1. INTRODUCTION

Article I, Section 2 of the U.S. Constitution requires that the House of Representatives "shall be apportioned among the several States---according to their respective Numbers," and that "each State shall have at least one Representative." That section also includes the requirement that an enumeration of the population for the purpose of apportioning the House be conducted every ten years. The quoted words obviously do not explicitly state what method should be used for apportionment, and for over 200 years the issues of which is the "best" method and which methods are constitutional have been debated. In fact, apportionment of the House was the subject of George Washington's first veto.

The "best" method issue is, in this author's opinion, unresolvable, since it depends on the criteria employed. However, the constitutional question was at least partially resolved on March 31, 1992, when Justice Stevens delivered an opinion for a unanimous Supreme Court upholding the constitutionality of the currently used apportionment method, equal proportions (EP), also known as the Hill or Huntington method.

The path to this resolution began in 1991 when the states of Montana and Massachusetts initiated separate lawsuits in federal court (*Montana v. United States Department of Commerce* 1991; *Massachusetts v. Mosbacher* 1992) challenging, for the first time in U.S. history, the constitutionality of the current method. Montana proposed two methods as alternatives to EP. Their preferred methods are the method of harmonic means (HM), also known as the Dean method, and the method of smallest divisors (SD), also known as the Adams method, both of which would have given Montana two seats instead of the single seat allocated by EP, but would have not increased Massachusetts' EP allocation of ten seats. Massachusetts proposed, using different arguments, the use of the method of major fractions (MF), also known as the Webster method, which would have allocated eleven seats to Massachusetts, and one to Montana.

The two cases were considered by separate three-judge panels. The panel in the Montana case, by a two-to-one vote, declared EP unconstitutional, while the judges in the Massachusetts case unanimously upheld the constitutionality of EP. The ruling in the Montana case was appealed to the Supreme Court (*United States Department of Commerce v. Montana* 1992), with Massachusetts filing a friend-of-the-court brief before the Supreme Court in order to present their position in favor of MF. On March 4, 1992, the Supreme Court heard the case and 27 days later unanimously overruled the decision of the three-judge panel in the Montana case.

This paper discusses the mathematical and statistical issues in these cases. This author wrote the declarations that served as a basis for many of the technical arguments used by the defense in these cases, and this paper is in part an outgrowth of that work. Section 2 of the paper provides an historical background on the apportionment issue and a discussion of the properties of the major apportionment methods. Balinski and Young (1982), the major source of the material in that section, provides a more detailed treatment of these matters. Sections 3 and 4 discuss the issues debated before the three-judge panels in the Montana and Massachusetts cases, respectively. Finally, the Supreme Court appeal is discussed in Section 5.

2. HISTORICAL BACKGROUND AND PROPERTIES OF METHODS

Six apportionment methods are considered here. They are the four methods mentioned in the Introduction, the method of greatest divisors (GD), also known as the Jefferson method, and the method of greatest remainders (GR), also known as the Hamilton or Vinton method.

All of these methods except GR are members of a class of apportionment methods known as divisor methods. Although there are an infinite number of possible divisor methods, only the five considered here have had any significant role in apportionment history. They will be referred to as the historical divisor methods. With a divisor method, the number of seats assigned to a state is a function of its population, p, and a divisor, λ, which can be thought of as a target district size. The same value of λ must be used for each state. If $\lfloor p/\lambda \rfloor = b$ (where $\lfloor x \rfloor$ denotes the integer portion of x), then the state receives either b or b+1 seats. It receives b+1 seats if $p/\lambda > \delta(b)$, and b seats if $p/\lambda < \delta(b)$, where δ, the function that determines the rounding, depends on the particular method. If $p/\lambda = \delta(b)$, the rounding is not unambiguously defined. δ is a strictly increasing function of b satisfying $b \le \delta(b) \le b+1$ for all nonnegative integers b. In Table 1, $\delta(b)$ is presented for each of the five historical divisor methods.

Table 1. Rounding Criteria for Historical Divisor Methods

Method	$\delta(b)$
SD	b
HM	$2b(b+1)/(2b+1)$
EP	$\sqrt{b(b+1)}$
MF	$b+.5$
GD	$b+1$

Thus, SD rounds up and GD rounds down all fractional remainders, while MF rounds up fractional remainders greater than .5. HM rounds up quotients that exceed the harmonic mean of b and b+1. This can be shown to be equivalent to rounding up if the absolute difference between λ and the state's average district size is minimized with b+1 seats, that is if $|p/(b+1) - \lambda| < |p/b - \lambda|$. This was Dean's original motivation for HM. Similarly, EP rounds up quotients that exceed the geometric mean of b and b+1, which is equivalent to minimizing the relative distance between λ and the average district size for the state. (The relative difference between two positive numbers x,y is $|x-y|/\min\{x,y\}$ or, equivalently, (max{x,y}/min{x,y}) -1.)

Note also that for MF and GD, the modification $\delta(0)=0$ is required to insure that all states, no matter how small, receive at least one representative.

A GR apportionment is obtained slightly differently. Begin with a fixed house size n, and a set of N states with populations p_i, i=1,...,N. Let $d=\sum_{i=1}^{N} p_i/n$, the national average district size;

$q_i = p_i / d$, the exact quota for state *i*; and a_i denote the number of seats allocated to state *i* under any method. Then for GR, either $a_i = \lfloor q_i \rfloor$ *or* $a_i = \lfloor q_i \rfloor + 1$, with $a_i = \lfloor q_i \rfloor + 1$ for the $n - \sum_{i=1}^{N} \lfloor q_i \rfloor$ states with largest fractional remainders, $q_i - \lfloor q_i \rfloor$. To illustrate how these six methods produce apportionments, consider the example in Tables 2 and 3 for which *N*=7 and the populations are as given in the p_i column of each of these two tables. Then with $d = 1000$, which corresponds to *n*=100, and with the divisor $\lambda = d$ used for each of the divisor methods, the allocations for each of the historical divisor methods are given in Table 2. Note that with $\lambda = 1000$, none of these divisor methods gives a total allocation of 100 seats. To obtain an allocation of 100 seats, it is necessary to adjust the divisor λ upward for SD, HM and EP, and downward for MF and GD. The allocations for each of the divisor methods for *n*=100 and the minimum and maximum integer values of λ which yield these allocations are presented in Table 3, along with the GR allocation.

Table 2. Divisor Methods Allocations for Example with $\lambda = 1000$

			a_i for Method				
	State	p_i	SD	HM	EP	MF	GD
	1	91,490	92	91	91	91	91
	2	1,660	2	2	2	2	1
	3	1,460	2	2	2	1	1
	4	1,450	2	2	2	1	1
	5	1,440	2	2	2	1	1
	6	1,400	2	2	1	1	1
	7	1,100	2	1	1	1	1
Totals		100,000	104	102	101	98	97

Table 3. Allocations for Six Methods for Example with *n*=100

a_i for Method

State	p_i	q_i	GR	SD	HM	EP	MF	GD
1	91,490	91.490	92	88	89	90	93	94
2	1,660	1.660	2	2	2	2	2	1
3	1,460	1.460	2	2	2	2	1	1
4	1,450	1.450	1	2	2	2	1	1
5	1,440	1.440	1	2	2	2	1	1
6	1,400	1.400	1	2	2	1	1	1
7	1,100	1.100	1	2	1	1	1	1
Totals	100,000	100.000	100	100	100	100	100	100
Min λ				1,040	1,023	1,011	979	964
Max λ				1,051	1,033	1,018	989	973

An alternative to adjusting λ to obtain an apportionment for the House of Representatives with a fixed number of seats *n*, for a divisor method based on the function δ, is to use the following recursive algorithm. Let a_{ik}, $i=1,...,N$, $k=N,N+1,...,n$ denote the allocation to state *i* with *k* seats. Then let $a_{iN}=1$, $i=1,...,N$. For $k>N$ choose i_k satisfying

$$p_{i_k}/\delta(a_{i_k(k-1)}) \ = \ \max\{p_i/\delta_{i(k-1)} \ : \ \ i=1,...,N\}\,,$$

and then let $a_{i_k k} \ = \ a_{i_k(k-1)}+1$, $a_{ik}=a_{i(k-1)}$ for $i\neq i_k$. a_{in}, $i=1,...,n$ is then a δ apportionment for *n* seats. Note that in the rare case when i_n is not unique then there is a "tie" for the *n*-th seat and the apportionment is not unique.

State *i* is said to satisfy quota if $\lfloor q_i\rfloor\leq a_i<\lfloor q_i\rfloor+1$. Note that a quota violation occurs for *i*=1 for each of the five divisor methods for the allocations in Table 3, since $a_1<91$ or $a_1>92$. GR, however, can never violate quota. (Actually, because of the minimum requirement of 1 representative per state, this last statement is theoretically only true for an apportionment of the

House of Representatives if the exact quota q_i is replaced by the modified exact quota $\tilde{q}_i = \max\{1, tq_i\}$, where t satisfies $\sum_{i=1}^{N} \max\{1, tq_i\} = N$, as discussed in Balinski and Young (1982)). Furthermore, although all five historical divisor methods can violate quota in theory, EP, HM and MF would never have violated quota for any of the 21 censuses through 1990, while SD and GD would have violated quota for at least one state for each census since 1820. For example, for California for 1990, a_i = 50 for SD and a_i = 54 for GD, while q_i = 52.124.

GD was used to apportion the House for the first five censuses through 1830. Eventually, Congress became dissatisfied with this method because it appeared to favor large states, allocating 40 seats to New York in 1830, for example, despite an exact quota of 38.593. SD, MF and HM, were developed as alternatives by John Quincy Adams, Daniel Webster and James Dean (a professor at the University of Vermont), respectively. MF was used in 1840. GR was the specified method from 1850-1900, although as Balinski and Young (1982) note, for some of these censuses the GR allocation was altered so that no method was really used. However, Congress became disenchanted with GR, because under this method, unlike any divisor method, it is possible, with a fixed set of state populations, for a state to lose seats if the House size is increased. This anomaly is known as the "Alabama paradox" because it was observed that for the 1880 census, Alabama would have received 8 seats with a House size of 299 and 7 seats with a House of 300. This occurred because Alabama, Illinois and Texas had exacts quotas of 7.646, 18.640 and 9.640, and allocations of 8, 18 and 9 seats, respectively, for a House size of 299, but these states had exact quotas of 7.671, 18.702 and 9.672, and allocations of 7, 19 and 10 seats, respectively, for a House size of 300. This was a particularly unpleasant property since the House was not automatically fixed by law during the period of use of GR, but was decided upon by Congress following each census, after reviewing the allocations with various House sizes.

Congress returned to MF for the 1910 census. Congress also passed legislation which, after New Mexico and Arizona became states in 1912, fixed the House size at 435. About the time of the 1920 census, Professor Edward Huntington of Harvard refined and became the principal champion of EP, which had first been developed by Joseph Hill of the Census Bureau in 1911.

Huntington (1921) is one of the earliest of his many papers on this subject. The case for EP rested primarily on the pairwise optimality tests. An apportionment is said to be pairwise optimal with respect to a particular measure of inequity if no transfer of representatives between any pair of states can decrease the amount of inequity between these states. HM is pairwise optimal with respect to absolute difference in average district sizes, that is with respect to the measure, $|p_i/a_i - p_j/a_j|$ between states *i* and *j*. MF is pairwise optimal with respect to the absolute difference in per capita shares of a representative, that is $|a_i/p_i - a_j/p_j|$. However, EP is pairwise optimal with respect to relative differences in both district sizes and shares of a representative, which became the key argument for EP. SD and GD are pairwise optimal with respect to two other tests, absolute representation surplus and absolute representation deficiency, respectively. (If $a_i/a_j > p_i/p_j$, then absolute representation surplus for the pair *i,j* is $a_i - (p_i/p_j)a_j$, that is the amount by which the allocation for state *i* exceeds the number of seats it would have if its allocation was directly proportional to the actual allocation for state *j*. Similarly, absolute representation deficiency is $(p_j/p_i)a_i - a_j$.) Furthermore, every transfer of seats between a pair of states from an apportionment obtained from the optimal method will actually strictly increase, as opposed to merely not decrease, the corresponding measure of inequity for the pair of states when the optimal method produces a unique apportionment, that is when there are no ties for the last seat.

The opposition to the views of Huntington was led by Professor Walter Wilcox of Cornell, who supported MF. He was of the opinion that EP was biased in favor of small states, while MF was mathematically neutral between small and large states. Huntington disagreed, contending that it is actually EP that is mathematically neutral in this respect. Huntington’s argument was based on the fact that among SD, HM, EP, MF and GD, all transfers of seats that result from the replacing of one method with a method further to the right on this list are to states that are larger than the states losing seats. Thus, in a relative sense, EP favors smaller states less then SD and HM, and larger states less then GD and MF. This result alone does not establish anything about bias beyond how the methods compare in relation to each other.

Congress failed to reapportion the House at all after the 1920 census, but in an attempt to resolve the technical dispute, the Speaker of the House requested that the National Academy of Sciences (NAS) review the mathematical aspects of the problem of reapportionment. A NAS committee issued a report in 1929 (Bliss et al.). The report considered the five divisor methods discussed in this paper and focused on the pairwise comparison tests described above. The committee adopted Huntington’s reasoning that EP is preferred on the basis of the pairwise tests for which it is optimal and also concluded that EP "occupies mathematically a neutral position with respect to emphasis on larger and smaller states."

The 1930 allocations for EP and MF were identical, so Congress took no further action after that census. Under the applicable law, the House was automatically apportioned under the method last used, MF.

In 1940, however, EP and MF differed, with Arkansas allocated 7 seats by EP and 6 by MF, while Michigan was allocated 17 by EP and 18 by MF. In 1941, on a mainly party line vote, legislation was enacted apportioning the House by EP. This method has been used ever since and, under the 1941 law, its continued use is automatic until superseding legislation is enacted.

In 1948, a new NAS committee revisited the apportionment issue and also endorsed EP (Morse et al.). Their report included the new argument that among the four pairwise comparison tests previously mentioned for which either EP, HM or MF are optimal, EP is always superior to each of the other four divisor methods for at least three of them. For example, it can be shown that EP is superior to MF with respect to absolute difference in district sizes, in the sense that no transfer of seats resulting from the use of MF instead of EP can ever lower this measure of inequity for any pair of states. In this sense EP is, of course, also superior to MF and all other methods, with respect to relative differences in district sizes and shares of a representative, while

MF is superior to EP with respect to absolute difference in shares of a representative. Analogously, EP is superior to HM for all of these tests except absolute difference in district sizes. The committee found the total score in favor of EP using this approach "decisive."

Much of the interest in the apportionment issue since the mid 1970s is a result of the work of Michel Balinski and H. Peyton Young. In their early writings on apportionment (Balinski and Young 1975), they expressed the view that an apportionment method should never violate quota and should not be subject to the Alabama paradox. None of the six methods considered in this paper meet both of these conditions. However, Balinski and Young (1975) developed a modification of GD that they called the quota method, which does satisfy both of these conditions. Still (1979), among others, generalized Balinski and Young's result and obtained modifications of all six methods considered in this paper which satisfy these two conditions. Unfortunately, all of these modified methods suffer from an unpleasant property that none of these methods possess in unmodified form. They allow a form of "population paradox" in which one state can have a population increase, while all other states and the total house size remained fixed, and yet the growing state can lose seats. Balinski and Young eventually abandoned their support of the quota method and became proponents of MF. Their main argument for MF was, like Wilcox's decades earlier, their belief that MF is the only divisor method that is not biased in favor of either large or small states. Their work, culminating in the book *Fair Representation* (Balinski and Young 1982), presented a number of new theoretical and empirical results to support their view.

For example, corresponding to a divisor λ and a divisor method based on δ, they considered intervals

$$[\delta(b-1)\lambda,\ \delta(b)\lambda], \qquad\qquad b=1,2,3..., \tag{2.1}$$

(where $[\alpha,\beta]$ denotes $\{x\text{: } \alpha\leq x\leq\beta\}$), that is populations for which b seats are assigned, and established that MF is pairwise unbiased in the sense that if states 1 and 2 have independent populations p_1 and p_2, respectively, uniformly distributed in intervals

$[\delta(b_1 - 1)\lambda,\ \delta(b_1)\lambda]$, $[\delta(b_2 - 1)\lambda,\ \delta(b_2)\lambda]$, respectively, for positive integers $b_2 > b_1$, then the probability is .5 that state 2 is favored over state 1 in the sense that $b_2/p_2 > b_1/p_1$. They also established that MF is the only proportional divisor method with this property, where proportional divisor methods are a set of "reasonable" divisor methods, defined in Balinski and Young (1982, p. 97), that include all five historical divisor methods. They then generalized this result from pairs of states to two groups of smaller and larger states, obtaining the result that MF is the unique unbiased proportional divisor method.

Their empirical results include comparisons of the historical divisor methods for the "bias ratio" and "percentage bias," two measures of apportionment method bias developed by these authors. For both measures they excluded states with exact quotas below .5 as their means of compensating for the constitutional requirement of at least one representative per state, a provision which in effect creates a constitutionally mandated bias in favor of the small states. The bias ratio was obtained by first computing for each census the number of pairs of non-excluded states *i*,*j* with $p_i < p_j$, where state *i*, the smaller state, is favored in the sense that $a_i/p_i > a_j/p_j$. The total of the number of pairs for which the smaller state was favored, summed over the 19 censuses through 1970, was then divided by the total number of pairs of non-excluded states in these 19 censuses to obtain the bias ratio. Balinski and Young's results for the five historical divisor methods are presented in Table 4. The ideal ratio is, of course, 50%.

Table 4. Bias Ratio of Censuses Through 1970

	SD	HM	EP	MF	GD
Bias ratio	77.2%	56.6%	54.6%	51.5%	25.0%

They computed percentage bias for each census by first dividing the non-excluded states into approximately equal classes of large (L), middle, and small states (S), with the middle class receiving the extra states when the number of non-excluded states was not divisible by three.

The percentage bias for each census is then $(\sum_S a_i/\sum_S p_i)/(\sum_L a_j/\sum_L p_j)\ -1$ expressed as a percentage. Balinski and Young's (1982) results, averaged over the 19 censuses through 1970, are presented in Table 5. A positive percentage indicates that small states are favored and a negative value indicates that large states are favored.

Table 5. Percentage Bias Averaged over Censuses Through 1970

	SD	HM	EP	MF	GD
Average bias	18.3%	5.2%	3.4%	0.3%	-15.7%

Balinski and Young (1982) also presented a secondary reason for their support of MF, the "near the quota" property that they developed. They defined an apportionment to be "near the quota" if no transfer of a seat from one state to another can bring both states nearer to their exact quotas. They proved that MF is the unique divisor method that is "near the quota" for all apportionments and noted that this result is true whether distance is measured in absolute or relative terms. (GR apportionments also always satisfy this property.) By absolute terms, they meant, of course, the measure $|a_i - q_i|$. By relative terms, they meant $|a_i - q_i|/q_i$, not the relative difference between a_i and q_i, which is $|a_i - q_i|/\min\{a_i, q_i\}$. Balinski and Young's result is equivalent to saying that MF is the only divisor method which can never produce an apportionment which rounds up q_i for a state i with $q_i - \lfloor q_i \rfloor < .5$, while rounding down q_j for a state j with $q_j - \lfloor q_j \rfloor > .5$. (Again, as these authors note, because of the minimum requirement of 1 representative per state, this result is actually only true for an apportionment of the House of Representatives if q_i is replaced by the modified exact quota, $\tilde{q}_i$.)

A final set of properties of apportionment methods are measures of total error of an apportionment. Let $d_i = p_i/a_i$, $d = (\sum p_j)/n$, $s_i = 1/d_i$, and $s = 1/d$. Three classes of error measures are, for $\rho \geq 1$,

$$\sum_{i=1}^{N} |a_i - q_i|^{\rho}, \tag{2.2}$$

$$\sum_{i=1}^{N} a_i |d_i - d|^{\rho}, \tag{2.3}$$

and

$$\sum_{i=1}^{N} p_i |s_i - s|^{\rho}. \tag{2.4}$$

(2.2), (2.3) and (2.4) are, respectively, the sum of the ρ-th power of each state's absolute deviation from its exact quota, each district's absolute deviation from the national average district size, and each person's absolute deviation from the national average share of a representative. The assumption that the districts within each state are of the same size are used in (2.3) and (2.4).

GR minimizes (2.2) for all $\rho \geq 1$ (Birkhoff 1976), while for $\rho = 2$, EP minimizes (2.3) (Huntington 1928) and MF minimizes (2.4) (Owen 1921). As observed by Gilford (1981), for $\rho = 1$, (2.3) and (2.4) are minimized by GR since they are constant multiples of (2.2) with $\rho = 1$.

Interestingly, the various measures of total error of an apportionment have generally not been a major focal point in the selection of a method for apportioning the House, perhaps because no apportionment method minimizes all these measures. As will be seen in the next two sections, these measures did become an issue in both apportionment cases.

3. THE MONTANA DISTRICT COURT CASE

The Montana lawsuit was primarily based on the following legal reasoning (Racicot et al. 1991). In *Wesberry v. Sanders* (1964), the U.S. Supreme Court had declared that the *intrastate* redistricting of congressional districts must be accomplished to provide "equal representation for equal numbers of people," that is the "one person, one vote" principle. That case did not set any test for meeting this principle, but in subsequent decisions, such as the *Karcher v. Dagget* (1983), the Supreme Court ruled that this principle required that "districts be apportioned to achieve population equality as nearly as practicable." The plaintiffs concluded from the intrastate redistricting cases they cited, that the courts required this principle be met in the intrastate context by minimizing "absolute population variances between districts" and that this requirement also applied to interstate apportionment. Of course, no court had previously ruled that the "one person, one vote" principle applied to interstate apportionment, much less that a certain test was superior to another for interstate apportionment. In fact, even for intrastate redistricting, no court had specifically ruled that a test based on district sizes is a better test than one based on shares of a representative, or that absolute difference is a better measure than relative difference. Furthermore, this issue of the best test would not even be relevant for intrastate redistricting since differences, at least in theory, can be made as close to zero as desired for any of these methods of measurement. Finally, the plaintiffs never offered any specific reasons why absolute difference in district sizes is the only appropriate test beyond citing these redistricting cases.

After using these prior cases as their rationale for their view that absolute difference between district sizes is the only appropriate test, the plaintiffs noted in their briefs and the affidavits of their experts, that the pairwise test for which HM is optimal and Dean’s original motivation for HM are both criteria that they considered consistent with the cited cases. In addition, in the affidavits of the plaintiffs’ experts, Hill (1991) and Tiahrt (1991), it was observed that for the 1990 census, among EP, HM and SD, HM produces the smallest variance while SD produces the smallest range, and the plaintiffs declared that either of these are appropriate tests of inequity among district sizes. Furthermore, the Hill affidavit included the formula used in computing the variances, namely, using the notation of Section 2,

$$\sum_{i=1}^{50} (d_i - d)^2/49, \qquad (3.1)$$

and also an alternative formula with $\sum_{i=1}^{50} [p_i/(50a_i)]$ replacing d in (3.1).

The defendants' reply to the plaintiffs' assertions (Gerson, Poppler et al. 1991) contained a number of legal arguments, including the argument that apportionment of the House is a political question to be decided by Congress, and that it should not be considered by the courts. It was also argued that in carrying out its constitutionally mandated duty to apportion the House, Congress should be allowed broad discretion by the courts even if the issue is considered justiciable. In addition, it was observed that interstate apportionment is very different from intrastate redistricting, since large differences in district sizes between states are inevitable because districts cannot cross state lines and each state must have at least one representative. Consequently, the defendants claimed that the redistricting cases cited by the plaintiffs are not applicable to interstate apportionments.

The (U.S.) Government used these arguments before each of the courts that considered the two apportionment cases. As will be seen, the success with these arguments varied except for the political question argument, which was unsuccessful. In addition to the above arguments, substantive arguments were presented to demonstrate the advantages of EP, based primarily on the declaration of this author (Ernst 1991a), which will be the focus in this paper. There was no attempt to demonstrate that EP is clearly superior to all other apportionment methods, or the only constitutional method, but instead that neither of these claims is true for any other apportionment method or set of apportionment methods which exclude EP.

After first reviewing the apportionment history, including the 1929 NAS report, we responded to the general argument that absolute differences in district sizes is the only proper criterion for evaluating an apportionment. We pointed out that it can be argued that a test involving differences in shares of a representative is a better test of "the one person, one vote" principle for interstate apportionment than a test involving differences in district sizes, since share of a representative measures the portion of a vote to which a person is entitled in the House. It was also observed that intrastate redistricting and interstate apportionment are conceptually very different, since in the former case, the people in each state are allocated to a fixed number of districts, while in the latter case, districts are allocated to the fixed number of people in the various states.

An artificial example was presented to illustrate the distinction between absolute and relative difference, which noted that if the national average district size is 600,000 then, as measured by relative differences, a district of size 1,200,000, twice as large as the ideal, and a district of size 300,000, twice as small as the ideal, are equally inequitable. In addition, while the relative difference between a district of size less than 300,000 and the ideal district of 600,000 is greater than the relative difference between the 1,200,000 and 600,000, the opposite relationship holds for absolute difference, even if the smaller district is of size 1.

It was noted that for 1990, as guaranteed by the optimality results for the pairwise difference tests, the relative difference between Washington’s and Montana’s average district sizes and average shares of a representative under EP (48.0%) is smaller than under HM (52.1%). It was also observed that the relative difference between Montana’s average district size and the national average district size is 40.4% under EP and 42.5% under HM, while Washington’s is 5.4% under EP and 6.7% under HM. EP always gives a higher priority to awarding a seat to a state that would be moved closer to the ideal, as measured by relative difference, than to a state for which the opposite is the case.

Although the plaintiffs declared the proper measure of inequity in an apportionment is absolute population variance among all districts and claimed that HM results in the smallest such variance, the defendants observed that it is actually EP that always minimizes this measure, since it minimizes (2.3) with $\rho=2$. The reason for the discrepancy in the claims is that the formula used by the plaintiffs, (3.1), did not take into account the number of districts in each state. Their formula measures variability among the mean district sizes of the 50 states, not the variance of the sizes of the 435 districts. Now, measuring variability among state average districts sizes is not necessarily an inappropriate criterion for comparing apportionments; it is simply that (2.3) with ρ=2, not (3.1), reflects the criterion actually stated in the plaintiffs' briefs.

As for SD, it is indeed true that SD minimizes the range of district sizes for the 1990 census among the three methods considered by the plaintiffs, and actually minimizes this measure for 1990 among all possible apportionments. The defense case against SD focused on its tendency to violate quota. We noted that while, for 1990, California's exact quota is 52.124 seats, SD only allocates it 50 seats and also results in quota violations for Illinois, New York and Ohio. It was also noted that if SD had been employed for all 21 censuses, quota violations would have resulted for every census since 1820, with a total of 47 violations.

There is a requirement in federal law that redistricting cases be heard before a three-judge panel, instead of a single judge, with decisions of these panels generally appealable directly to the Supreme Court. Although the Government contended that this law does not apply to interstate apportionment cases, such panels did hear both the Montana and Massachusetts cases.

By a two-to-one majority, the court in the Montana case upheld Montana's position that equal proportions was unconstitutional (Lovell 1991). Judges Lovell and Battin, both from Montana, constituted the majority. They agreed with the plaintiffs' argument that the "one person, one vote" principle applies to interstate apportionment and, citing prior intrastate redistricting cases, that absolute difference in district sizes is the only proper standard for testing this principle. The judges provided their rationale for rejecting tests involving representatives per person or using relative differences in two footnotes. First they dismissed relative difference (footnote 3) stating:

> By arguing that proportions and percentages are the proper criteria, rather than absolute numbers, Defendants ignore the fact that each number represents a person whose voting rights are potentially impacted by the population disparities.

Concerning share of a representative, they observed (footnote 7):

> The Constitution decreed that one house should be chosen on the basis of population (persons per representative) and Congress cannot ignore that mandate by choosing a method which considers each person’s share of a representative.

This author does not fully comprehend either of these quotes.

The majority never made clear what specific tests involving absolute differences in district sizes should be used. They quoted various tests from previous opinions, including: "variances between the actual district and the district size," "range," "average deviation from the ideal district size," and "maximum deviations above and below the mean district size," but never stated which, if any, of these tests they were adopting.

However, the majority did state: "Courts traditionally look to variances from the ideal district size to determine whether a district is under or over represented" and, additionally, "absolute difference from the ideal is the proper criterion... ," statements which would appear to rule out range and pairwise difference tests. The majority did not directly address the defense point that it is actually EP that minimizes variance among all districts, although they may have been indirectly referring to it when they stated: "The Hill method can never meet the criteria proposed by Plaintiffs, because its express objective is to minimize the relative difference between the number of persons per representative and the relative difference between each person’s share of a representative." It is not clear if the majority meant by that statement that EP cannot possibly minimize (2.3) with $\rho=2$.

In any case, whatever specific tests the majority had in mind, they considered that HM comes closer than EP to satisfying the "one person, one vote" principle, and concluded that the use of EP is unconstitutional. They did, however, reject SD from consideration based on the quota violations.

Circuit Court Judge O'Scannlain of Oregon, while agreeing with the majority on the justiciability of the case, dissented on the merits, noting several points (O'Scannlain 1991). He first found, as did the majority, that SD is inconsistent with the constitutional requirement of allocating House seats by population, since it results in quota violations for four states in 1990. He cited the fact, from the defense declaration, that the relative difference between Montana's and Washington's average district size is larger under HM then EP. Judge O'Scannlain also stated that range of district sizes is not the best test of disparity, employing the most common argument against a range test, that it only considers the largest and smallest of the 435 congressional districts.

Judge O'Scannlain's opinion focused on measures of total error, which he referred to generically as variance. He cited Gilford's (1981) testimony in which it was noted that different apportionment methods optimize (2.2), (2.3) and (2.4) with $\rho=2$ and $\rho=1$, as evidence that this is not a straightforward issue. Citing the defense declaration, he observed that (2.3) with $\rho=2$, not (3.1), measures variance among all districts, and that EP, not HM, minimizes the appropriate variance. Judge O'Scannlain also quoted the majority statement that "absolute difference from the ideal district is the proper criterion... ." He interpreted this as requiring the test (2.3) with $\rho=1$. He calculated that EP for 1990 produces a lower value for this measure than HM. (As noted in Section 2, it is actually GR that minimizes (2.3) with $\rho=1$, but Judge O'Scannlain prefaced his discussion by stating that three methods were before the court, EP, SD and HM.) The Judge concluded: "In sum, neither of the formulae proposed by the State lead to less population variance than the Hill equal proportions formula in use for the last fifty years. The State, in my view, has failed to demonstrate that a better formula exists than the one chosen by Congress."

A number of questions naturally arise from the issues raised in this case that were not answered in the court documents. For example, the plaintiffs noted that for 1990, HM has a smaller value for (3.1) than EP, and both HM and SD have a smaller range of district sizes than EP. The example in Table 6 with N=3, n=5 illustrates that none of these relationships always hold.

Table 6. Example for Which EP Minimizes Both Range of District Sizes and Variance Formula (3.1)

			HM and SD		EP	
State	p_i	q_i	a_i	d_i	a_i	d_i
1	2370	2.37	2	1185	3	790
2	1320	1.32	2	660	1	1320
3	1310	1.31	1	1310	1	1310

For this example, the EP allocation results in values of 121,300 for (3.1) and 530 for the range, while the HM and SD allocation results in values of 122,962.5 for (3.1) and 650 for the range. The EP allocation is actually optimal among all apportionments for these two measures for this example.

However, if the comparisons are limited to the 21 actual censuses, then among the nine censuses for which HM and EP did not produce identical apportionments, HM always yielded a lower value of (3.1) than EP, and a range of district sizes less than or equal to EP. EP did yield a smaller range of district sizes then SD for the 1810 and 1840 censuses. Furthermore, it can be argued that states with exact quotas less than 1 should be excluded from range computations, since they must receive one seat due to the constitutional provision of at least one representative per state. With such states excluded, EP produced a smaller range of district sizes than SD as recently as the 1940, 1950 and 1960 censuses.

Judge O'Scannlain computed that (2.3) with ρ=1 is smaller for EP than HM for 1990. This is also true for the other eight censuses for which these two methods did not yield identical apportionments. However, for the example in Table 7 with N=3, n=5, the EP allocation has a value of 1344 for (2.3) with ρ=1, while the corresponding value for the HM and SD allocation is 1320. Furthermore, since HM, SD and GR have the same allocation for this example, HM and SD are optimal for this measure.

Table 7. Example for Which HM and SD Minimize (2.3) with ρ=1

State	p_i	q_i	HM and SD a_i	HM and SD d_i	EP a_i	EP d_i
1	2328	2.328	2	1164	3	776
2	1340	1.340	2	670	1	1340
3	1332	1.332	1	1332	1	1332

Finally, a measure of total error that the plaintiffs never mentioned is

$$\sum_{i=1}^{N} |d_i - d|. \tag{3.2}$$

Because of the pairwise difference test that HM optimizes, it might appear that HM is optimal for this measure. This is true with the qualifications given in the following theorem. (The proofs of all theorems are given in the Appendix.)

Theorem 3.1. If HM produces an apportionment for which there are no quota violations, then HM minimizes (3.2) among all apportionments which do not violate quota.

In particular, since HM has not violated quota for any of the 21 censuses, this result is applicable to all the actual censuses.

However, the example in Table 8, with N=9, n=20, illustrates that without the quota restriction, HM does not always optimize (3.2). The value of (3.2) is 1210 for HM and 1095 for SD. However, SD violates quota for state 1.

Table 8. Example for Which (3.2) is Smaller for SD than HM

			HM		SD	
State	p_i	q_i	a_i	d_i	a_i	d_i
1	4320	4.32	4	1080	3	1440
2	2970	2.97	2	1485	3	990
3-6	1820	1.82	2	910	2	910
7-9	1810	1.81	2	905	2	905

4. MASSACHUSETTS DISTRICT COURT CASE

The Massachusetts case was much more complex than the Montana case in terms of the technical issues involved. The EP 1990 allocation of ten seats to Massachusetts is one less than its 1980 allocation. Massachusetts would have received eleven seats for 1990 if either MF, GD or GR had been used. The plaintiffs chose only to claim that MF is constitutionally superior to EP (Harshbarger et al. 1991). MF, in addition to increasing Massachusetts' EP allocation, would reduce Oklahoma's EP allocation of six seats to five seats, but would produce the same apportionment for the remaining 48 states as EP.

The plaintiffs claimed that EP is unconstitutional for three separate reasons, the first two of which were based on the work of Balinski and Young (1982). Their major claim was that EP is unconstitutionally biased on favor of small states. They also found EP lacking because it, unlike MF, can yield apportionments which violate the "near the quota" principle. Finally, the plaintiffs claimed that the "one person, one vote" principle in interstate apportionment is best met by the pairwise test for which MF is optimal, absolute difference in shares of a representative.

Not surprisingly, the plaintiffs retained as their expert, H. Peyton Young, who wrote three affidavits in support of Massachusetts' claims (Young 1991), which formed the heart of their case.

The plaintiffs and their expert made the following key points on the bias issue. They described the percentage bias test, mentioned in Section 2, and presented the percentage bias figures averaged over all 21 censuses for EP and MF, which are virtually the same as those in Table 5 for the first 19 censuses. They stated that the percentage bias (in absolute value) for MF was less than or equal to the percentage bias for EP for each of the 21 censuses. They also noted that while the fractional part of $\delta(b)$, which they called the "rounding threshold," increases for EP as b increases, the "rounding threshold" for MF is .5 regardless of the value of b, and stated that this provides an intuitive reason why EP is biased in favor of small states and MF is unbiased in its treatment of small and large states. They also referred to Balinski and Young's theoretical result on the unbiasedness of MF mentioned in Section 2.

Finally, the plaintiffs described computer simulations of Balinski and Young (1984) in which the allocations for each state were averaged over 1000 randomly generated populations and compared to the state's modified exact quota. They stated that the results showed that EP tended to allocate more seats on average than the modified exact quotas for smaller states and to produce the opposite result for larger states, while MF produced no pattern of favoritism towards the smaller or larger states.

On the "near the quota" property, the plaintiffs, in addition to explaining this property, provided illustrative examples from the 1970 and 1920 censuses. They noted that in 1970, EP rounded up South Dakota's exact quota of 1.435 while rounding down Connecticut's exact quota of 6.503. MF produced the opposite results for these states, and thus produced an allocation which brought both states closer to their exact quotas in absolute terms and relative terms (that is, with respect to the measures $|a_i - q_i|$ and $|a_i - q_i|/q_i$, as noted in Section 2) (Actually, this is not a good example since Connecticut's modified exact quota was 6.493). In 1920, they observed, three states with exact quotas with fractional part less than .5 would have been rounded up by EP and down by MF, while three states with exact quotas with fractional part greater than .5 would have been rounded down by EP and up by MF. (This also would have been true for modified exact quotas.)

To support their claim that absolute difference in shares of a representative is the best pairwise test, the plaintiffs essentially used the same reasoning that the defendants had used in the Montana district court to argue the superiority of share of a representative over district size as a test of the "one person, one vote" principle. However, in their initial brief they had no real argument to support the claim that absolute difference is a better measure of inequity than relative difference. They were handicapped on this point because their expert was unable to provide support, since his view had always been that the choice between absolute and relative differences in pairwise comparisons is a "question of preference" (Balinski and Young 1982, p. 102). The plaintiffs initially were only able to argue that their preferred pairwise test is best since MF is optimal for it and MF is, in their opinion, unbiased. The defendants, criticized the logic of this argument. In their final brief, the plaintiffs developed a new argument that will be described later in this section.

The technical arguments used by the defense (Gerson, Budd et al. 1991) were based primarily on three declarations written by this author (Ernst 1991b). Several points were raised in response to the plaintiffs' claims on the bias issue. The key point of contention was the plaintiffs' assumption that only states with exact quotas less than .5 should be excluded in bias measures, an assumption on which much of their empirical and theoretical results were based. The defense claimed that it would be more appropriate to exclude all states with exact quotas less than 1, since even though all such states are overrepresented, this is an overrepresentation mandated by the Constitution.

The following are some of the changes in the empirical results that we noted occurred with this change in the set of excluded states. While the bias ratio in Table 4 is 54.6% for EP and 51.5% for MF, this ratio for the same 19 censuses with all states with exact quotas below 1, instead of only those below .5, excluded is 50.8% for EP and 47.4% for MF. Similarly, the average percentage bias in Table 5 is 3.4% for EP and .3% for MF, while with all states with exact quotas below 1 excluded, it is 1.8% for EP and -.9% for MF. Furthermore, with all states with exact quotas less than 1 excluded, the plaintiffs assertion that the percentage bias for MF never exceeded the percentage bias for EP for each of the 21 censuses does not hold. In fact, with these states excluded, the percentage bias for the 1990 census is -.6% for EP and -1.0% for MF. That is, by this measure, the 1990 EP apportionment favors the large states and substitution of MF would simply increase the magnitude of the favoritism. Finally, we noted that, with states with exact quotas below 1 excluded, EP favored small states 13 times among the 21 censuses through 1990 using the percentage bias test, but MF favored large states 15 times.

Two new theoretical results were obtained by the defendants. First it was observed that Balinski and Young's (1982) result that MF is pairwise unbiased is dependent on use of the partition (2.1) (ignoring the overlap of endpoints), which for MF reduces to

$$[.5\lambda,\ 1.5\lambda],\ [1.5\lambda,\ 2.5\lambda],\ [2.5\lambda,\ 3.5\lambda],\ \dots\ . \tag{4.1}$$

However, we argued that the alternate partition

$$[\lambda,\ 2\lambda],\ [2\lambda,\ 3\lambda],\ [3\lambda,\ 4\lambda],\ \dots \tag{4.2}$$

would be more consistent with the exclusion of all states with exact quotas less than 1. Partition (4.2) leads to the following, very different result than (4.1) on the pairwise bias of MF.

Theorem 4.1 For a divisor λ, if states 1 and 2 have independent populations p_1, p_2 uniformly distributed in intervals $[b_1\lambda,\ (b_1{+}1)\lambda]$ and $[b_2\lambda,\ (b_2{+}1)\lambda]$, respectively, for positive integers $b_2 > b_1$, and the states have allocations a_1 and a_2, respectively, then for an MF apportionment the probability is greater than .5 that $a_2/p_2 > a_1/p_1$.

Thus, in the sense of Theorem 4.1, MF is pairwise biased in favor of large states.

The defendants' second theoretical result on bias is:

Theorem 4.2 With the assumptions and notation of Theorem 4.1, $E(d_2) = E(d_1)$ for EP and $E(d_2) < E(d_1)$ for MF.

Thus, in the sense of Theorem 4.2, EP is unbiased and MF is biased in favor of large states.

We also developed an alternate approach to measuring percentage bias consistent with Theorem 4.2, namely

$$(\sum_S d_i / \sum_L d_i) - 1, \qquad (4.3)$$

where S and L are as in Section 2, but with all states with exact quotas below 1 excluded. Positive values for (4.3) indicate that large states are favored and negative values, that small states are favored. Averaged over all 21 censuses through 1990, (4.3) is -1.0% for EP and 2.7% for MF. For 1990 alone, (4.3) is 3.9% for EP and 4.5% for MF. Among all 21 censuses, EP favored small states by this measure 13 times, while MF favored large states 17 times.

The defendants responded to the plaintiffs' argument that the "rounding thresholds" provided intuitive evidence to support their bias claims by simply pointing out that "rounding thresholds" are not always indicative of how the exact quota of a state would be rounded. It was noted that for 1990, New York, New Jersey, Massachusetts and Oklahoma have exact quotas of 31.521, 13.536, 10.532 and 5.516, respectively. The fractional portion of the exact quota for each of these states is above the "rounding thresholds" for both MF and EP. Yet MF and EP round down the exact quota for each of these states, with the exception of Massachusetts for MF and Oklahoma for EP.

The defendants' response to the claims concerning the computer simulations of Balinski and Young (1984) was to note that the results of the original simulations done by these authors, which were based on random variations from the modified exact quotas computed from Census Bureau projections of the 1990 data, showed that for MF the average allocation exceeded the modified exact quota for each of the largest 22 states, while the opposite was true for each of the 6 smallest states. These results did not support the claim that MF is unbiased. They then rounded the modified exact quota of each state to the nearest integer and excluded the 6 smallest states, which were those that originally had modified exact quotas less than 1.5. New simulations based on random variations from the rounded values for the remaining 44 states then yielded results more in accordance with their bias claims. It was also noted that Balinski and Young's exclusion of states with modified exact quotas less than 1.5 was not consistent with their usual exclusion criterion of .5.

The plaintiffs rebutted the defendants' claim that the exact quota cutoff for excluding states from bias computations should be 1, by citing a hypothetical example of a state with exact quota of .9 in one census that grew in the next census to have an exact quota of 1.1 and received one seat each time. Such a state would be fairly treated on average for the two censuses they argued, but if it was excluded from the bias calculations for the first census, then one would conclude incorrectly that it had been unfairly treated because in the second census, its allocation was less than its exact quota.

The defendants responded to this rebuttal by noting that the plaintiffs' position, unlike the defendants', was based on averaging of exact quotas for a state over more than one census and assumed growth of a state to the point where its exact quota was over 1. We noted that the Constitution clearly requires that an apportionment be based solely on the current census numbers, not on averaging over past censuses or hypothesizing about future censuses. We also noted that two of the three states with exact quotas below 1 for the 1990 census, Alaska and Wyoming, have never had exact quotas above 1.

The defendants responded to the plaintiffs' "near the quota" claims in several ways. We first noted that while it was true that MF was the only divisor method for which a transfer of a seat between states can never bring both states' allocations closer to their exact quotas, as measured by either $|a_i - q_i|$ or $|a_i - q_i|/q_i$ for state *i*, it is actually EP that is the only divisor method with this property if relative difference, that is $|a_i - q_i| / \min\{a_i, q_i\}$, is used as the measure of discrepancy.

We also noted that the Balinski and Young's concept of "near the quota" can be generalized to "near the ideal." An apportionment method is said to be "near the ideal" if a transfer of a seat between two states can never bring both states' allocations closer to the ideal. We considered the three ideals q_i, d and s, and proved the following results.

Theorem 4.3 For the ideals q_i, d and s:

a. MF is the only divisor method that is "near the ideal" for q_i and s as measured by $|a_i - q_i|$ and $|s_i - s|$, or, equivalently, by $|a_i - q_i|/q_i$ and $|s_i - s|/s$.

b. HM is the only divisor method that is "near the ideal" for d as measured by $|d_i - d|$, or, equivalently, by $|d_i - d|/d$.

c. EP is the divisor method that is "near the ideal" for all three ideals, q_i, d and s as measured by relative difference, that is $|a_i - q_i| / \min\{a_i, q_i\}$, $|d_i - d| / \min\{d_i, d\}$ and $|s_i - s| / \min\{s_i, s\}$.

It was noted that the plaintiffs' examples from the 1920 and 1970 censuses of "near the quota" violations would not hold if relative difference is used as the measure of discrepancy. In addition, the 1870 census was used to illustrate that MF was subject to "near the ideal" violations. For that census, MF would have allocated Illinois 20 seats and Florida 1. A transfer of a seat from Illinois to Florida, as would have occurred under EP, would have brought both states closer to the ideals q_i, d and s as measured by relative difference and also brought both states closer to d as measured by absolute difference.

The defendants also remarked that the term "near the quota" was somewhat of a misnomer since these words can be misinterpreted to imply more than the actual property. It was noted that it is actually GR that minimizes the overall measure of discrepancy of an apportionment from exact quotas, (2.2), and that EP can sometimes produce a smaller value for (2.2) than MF.

The plaintiffs responded to the defendants' arguments concerning "near the quota" by claiming that relative difference is not appropriate when comparing deviations from an ideal, with examples to illustrate this point.

A key example concerned a city with an average annual rainfall of 20 inches, which had a rainfall of 15 inches one year. The plaintiffs noted that one might say that the rainfall was 5 inches below normal (analogous to the measure $|a_i - q_i|$) or 25% below normal (analogous to the measure $|a_i - q_i|/q_i$). However, they noted that relative difference between 20 and 15 is 33 1/3%, which says that normal rainfall was 33 1/3% above the observed value. They stated that if one is interested in deviations from the mean, then the 33 1/3% relative difference is not relevant.

The plaintiffs also observed, continuing this example, that if the following year the rainfall was 25 inches, then the rainfall that year was 5 inches or 25% above normal, and that the differences above and below the mean balance out. However, the relative difference between 25 and 20 is 25% and hence the relative differences do not balance out.

This example and similar examples accompanied the last round of briefs and consequently, the defendants did not have an opportunity to fully respond to them. However, given the opportunity, the defendants could have observed, concerning the rainfall example, that in addition to talking about absolute amounts or percentages above or below normal, it is common to say that if the annual rainfall was 40 inches, it was twice the normal amount and if it was 10 inches, it was one-half the normal amount. Such phraseology corresponds to thinking in terms of ratios, which is the whole point of relative differences. That is, to measure discrepancy by dividing the smaller number into the larger number, as relative difference does, is as valid as measuring discrepancy by subtracting the smaller number from the larger, as absolute difference does.

The defendants also could have noted, that the fact that the absolute differences over two years balance out while the relative differences do not, just establishes that absolute differences are different from relative differences, not that they are better. To illustrate, two numbers above and below 20 have the same absolute difference from 20 if and only if the arithmetic mean of the numbers is 20. Similarly, the relative distance between 20 and each of two numbers above and below 20 is the same if and only if the geometric mean of the two numbers is 20. (For example, the relative difference between 40 and 20 is 100% as is the relative difference between 10 and 20, and the geometric mean of 40 and 10 is 20.) Thus, the plaintiffs' claim that relative difference is not appropriate when comparing deviations from an ideal, is essentially equivalent to the claim that the geometric mean is an inappropriate statistic for this purpose.

In addition, the defendants could have noted that Balinski and Young (1982) had observed, without criticism, that EP minimizes the relative difference between each state's average district size and the divisor λ.

The criticism of relative difference, which was in Young's last affidavit, only applied to deviations from an ideal, not pairwise comparison tests. The distinction was not made in the plaintiffs' accompanying brief, however. They used their expert's claims concerning deviations from an ideal to argue that absolute difference between average shares of a representative is the best pairwise test. The defendants pointed out in oral arguments that the plaintiffs had failed to note this distinction.

The defendants responded to the plaintiffs' claim that absolute difference in average shares of a representative is the best test of the "one person, one vote" principle in several ways. We noted that for 1990, as guaranteed by the theory, the relative difference between Oklahoma's and Massachusetts' average district sizes and average shares of a representative, and the absolute difference between the two states' average district sizes are smaller under EP than MF. (It is 14.6% under EP and 15.2% under MF for the two relative difference tests, and 76,638 under EP and 83,425 under MF for the absolute difference in average district sizes.)

The defendants noted the symmetry in the fact that among the four pairwise tests for which either EP, MF or HM are optimal, the plaintiffs in this case consider the one test for which MF is superior to EP to be the only appropriate test, just as the plaintiffs in the Montana case consider absolute difference in average district sizes to be the only appropriate test since it the only one of these four tests for which HM is superior to EP. The defendants expressed concurrence with Balinski and Young's (1975, p. 709) rhetorical question: "Why choose.......one divisor criterion [rather] then another?"

The three-judge panel in this case, in a unanimous decision, written by Judge Woodlock (1992) of Massachusetts, upheld the constitutionality of EP. Although the judges agreed with the plaintiffs that the "one person, one vote" principle applies to interstate apportionment, they rejected each of the three major substantive issues raised by the plaintiffs.

On the bias issue, the court observed that with states with exact quotas below 1 excluded "the historical bias showing made by plaintiffs all but evaporates," and that for the 1990 census, EP yields an apportionment with a percentage bias closer to 0 than MF would. The court, noting the exclusion cutoff of 1.5 in Balinski and Young (1984), stated that any exclusion, at least up to 1.5, is a reasonable means of accounting for the special constitutional treatment of very small states.

Judge Woodlock observed with regards to the plaintiffs' "near the quota" claims, that the showing was "at best mixed." He noted that it is GR that satisfies (2.2), and while MF satisfies the "near the quota" property with respect to the measures $|a_i - q_i|$ and $|a_i - q_i|/q_i$, it is EP that possesses this property with respect to relative difference.

The opinion explicitly addressed not only Massachusetts' claim that the pairwise test that best meets the "one person, one vote" principle in interstate apportionment is absolute difference in average shares of a representative, but also the claim in the Montana case that absolute difference in average district sizes is the only constitutional test. Judge Woodlock stated simply that "we can find nothing in the Constitution mandating a particular mathematical formula be employed to the exclusion of others." He expressed agreement with Balinski and Young's (1975) view that there is no reason to choose one divisor criterion over another.

The decision also noted that courts in the intrastate context have consistently measured equity by relative departures from the ideal district. (It is not clear whether that meant $|d_i - d|/d$ or $|d_i - d|/\min\{d_i, d\}$. In the intrastate context, the ratio of these two measures would generally be near 1 anyway, since d_i and d should always be close.) The judges found relative measurement to be a mathematically acceptable means of making equity comparisons and that nothing in case law or the Constitution prohibited its use.

The court, summarizing their views, stated: "The Constitution does not prescribe a particular formula, a specific methodology or a set standard to embody the 'one person, one vote' principle in this complex setting." The judges concluded that EP does satisfy this principle and hence the courts have no authority to interpose a different method than the one adopted by Congress. It is clear from the opinion that their ruling would have been the same if they had the Montana case before them.

Massachusetts did not appeal this decision to the Supreme Court. The Supreme Court's decision in the Montana case had made an appeal on this issue futile. However, as detailed in the next section, Massachusetts did present their views on apportionment methods to the Supreme Court through a friend-of-the-court brief in the Montana case.

Massachusetts also argued a second issue before the district court. They contended, using several arguments, that U.S. government employees working overseas, military and civilian, and their dependents, should not have been included in the apportionment counts. Massachusetts won on this issue before the district court on one of these points, namely that the decision to allocate overseas military personnel to the state designated as their "home of record" was arbitrary and capricious under the standards of the Administrative Procedures Act (APA). This decision, if upheld in its entirety, would have increased Massachusetts' allocation to 11 seats and reduced Washington's to 8 seats. However, the Government appealed this decision to the Supreme Court, which unanimously reversed the district court's decision to exclude overseas federal employees from the apportionment counts (*Franklin v. Massachusetts* 1992).

In overturning the district court ruling on the overseas employees issue, five of the justices, in an opinion by Justice O'Conner, held that the APA was not applicable, since the apportionment law required an action by the President, namely the transmittal of the apportionment to Congress. These justices found that the APA did not apply to the President and hence never reached the merits of this issue. The other four justices, in an opinion by Justice Stevens, did find the APA to be applicable, reasoning that it is the Secretary of the Commerce, who is covered by the APA, that is authorized to conduct the decennial census, and that the President's role is purely ministerial. However, these justices did not find the "home of record" method of allocating the overseas military personnel to be arbitrary and capricious.

As in the Montana case, some issues arose in the Massachusetts case that were not answered in the court documents. For example, although the defendants noted that EP can sometimes produce a lower value for (2.2) than MF, no illustrative example was provided. We were well aware of a very important example. For the 1940 census, the one which resulted in the adoption of EP, Arkansas' exact quota was 6.473 and Michigan's was 17.453. The EP allocation of 7 seats and 17 seats, respectively, to these two states yielded a lower value for (2.2) than the MF allocation of 6 seats and 18 seats, since the two apportionments were otherwise identical. However, this example was not mentioned by the defendants, because (2.2) for all $\rho>1$ would have been lower for MF than for EP for each of the other 11 censuses for which these two methods produced different apportionments.

In the plaintiffs' briefs, the assertion that MF is optimal with respect to absolute difference in shares of a representative, was generally not qualified to be limited to the pairwise comparison test. Some other possible interpretations of the plaintiffs' statements include minimization of the following: range of s_i, (2.4) with $\rho=1$,

$$\sum_{i=1}^{N} |s_i - s| \tag{4.4}$$

and

$$\max_{i} |s_i - s|. \tag{4.5}$$

MF is not always optimal for any of these measures. EP would have produced a smaller range for s_i than MF for the first census in 1790. (Delaware had the maximum value for s_i under either method. However, the minimum value for s_i was .00002772 for Pennsylvania under EP and .00002338 for Vermont under MF.) As noted in Section 2, GR minimizes (2.4) with ρ=1. The EP and GR apportionments for 1940 coincided and hence both yielded a lower value for this measure than MF. The optimality of MF for (4.4) is analogous to the optimality of HM for (3.2), as stated in the following theorem, which is applicable to all 21 censuses.

Theorem 4.4. If MF produces an apportionment for which there are no quota violations, then MF minimizes (4.4) among all apportionments which do not violate quota.

The proof of this theorem is essentially identical to the proof of Theorem 3.1.

However, without the quota restrictions, MF does not always minimize (4.4). The value of (4.4) for the example in Table 8 is .00111 for MF and .00103 for SD. The MF apportionment is identical to the HM apportionment for this example.

Although MF minimizes (4.5) among all six apportionment methods for all 21 censuses, for the example in Table 9 with N=3, n=7, the value of (4.5) is .00026 for EP, the optimal value, and .00037 for MF.

Table 9. Example for Which EP Minimizes (4.5)

		MF		EP	
State	p_i	a_i	s_i	a_i	s_i
1	2710	3	.00111	3	.00111
2	2690	3	.00112	2	.00074
3	1600	1	.00063	2	.00125

5. THE SUPREME COURT CASE

The (U.S.) Government appealed the decision of the three-judge district court in the Montana case to the Supreme Court, which granted an expedited review.

Generally, new factual information is not introduced on appeal, and for the most part, both sides did adhere to this rule. The Government did note (Starr et al. 1992) that for 1990, HM rounded up Montana's exact quota of 1.404 while rounding down Washington's exact quota of 8.538, even though Washington had the higher fractional remainder. This issue had not been brought up previously, because in the Massachusetts case for the two states for which EP and MF disagree, Oklahoma, with an exact quota of 5.516, and Massachusetts, with an exact quota of 10.532, it is MF that rounds up the state with the higher fractional remainder.

The Government did reiterate most of the issues raised with the district court. In addition, the Government vigorously argued that EP unquestionably apportions representatives among the states "according to their respective Numbers," which is all that the Constitution requires.

Montana, while raising no real new points, did clarify their position on the issue of variance among district sizes (Racicot et al. 1992). They stated that by "variance" in redistricting cases, the courts have not meant mathematical variance at all, in the sense of either (2.3) with ρ=2, or (3.1), but instead have meant

$$\max_i \; |d_i - d|. \tag{5.1}$$

This had indeed been mentioned as one criterion in Montana's district court briefs, but this measure has not been clearly referred to as a measure of variance previously. Furthermore, it does not appear from the affidavits of Montana's experts that either of them understood that (5.1) is the proper measure of variance in this context, since they had only associated the word "variance" with (3.1).

HM does indeed minimize (5.1) among the six apportionment methods considered in this paper, not only for 1990, but for all 21 censuses. However, for the example in Table 6, the value of (5.1) is 340 for HM and 320 for EP.

Montana responded to the Government's observation on the fractional remainders of Montana and Washington with their central argument that quota, like share of a representative and all factors other than absolute difference in district sizes, is an inappropriate criterion for measuring adherence to the "one person, one vote" principle.

Massachusetts participated in the Supreme Court case through a friend-of-the-court brief (Harshbarger et al. 1992), after an unsuccessful request that the Supreme Court delay hearing the Montana case until a district court decision had been issued in the Massachusetts case, and then hear appeals of these two cases in tandem.

Massachusetts' friend-of-the-court brief was filed shortly before the district court ruling in their own case. A new affidavit from their expert, Peyton Young (1992), accompanied it. They raised the same three issues as had been raised in the district court. Their argument on the bias and the "near the quota" issues were similar to those in the district court. However, perhaps in recognition of the focus of Montana's arguments, Massachusetts presented new arguments to support their contention that absolute deviations of shares of a representative is the only proper measure of equity in the interstate context.

Massachusetts presented an interesting verbal argument for the superiority of share of a representative over district size as a test. They observed that: "To bring all district sizes as near equity as possible is to treat all representatives as equally as possible. The relevant principle is to treat all citizens as equally as possible."

Massachusetts also provided an example that they stated showed a problem with average district size as a measure of equity. They considered a state with a population of 750,000 in each of two censuses, with *d*=500,000 in both censuses. The state's allocations were 1 seat in census 1, and 2 seats in census 2. They observed that averaged over the two censuses, the state's allocations and its exact quotas are both 1.5. Also, the average share of a representative for the state averaged over the two censuses and *s*=1/500,000 are both the same. However, the state's average district size averaged over the two censuses is greater than *d*=500,000.

This example implied that exact quota and average share of a representative are consistent measures and that average district size is inconsistent with the other two. The implication that average share of a representative is consistent with exact quota in this respect is false. To see this, simply consider the same example except suppose that the state's population is 600,000 in census 1 and 900,000 in census 2 (or any two numbers $p_1,\ p_2$, respectively, with arithmetic mean 750,000, for which $500{,}000 < p_1 < 750{,}000)$. Then the state's allocations and exact quotas averaged over the two censuses remain the same, but the average share of a representative averaged over these two censuses is less than 1/500,000.

Massachusetts approached the absolute difference versus relative difference issue somewhat differently than previously. Citing case law, they claimed equity should be measured by deviations from the ideal, not pairwise comparisons. They repeated their claim that relative difference is inappropriate in measuring deviations from an ideal. They observed that MF minimizes (2.4) with ρ=2 and asserted that this measure is superior to (2.3) with ρ=2.

Massachusetts presented one new idea relating to the "near the ideal" principle. EP for 1990 does not violate this principle in any of the senses of Theorem 4.3. However, for states *i*,*j*, if $s_{ij} = (a_i + a_j)/(p_i + p_j)$, that is, the average share of a representative for the two states combined, then the following result holds:

Theorem 5.1. For a unique MF apportionment, every transfer of seats between states *i*,*j* increases both $|s_i - s_{ij}|$ and $|s_j - s_{ij}|$ (and, equivalently, $|s_i - s_{ij}|/s_{ij}$ and $|s_j - s_{ij}|/s_{ij}$.)

This is equivalent to saying that any apportionment not agreeing with the MF apportionment for states *i*,*j* will violate the "near the ideal" principle for the ideal s_{ij} with respect to the measures given in the theorem.

Massachusetts did not actually state this theorem, but instead illustrated this result by demonstrating that for the 1990 census, MF brings both Massachusetts and Oklahoma closer to their combined average share of a representative than EP does with respect to the measures of Theorem 5.1.

The difficulty with this result is that it appears inconsistent to argue that deviations should be measured from an ideal, not by pairwise comparisons, and then use as an ideal s_{ij}, which is a function of a pair of states, instead of the traditional ideal *s*.

Interestingly, the result that would be analogous to Theorem 5.1 for district size (that is with s_i, s_j and s_{ij} replaced by d_i, d_j and $d_{ij} = 1/s_{ij}$) does not hold for every HM apportionment. For example, for the 1990 HM apportionment, a transfer of a seat from Montana to Washington would bring Washington's district size closer to the average district size for the two states combined as measured by $|d_i - d_{ij}|$ or $|d_i - d_{ij}|/d_{ij}$.

The district court decision in the Massachusetts case was released after the filing of Massachusetts' brief, but before the Government's reply brief was due. The Government chose not to respond to the Massachusetts claims in detail, but simply to note that these claims had been answered in the district court and had been rejected by that court. The Government in response to the claims of both Montana and Massachusetts, did review the expert support that EP has historically had, in particular the reasoning in the 1948 National Academy of Science report. The Government also cited the example of Arkansas and Michigan for the 1940 census, for which EP but not MF awarded the seat in dispute to the state with the higher fractional remainder.

On March 31, 1992, only 27 days after oral arguments, the Supreme Court unanimously upheld the constitutionality of EP, in an opinion written by Justice John Paul Stevens.

Justice Stevens first considered and rejected, as had both district courts, the Government's argument that Congress' selection of an apportionment method is a "political question" not subject to judicial review.

Justice Stevens then discussed the constitutional issues in dispute. He observed that while the same principle of equity that the Supreme Court requires in intrastate districting might apply to interstate apportionment, he did not find that the facts constituted a violation of the Wesberry standard. He noted that there is no incompatibility within a state in minimizing both absolute and relative differences, and that all districts within a state can be brought closer to the ideal simultaneously. However, for 1990, HM, while bringing Montana's average district size closer to the ideal district size as measured by absolute difference, brings Washington's average district further away from the ideal district size with respect to absolute difference, and moves both states further from this ideal with respect to relative difference.

Justice Stevens also noted that it can be argued, as in Judge O'Scannlain's dissent in the Montana district court case, that a measure of deviation from the ideal district size should take into account the number of districts in each state.

Justice Stevens then made the critical observation that "neither mathematical nor constitutional interpretation provides a conclusive answer" to the question of the best measure of inequality among the four measures obtained by pairing either absolute or relative difference with either district size or share of a representative. As had Judge Woodlock in the Massachusetts district court case, he concluded: "The polestar of equal representation does not provide sufficient guidance to allow us to discern a single constitutionally permissible course." These comments amounted to a complete rejection of Montana’s entire argument and a rejection of one of Massachusetts’ key issues.

The opinion further observed that the goal of mathematical equality, while appropriate in the intrastate context, is illusory for interstate apportionment, since each state must have at least one representative and districts cannot cross state lines. In addition, since the Constitution expressly authorizes Congress to enact legislation to carry out its delegated responsibilities, its choice of a method that apportions representatives "according to their respective Numbers" commands far more deference than a state redistricting decision that can be required to meet a rigid mathematical standard.

The "near the quota" issue raised by Massachusetts was not mentioned in the opinion. The bias issue was discussed in a footnote, which first described Balinski and Young’s (1982) views and then simply noted, citing the opinion of the Massachusetts district court, that this contention has been disputed. Later in the opinion, Justice Stevens returned to this issue, stating that a fair apportionment required some compromise between the interests of the smaller and larger states, and indicating that Congress had been delegated the authority in the Constitution to reach this compromise.

In addition to challenging the constitutionality of EP, Montana had challenged the constitutionality of the automatic apportionment law, claiming it deprived their congressional delegation of their right to vote on an apportionment. While not formally ruling on this claim, the district court majority stated that it had merit. Justice Stevens disagreed, stating that this claim had no merit. He found that an automatic use of an otherwise constitutional apportionment method is a sensible procedure that removed apportionment from political controversy. He found nothing in the Constitution that prevented the adoption of an automatic procedure.

Justice Stevens concluded his answer to this 200 year old constitutional question, stating:

> The decision to adopt the method of equal proportions was made by Congress after decades of experience, experimentation, and debate about the substance of the constitutional requirement. Independent scholars supported both the basic decision to adopt a regular procedure to be followed after each census, and the particular decision to use the method of equal proportions. For a half century the results of that method have been accepted by the States and the Nation. That history supports our conclusion that Congress had ample power to enact the statutory procedure in 1941 and to apply the method of equal proportions after the 1990 census.

APPENDIX

Proof of Theorem 3.1. Let d_i, i=1,...,N, denote the HM apportionment and d_i', i=1,...,N, any other apportionment satisfying quota. Let $D=\{i\colon\ d_i\neq d_i'\}$, and m denote the number of elements in D. If $i,j\in D$, $i\neq j$, then $\min\{d_i,d_j\}\leq d\leq \max\{d_i,d_j\}$ since both apportionments satisfy quota. Consequently, $|d_i-d|+|d_j-d|=|d_i-d_j|$. Therefore, for $i,j\in D$, $i\neq j$,

$$|d_i-d|+|d_j-d|\leq|d_i'-d_j'|\leq|d_i'-d|+|d_j'-d|,$$

from which it follows that

$$\sum_{i\in D}|d_i-d|=\frac{1}{m(m-1)}\sum_{\substack{i,j\in D\\ i\neq j}}(|d_i-d|+|d_j-d|)\leq\sum_{i\in D}|d_i'-d|.$$

Proof of Theorem 4.1. The ordered pair $(p_1,\ p_2)$ is uniformly distributed in a square with vertices $(b_1\lambda,\ b_2\lambda)$, $((b_1+1)\lambda,\ b_2\lambda)$, $(b_1\lambda,\ (b_2+1)\lambda)$, $((b_1+1)\lambda,\ (b_2+1)\lambda)$ and area λ^2. The region within the square in which state 2 is favored over state 1, that is, for which $a_2/p_2>a_1/p_1$, is the union of the following four regions: (1) A triangle with vertices

$$(b_1\lambda,b_2\lambda),\ (b_1(b_2+.5)\lambda/b_2,b_2\lambda),\ (b_1(b_2+.5)\lambda/b_2,\ (b_2+.5)\lambda),$$

and area .125 $b_1\lambda^2/b_2$, for which $a_1=b_1$, $a_2=b_2$. (2) A rectangle with vertices

$$(b_1(b_2+.5)\lambda/b_2,\ b_2\lambda),\ ((b_1+.5)\lambda,\ b_2\lambda),\ (b_1(b_2+.5)\lambda/b_2,\ (b_2+.5)\lambda),\ ((b_1+.5)\lambda,\ (b_2+.5)\lambda),$$

and area .25 $(1-b_1/b_2)\lambda^2$, for which $a_1=b_1$, $a_2=b_2$. (3) A square with vertices

$$(b_1\lambda,\ (b_2+.5)\lambda),\ ((b_1+.5)\lambda,\ (b_2+.5)\lambda),\ (b_1\lambda,\ (b_2+1)\lambda),\ ((b_1+.5)\lambda,\ (b_2+1)\lambda),$$

and area $.25\lambda^2$, for which $a_1 = b_1$ and $a_2 = b_2 + 1$. (4) A triangle with vertices

$$((b_1 + 1)(b_2 + .5)\lambda / (b_2 + 1),\ (b_2 + .5)\lambda),\ ((b_1 + 1)\lambda,\ (b_2 + .5)\lambda),\ ((b_1 + 1)\lambda,\ (b_2 + 1)\lambda)$$

and area $.125\ (b_1 + 1)\lambda^2/(b_2 + 1)$, for which $a_1 = b_1 + 1$, $a_2 = b_2 + 1$. The total area of these four regions is $(.5 + .125(b_2 - b_1)/[b_2(b_2 + 1)])\lambda^2$ and the probability that state 2 is favored is, therefore, $.5 + 125(b_2 - b_1)/[b_2(b_2 + 1)] > .5.$

Proof of Theorem 4.2. We drop all subscripts to obtain results that apply to both states. Since for EP, $d = p/b$ if $p < \sqrt{b(b+1)}\,\lambda$ and $d = p/(b+1)$ if $p > \sqrt{b(b+1)}\,\lambda$, it follows that

$$E(d) = \frac{1}{\lambda}\left(\int_{b\lambda}^{\sqrt{b(b+1)}\,\lambda} \frac{p}{b}\,dp + \int_{\sqrt{b(b+1)}\,\lambda}^{(b+1)\lambda} \frac{p}{b+1}\,dp\right) = \lambda \qquad \text{(A.1)}$$

for all b, where the $1/\lambda$ term in (A.1) arises from the fact that $[b\lambda,\ (b+1)\lambda]$ is an interval of length λ.

Similarly, for MF,

$$E(d) = \frac{1}{\lambda}\left(\int_{b\lambda}^{(b+.5)\lambda} \frac{p}{b}\,dp + \int_{(b+.5)\lambda}^{(b+1)\lambda} \frac{p}{b+1}\,dp\right) = \left(1 + \frac{1}{8b(b+1)}\right)\lambda,$$

which is a decreasing function of b.

Proof of Theorem 4.3.

(a) The results for the ideal q_i were established in Balinski and Young (1982, p. 132-133). The results for s follow from these results and the fact that since $|s_i - s| = |a_i - q_i|/p_i$, bringing s_i closer to s and a_i closer to q_i are equivalent.

(b) Clearly it suffices to establish this result for the measure $|d_i - d|$.

Let b be a positive integer, $\delta(b) = 2b(b+1)/(2b+1)$, the rounding function for HM, and p the population of a state. To prove that an HM apportionment is always "near the ideal" d for the measure $|d_i - d|$, first establish that

$$\text{if } p/\delta(b) \geq d \text{ then } |p/b - d| \geq |p/(b+1) - d|. \tag{A.2}$$

To do this, observe that

$$\frac{p}{b} + \frac{p}{b+1} = \frac{2p}{\delta(b)} \geq 2d,$$

and hence

$$\frac{p}{b} - d \geq d - \frac{p}{b+1}. \tag{A.3}$$

In addition, clearly $p/b - d > p/(b+1) - d$, and (A.2) then follows.

Similarly, it can be established that if both inequality signs in (A.2) are replaced by either "≤" ">" or "<" the statement remains true. Furthermore, (A.2) remains true if b is replaced by 0, with the convention that $p/0 = \infty$, a convention that is used throughout this proof.

Then consider an HM apportionment for which states i and j have populations p_i, p_j and allocations a_i, a_j. Then $p_i/\delta(a_i - 1) \geq p_j/\delta(a_j)$ by the min-max inequality (Balinski and Young 1982, p. 100). Consequently, if a seat is transferred from state *i* to state *j* then either $p_i/\delta(a_i - 1) \geq d$, in which case the transfer does not bring the allocation for state *i* closer to the ideal by (A.2) with $b = a_i - 1$, or $p_j/\delta(a_j) \leq d$, in which case the allocation for state *j* is not brought closer to the ideal by A.2, with the inequality signs reversed and $b = a_j$.

To prove that any other divisor method, with rounding function $\delta^*(b)$, is not always "near the ideal" *d* for the measure $|d_i - d|$, first note that for some positive integers $b_1 \neq b_2$,

$$\frac{\delta(b_2)}{\delta^*(b_2)} > \frac{\delta(b_1)}{\delta^*(b_1)}.$$

(For otherwise, $\delta^*(b) = \delta(b)$ for all $b > 0$. Then $\delta^*(0) > 0$, in which case δ^* can produce apportionments in which some states have 0 representatives and other states have more than 1 representative, which would be a "near the ideal" violation.)

Then consider a three state problem, for which

$$p_1 = \delta(b_1) + \varepsilon, \quad p_2 = \delta(b_2) - \varepsilon, \quad p_3 = b_1 + b_2 + 2 - \delta(b_1) - \delta(b_2),$$

where $\varepsilon > 0$ satisfies the relations

$$\frac{p_2}{\delta^*(b_2)} > \frac{p_1}{\delta^*(b_1)}, \tag{A.4}$$

$$\frac{p_2}{\delta(b_2 - 1)} > \frac{p_1}{\delta(b_1)}, \tag{A.5}$$

$$\frac{p_1}{\delta(b_1+1)} < 1. \tag{A.6}$$

Note that since $\delta(1)-1=1/3$ and $1/3<\delta(b)-b<1/2$ for $b>1$, it follows that

$$1<p_3<4/3. \tag{A.7}$$

Furthermore, by (A.5),

$$\frac{p_2}{\delta(b_2-1)} > 1. \tag{A.8}$$

Let a_i, i=1,2,3, denote the apportionment corresponding to the rounding function δ for this set of populations and a house size of b_1+b_2+2. Since by the definitions of p_1 and p_2, and (A.7),

$$\frac{p_1}{\delta(b_1)} > 1 > \max\left\{\frac{p_2}{\delta(b_2)},\ \frac{p_3}{\delta(1)}\right\},$$

state 1 has a higher priority for b_1+1 seats then either state 2 has for b_2+1 seats or state 3 has for 2 seats. Consequently $a_1 \geq b_1+1$. Similarly, since by (A.5), (A.7) and (A.8),

$$\frac{p_2}{\delta(b_2-1)} > \max\left\{\frac{p_1}{\delta(b_1)},\ \frac{p_3}{\delta(1)}\right\},$$

it follows that $a_2 \geq b_2$. Finally, since $a_3 \geq 1$, the HM apportionment must be $a_1 = b_1+1,\ a_2 = b_2,\ a_3 = 1$. Furthermore, since $p_i/\delta(a_i-1)\ >\ 1=d\ >\ p_i/\delta(a_i),\ i=1,2,3$, it follows from the variations of (A.2) that $|d_i - d|$ is uniquely minimized for each i by this apportionment.

Now for the $\delta^*(b)$ apportionment, by (A.4) state 2 has a higher priority for b_2+1 seats than state 1 has for b_1+1 seats. Consequently, the two apportionments are not identical, and hence the $\delta^*(b)$ apportionment must switch seats between at least two states from the HM apportionment, increasing $|d_i - d|$ for these states.

(c) Since $s_i = a_i/p_i,\ s = q_i/p_i,\ d_i = p_i/a_i,\ d = p_i/q_i$, all three measures are identical. Therefore, it suffices to prove these results for any one of the ideals, say d.

To prove that an EP apportionment is always "near the ideal" d for the measure $|d_i - d|/\min\{d_i, d\}$, simply establish that, with $\delta(b)\ =\ \sqrt{b(b+1)}$,

$$\text{if } p/\delta(b)\ \geq\ d \text{ then } \frac{|p/b - d|}{\min\{p/b, d\}}\ \geq\ \frac{|p/(b+1) - d|}{\min\{d/(b+1), d\}} \tag{A.9}$$

and then proceed as in (b). To prove (A.9), note that

$$\frac{|p/b - d|}{\min\{p/b, d\}}\ =\ \max\{p/(bd),\ bd/p\} - 1\ \geq\ p/(bd) - 1$$

$$\frac{|p/(b+1) - d|}{\min\{p/(b+1), d\}}\ =\ \max\{p/[(b+1)d],\ (b+1)d/p\} - 1.$$

Now, $p/(bd) \geq (b+1)d/p$ can be obtained by algebraic manipulation of $p/\delta(b) \geq d$, while clearly $p/(bd) > p/[(b+1)d]$. (A.9) can then be obtained by combining all the above relations.

To prove the converse, that is that no other divisor method is always "near the ideal" d for the measure $|d_i - d| / min\{d_i, d\}$, proceed as in ($b$).

Proof of Theorem 5.1. Let a_i, a_j be the MF allocation for states i and j, and b_i, b_j be an allocation obtained by transferring seats between these two states. Then $b_i + b_j = a_i + a_j$, and hence

$$\left|\frac{b_i}{p_i} - s_{ij}\right| = \left|\frac{b_i}{p_i} - \frac{b_i + b_j}{p_i + p_j}\right| = \frac{|b_i p_j - b_j p_i|}{p_i(p_i + p_j)}$$

$$= \frac{p_i p_j |b_i/p_i - b_j/p_j|}{p_i(p_i + p_j)} > \frac{p_i p_j |a_i/p_i - a_j/p_j|}{p_i(p_i + p_j)} = \left|\frac{a_i}{p_i} - s_{ij}\right|.$$

ACKNOWLEDGEMENT

The author thanks Bev Causey who did the programming that produced most of the new empirical results in Section 4.

REFERENCES

Balinski, M.L., and Young, H.P. (1975), "The Quota Method of Apportionment," *American Mathematical Monthly*, 82, 701-730.

_____________ (1977), "Apportionment Schemes and the Quota Method," *American Mathematical Monthly*, 84, 450-455.

_____________ (1982), *Fair Representation*, New Haven: Yale University Press.

_____________ (1984), "Evaluation of Apportionment Methods," report prepared under Contract No. CRS 84-15 for the Congressional Research Service of the Library of Congress, Washington, D.C.

Birkhoff, G. (1976), "House Monotone Apportionment Schemes," *Proceedings of the National Academy of Sciences*, U.S.A. 73, 684-686.

Bliss, G.A., Brown, E.W., Eisenhart, L.P., and Pearl, R. (1929), report to the President of the National Academy of Sciences, February 9, 1929. In *Congressional Record*, 70th Congress, 2nd Session, 70, 4966-4967, March 2, 1929.

Ernst, L.R. (1991a), declaration in *Montana v. United States Department of Commerce*, CV 91-22-H-CCL, Helena, Montana.

___________ (1991b), declarations in *Massachusetts v. Mosbacher*, CA 91-11234-WD, Massachusetts.

Gerson, S.M., Poppler, D.M., Linder, D.G., McLean, K.A., Schraibman, S.M., Korytkowski, S.L., and Murphy, Mark H. (1991), defendants' briefs in *Montana v. United States Department of Commerce*, CV 91-22-H-CCL, Helena, Montana.

Gerson, S.M., Budd, W.A., Yogman, J.S., Schraibman, S.M., and Murphy, M.H. (1991), defendants' briefs in *Massachusetts v. Mosbacher*, CA 91-11234-WD, Massachusetts.

Gilford, L. (1981), testimony at hearing before the Subcommittee on Census and Population, 97th Congress, 1st Session, on H.R. 1990, 97-18, 58-70.

Harshbarger, S., Golann, D., Berenson, S., Driscoll, J.P., Jr., Leibensperger, E.P., and Motenko, N.P. (1991), plaintiffs' briefs in *Massachusetts v. Mosbacher*, CA 91-11234-WD, Massachusetts.

_________ (1992), Massachusetts' amicus curiae brief in *United States Department of Commerce v. Montana*, U.S. Supreme Court, 91-860.

Hill, D.M. (1991), affidavit in *Montana v. United States Department of Commerce*, CV 91-22-H-CCL, Helena, Montana.

Huntington, E.V. (1921), "A New Method of Apportionment of Representatives," *Quarterly Publication of the American Statistical Association*, 17, 859-870.

_________ (1928), "The Apportionment of Representatives in Congress," *Transactions of the American Mathematical Society*, 30, 85-110.

Lovell, C.C. (Judge) (1991), majority opinion in *Montana v. United States Department of Commerce*, 775 F. Supp. 1358 (U.S.D.C. Mt.).

Morse, M., von Neumann, J., and Eisenhart, L.P. (1948), report to the President of the National Academy of Sciences, May 28, 1948.

O'Scannlain, D.F. (Judge) (1991), minority opinion in *Montana v. United States Department of Commerce*, 775 F. Supp. 1358 (U.S.D.C. Mt.).

Owens, F.W. (1921), "On the Apportionment of Representatives," *Quarterly Publication of the American Statistical Association*, 17, 958-968.

Racicot, M., Smith, C.R., and Baker, E.S. (1991), plaintiffs' briefs in *Montana v. United States Department of Commerce*, CV 91-22-H-CCL, Helena, Montana.

_________ (1992), appellees' briefs in *United States Department of Commerce v. Montana*, U.S. Supreme Court, 91-860.

Still, J.W. (1979), "A Class of New Methods for Congressional Apportionment," *SIAM Journal on Applied Mathematics* (1979), 37, No. 2, 401-418.

Starr, K.W., Gerson, S.M., Roberts, J.G., Jr., Kneedler, E.S., Singer, M.J., Stern, M.B., and Raab, M.S. (1992), appellants' briefs in *United States Department of Commerce v. Montana*, U.S. Supreme Court, 91-860.

Stevens, J.P. (Justice) (1992), opinion in *United States Department of Commerce v. Montana*, 112 S. Ct. 1415.

Tiahrt, K.J. (1991), affidavit in *Montana v. United States Department of Commerce*, CV 91-22-H-CCL, Helena, Montana.

Woodlock, D.P. (Judge) (1992), opinion in *Massachusetts v. Mosbacher*, 785 F. Supp. 230 (U.S.D.C. Ma.).

Young, H.P. (1991), affidavits in *Massachusetts v. Mosbacher*, CA 91-11234-WD, Massachusetts.

____________ (1992), affidavit in *United States Department of Commerce v. Montana*, U.S. Supreme Court, 91-860.

CASES CITED

Franklin v. Massachusetts (1992), 112 S. Ct. 2767.

Karcher v. Dagget (1983), 426 U.S. 725.

Massachusetts v. Mosbacher (1992), 785 F. Supp. 230 (U.S.D.C. Ma.).

Montana v. United States Department of Commerce (1991), 775 F. Supp. 1358 (U.S.D.C. Mt.).

United States Department of Commerce v. Montana (1992), 112 S. Ct. 1415.

Wesberry v. Sanders (1964), 376 U.S. 1.