are derived from an estimated ARIMA model for the observed series (perhaps after a data transformation).

As with the DSP approach, some basic features of this optimal filtering approach to seasonal adjustment are best revealed by considering the idealized situation in which bi-infinite data are available and a generalization of (3.2) due to Bell (1984) is applicable. In the ARIMA case, this generalization requires the components S_t and N_t to follow known nonstationary ARIMA models whose differencing polynomials $d_S(B)$ and $d_N(B)$ have no common zero (e.g. monthly airline models with $d_S(B) = 1 + B + \cdots + B^{11}$ and $d_N(B) = (1-B)^2$). Then $d_Z(B) = d_S(B)d_N(B)$ transforms Z_t into a stationary ARMA process, $z_t = d_Z(B)Z_t$ (whose spectral density $f_z(l)$ we assume to be strictly positive). Concerning the spectral densities

$f_s(\lambda)$ and $f_n(\lambda)$ of the ARMA processes $s_t = \delta_S(B) S_t$ and $n_t = \delta_N(B) N_t$, it is assumed that when $f_s(\lambda)$ or $f_n(\lambda)$ is zero for some λ_0, the associated value of its differencing polynomial, $\delta_S\left(e^{-i\frac{2\pi}{12}\lambda_0}\right)$ or $\delta_N\left(e^{-i\frac{2\pi}{12}\lambda_0}\right)$, respectively, is nonzero. Then, when the initializing values of the ARIMA model equation for Z_t are independent of the series z_t, the nonnegative function

$$\frac{\left|\delta_N\left(e^{-i\frac{2\pi}{12}\lambda}\right)\right|^{-2} f_n(\lambda)}{\left|\delta_S\left(e^{-i\frac{2\pi}{12}\lambda}\right)\right|^{-2} f_s(\lambda)+\left|\delta_N\left(e^{-i\frac{2\pi}{12}\lambda}\right)\right|^{-2} f_n(\lambda)}$$
$$=\frac{\left|\delta_S\left(e^{-i\frac{2\pi}{12}\lambda}\right)\right|^{2} f_n(\lambda)}{\left|\delta_N\left(e^{-i\frac{2\pi}{12}\lambda}\right)\right|^{2} f_s(\lambda)+\left|\delta_S\left(e^{-i\frac{2\pi}{12}\lambda}\right)\right|^{2} f_n(\lambda)} \tag{3.4}$$

is the frequency response function (and the gain function) of the symmetric filter that provides the minimum mean square linear estimator of N_t from Z_t.

The gain function (3.4) is a nonnegative rational function bounded above by one. It can take on the values one and zero (and any value in between) only finitely many times (not more than the sum of the degrees of the numerator and denominator polynomials). Thus it can achieve the ideal of the DSP approach only approximately (see Subsection 4.2.4 for a "close" example). For seasonal time series models, (3.4) is zero at $\lambda = 1,2,\ldots,6$ because $\left|\delta_S\left(e^{-i\frac{2\pi}{12}\lambda}\right)\right|^2$ is zero. Equation (3.4) has the value one at $\lambda = 0$, where $\left|\delta_N\left(e^{-i\frac{2\pi}{12}\lambda}\right)\right|^2$ is zero, and differs from one elsewhere unless the model for S_t is canonical in the sense of Hillmer and Tiao (1982), meaning $f_s(\lambda_0)=0$ for at least one λ_0. If this happens at *m* positive frequencies, then the gain function (3.4) has the value one at $m+1$ nonseasonal frequencies. In this sense, the Hillmer and Tiao canonical approach of SEATS is the optimal filtering approach most compatible with the DSP approach.

To plausibly support a claim of mean square optimality, the optimal filtering approach requires the assumed model form for Z_t to be nearly correct for the data, and its estimated coefficients to be close to their asymptotic values. Then, an extreme perspective can be formulated as follows. Whatever gain and phase delay functions the model-based filter has are ideal. Deviations from them, e.g. filters with gain functions closer to one away from seasonal frequencies or phase delays closer to zero in $0 \le \lambda < 1/3$, are less desirable (unless they are optimal for another admissible decomposition of Z_t). This extreme position is untenable for various reasons, not least because, for any observed economic time series, several plausible models can usually be found. Also, sometimes the best fitting model among those under consideration lacks a decomposition into seasonal and nonseasonal models, i.e. is not admissible in the sense of Hillmer and Tiao (1982); see Findley, Martin and Wills (2002) for examples. In any case, mean square optimality, while attractive in a general sense, does not directly address properties of interest to many users, such as small phase delay at low frequencies or smoothness of the adjusted series.

The model-based approach has attractive features that are not dependent on optimality. Some of these will be described below.

3.3 Some Implications for Comparing Methods and Filters

The X-11/12-ARIMA seasonal adjustment filters are very close to model-optimal filters of certain ARIMA models; see Cleveland and Tiao (1976) and, especially, Bell, Chu, and Tiao (2002). Nevertheless, the perspective of the Subsection 3.2 is not appropriate for these filters because most of these ARIMA models are usually much more complex than those used in practice, and the filter selection procedures of X-11/12-ARIMA are not based on ARIMA model goodness-of-fit diagnostics. The DSP perspective is somewhat more natural for the analyses of these filters because the symmetric Henderson trend filters of X-11/12-ARIMA, which are components of the seasonal adjustment filters, have close-to-ideal gain functions relative to their lengths from the DSP perspective (see Dagum et al. 1996 and Fig. 3 and Appendix B of Findley et al. 1998 for more on Henderson filters and their optimal smoothness properties.)

However, the DSP perspective of Subsection 3.1 is not entirely natural for the seasonal adjustment of some kinds of series, for example, a series whose irregular component is strong and whose seasonal component is evolving significantly over time. For such a series, the irregular component, regarded as a broadband or white noise component of the nonseasonal component of the series, will have significant spectral amplitudes at all frequencies, and the seasonal component will have significant spectral amplitudes in a relatively wide interval around one or more seasonal frequencies. Thus, there will be intervals near the seasonal frequencies in which both seasonal and nonseasonal components have significant amplitudes. Heavy suppression of all amplitudes in these intervals, following the DSP approach, would suppress nonseasonal as well as seasonal components. By contrast, if a reasonably well-fitting model can be found (often difficult for noisy series), then one would expect its optimal filter to perform better than a DSP filter on average because it would provide a systematically limited suppression of amplitudes in frequency intervals with significant seasonal and nonseasonal variance components. Also, the noise-to-signal ratios used to determine the choice of the X-11/12-ARIMA filter may lead to better filter selection than the DSP perspective in this situation. Concerning the comparison of X-11/12-ARIMA and SEATS for such series, more research is needed. In the very limited results reported in Hood, Ashley, and Findley (2000), for most series with a very large irregular component, X-11/12-ARIMA gave better adjustments for segments of the series of lengths 48-84, whereas for segments of the series of lengths 108-120, the model-based adjustments from SEATS were more often better. (For these difficult series, many of the “better” adjustments were still unacceptably bad.)

The limited applicability of the DSP perspective does not severely limit the diagnostic value of the squared gain and phase delay functions of adjustment filters. When these functions suggest deficiencies in the filter from the DSP perspective or another perspective such as smoothing or business cycle analysis, it will sometimes be possible to find another adequately fitting model, or an alternative X-11/12-ARIMA option choice with good diagnostics, whose adjustment filters have more appealing gain or phase delay properties. See Section 5 for an example.

3.4 Missing Theory for Interpretation with Nonstationary Data

In the discussion above and in later sections, for simplicity we refer to effects of filters on frequency components of nonstationary series as if such components were well defined and subject to the same filtering effects as stationary time series. However, for nonstationary series, no theoretical connection has been established between the periodogram, which is the basic empirical statistical estimator of frequency components, and the functions we are considering, except in the special case in which $z_t = d_Z(B)Z_t$ is white noise and $d_Z\left(e^{-i\frac{2p}{12}l}\right)$ has only simple zeros (for example, $d(B) = 1 - B^{12}$). For this very special case, a slight generalization of a discrete-time analogue of the continuous-time Theorem 1 of Solo (1992) would state that the large-sample limit of the expected value of the periodogram of Z_t is equal to $\left|d_Z\left(e^{-i\frac{2p}{12}l}\right)\right|^{-2} f_z(l)$ at all frequencies for which $d_Z\left(e^{-i\frac{2p}{12}l}\right) \neq 0$. It is not obvious from Solo's analysis that this result can be generalized to other cases.

The function $\left|d_Z\left(e^{-i\frac{2p}{12}l}\right)\right|^{-2} f_z(l)$ and similar functions such as $\left|d_N\left(e^{-i\frac{2p}{12}l}\right)\right|^{-2} f_n(l)$ and $\left|d_S\left(e^{-i\frac{2p}{12}l}\right)\right|^{-2} f_s(l)$ are commonly called the pseudo-spectral densities (of the models) of their respective series, Z_t, S_t and N_t. Some graphical diagnostics of SEATS, and many SEATS users, treat pseudo-spectral densities as if they describe frequency components of the nonstationary series they are associated with. As we just explained, there is at present no theoretical justification for this except in the special case described, which is of limited interest for seasonal adjustment.

Rather than attempting to use graphs of pseudo-spectral densities as diagnostics, we think it is better (and simpler, because infinite values are avoided) to look at graphs of squared gain and phase functions of filters, for two basic reasons. First, there is empirical experience offering general if not specific support for standard interpretations of these filter diagnostics even with nonstationary time series. For example, seasonal adjustment filters with smaller gain function values at higher frequencies are consistently observed to yield smoother seasonally adjusted series. Similarly, the limited experiment in Findley (2000) indicates that greater phase delay at low frequencies is associated with greater delay in revealing a turning point. Second, even if a theoretical connection can be found between finite values of the pseudo-spectral density of an estimated model and frequency components of a seasonal time series, this connection is likely to assume that the estimated model has the correct form, and therefore, implicitly, that among competing models for the series, the best fitting model has been used to obtain the pseudo-spectral density. In the practice of model-based seasonal adjustment, however, a less well-fitting model with more desirable filter properties may be preferred. Section 5 presents a series and models for which the filter diagnostics we consider would lead some adjusters to make such a decision. Also, users of TRAMO/SEATS experience situations in which a model passed by TRAMO is replaced by SEATS with a less well-fitting model whose seasonal adjustment, and therefore whose seasonal adjustment filters, are expected to be better.

4. SQUARED GAIN AND PHASE DELAY FUNCTIONS OF VARIOUS X-11/12-ARIMA AND SEATS SEASONAL ADJUSTMENT FILTERS

As mentioned before, the filters we shall consider are those that arise when an "airline" model is used to extend the data for X-11/12-ARIMA or to determine the canonical model-based seasonal adjustment filters of SEATS. Airline models have the form

$$(1-B)(1-B^p)Z_t = (1-qB)(1-\Theta B^p)a_t \ , \qquad (4.1)$$

where a_t denotes a series of independent random variables with mean zero and constant variance. For monthly data, $p = 12$. The coefficients q and Θ, the *nonseasonal* and *seasonal* moving average parameters, respectively, can be taken to have magnitudes not exceeding one. For seasonal economic time series, they are usually nonnegative, so we only consider parameter values $0 \le q \le 1$ and $0 \le \Theta \le 1$. Given initial values $Z_1, \ldots, Z_{p+1}$, the solutions $Z_t, t \ge p+2$, of the difference equation (4.1) can be written in the form

$$Z_t = b + ct + s_t + v_t, \tag{4.2}$$

with s_t satisfying the "fixed seasonality" condition $s_t + s_{t-1} \cdots + s_{t-p+1} = 0;\ t \ge p$ (and therefore $s_{t-p} = s_t$). The quantities *b*, *c*, and s_t are functions of the initial values of Z_t and of $a_1, \ldots, a_{p+1}$ and (q, Θ). The component v_t is a function of (q, Θ) and of $a_{p+2}, \ldots, a_t$. To illuminate the range of behaviors encompassed by (4.1), in the Appendix we provide explicit formulas for v_t for the four extreme cases in which q and Θ have the values zero or one.

We obtained the filter coefficients of (1.2) of SEATS and X-11/12-ARIMA as follows. For fixed values of the airline model coefficients, SEATS[3] was applied to each of the T impulse series,

$$d_t^{(T-u+1)} = \begin{cases} 1, & t = T-u+1 \\ 0, & t \ne T-u+1 \end{cases}, \quad 1 \le t \le T,$$

specified by $1 \le u \le T$. If we designate the resulting seasonally adjusted series by $N_t^{(u)}, 1 \le t \le T$, then the coefficients of (1.2) arise as $c^T_{-(T-t)+u-1,t} = N_t^{(u)},\ 1 \le u \le T$. The X-11/12-ARIMA seasonal adjustment filter coefficients were obtained analogously, by applying X-11/12-ARIMA (with fixed trend and seasonal filters) to the impulse series extended by twelve backcasts and forecasts from the airline model.

We examined the squared gain and phase delay function plots of the concurrent adjustment filters (1.5) and the central adjustment filters (1.3) for series of lengths 37, 49, 61 and 109 months associated with the twelve airline model coefficient pairs (q, Θ) with q = 0.2, 0.4, 0.6, 0.8 and Θ = 0.4, 0.6, 0.8. For reasons of space, only plots for lengths 49 and 109 are shown below. Cholette (1979) emphasized that seasonal adjustment of series shorter than five years should generally be avoided, if possible. Our results will further illuminate why this is so. Thus the length 49 filters that we analyze should be regarded as somewhat extreme. In fact, the plots for length 61 months (not presented) are less erratic than the plots provided for length 49 in some frequency intervals but more erratic in others and therefore not convincingly better. The plots for length 37 (not presented) are uniformly more erratic.

X-11/12-ARIMA includes options that allow the user to specify various X-11 seasonal adjustment and trend filters. For this paper, we always used the 13-term Henderson trend filter (after an initial trend estimate obtained with the centered 12-term moving average as usual) and, except for Figs. 1 and 2, always the 3×5 seasonal filter (after an initial seasonal factor estimate was obtained with a 3×3 seasonal moving average as usual; see Ladiray and Quenneville 2000)[4].

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[3] With SEATS, it is necessary to set IMEAN = 0 to prevent mean correction of the input series $d_t^{(T-u+1)}$, and to assign QMAX a high value to make certain that the program does not, because of poor model fit to the input series, override the TRAMO command (INIT=2) to hold fixed the input ARIMA coefficients.

[4] For simplicity, we refer hereafter to this filter, and its modifications by model forecasts and backcasts, as the X-11/12-ARIMA default filter. More precisely, it is the most frequent choice of the default filter selection procedure of both the X-11-ARIMA and X-12-ARIMA programs.

4.1 Different Choices of X-11 Seasonal Filters Yield Seasonal Adjustment Filters Close to Those of SEATS for a Variety of Airline Models

Planas and Depoutot (2002) and Bell, Chu and Tiao (2002) provide interesting analyses of distances between most of the possible symmetric monthly X-11 filters and their closest infinite-length airline-model-based filters. Distance is measured in the first paper by the average squared difference of the frequency response functions and in the second by the theoretical mean square difference of the seasonal adjustments of a series that obeys the airline model that determines the model-based filter. Both articles show that there are good X-11 approximations to the canonical seasonal adjustment filters of all of the airline models considered in this paper but not to the filters determined by more extreme values of Θ, e.g. 0.1 or 0.9. The points we wish to make about finite filters do not require consideration of such extreme values. Also, whereas these authors compare infinite model-based filters with maximal length X-11 filters, we are concerned with the comparison of finite filters of the same length, the relevant comparison for seasonal adjustment practice. Before proceeding to our main analyses in the next subsection, we present a few examples to illustrate that common choices of X-11 seasonal filters (that are not optimal approximations in the senses of these papers) yield X-11/12-ARIMA seasonal adjustment filters whose frequency domain diagnostics are quite close to those of model-based filters from a variety of airline models.

Fig. 1 contains plots of the squared gain function for series of length $T = 109$ of concurrent (1.5) (left side of figure) and symmetric (1.4) (right side) SEATS and X-11/12-ARIMA filters resulting from a model with q = 0.6 and either Θ = 0.4 (top), Θ = 0.6 (middle) or Θ = 0.8 (bottom). The default X-11/12-ARIMA filter was used for the middle plots, whereas a non-default seasonal filter was chosen for the top and bottom plots (the choice being based on whether $\Theta = 0.4$ (significantly changing seasonality) or $\Theta = 0.8$ (quite stable seasonality). The non-default filters used incorporated the 3×3 seasonal filter for the top plots and the 3×9 seasonal filters for the bottom plots. (More details on the filter options of X-11/12-ARIMA can be found in Dagum et al. 1996, Findley et al. 1998 and, especially, in Ladiray and Quenneville 2001). Fig. 2 shows the phase delays associated with the concurrent filters whose squared gains appear on the left in Fig. 1. We also examined the corresponding gain and phase delay plots (not shown) of filters of length 49. The phase delay graphs for X-11/12-ARIMA (of the non-default filters for Θ = 0.4, 0.8) are as close to those of SEATS as are the corresponding graphs of Fig. 2. The squared gain graphs are even closer than in Fig. 1.

4.2 Comparing SEATS Functions to Default X-11/12-ARIMA Functions For Various Airline Models

4.2.1. Squared gain functions. We now compare properties of SEATS filters for different models and series lengths, always using the more fixed properties of the default X-11/12-ARIMA filters as a reference to make differences between SEATS filters more apparent. Squared gain function graphs (with $0 \le l \le 6$ cycles per year) for Θ = 0.4, 0.6, and 0.8 are shown for representative values of q in Figs. 3, 4 and 5 respectively, for series of length 49, and in Fig. 6 for length 109. We shall offer analytic explanations for some of the visual features we identify.

When $T = 49$, the figures show that the impact of q is greater on the concurrent filters than on the symmetric filters (and greater on SEATS than on X-11/12-ARIMA as would be expected since only the effects of the forecast and backcast functions change for the latter). For the concurrent squared gains, over the intervals between the seasonal frequencies, for a fixed value of Θ, the average level in all three figures decreases with increasing q, indicating greater suppression of frequency components on average for frequencies $l > 1$ (components with periods shorter than a year). However, this decrease becomes less pronounced as Θ increases, being quite modest when $\Theta = 0.8$. For the symmetric squared gains, as q increases, so do the sizes of the oscillations between seasonal frequencies, but there is relatively less decrease in the average levels of the graphs between seasonal frequencies that stay near one for $\Theta = 0.6, 0.8$.

As Θ increases, resulting in an increasing contribution of the periodic component s_t in (4.2) as noted the Appendix, the troughs at seasonal frequencies of the squared gain function of both concurrent and symmetric filters of SEATS become narrower, and there are increasingly rapid oscillations between seasonal frequencies. We offer explanations for the oscillation phenomena in the next subsection. Concerning the narrowing, meaning an increase in the amplitude of the squared gain function near the seasonal frequencies $l = 1, 2, \ldots, 6$ as Θ increases to one, an explanation is available for squared gains of infinite filters. For these (not shown), for symmetric filters, the narrowing is tied to decreases near these frequencies of the denominator on the right in (3.4), which for the airline model is equal to $\left|\left(1 - q e^{-i\frac{2p}{12}l}\right)\left(1 - \Theta e^{-i\frac{2p}{12}l}\right)\right|^2$. There is a similar denominator expression for the concurrent filter in the corresponding formula of Bell and Martin (2002). For the infinite symmetric airline model filters, a more complete analysis of this narrowing that accounts for the dependence of the numerator on the right in (3.4) on Θ, can be performed with the aid of the formulas of Section 6 of Hillmer and Tiao (1982).

The effects in Figs. 3 - 5 of increasing Θ just noted for series of length 49 are visible in the squared gains for series of length 109 in Fig. 6. The narrowness of the troughs at seasonal frequencies for the concurrent SEATS filter from $\Theta = 0.8$ is striking, as are the rapid, tightly focused oscillations between these frequencies. The oscillatory patterns between seasonal frequencies of the squared gain functions of the symmetric filters of length 109 are tighter and more regular than happens with length 49, and their dips at the seasonal frequencies resemble the dips of the concurrent filters of length 49 (almost the half-length of the symmetric filter) at least as much they do the broader dips of the squared gain functions of the symmetric filters of length 49. This fact highlights the exceptional broadness of the dips at seasonal frequencies of the symmetric filters of length 49.

The broad dips are just one example of the tendency of the squared gains of the symmetric filters for short series to have rapid changes of amplitude that are not limited to frequency intervals very close to the seasonal frequencies. For the squared gains of all seasonal adjustment filters, sudden changes in amplitude close to the seasonal frequencies are the inevitable and desirable consequence of the suppression of seasonal components. But away from the seasonal frequencies, such changes seem undesirable because they could mean that nonseasonal frequency components of the data with frequencies close to one another are being treated substantially differently by the filter, a somewhat inconsistent treatment of neighboring frequency components that might make certain aspects of the seasonally adjusted series misleading. (This would be most problematic at low frequencies.) Thus, a strong reason to consider the series length 49 to be quite short for seasonal adjustment, in agreement with Cholette (1979), is that large rapid movements occur away from seasonal frequencies in the squared gains of many filters for such series, especially of the symmetric filters (which might offset the latter's advantage over the concurrent filter of having zero phase delay.)

It can be observed in all of these figures that for a given pair of model coefficients, the symmetric filter's squared gain has broader (deeper) troughs at the seasonal frequencies than the squared gain of the concurrent filter. For infinite filters, this phenomenon can be understood by comparing the formula (3.4) for symmetric bi-infinite seasonal adjustment filters with the formula for the frequency response function of the infinite concurrent filters derived in Bell and Martin (2002). These two formulas show that for ARIMA models with the differencing operator of (4.1), the zeros of the squared gains of infinite model-based symmetric filters at seasonal frequencies have order (multiplicity) four, whereas the zeros at these frequencies of the squared gains of the one-sided filters have order two. Consequently, the squared gain functions of the symmetric filter must have lower amplitudes near these frequencies, and therefore broader troughs. For the finite filters, however, Bell has shown in unpublished work that the zeros at seasonal frequencies of the squared gain functions of both the symmetric and asymmetric filters are of order two. Regarding the fact that the finite symmetric filters we are considering have broader squared gain troughs than the concurrent filter, Bell suggests that this may be connected to the fact that the gain functions of the finite filters converge to those of the infinite filters with increasing lengths and so must have shapes similar to the latter past a certain length.

Whenever the concurrent filter has conspicuously narrower squared gain troughs at seasonal frequencies than the symmetric filter, its adjusted series will usually be obtained by removal of an estimated seasonal component that will tend to change less from one year to the next. In this case, the concurrent adjustment will be a more conservative adjustment in the sense that less smoothing is being done. These concurrent adjustment filters will be less able than the symmetric filters to track changing seasonal patterns.

4.2.2. Sources of Oscillatory and Related Behavior. Often the rapid changes discussed above are oscillatory movements. Throughout the graphs in Figs. 1, 3-6, one sees in the central area of the intervals between the frequencies $l = 0,1,\ldots,6$ that when there are squared gain values near one there are also oscillations in the graphs which are usually larger for the symmetric filters, often much larger. The magnitudes of these oscillations decrease with increasing T and vanish at $T = \infty$; see Subsection 4.2.4 and Fig. 9. Thus such oscillations are a series-length phenomenon rather than a reflection of intrinsic properties of airline model data. They resemble the oscillatory behavior of the well-known Gibbs phenomenon, which arises when a (necessarily infinite) filter whose frequency response function has discontinuous jumps, such as an ideal DSP gain that jumps between zero and one, is approximated by truncating (or tapering) its filter coefficient sequence to be zero past some lead and/or lag index; see Bloomfield (2000, p. 110). It is difficult to make quantitative statements about the squared gains of seasonal adjustment filters like those available for the Gibbs phenomenon, but we can offer a qualitative conjecture concerning the phenomenon that, for a given model coefficient pair and series length T, the symmetric filter's squared gain has larger oscillations than the squared gain of the concurrent filter. Two sources for this phenomenon suggest themselves. First and most obvious, the many more constraints on the symmetric filter's coefficients restrict the flexibility of its gain function: the symmetric filters have only $(T+1)/2$ distinct coefficients, whereas the concurrent filters have T. Second, the symmetric filters have amplitudes between seasonal frequencies closer to one on average, and therefore farther from zero, resulting in a persistently larger range of movement. With fewer distinct coefficients than the concurrent filter, this must necessitate larger ancillary movements.

4.2.3. Phase delay functions. Fig. 7 shows concurrent filter phase delay functions for $q = 0.2$ and $\Theta = 0.4, 0.6,$ and 0.8 for series of lengths 49 and 109 over the frequency interval $0 < l < 1$. As noted earlier, this interval includes the frequencies usually associated with trend and business cycles. Fig. 8 shows the corresponding plots for $q = 0.8$. For small values of T, concurrent filter phase delay functions can also have large oscillations that suggest somewhat inconsistent delays associated with neighboring frequency components. The functions are smoother for length 109, but, for a given pair of coefficient values, the basic features are otherwise the same. For fixed values of q, the phase delay at low frequencies increases as Θ decreases toward zero as predicted by the analysis of the solutions of (4.1) in the Appendix. For fixed values of Θ, the phase delay increases with increasing q. For the least favorable combination ($q = 0.8$, $\Theta = 0.4$), the phase delay is approximately two months at the four-year cycle frequency $l = 0.25$. It can exceed three months for very small Θ (not shown). The most obvious pattern in these graphs seems to be that more suppression of higher frequencies in the gain function graphs is associated with more phase delay at low frequencies. There is no such obvious connection with the squared gain behavior at lower frequencies.

4.2.4. Comparison of finite and infinite length concurrent and symmetric filter functions. The squared gain plots of filters of infinite length and length 109 of the airline model with $(q, \Theta) = (0.4, 0.8)$ are shown in Fig. 9. By comparing the graphs for this model in Figs. 5, 6 and 9, one sees that squared gains of the infinite filters often give no indication of where the finite filters will have broad troughs, rapid movements or oscillations. (In fact, the squared gains of the infinite filters never have the rapid oscillations like those of the finite filters.)

In particular, in Subsection 4.2.1, we noted that the dips at seasonal frequencies of the squared gain of the concurrent filters (especially for SEATS) in the graph of Fig. 6 for the filters associated with the parameters (0.4, 0.8) are much narrower than the dips of the symmetric filter. Agustin Maravall suggested that this might be due to the fact that we considered a finite filter of length 109 rather than the infinite length filter. Fig. 9 shows that this is indeed a finite filter phenomenon: near the seasonal frequencies, the infinite concurrent filter has only slightly narrower dips in the squared gain than the infinite symmetric filter.

These observations lead to the conclusion that the squared gains of the infinite filters are inadequate diagnostics: they cannot be used to predict which trend or cyclical components of the observed series of possible interest or concern to the data user are likely to be magnified or suppressed by the finite length filters. Similarly, they should not be used to make inferences about differences between X-11/12-ARIMA and SEATS adjustments.

There appears to be more latitude with phase delays. Fig. 10 provides the infinite filter phase delay plots corresponding to some of the plots of Figs. 7 - 8. The plots of Fig. 10 are much more indicative of the shapes of the finite filter plots than was the case for squared gains. They suggest that for moderate series lengths like $T = 109$, the infinite-filter phase delay plots might be a useful diagnostic when finite-filter diagnostics are not available.

As a final comment about the infinite symmetric filter's squared gain in Fig. 9, we note that this function has the appearance of a continuous approximation to the squared gain of an ideal DSP filter described in Subsection 3.1 because its values are very close to one except near seasonal frequencies, where they drop quickly to zero.

We used the algorithms of Bell and Martin (2002) to obtain the gain and phase delay functions of the infinite concurrent filter for Figs. 9-10.

5. A “BETTER” MODEL WITH “MORE PROBLEMATIC” FILTERS

Equipped with the insights provided by the preceding section, we now consider a data example that illustrates the point made in Subsection 3.3 that the DSP perspective on properties of squared gain and phase delay functions is important for users of the Optimal Filtering approach to seasonal adjustment.

The series X41020 of monthly U.S. Exports of Cookware, Cutlery, House and Gardenware from January, 1989 through November 2001 is one for which the BIC model selection criterion of Schwarz (1978) used by TRAMO to select models for SEATS prefers the model

$$(1-B)(1-B^{12})Z_t = (1-q_1 B - q_{12} B^{12} - q_{13} B^{13}) a_t , \qquad (5.1)$$

with $BIC = 4803.96$, over the airline model (4.1) which imposes the constraint $q_{13} = -q_1 q_{12}$ and has $BIC = 4804.73$. For this series Z_t represents the log of the trading day and outlier adjusted data. The parameter estimates from X-12-ARIMA are $q_1 = 0.3569$, $q_{12} = 0.6589$, $q_{13} = -0.4245$ for model (5.1) and $q = 0.4182$, $\Theta = 0.6351$ for model (4.1). The Box-Ljung Q statistic at lag 24 has the *p*-values 0.275 for model (5.1) and 0.304 for model (4.1), so both models are acceptable by the criteria of TRAMO. Within the lag range $l \le 24$, there are three lags, $l = 6,9,10$, for which the model (5.1) has Q's with *p*-values less than 0.05 (the respective *p*-values are 0.041, 0.043 and 0.037), something that does not occur for model (4.1), a fact that could lead some modelers to prefer model (4.1) for this series. The models are therefore competitive.

Fig. 11 shows that their seasonal adjustment filters have quite different frequency domain characteristics. The phase delays are comparable for long-term trend movements associated with quite small l, but for model (5.1) (which we call the 1-12-13 model) the phase delay becomes increasingly more than one month for cyclical components whose period is shorter than six years, in contrast to the phase delay of model (4.1). The squared gain functions of model (5.1) change much more rapidly than those of (4.1) away from seasonal and long-term trend frequencies, with much more suppression of higher frequency components and of medium to short term business cycle components. The concurrent filter of model (5.1) has a greater tendency to emphasize and perhaps exaggerate long period components. These features suggest that the filters of model (5.1) will effect more smoothing than those of (4.1), but in a somewhat erratic way and with somewhat greater phase delays of frequency components that may be of interest to some data users. Given the information provided by the diagnostics of Fig. 11, we would expect a number of seasonal adjusters to prefer seasonal adjustments from model (4.1) even though they will be less smooth than those from the model (5.1) preferred by BIC, and few seasonal adjusters would be indifferent about which of these two models was used for seasonal adjustment. Thus it is important that the frequency domain diagnostics discussed in this article be made available to users of model-based seasonal adjustment software. Additional examples of competitive model pairs whose seasonal adjustment filters have different squared gain and phase delay properties are presented in Findley, Martin and Wills (2002).

6. CONCLUDING REMARKS

Our main findings are the following. (1) Non-default X-11/12-ARIMA filter choices can lead to filters with frequency domain properties like those of the various SEATS filters considered, as other analyses suggest. (2) With quite short series, the frequency domain analysis indicates that seasonal adjustments near the center of the observation interval may be even more problematic than concurrent adjustments. (3) As would be expected, the values of both coefficients of the airline model strongly influence the squared gains of the SEATS filters. They also have significant influence on the squared gains of the default X-11/12-ARIMA filters for short series, more than might have been expected. (4) The squared gains of the finite-length concurrent filters dip more sharply at seasonal frequencies than the squared gains of the associated symmetric filters, indicating that concurrent and near-to-concurrent adjustments will tend be more conservative, i.e. effect less smoothing, than adjustments for the central part of the series. (5) The phase delays of concurrent SEATS filters indicate that the extent to which they delay the detection of turning points depends significantly on the seasonal as well as on the nonseasonal moving average parameter of the airline model. (6) The squared gain and phase delay of the concurrent adjustment filter provide information that is different from and equal in value to that provided by the squared gain of the symmetric filters. (7) The squared gains of the infinite filters are not reliable diagnostics for series of the lengths considered in this article. In their place, squared gains of the finite filters should be used. Similarly, for comparing filter properties of SEATS and X-11/12-ARIMA, finite (series-length) filter diagnostics should be used. These filters can be calculated by the method of Section 4 or, for model-based filters, by the method of Koopman and Harvey (2000).

Finally, we note that even though turning points are regarded as properties of an underlying trend, we have investigated phase delay properties of seasonal adjustment filters rather than trend filters because most statistical offices publish seasonal adjustments and not trends, with the expectation that the data user will utilize visual clues and multiple sources of information to decide which oscillations of the seasonally adjusted series indicate economic turning points and which do not. Statistical agencies that publish trends could carry out a similar investigation of trend filters.

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DISCLAIMER

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APPENDIX: SERIES FROM AIRLINE MODELS WITH EXTREME PARAMETER VALUES

To understand the range of behaviors encompassed by time series Z_t that obey the airline model difference equation (4.1), it is helpful to consider explicit formulas for the component v_t of (4.2), because this component captures the cumulative effect of the random input to the series after the start. Such formulas are not difficult to find for the four extreme cases in which q and Θ have the values zero or one.

When $(q,\Theta)=(1,1)$, then

$$v_t = a_t ,$$

i.e. v_t is *white noise*, and Z_t has a well-defined linear trend and a perfectly repetitive seasonal component that will be detectable against a white noise background. When $(q,\Theta)=(0,1)$, then v_t is a *random walk*,

$$v_t = \sum_{u=p+2}^{t} a_u ,$$

so the seasonal component of Z_t is perfectly periodic, but its nonseasonal component is a random walk with drift, whose changes from one sample to the next, $c+\left(v_{t+1}-v_t\right)=c+a_{t+1}$, will be erratic unless the variability of a_t is small relative to *c*. When $(q,\Theta)=\left(1,0\right)$, then for each of the *p* sampling periods in the year (months if $p=12$), v_t is a *seasonal random walk*,

$$v_t = \sum_{j=0}^{\left[\frac{t-1}{p}\right]-1} a_{t-jp} ,$$

whose values from the different sampling periods of the year are independent. (For any number *w*, we use [*w*] to denote the greatest integer less than or equal to *w*.) Thus, over time, Z_t will show few signs of movements that suggest repetition from one year to the next, i.e. its movements will not appear seasonal. Also, yearly sums follow a random walk, $\sum_{j=0}^{p-1} v_{t-j} = \sum_{k=p+2}^{t} a_k$, so the nonseasonal component will be erratic. Finally, when $(q,\Theta)=\left(0,0\right)$,

$$v_t = \sum_{j=0}^{t-p-2}\left(\left[\frac{j}{p}\right]+1\right)a_{t-j} = \sum_{u=p+2}^{t}\left(\left[\frac{t-u}{p}\right]+1\right)a_u .$$

In this case, v_t is a form of *integrated noise process* whose variance s_v^2 increases cubically in [*t/p*]. Hence $s_v \sim [t/p]^{3/2}$, with the result that the mean zero process v_t will ultimately dominate $b+ct+s_t$ in (4.2).

When $\left(q,\Theta\right)$ is close to one of these extremes, the behavior of Z_t will be similar to the behavior of the nearby extreme solution. For example, from the solutions given for $\left(q,\Theta\right)$ equal to (1,0) and (0,0), we expect small values of Θ to be associated with erratic nonseasonal movements at both extremes of q values and thus perhaps for all values. The phase delay analyses presented in Subsection 4.2.2 show that for each value of q considered, turning points in the nonseasonal component are indeed more difficult to detect quickly with airline model-based filters for (4.1) when Θ is small. By similar reasoning, as Θ increases to one, we expect the impact of the fixed seasonal component s_t of the solution of (4.1), whose frequency components are restricted to the exact seasonal frequencies, to be greater. And, indeed, near seasonal frequencies, the shapes of the squared gain functions from $\Theta = 0.8$ presented in Section 4 are narrower than happens with smaller values of Θ, showing that, when Θ is large, optimal suppression of frequency components is more focused on seasonal frequencies.

In summary, small values of Θ yield series with erratic seasonal and level movements, most extremely when q is also small, the opposite of what happens when both these coefficients are close to one.

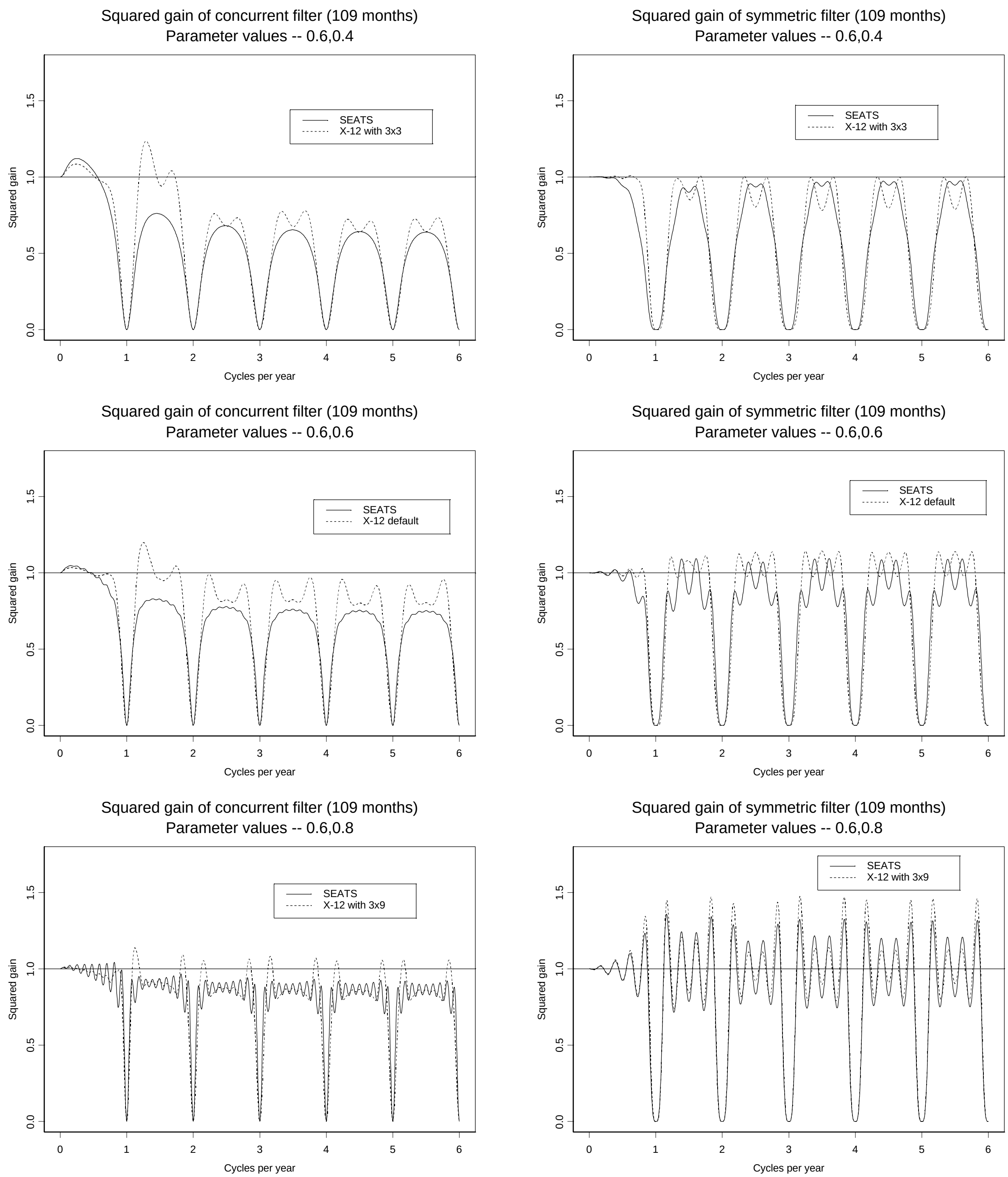


Fig. 1. Squared gain functions of concurrent (left) and symmetric (right) SEATS and X-11/12-ARIMA filters for θ = 0.6 and Θ = 0.4, 0.6, 0.8. Non-default X-11/12-ARIMA filters are used in the top and bottom graphs, the default X-11/12-ARIMA filter in the middle graph. The squared gains of non-default X-11/12-ARIMA filters resemble those of the corresponding SEATS filters.

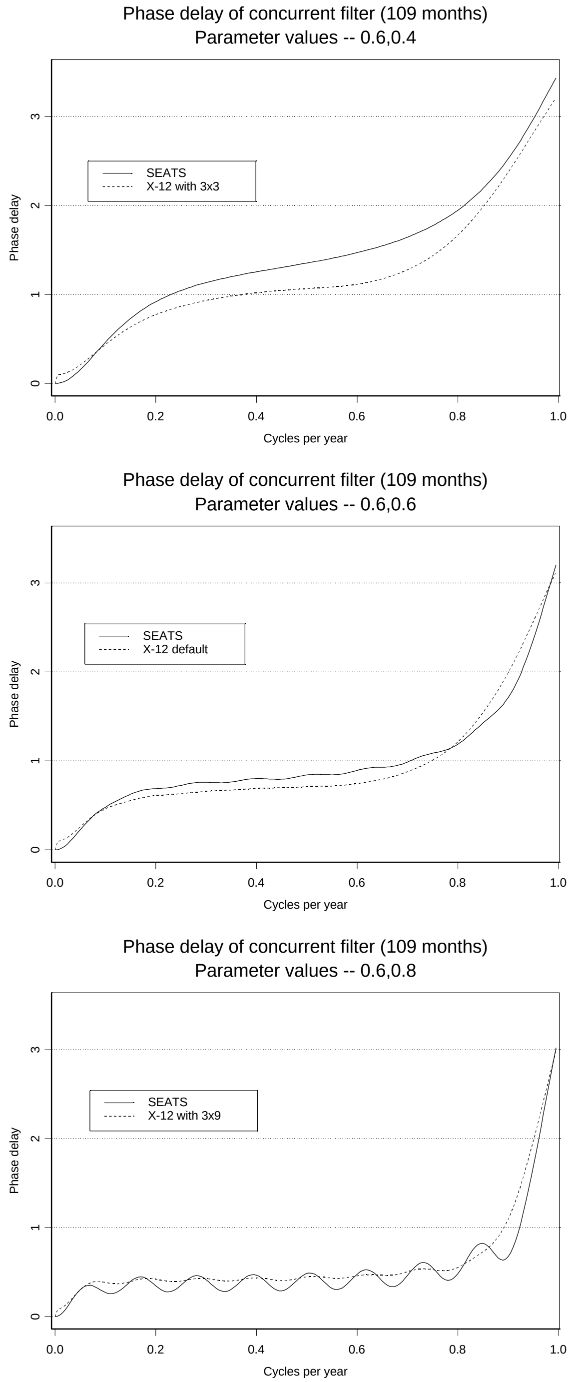


Fig. 2. Concurrent phase delay functions of SEATS and X-11/12-ARIMA filters for θ = 0.6 and Θ = 0.4, 0.6, 0.8. Non-default X-11/12-ARIMA filters are used in the top and bottom graphs, the default X-11/12-ARIMA filter in the middle graph. The phase delays of non-default X-11/12-ARIMA filters resemble those of the corresponding SEATS filters.

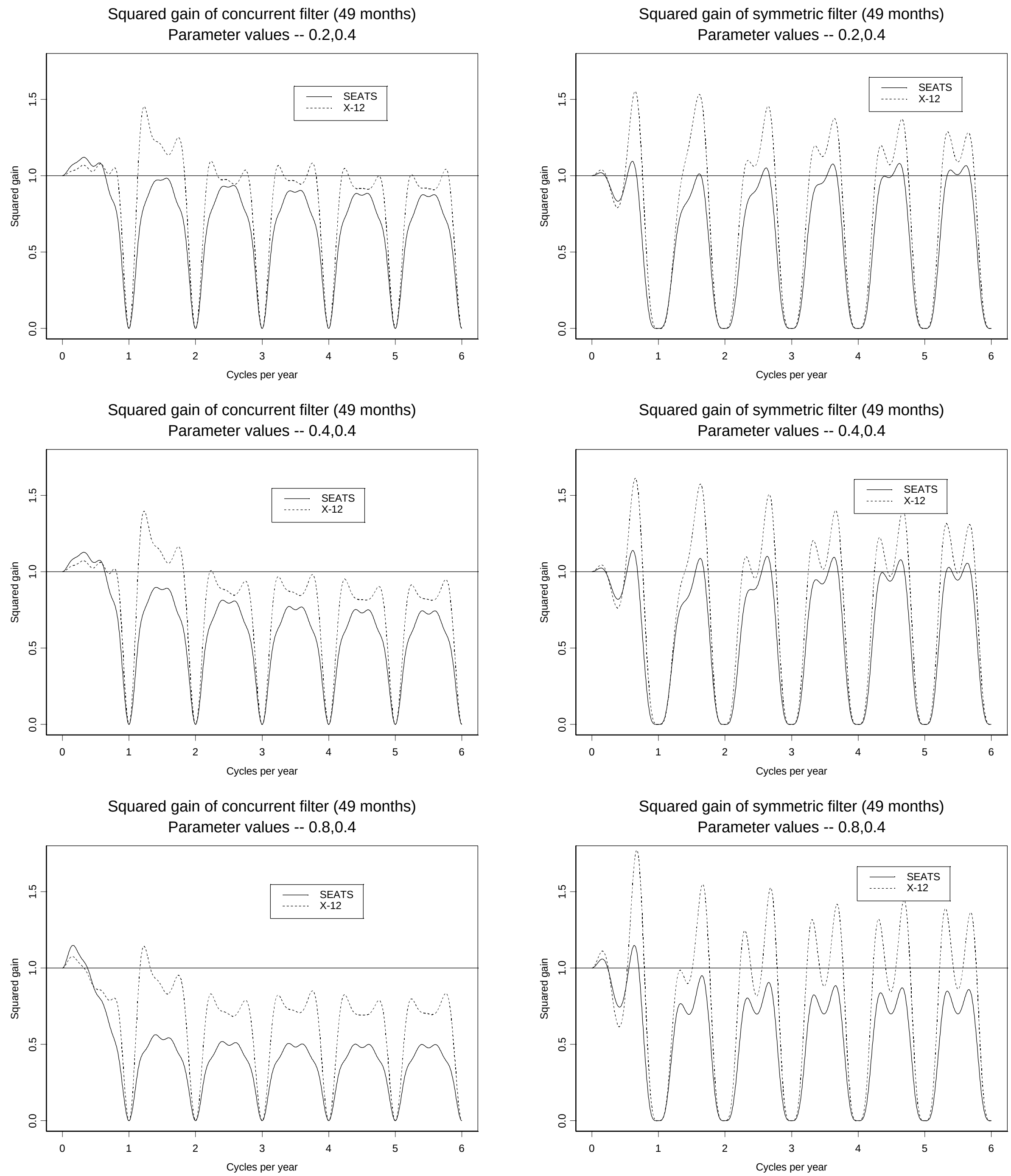


Fig. 3. Squared gain functions for $\Theta = 0.4$ and series of length 49. From top to bottom, $q = 0.2, 0.4, 0.8$. The impact of θ is greater on the concurrent filters (left) than on the symmetric filters (right), and greater on SEATS than on X-11/12-ARIMA as would be expected since only the contribution of the forecasts is changing for the latter.

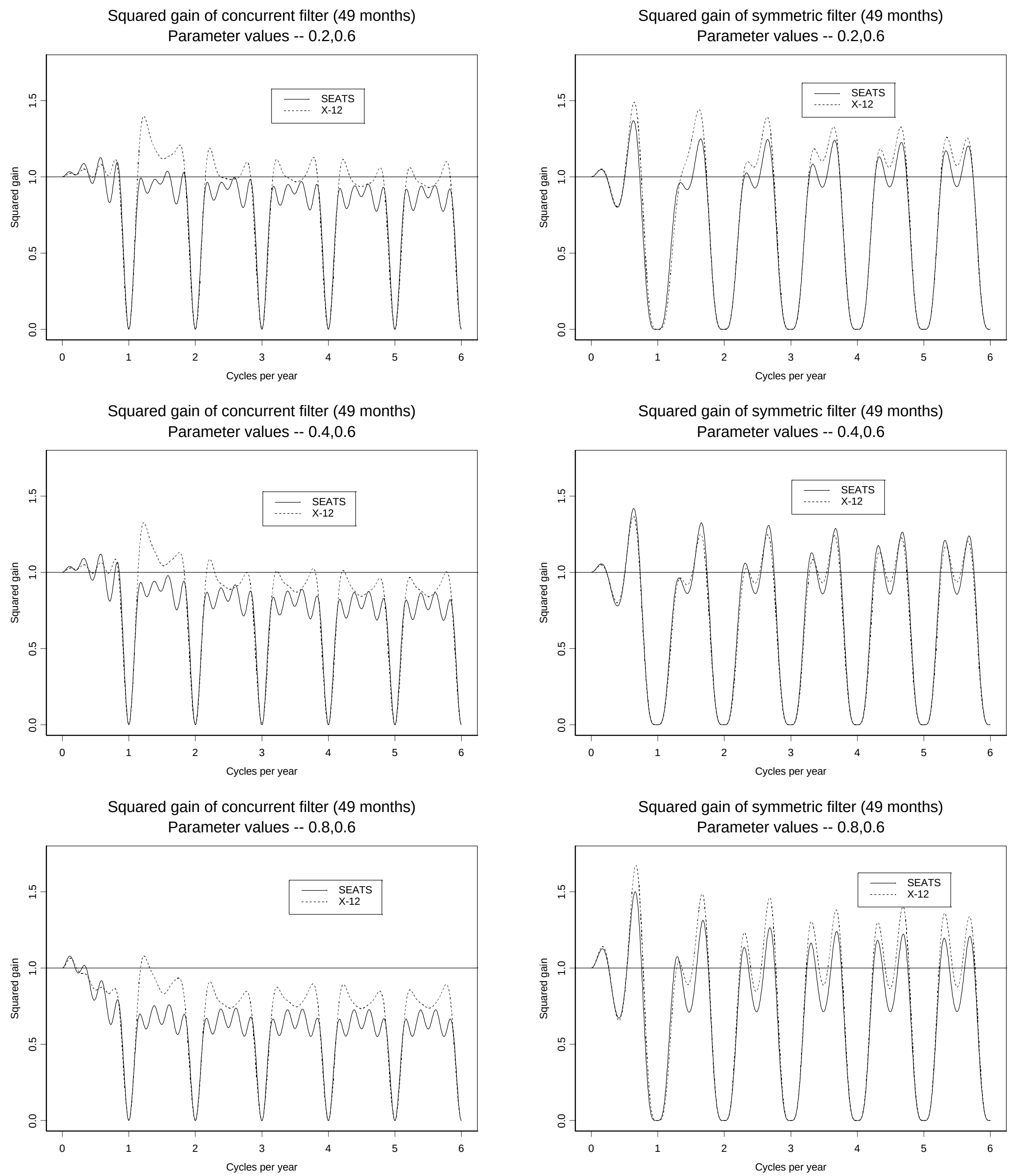


Fig. 4. Squared gain functions for $\Theta = 0.6$, series of length 49. From top to bottom, $q = 0.2, 0.4, 0.8$. These are rather similar to Fig. 3 except that here the SEATS functions for concurrent filters are generally closer to one away from seasonal frequencies.

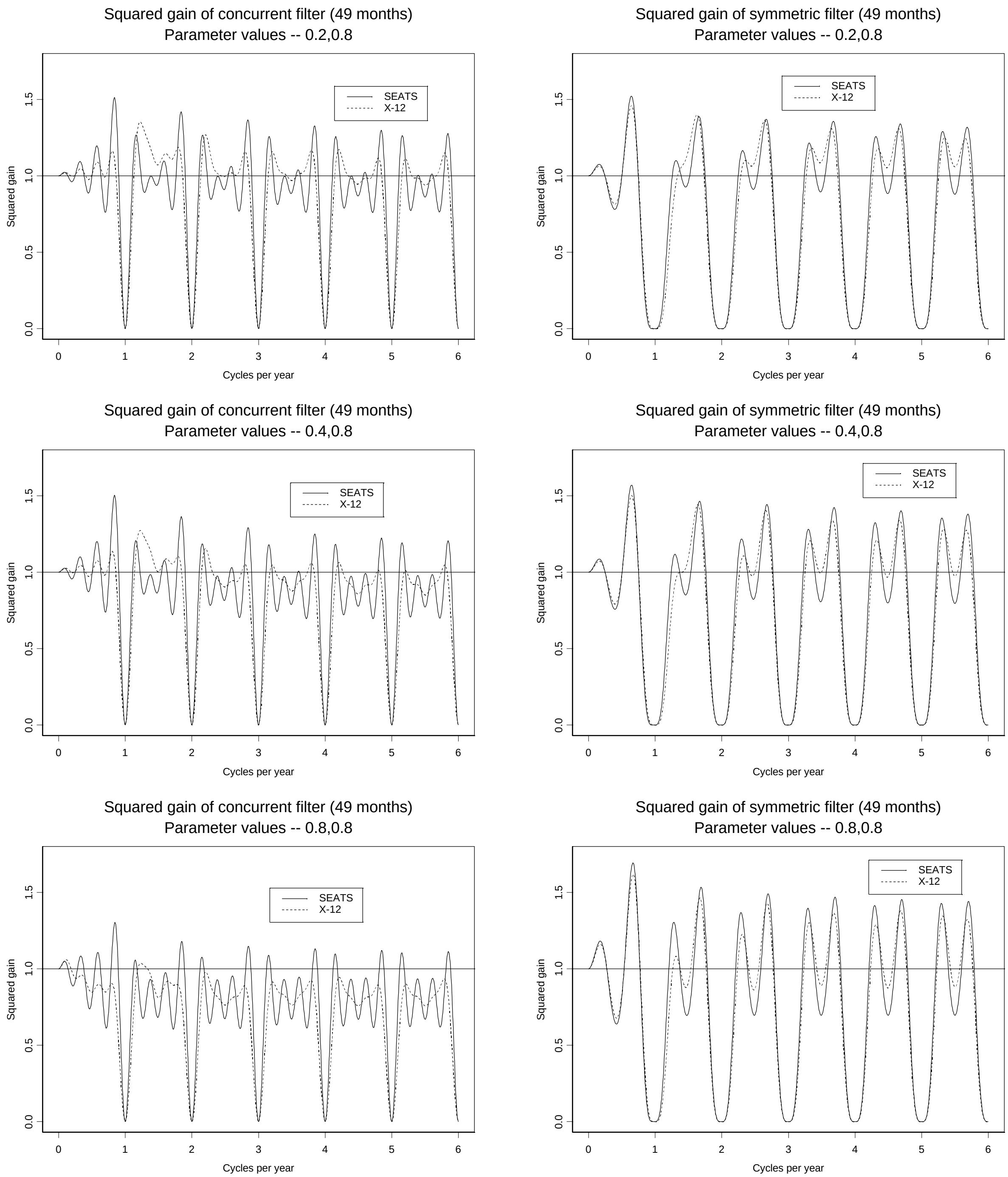


Fig. 5. Squared gain functions for $\Theta = 0.8$ and series of length 49. From top to bottom, $q = 0.2, 0.4, 0.8$. These are rather similar to Fig. 4 with SEATS concurrent functions even closer to one away from seasonal frequencies, while being more oscillatory. Note that in Figs. 3-5, the troughs around seasonal frequencies are narrower for the concurrent filters.

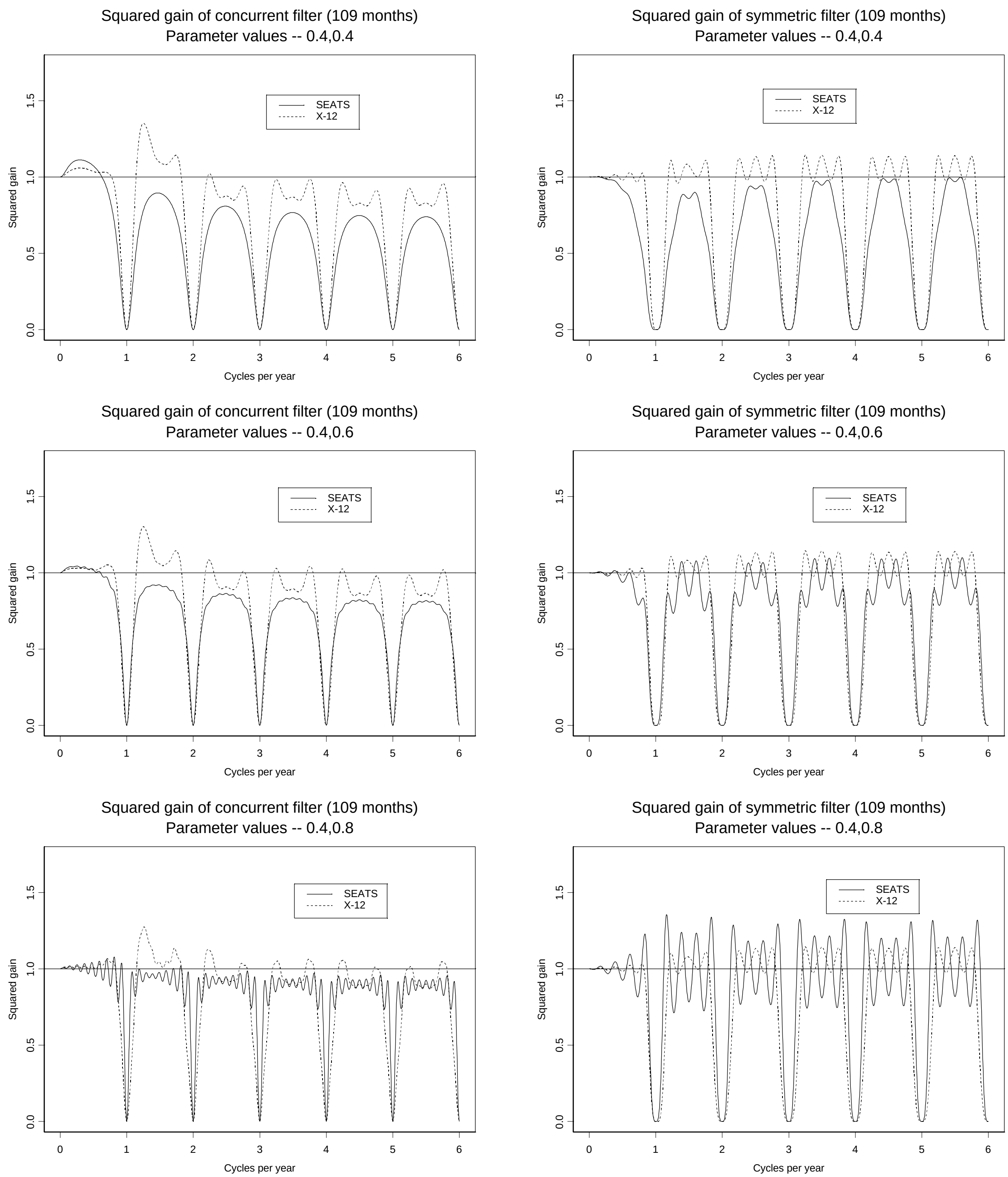


Fig. 6. Squared gain functions for $q = 0.4$ and series of length 109. From top to bottom, $\Theta = 0.4, 0.6, 0.8$. All the gain functions are closer to one than in Figs. 3-5 for the same Θ. The narrowness of the troughs at seasonal frequencies for the concurrent SEATS filter from $\Theta = 0.8$ is striking.

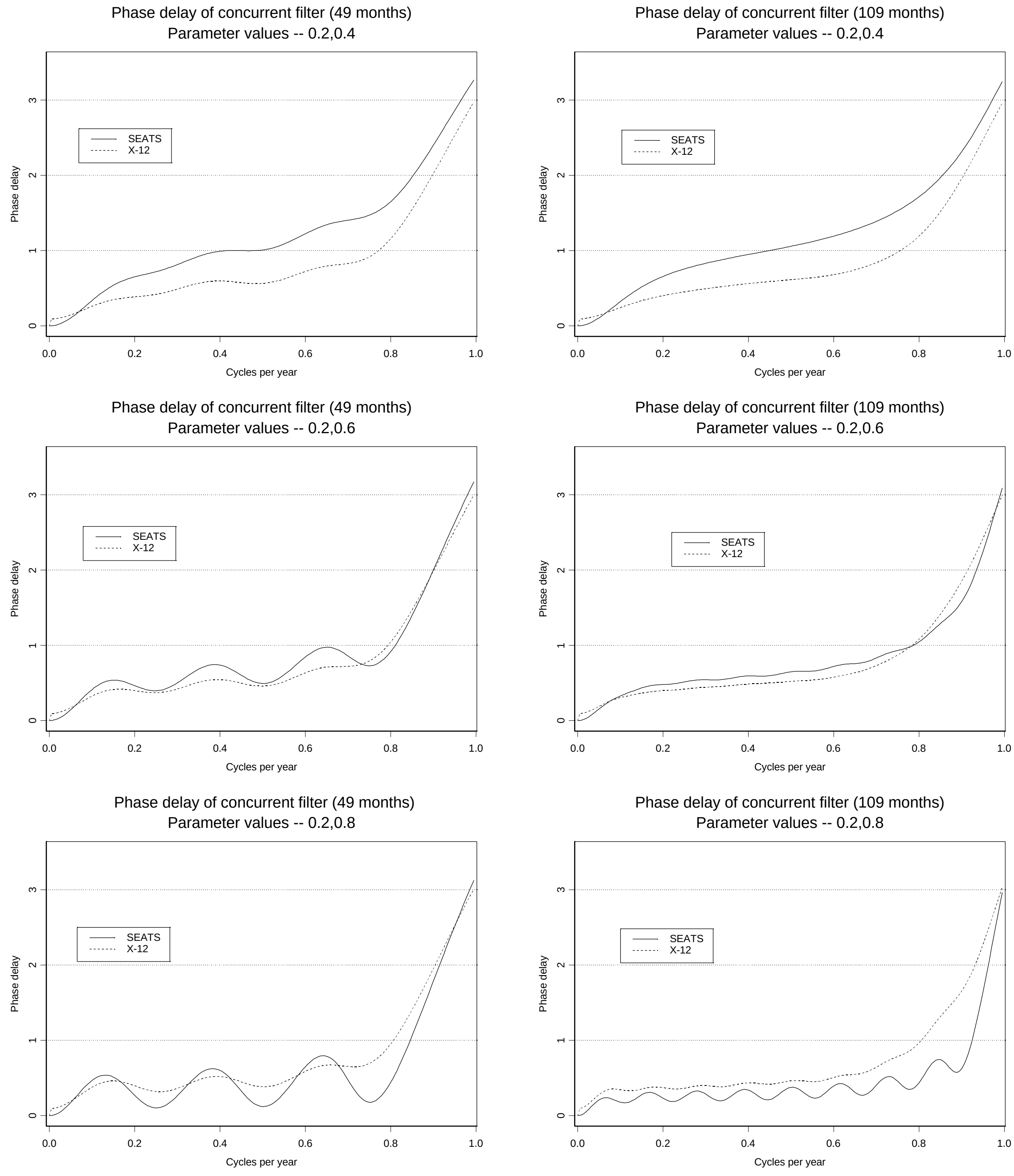


Fig. 7. Concurrent filter phase delay functions for $q = 0.2$. From top to bottom, $\Theta = 0.4, 0.6, 0.8$ for series lengths 49 (left) and 109 (right). Except at the lowest frequencies, the delay decreases as Θ increases.

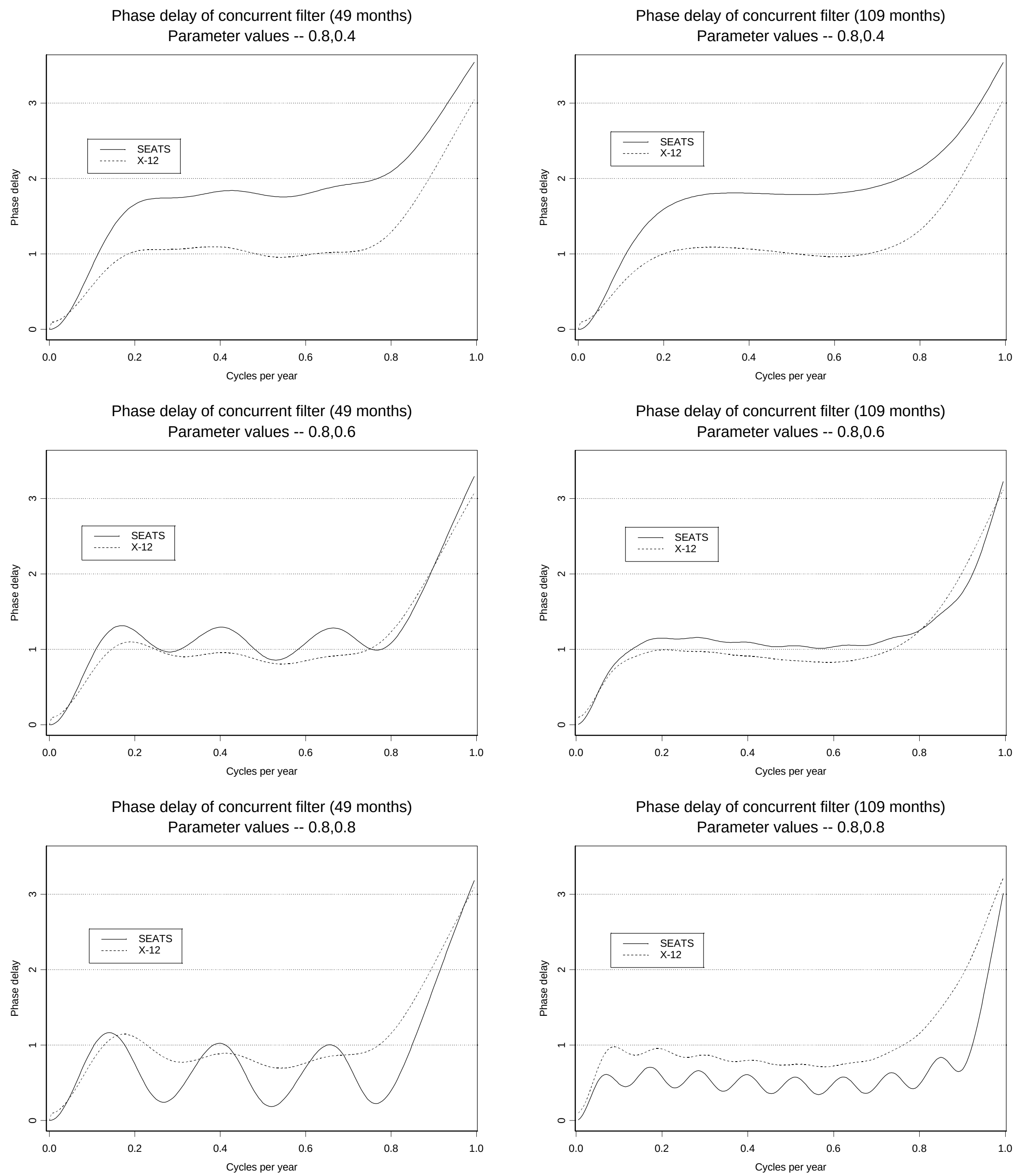
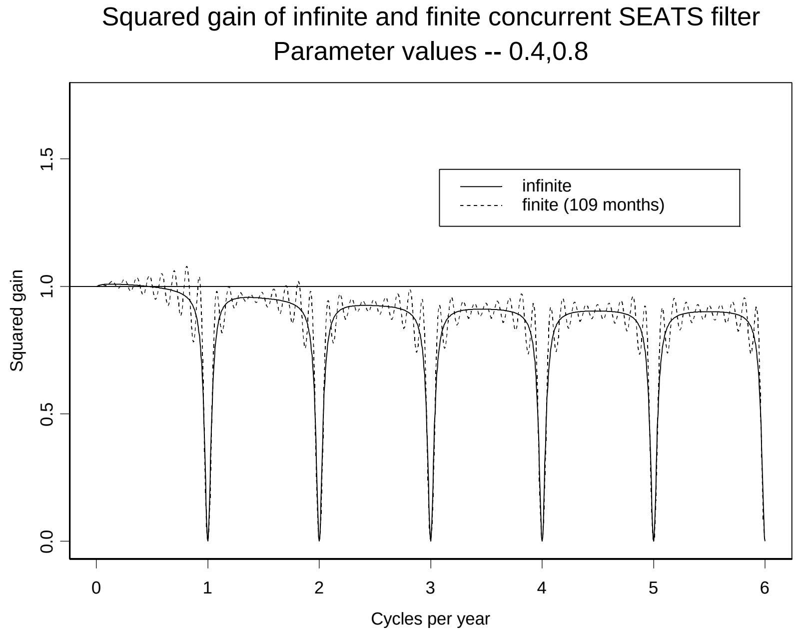


Fig. 8. Concurrent filter phase delay functions for $q = 0.8$. From top to bottom, $\Theta = 0.4, 0.6, 0.8$ for series lengths 49 (left) and 109 (right).



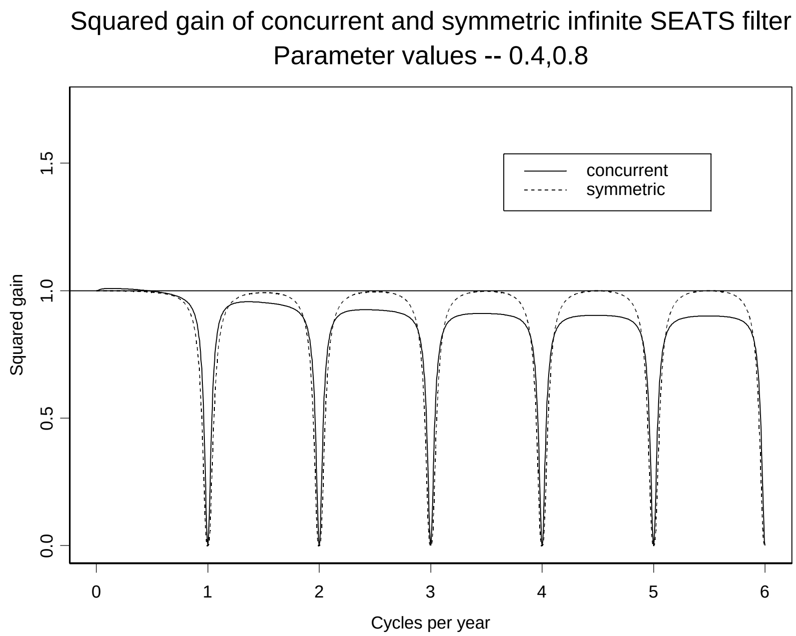


Fig. 9. Squared gain functions of infinite and finite (length 109) concurrent SEATS filters (top) and of infinite concurrent and symmetric filters (bottom). The parameter values are $(q,\Theta)=(0.4,0.8)$ in each case. The troughs at seasonal frequencies of the finite concurrent filter are slightly narrower than those of the infinite concurrent filter, which in turn are slightly narrower than the troughs of the infinite symmetric filter.

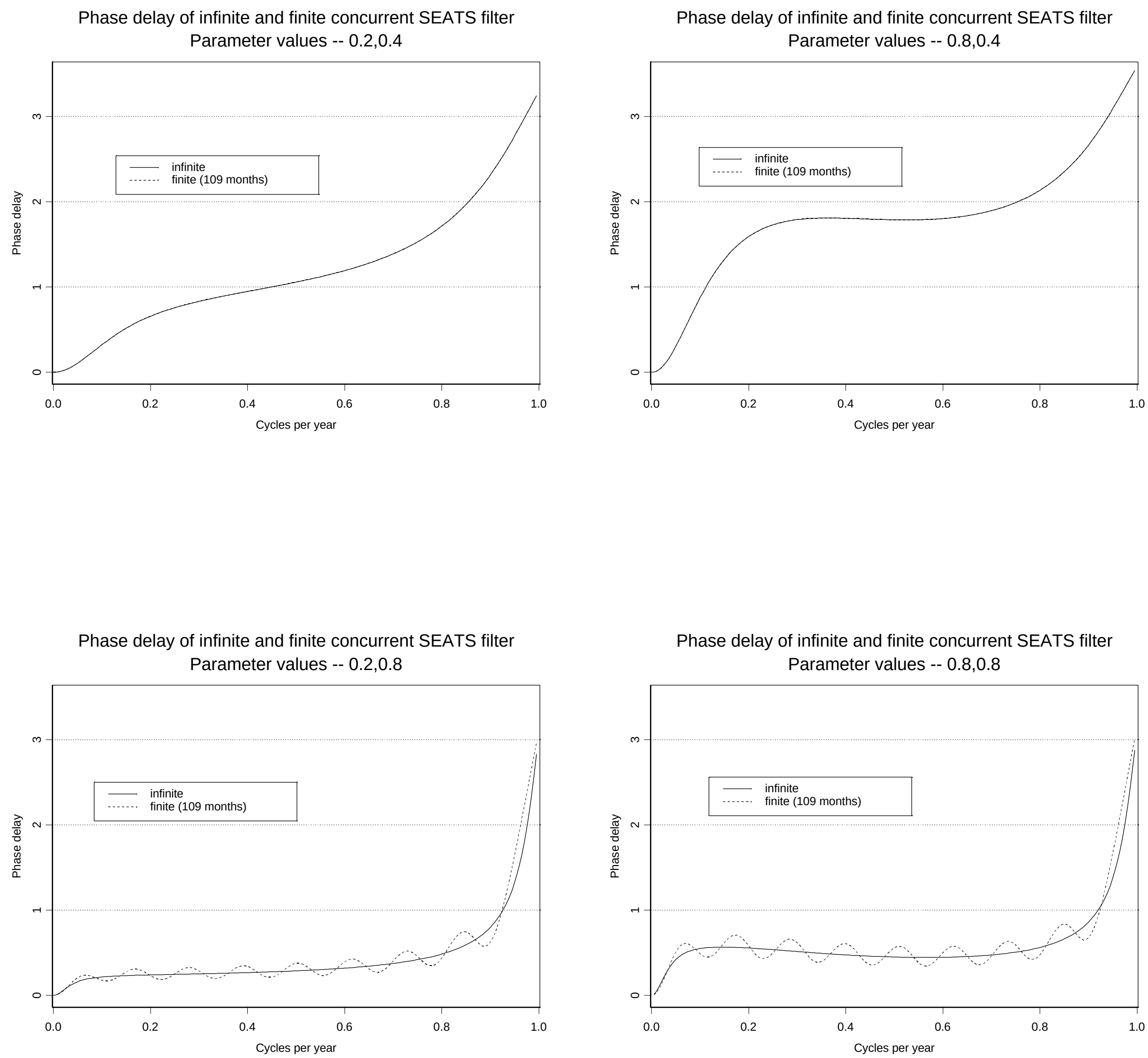


Fig. 10. Phase delay functions of infinite and finite (length 109) concurrent SEATS filters for $q = 0.2, 0.8$ and $\Theta = 0.4, 0.8$. The infinite filters' phase delays closely approximate those of the finite filters, essentially coinciding with them in the top graphs when $\Theta = 0.4$.

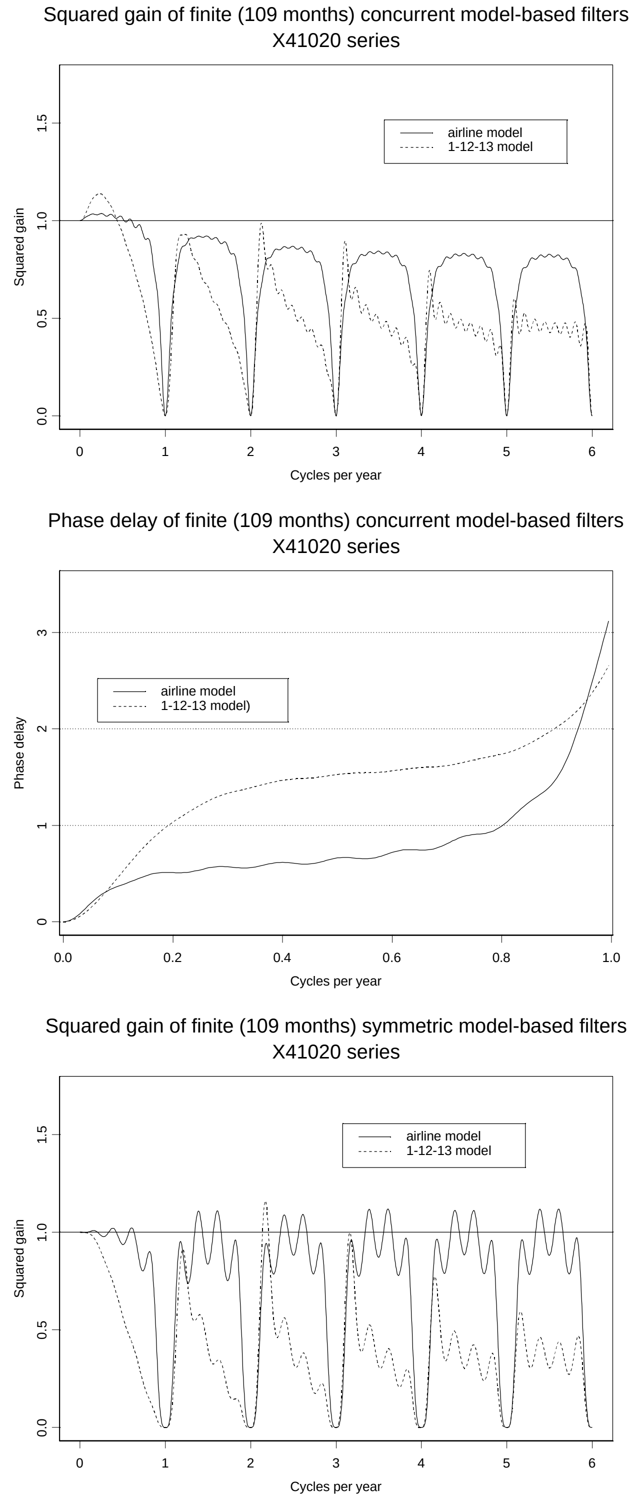


Fig. 11. Squared gains and phase delays of filters of length 109 for the models (4.1) (solid line) and (5.1) (dashed line) for an Exports series. Although the models have similarly good fits to the data, the plots reveal that their seasonal adjustment filters have substantially different properties.