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X-11 Symmetric Linear Filters and their Transfer Functions

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1. Introduction

The X–11 program (Shiskin, Young, and Musgrave 1967) is widely used for seasonal adjustment of economic time series. One can view additive X–11 as a linear filtering operation produced by successive application of simple linear filters — seasonal and nonseasonal moving averages and Henderson trend moving averages. In the additive decomposition this view is exact except for modifications made to deal with extreme values. This same linear filter view applies to the log–additive decomposition of X–11–ARIMA (Dagum 1983), except that it applies to the logarithms of the original time series. In the multiplicative decomposition the linear filter view differs from reality in that results obtained from applying a moving average at any stage are then divided into the series at hand, rather than subtracted from it, to produce results for the next stage (hence the name "ratio to moving average method"). Young (1968), however, assessed the differences between a linear approximation to and actual multiplicative X–11, and argued that the differences due to nonlinearities are generally unimportant. To this degree of approximation, therefore, the X–11 linear filters are also relevant to the multiplicative decomposition.

This note presents graphs of filter weights and transfer function squared gains corresponding to symmetric linear X–11 seasonal, seasonal adjustment, trend, and irregular filters, under the various choices of filter options allowed. We focus exclusively on the symmetric filters for two reasons: (1) the practice of revising initial seasonal adjustments to the final adjustments obtained from the symmetric filters suggests that these filters should be of prime interest; and (2) applying the symmetric filters to a series extended with as many minimum mean squared error forecasts and backcasts as needed (in the spirit of X–11–ARIMA (Dagum 1983)) minimizes mean squared seasonal adjustment revisions, as was shown by Geweke (1978) and Pierce (1980). If desired, one could obtain representations of asymmetric X–11 filters in the same way that we obtain the symmetric filters. However, a more convenient approach to obtaining asymmetric filters is that of Wolter and Monsour (1981), which uses expressions involving matrices whose rows contain the filter weights for all the symmetric and asymmetric moving averages. We do not pursue this here, as the possible number of asymmetric filters under the various options would be quite large.

2. Computation of X–11 Symmetric Linear Filters and Their Transfer Functions

Young (1968) and Wallis (1974) have offered linear filter approximations to X–11. We follow the approach of Wallis, whose approximation to X–11 is the more complete. (As noted by Wallis, Young omits two steps from X–11.) Let $Z_t = S_t + T_t + I_t$ be an additive, or log–additive, decomposition of an observed time series Z_t as seasonal + trend + irregular. Henceforth, let "MA" denote "moving average". Wallis (1974) lists the filtering steps used by X–11 in estimating this decomposition, a summary description of which is as follows:

1.) Detrend Z_t by subtracting a 2×12 MA (a 2–term MA of a 12–term MA).
2.) Take a first seasonal MA (default = 3×3) of the result as a preliminary estimate of S_t.
3.) Adjust the preliminary seasonal to sum more nearly to 0 over 12 months by subtracting a 2×12 MA.
4.) Subtract the result of 3. from Z_t to get a preliminary seasonally adjusted series.
5.) Subtract a Henderson trend MA of this from Z_t for a more refined detrending.
6.) Apply a second seasonal MA (default = 3×5) to the result of 5.
7.) Adjust the result of 6. as in 3. by subtracting a 2×12 MA — the result, $\hat{S}_t$, estimates S_t.
8.) The seasonally adjusted series (nonseasonal estimate) is then $\hat{N}_t = Z_t - \hat{S}_t$, the trend estimate, $\hat{T}_t$, is obtained by applying a Henderson trend MA to $\hat{N}_t$, and the irregular estimate is $\hat{I}_t = \hat{N}_t - \hat{T}_t$.

In the X–11 program the above sequence of steps can be applied 3 times, but the first two cycles are used only as part of adjusting the series for extreme values and for trading day variation. Thus, the above gives the basic filtering steps of (additive) X–11.

We can express each of the above steps in terms of polynomials in the backshift operator, B ($BZ_t = Z_{t-1}$), and then write down the resulting backshift operator polynomial representations of the X–11 seasonal, seasonal adjustment, trend, and irregular filters. Doing this produces the following expression for the X–11 seasonal filter, $\omega_S(B)$:

$$\omega_S(B) = [1 - \mu(B)]\lambda_2(B)[1 - H(B)\{1 - [1 - \mu(B)]\lambda_1(B)[1 - \mu(B)]\}] \tag{2.1}$$

where, letting $U(B) = 1+B + \ldots + B^{11}$ and $F = B^{-1}$, we write

$$\begin{aligned}
\mu(B) &= 2{\times}12 \text{ trend moving average} \\
&= (1/24)(F^6 + F^5)U(B) \\
\lambda_1(B) &= \text{first seasonal moving average, the default is the } 3{\times}3 \text{ seasonal MA:} \\
&\qquad (1/9)(F^{12} + 1 + B^{12})(F^{12} + 1 + B^{12}) \\
\lambda_2(B) &= \text{second seasonal moving average, the default is the } 3{\times}5 \text{ seasonal MA:} \\
&\qquad (1/15)(F^{12} + 1 + B^{12})(F^{24} + F^{12} + 1 + B^{12} + B^{24}) \\
H(B) &= \text{Henderson trend moving average.}
\end{aligned}$$

For the quarterly filter change 12 to 4 and 24 to 8 in the above expressions. The seasonal adjustment ($\omega_N(B)$), trend ($\omega_T(B)$), and irregular ($\omega_I(B)$) filters are obtained from the seasonal filter, $\omega_S(B)$, as follows:

$$\omega_N(B) = 1 - \omega_S(B) \tag{2.2}$$

$$\omega_T(B) = H(B)\omega_N(B) \tag{2.3}$$

$$\omega_I(B) = [1 - H(B)]\omega_N(B)\,. \tag{2.4}$$

The expressions (2.1) – (2.4) provide a convenient means of computing the weights and transfer functions for the various filters resulting from selection of default or optional choices of $\lambda_1(B)$, $\lambda_2(B)$, and $H(B)$. If $\omega(B) = \sum_k \omega_k B^k$ is an X–11 filter (seasonal, adjustment, trend, or irregular), the filter weights ω_k can then be computed using a mathematical programming language that will multiply polynomials; GAUSS (Edlefsen and Jones 1986) was used here. The transfer function of $\omega(B)$ is then $\omega(e^{2\pi i\lambda}) = \sum_k \omega_k e^{2\pi i\lambda k}$ for $\lambda \in [-.5, .5]$. For symmetric $\omega(B)$, i.e. $\omega_k = \omega_{-k}$, it turns out that $\omega(e^{2\pi i\lambda})$ is real valued with $\omega(e^{2\pi i\lambda}) = \omega(e^{-2\pi i\lambda})$ and squared gain $[\omega(e^{2\pi i\lambda})]^2$. Thus, for the symmetric filters, we only need plot the weights ω_k for $k \geq 0$, and we only need plot $[\omega(e^{2\pi i\lambda})]^2$ for $\lambda \in [0, .5]$.

The expressions (2.1) to (2.4) also facilitate study of the properties of the X–11 filters. This is pursued in Bell (1992).

The optional filters we present here are those available in the version of X–11 currently being distributed by the Census Bureau, the X–11.2 program (Monsell 1989). The default seasonal MA choices are as given above; optional 3–term, 3×3, 3×5, or 3×9 seasonal MAs can also be chosen. The choice of an optional seasonal MA implies that it is used for both $\lambda_1(B)$ and $\lambda_2(B)$. A "fixed seasonal filter" with equal weights can also be used, but as this is more akin to regression on seasonal dummies, and since the weights obtained under this filter depend on the length of the series, it shall also not be considered here.

For monthly series the X–11.2 user can optionally select a 9–term, 13–term, or 23–term Henderson trend MA, and for quarterly series a 5–term or 7–term Henderson trend MA. Weights for the n–term Henderson MA, $H_n(B) = \sum_j h_j^{(n)} B^j$, are $h_j^{(n)}$, $j = 0, \pm 1, \ldots, \pm(n-1)/2$ obtained from the following formula, with $m = (n+3)/2$, given by Macaulay (1931, p. 54) and Dagum (1985):

$$h_j^{(n)} = \frac{315[(m-1)^2-j^2][m^2-j^2][(m+1)^2-j^2][(3m^2-16)-11j^2]}{8m(m^2-1)(4m^2-1)(4m^2-9)(4m^2-25)} \quad . \tag{2.5}$$

The default option is for the X–11 program to automatically choose one of the available Henderson trend MAs, with the choice depending on the relative amounts of variation in preliminary estimates of the irregular and trend. We made a quick check of 30 Census Bureau monthly time series and found that X–11 picked the 9–term, 13–term, and 23–term filters 10, 18, and 2 times, respectively. It is possible for the automatic choice of H(B) used in (2.3) and (2.4) to differ from that used in (2.1), since the Henderson trend filter used in (2.3) and (2.4) is determined independently, though in the same fashion as for (2.1). For the 30 series mentioned earlier, a different choice of H(B) in (2.1) versus (2.3)–(2.4) was made only once. To avoid a large increase in the number of graphs required, while probably risking little loss of relevant information, we present graphs only for the case where H(B) is the same in (2.3) and (2.4) as it is in (2.1).

The options provided by X–11.2 differ from the other well–known incarnations of X–11: the original X–11 program described by Shiskin, Young, and Musgrave (1967); and the X–11–ARIMA program (Dagum 1983). X–11–ARIMA drops the optional 3–term seasonal MA, but also allows an optional 2×24 MA (monthly) or 2×8 MA (quarterly) in place of $\mu(B)$. The original X–11 program does not provide optional seasonal or Henderson trend MAs for quarterly series. (The 5–term Henderson is always used.) Original X–11 also differs from X–11.2 and X–11–ARIMA in one other arbitrary way, in that it stores the various MAs used to only 3 decimals, whereas X–11.2 and X–11–ARIMA use more accurate representations. This difference should not materially affect the graphs presented here.

3. Graphs of Filter Weights and Squared Gains for the X–11 Symmetric Linear Filters

The graphs of the X–11 symmetric linear filter weights and the squared gains of the filters are organized as follows. We first present the graphs of all the various monthly filter weights, followed by the graphs of all the squared gains of the monthly filters. This pattern is then repeated for all the quarterly filters. Within each of these sets, the graphs are grouped in the order of seasonal, adjustment, trend, and irregular filters. For each of these groups of graphs, there is one page of graphs for each choice of Henderson trend MA, in the order of 9–term, 13–term, and 23–term for the monthly graphs, and 5–term and 7–term for the quarterly graphs. Each page contains four graphs corresponding to alternative choices of seasonal MAs: default, 3×3, 3×9, and 3–term, in this order, clockwise from upper left. This ordering is given in abbreviated outline form on pages 7 and 8.

Upon examining graphs for the "optional" 3×5 seasonal MA we discovered that these were virtually identical to those for the default seasonal MA. Thus, the graphs presented for the default seasonal MA can also serve as graphs for the optional 3×5. Another way to look at this is that the 3×5 is not really a distinct option from the default seasonal MA, at least as far as the symmetric filter is concerned. (We have not compared asymmetric filters resulting from the default and optional 3×5 seasonal MAs.)

A few of the graphs of filter weights and gains that follow have been previously produced by other authors. Wallis (1974) plotted weights for the monthly and quarterly adjustment filters for the default seasonal MA and 13–term Henderson trend MA (5–term for the quarterly filter). He also graphed the squared gain of the monthly adjustment filter. Some of these plots are repeated in Wallis (1982) and Burridge and Wallis (1984), along with plots for some asymmetric filters. Young (1968) plotted monthly seasonal, trend, and irregular filter weights from his approximation to X–11 for the default seasonal MA with 9–term, 13–term, and 23–term Henderson filters. He also plotted end weights for the seasonal and trend filters. Those plots of Young and Wallis that correspond to cases treated here do appear to agree with our plots to the resolution permitted by the graphs.

Organization of Graphs — Outline

1. Monthly Filter Weights

    1.1 Seasonal Filters

        1.1.1 9–term Henderson Trend MA

            1.1.1.1 Default Seasonal MA
            1.1.1.2 Optional 3×3 Seasonal MA
            1.1.1.3 Optional 3×9 Seasonal MA
            1.1.1.4 Optional 3–term Seasonal MA

        1.1.2 13–term Henderson Trend MA

            1.1.2.1 Default Seasonal MA

            ⋮

        1.1.3 23–term Henderson Trend MA

        ⋮

    1.2 Adjustment Filters

    ⋮

    1.3 Trend Filters

    ⋮

    1.4 Irregular Filters

    ⋮

2. Monthly Filter Transfer Functions — Squared Gains

    2.1 Seasonal Filters

    ⋮

    2.2 Adjustment Filters

    ⋮

    2.3 Trend Filters

    ⋮

    2.4 Irregular Filters

    ⋮

3. Quarterly Filter Weights

    3.1 Seasonal Filters

        3.1.1 5–term Henderson Trend MA

            3.1.1.1 Default Seasonal MA
            3.1.1.2 Optional 3×3 Seasonal MA
            3.1.1.3 Optional 3×9 Seasonal MA
            3.1.1.4 Optional 3–term Seasonal MA

        3.1.2 7–term Henderson Trend MA

            3.1.2.1 Default Seasonal MA

            ⋮

    3.2 Adjustment Filters

    ⋮

    3.3 Trend Filters

    ⋮

    3.4 Irregular Filters

    ⋮

4. Quarterly Filter Transfer Functions — Squared Gains

    4.1 Seasonal Filters

    ⋮

    4.2 Adjustment Filters

    ⋮

    4.3 Trend Filters

    ⋮

    4.4 Irregular Filters

    ⋮

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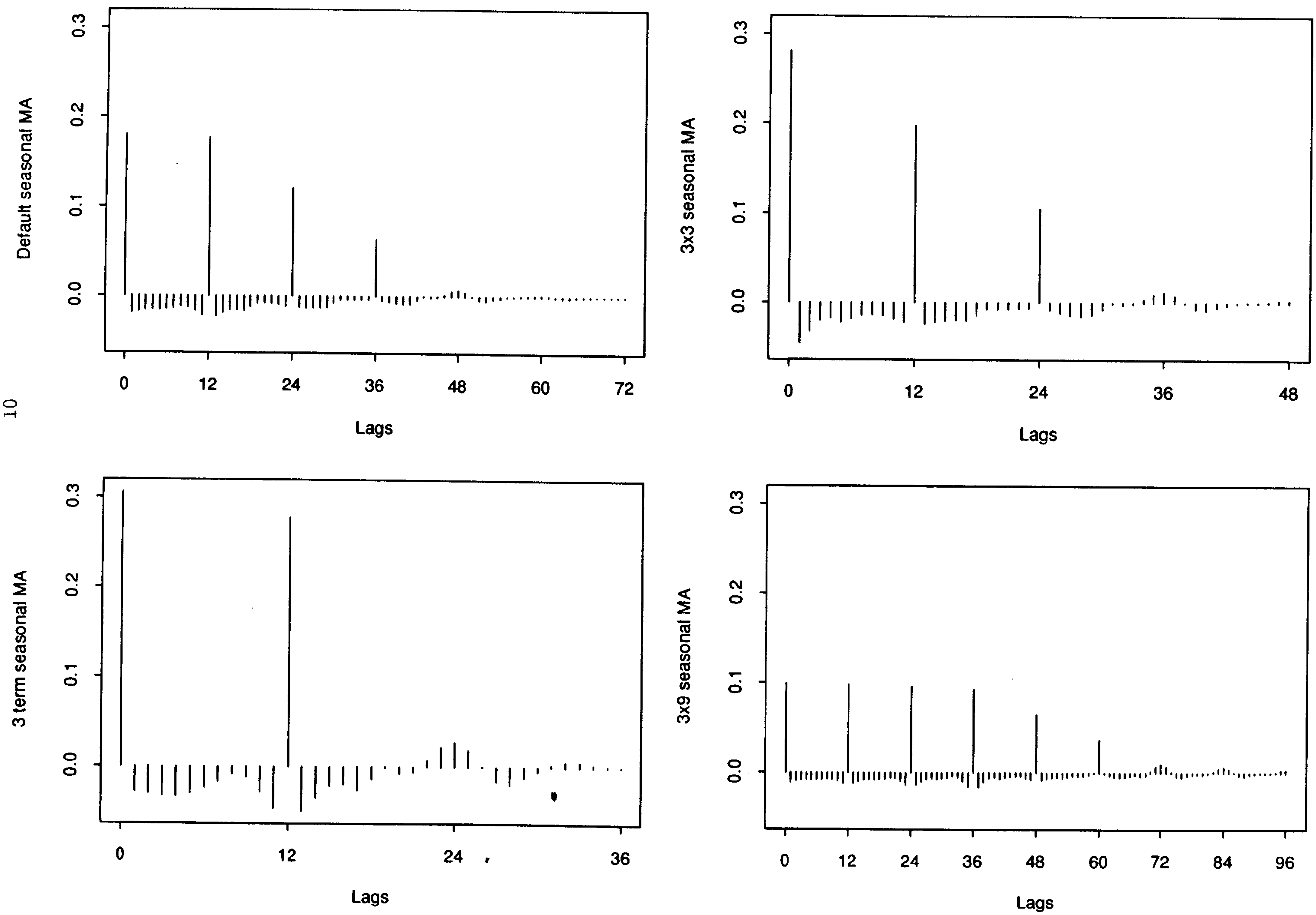
Monthly seasonal filter weights, 9-term Henderson



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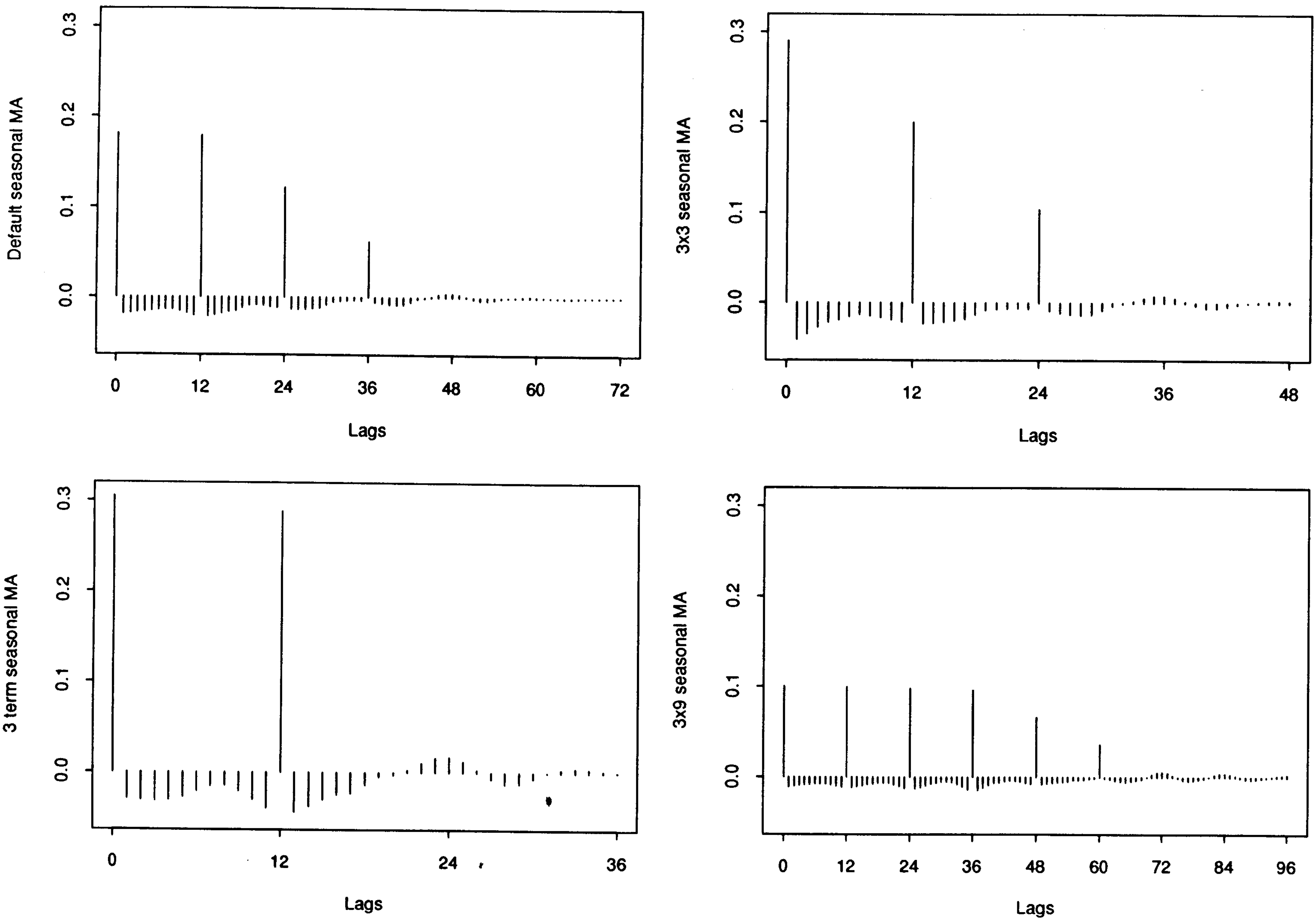

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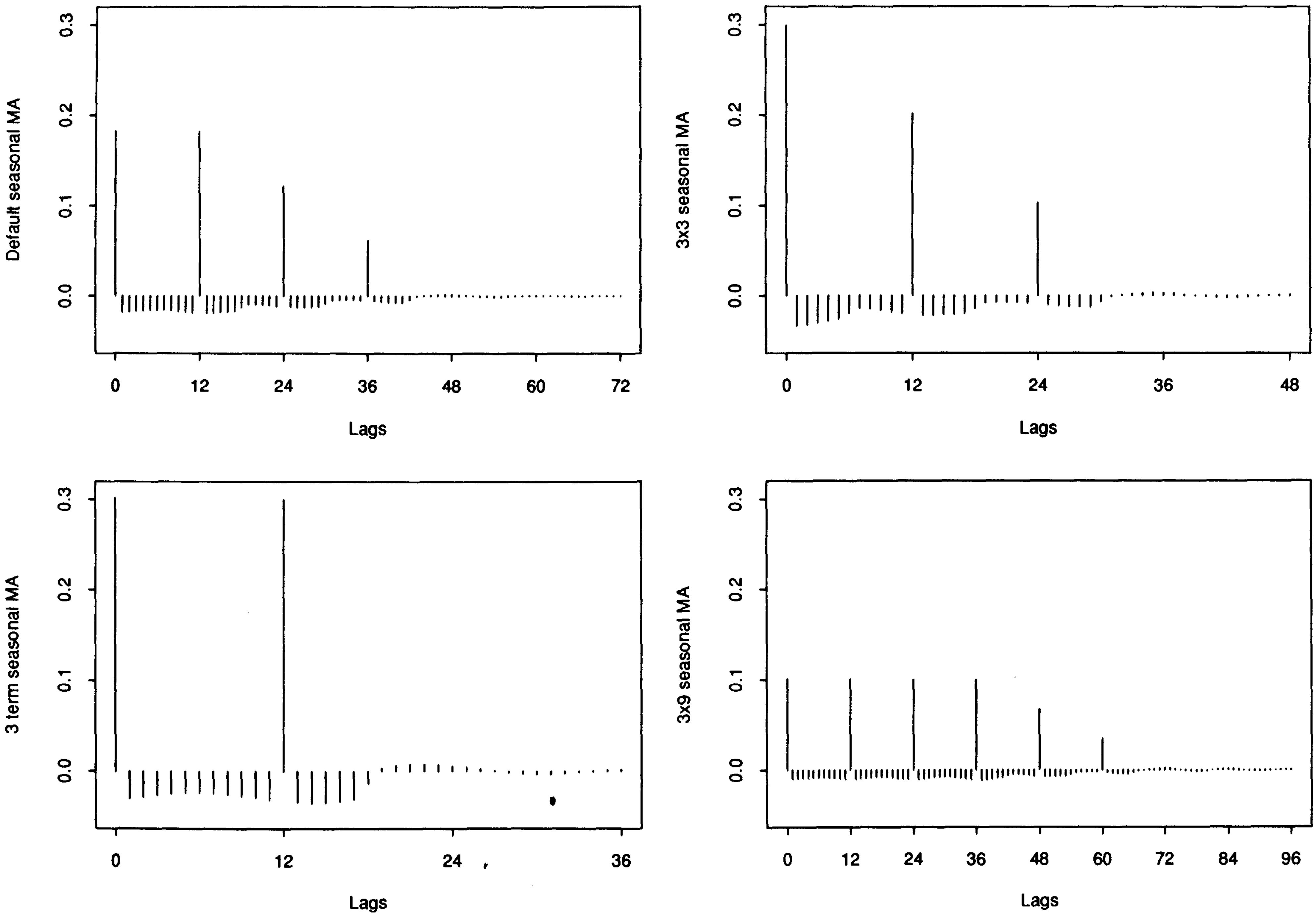
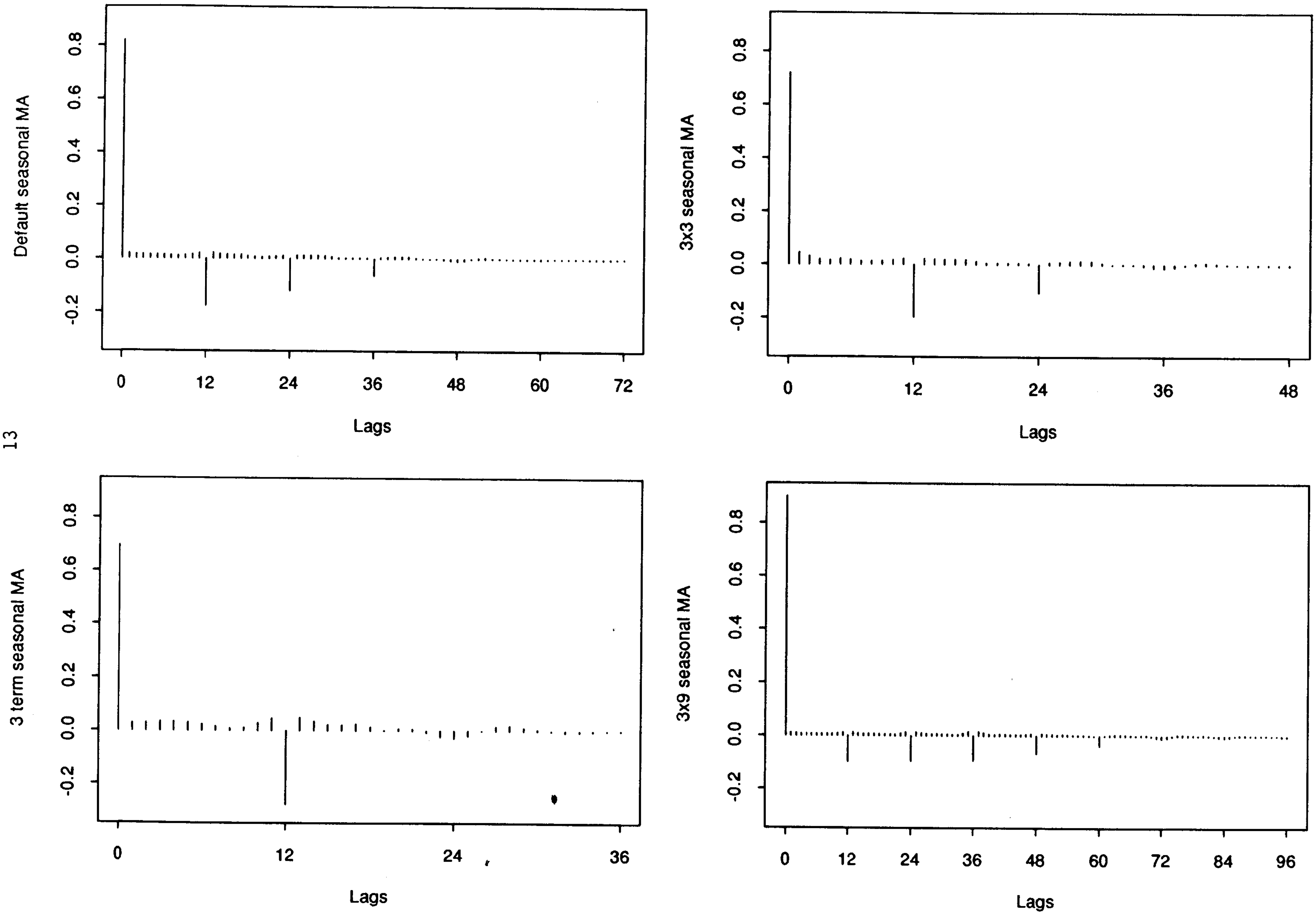


Monthly adjustment filter weights, 9-term Henderson



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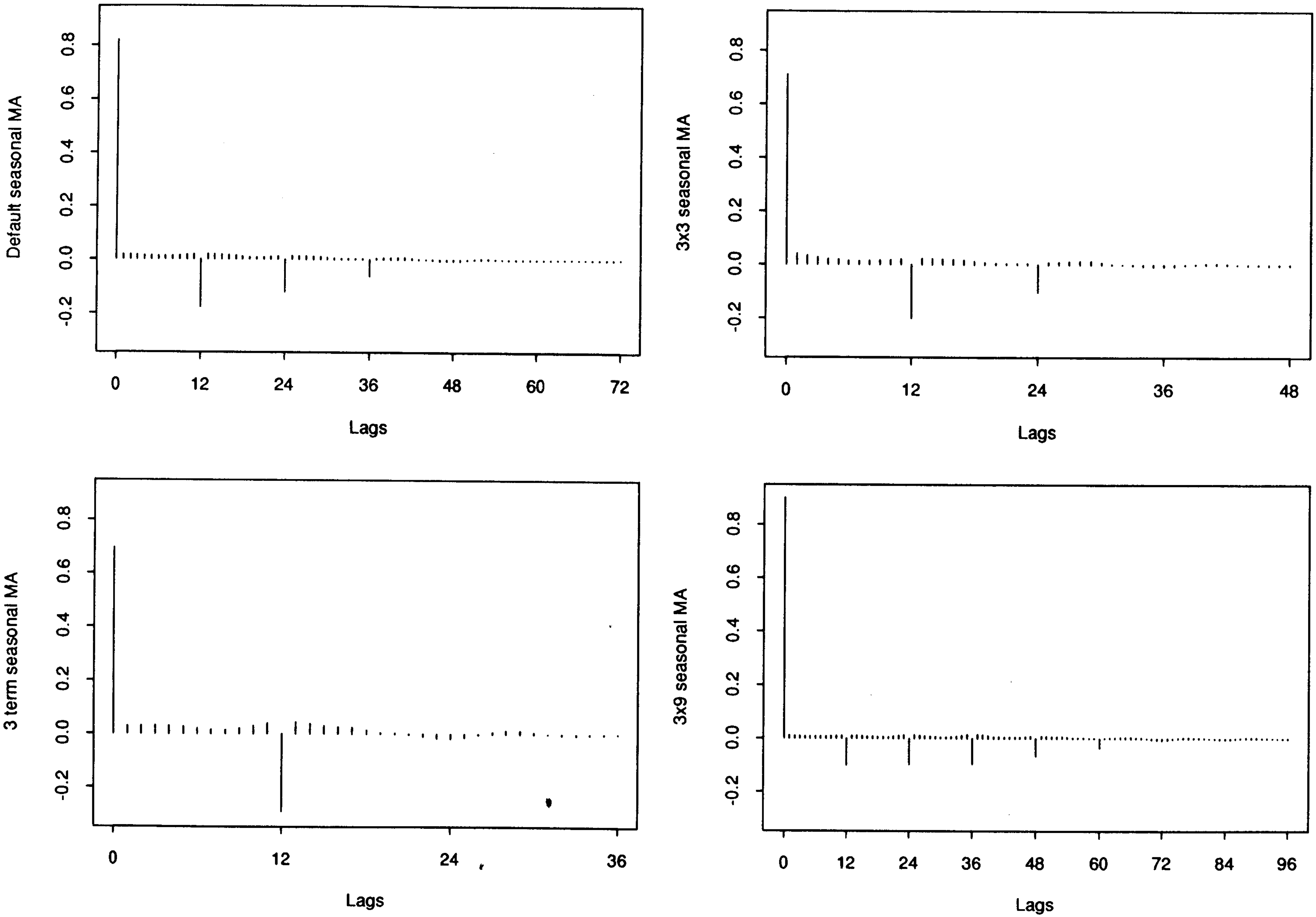

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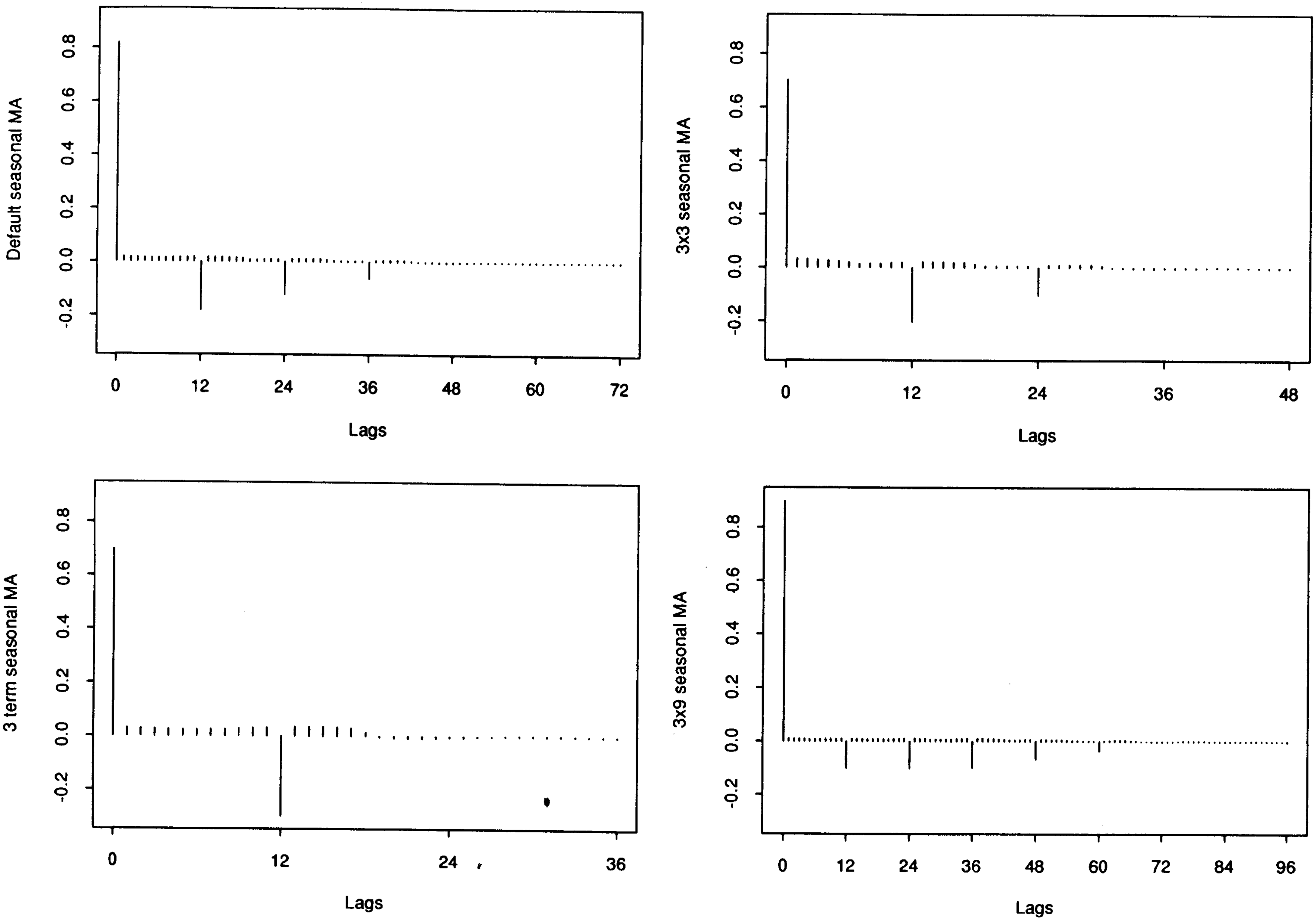
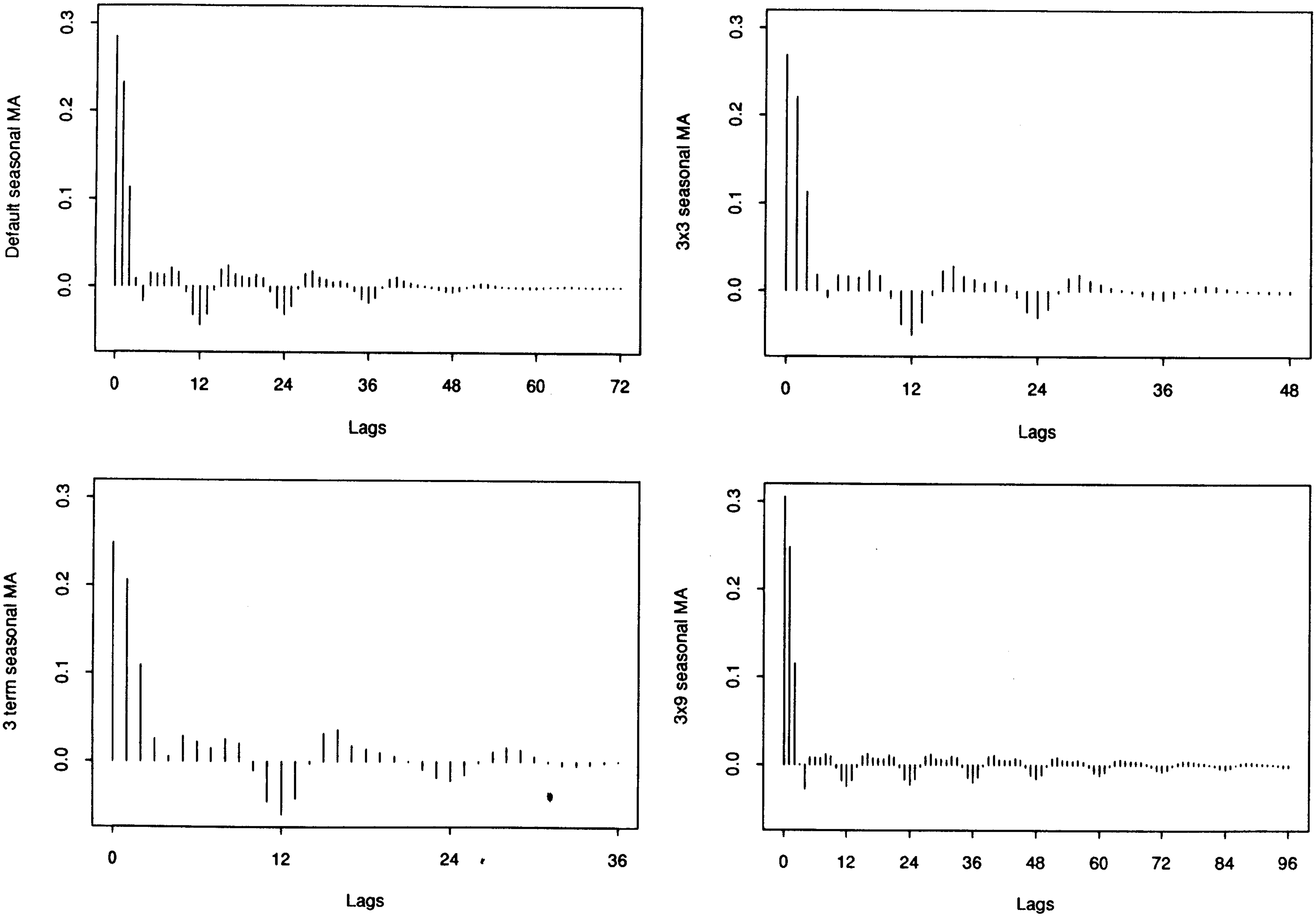


Monthly trend filter weights, 9-term Henderson



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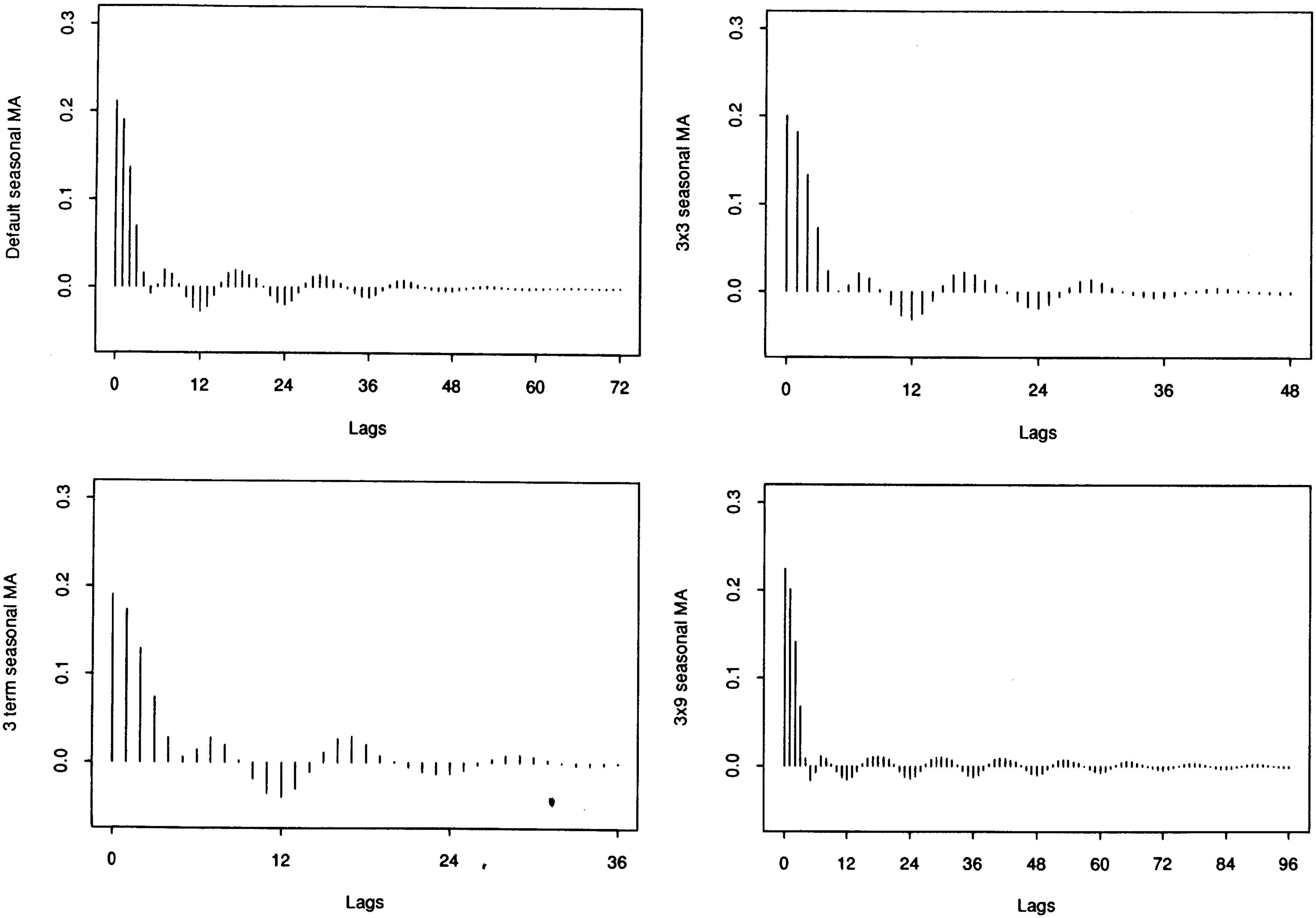

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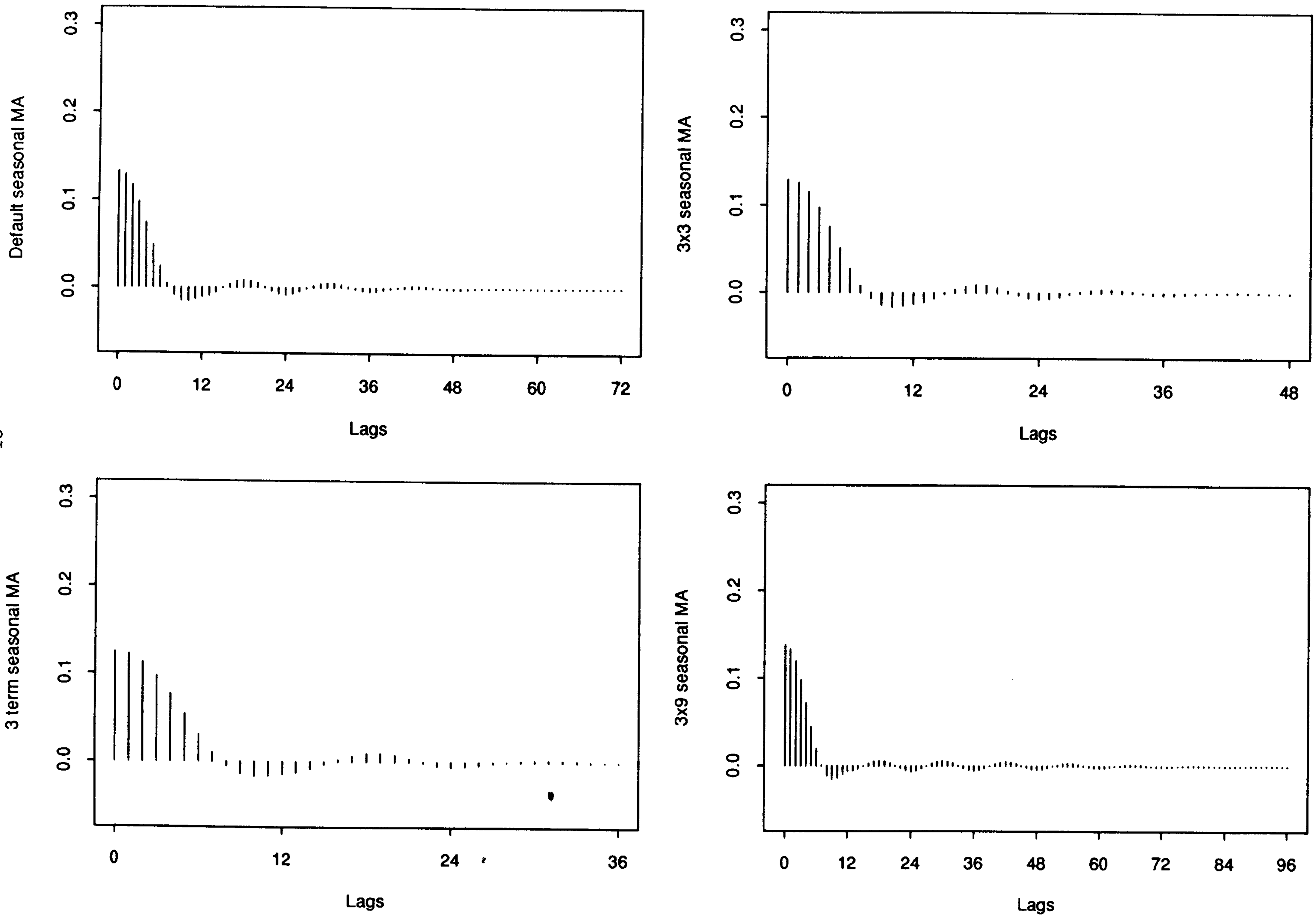
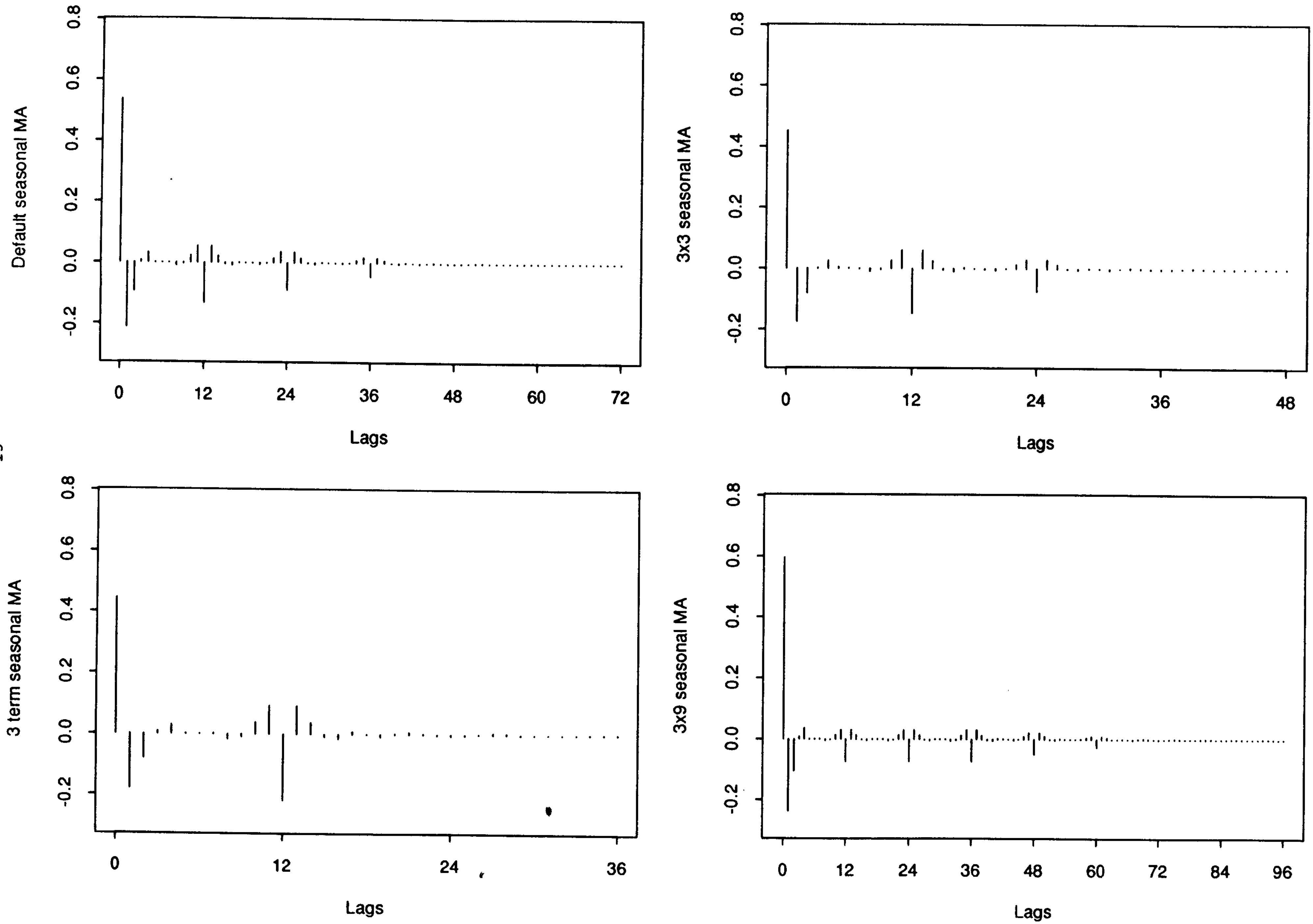


Monthly irregular filter weights, 9-term Henderson



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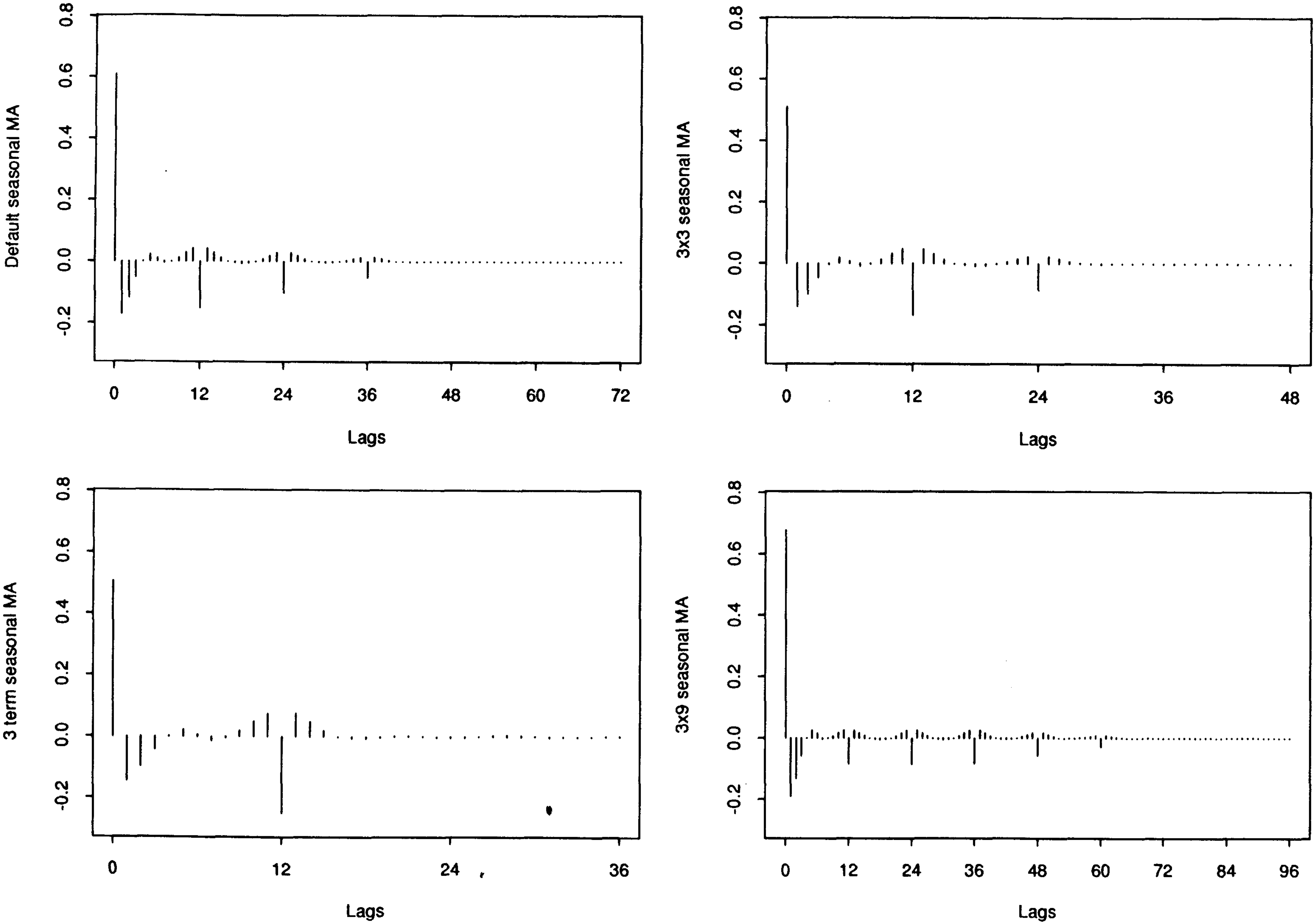

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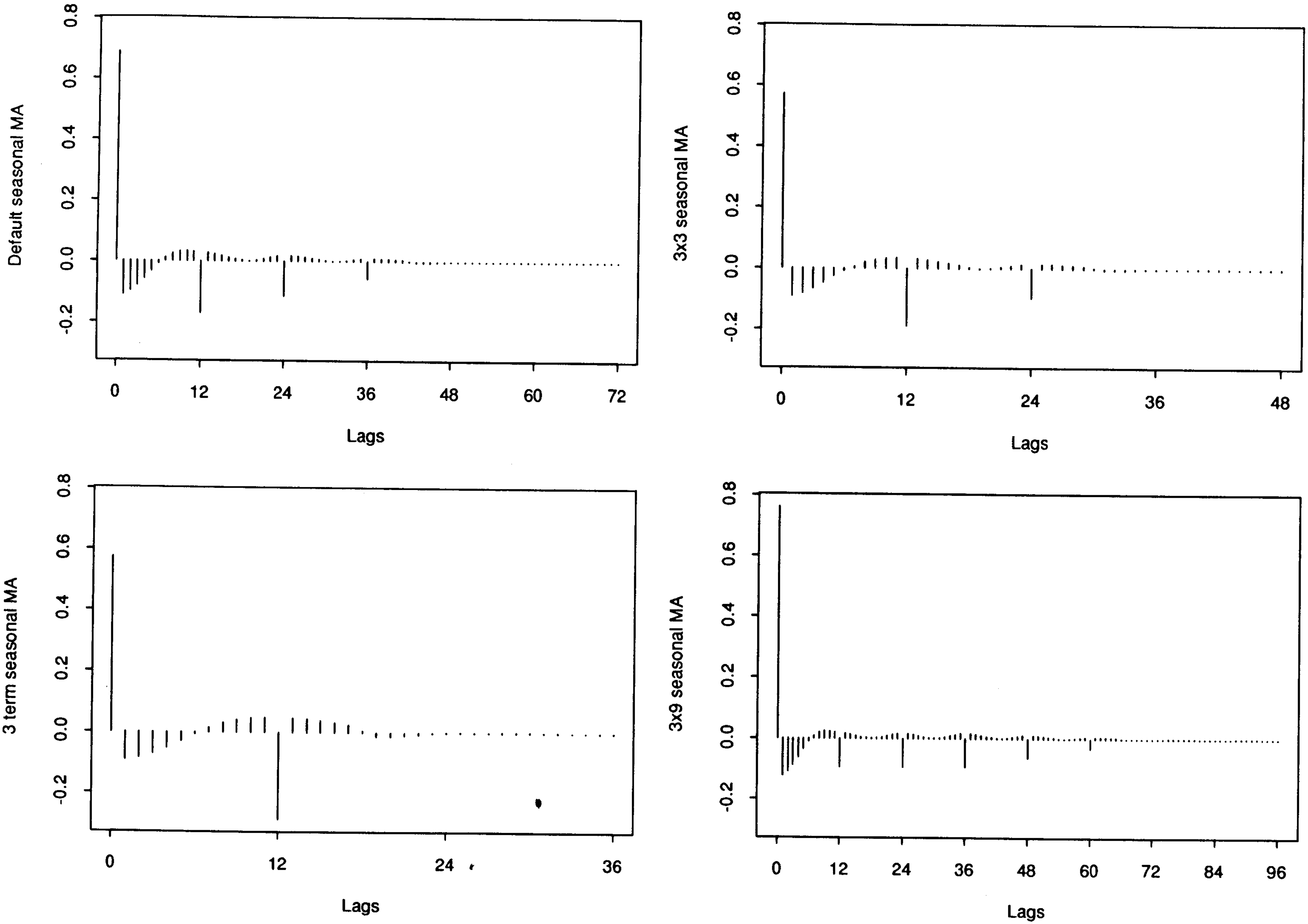
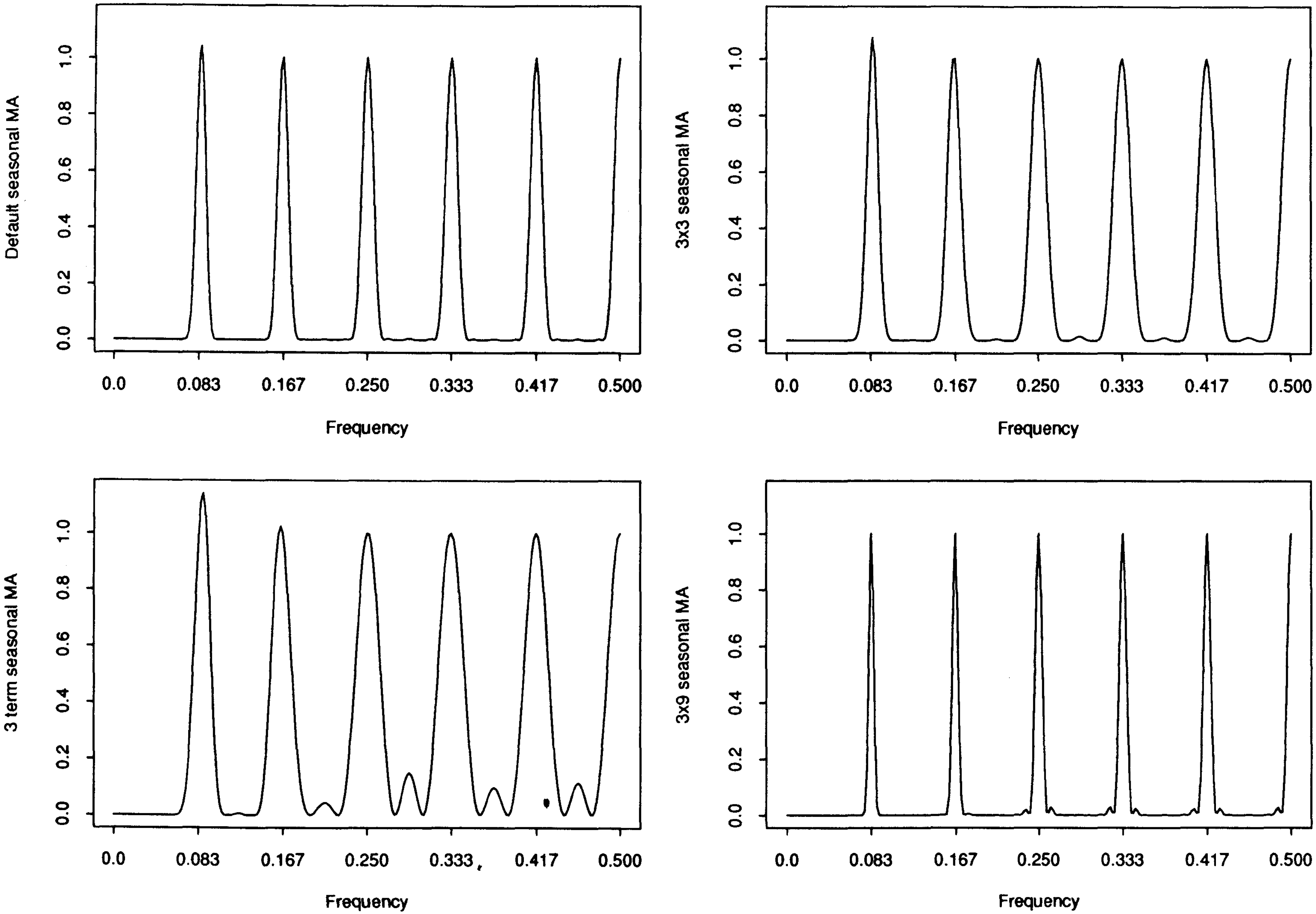


Monthly seasonal filter squared gain functions, 9-term Henderson



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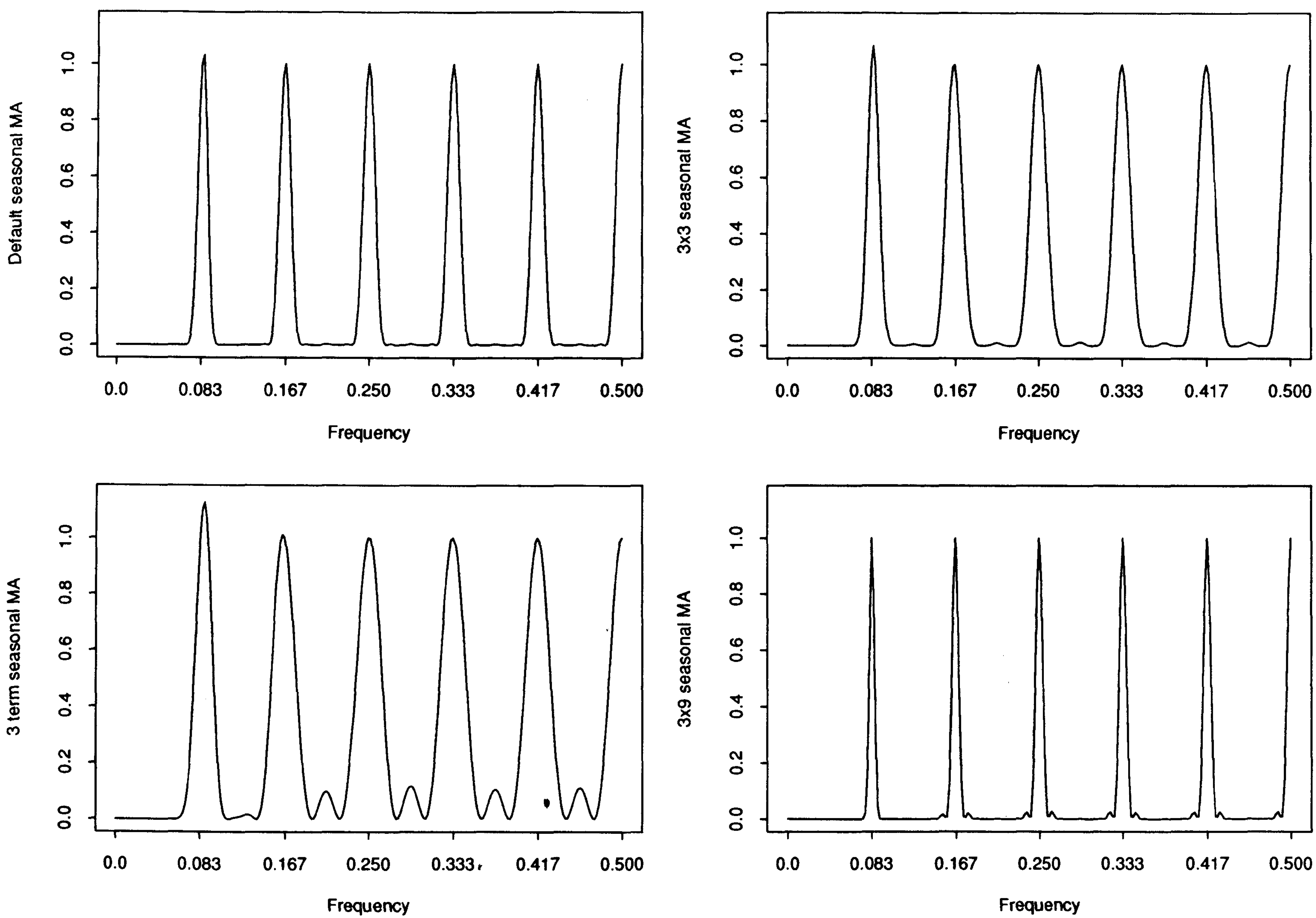

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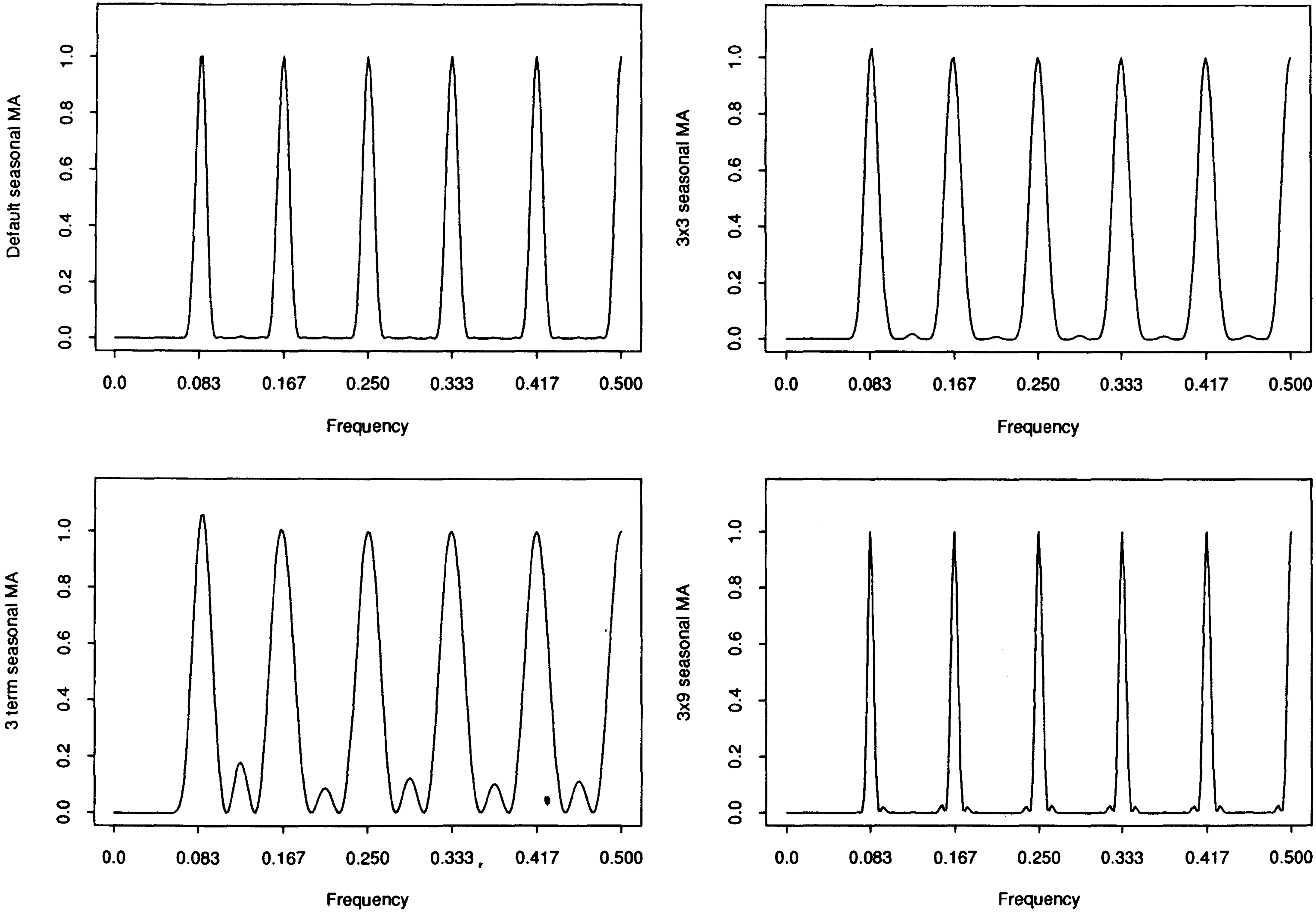
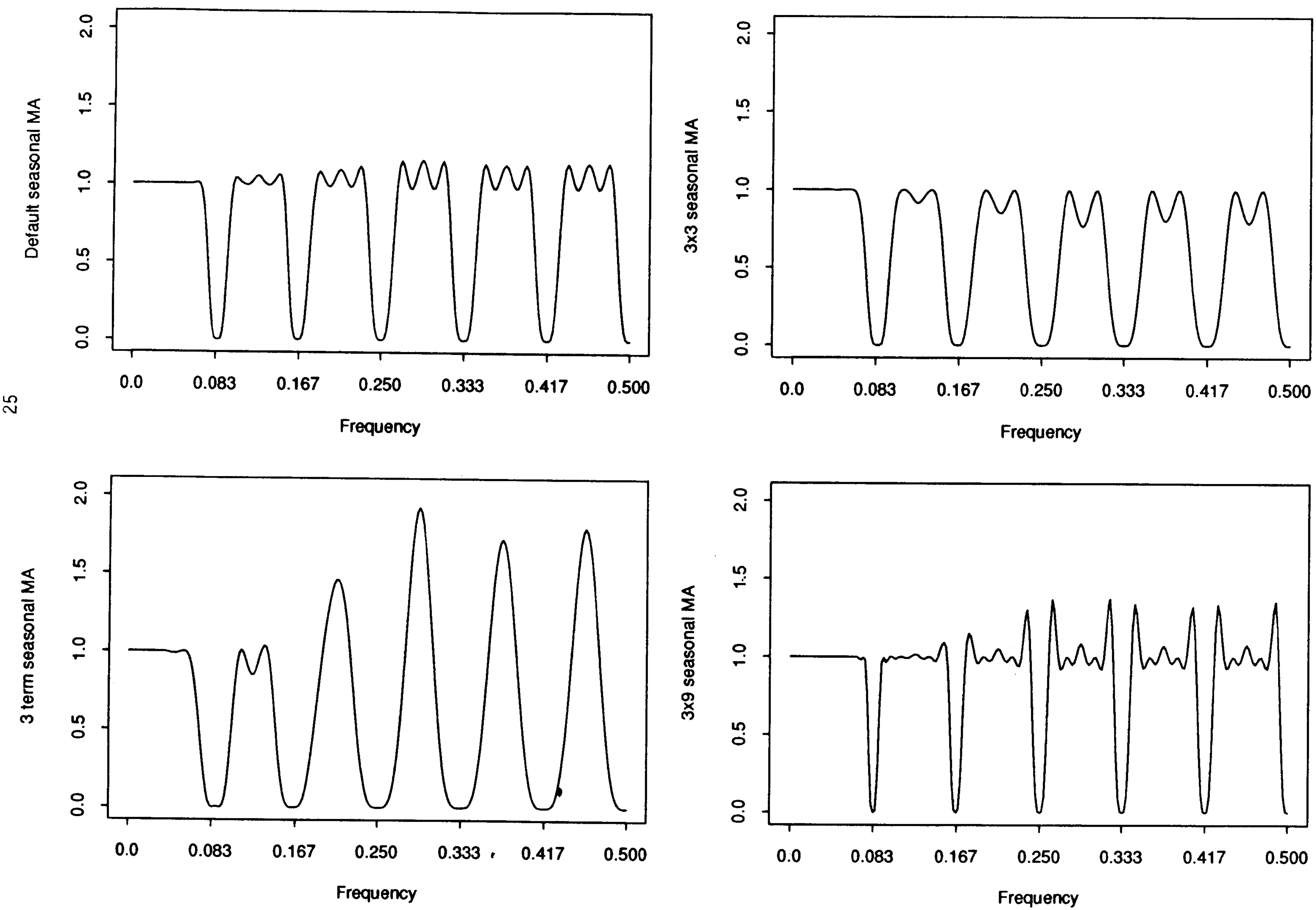


Monthly adjustment filter squared gain functions, 9-term Henderson



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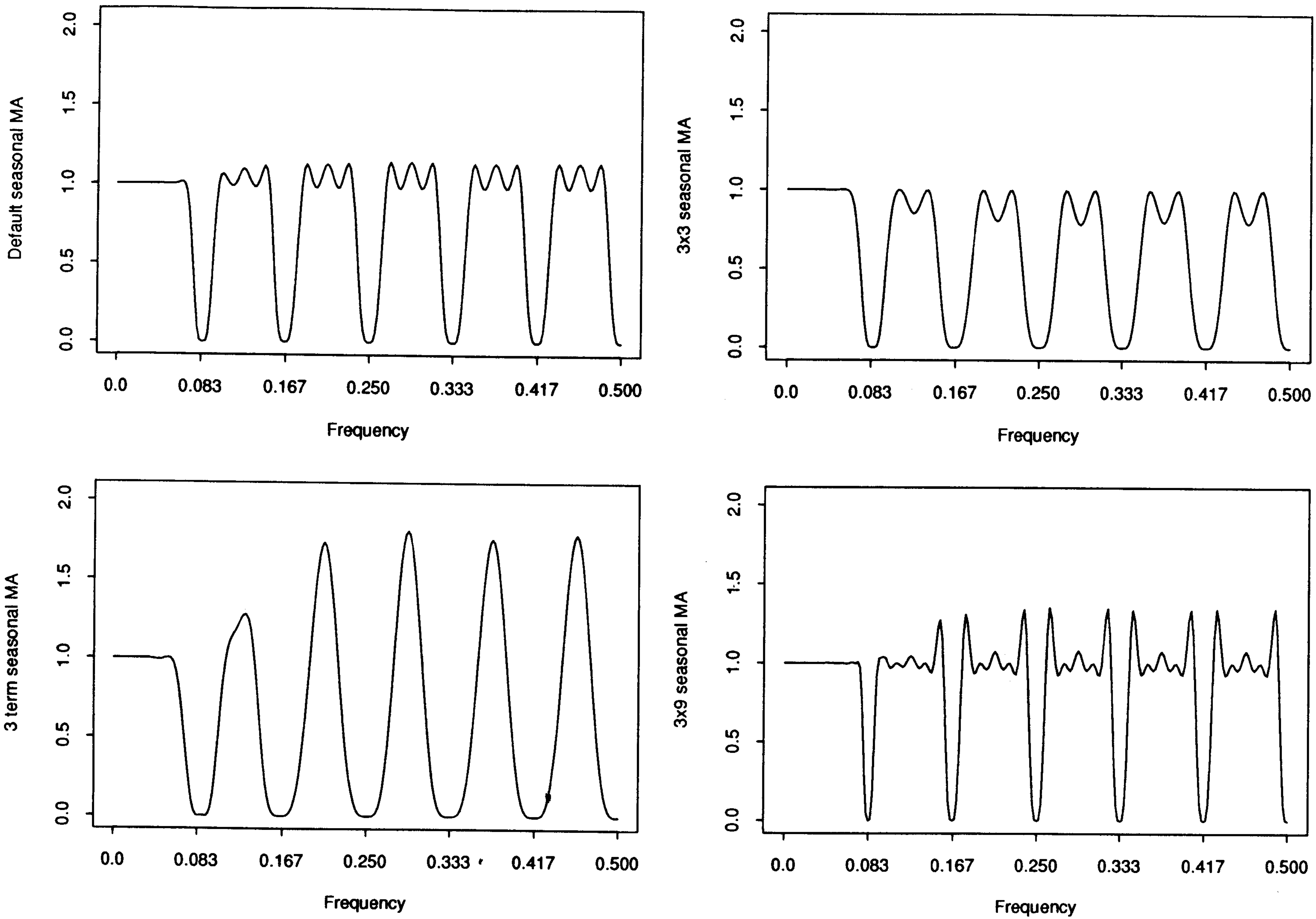

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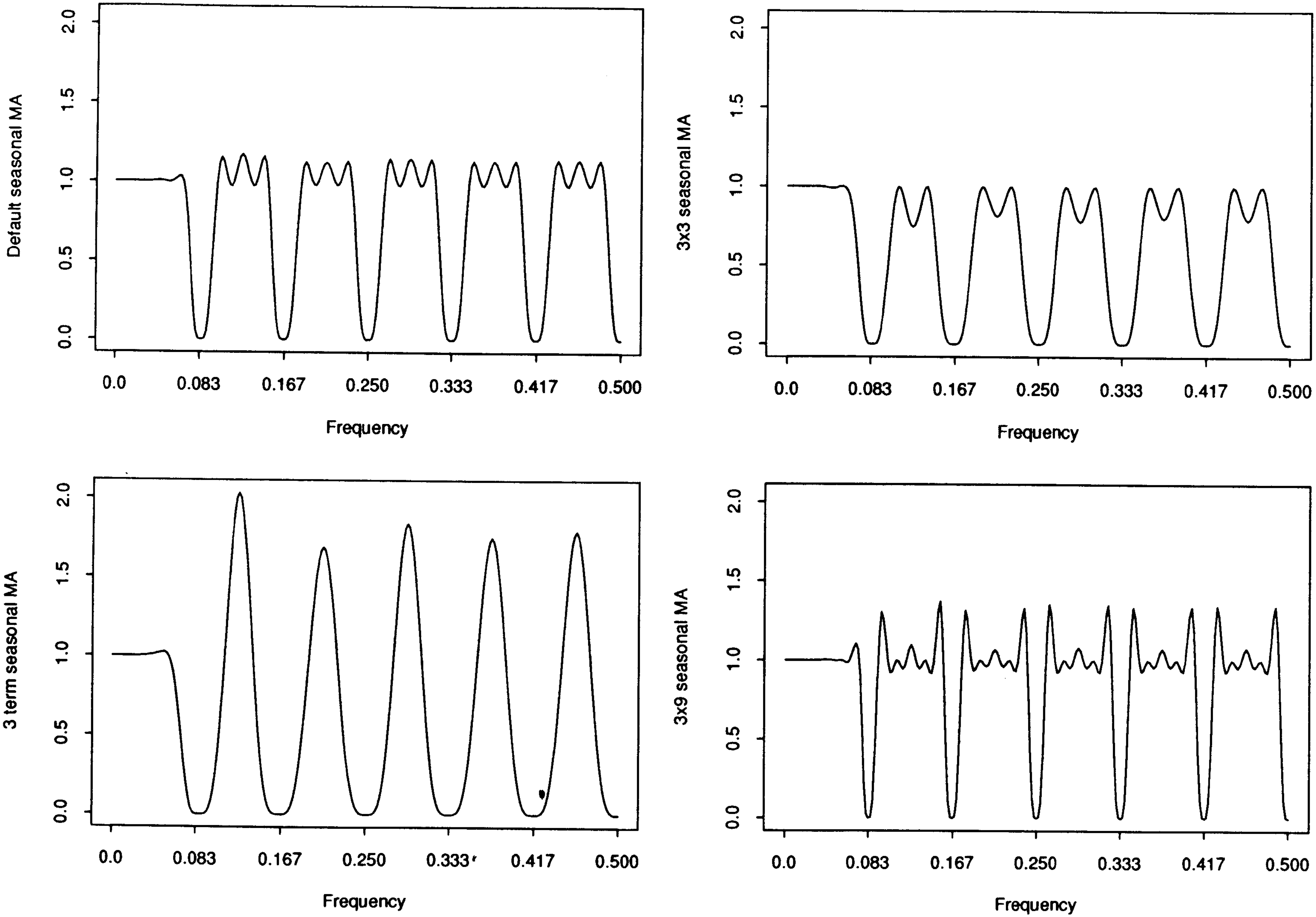

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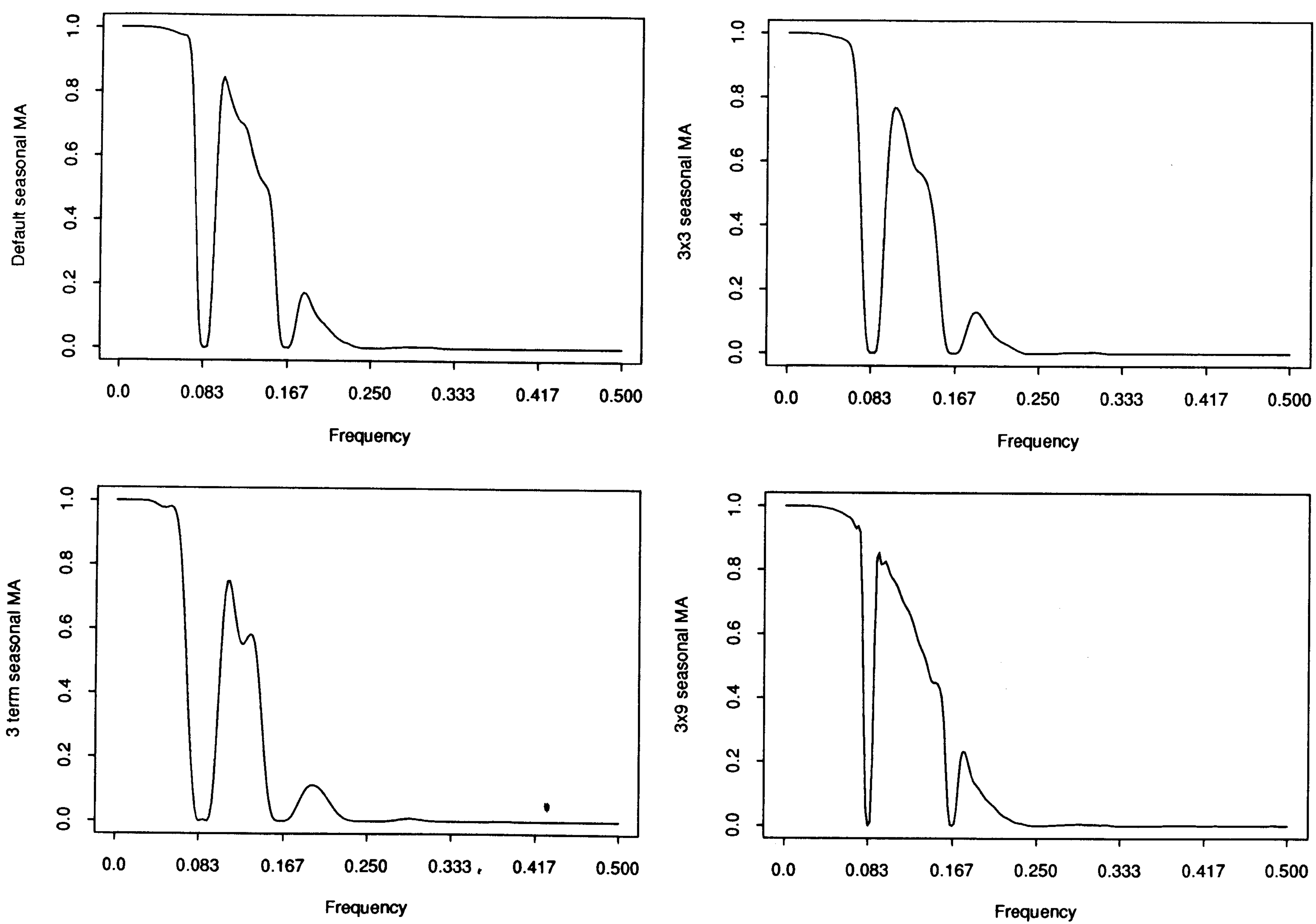

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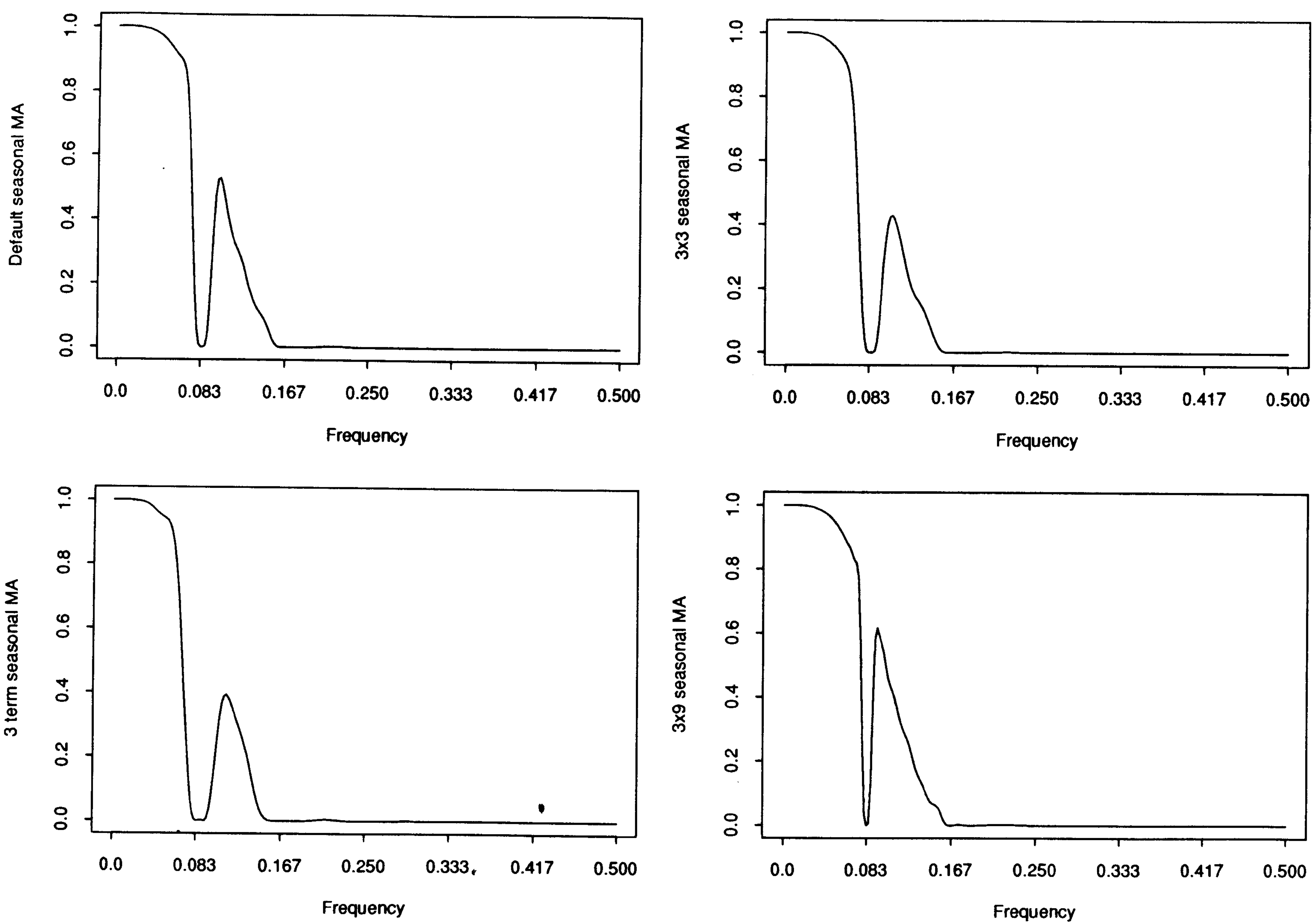

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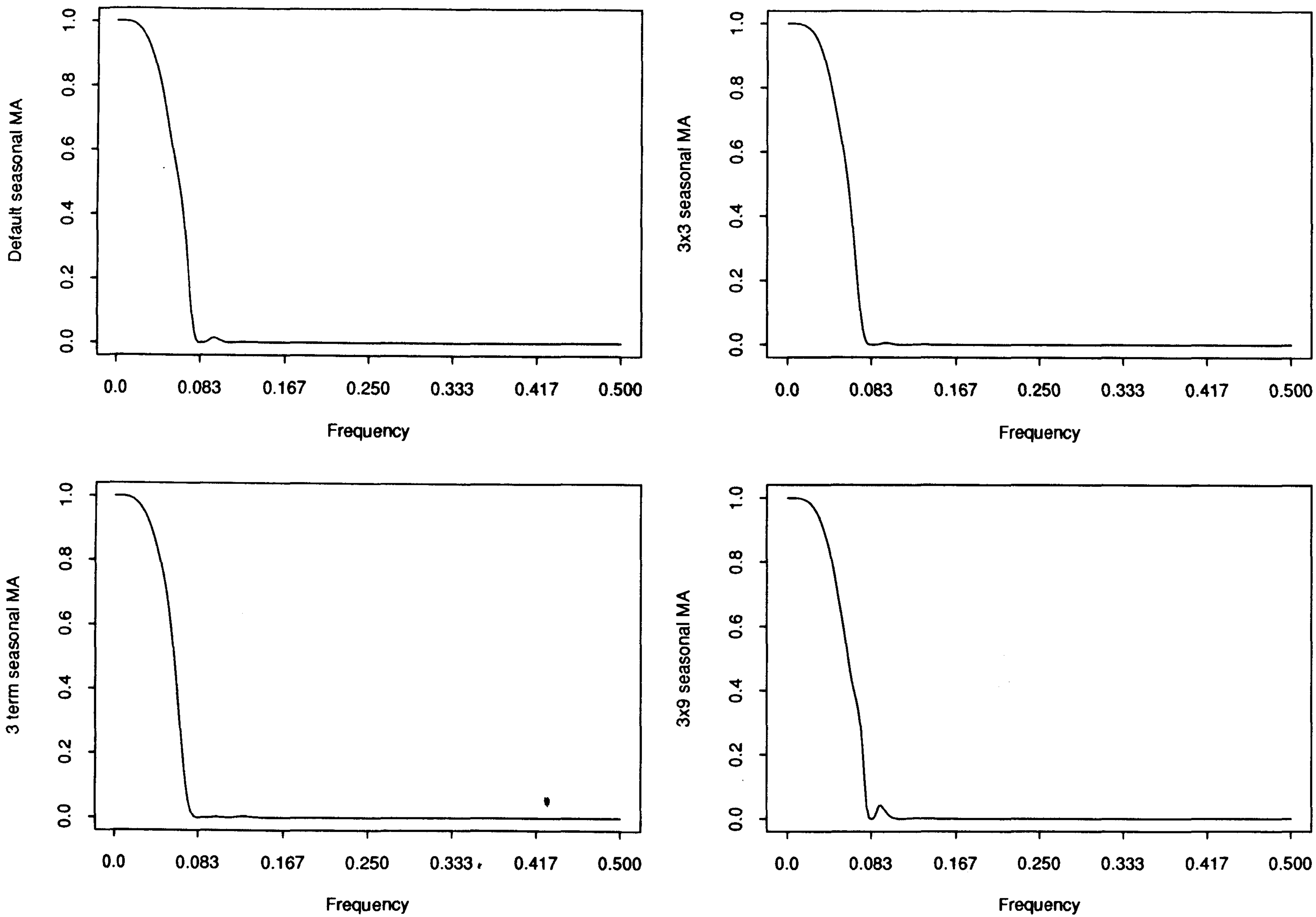
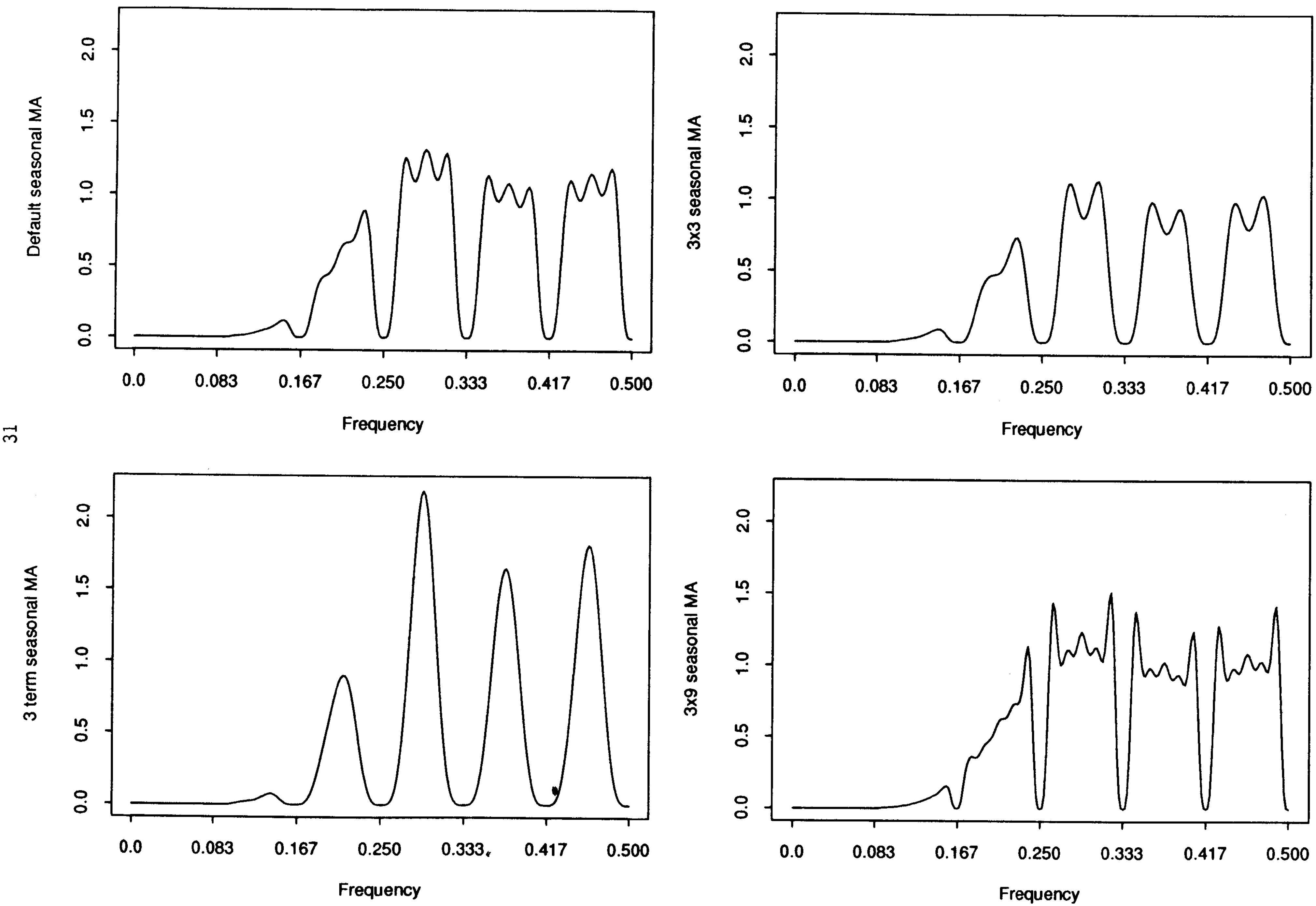


Monthly irregular filter squared gain functions, 9-term Henderson



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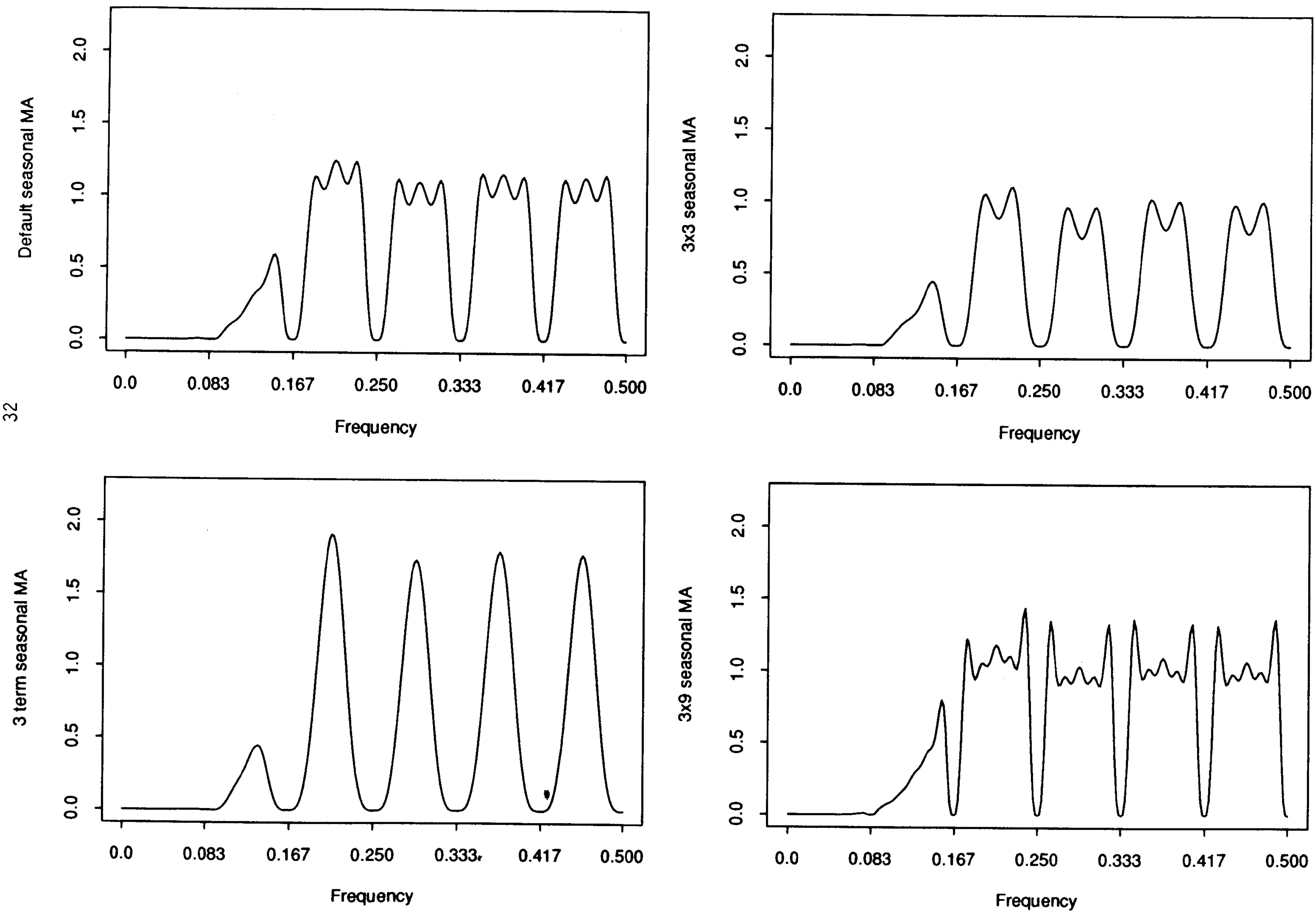

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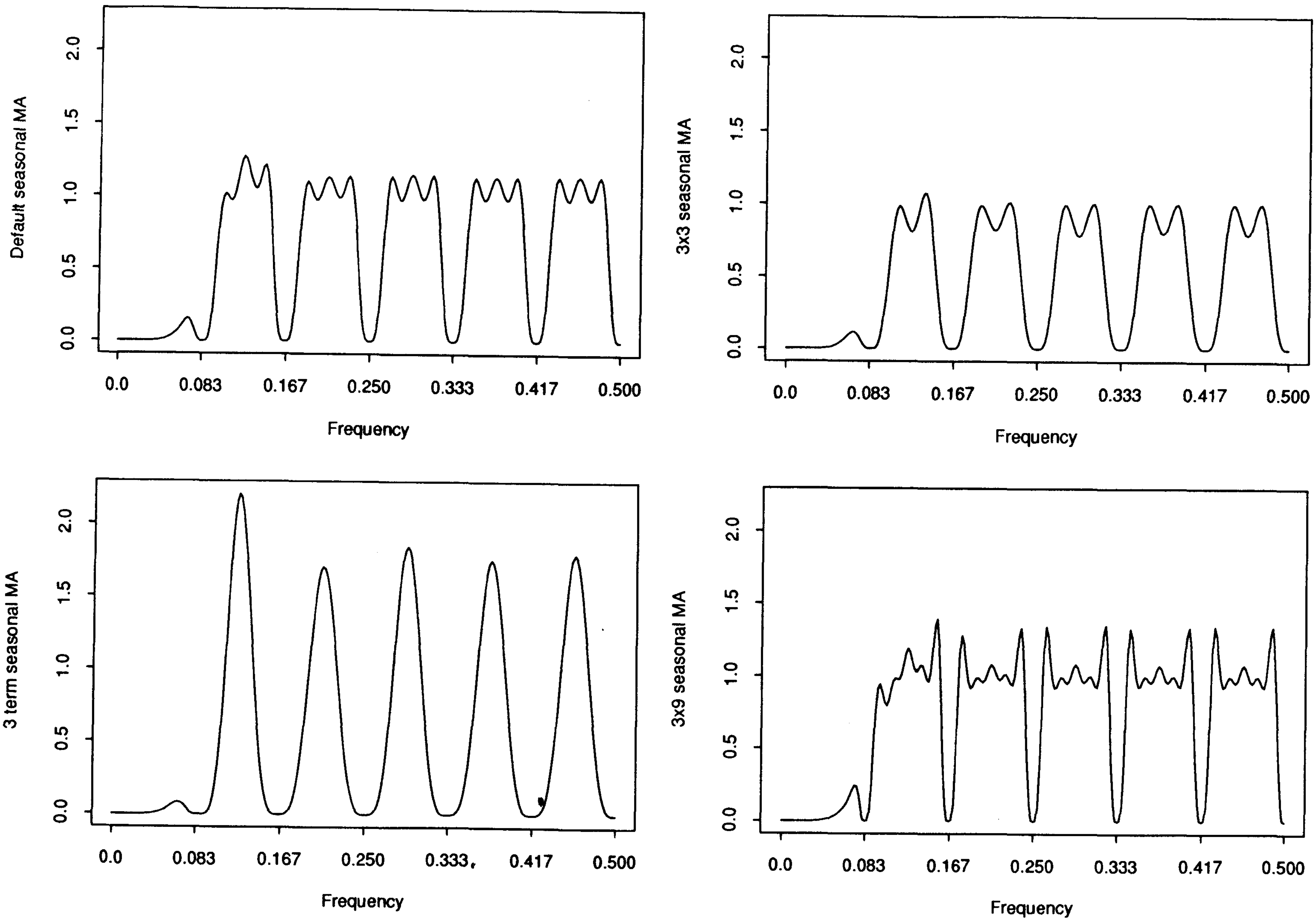
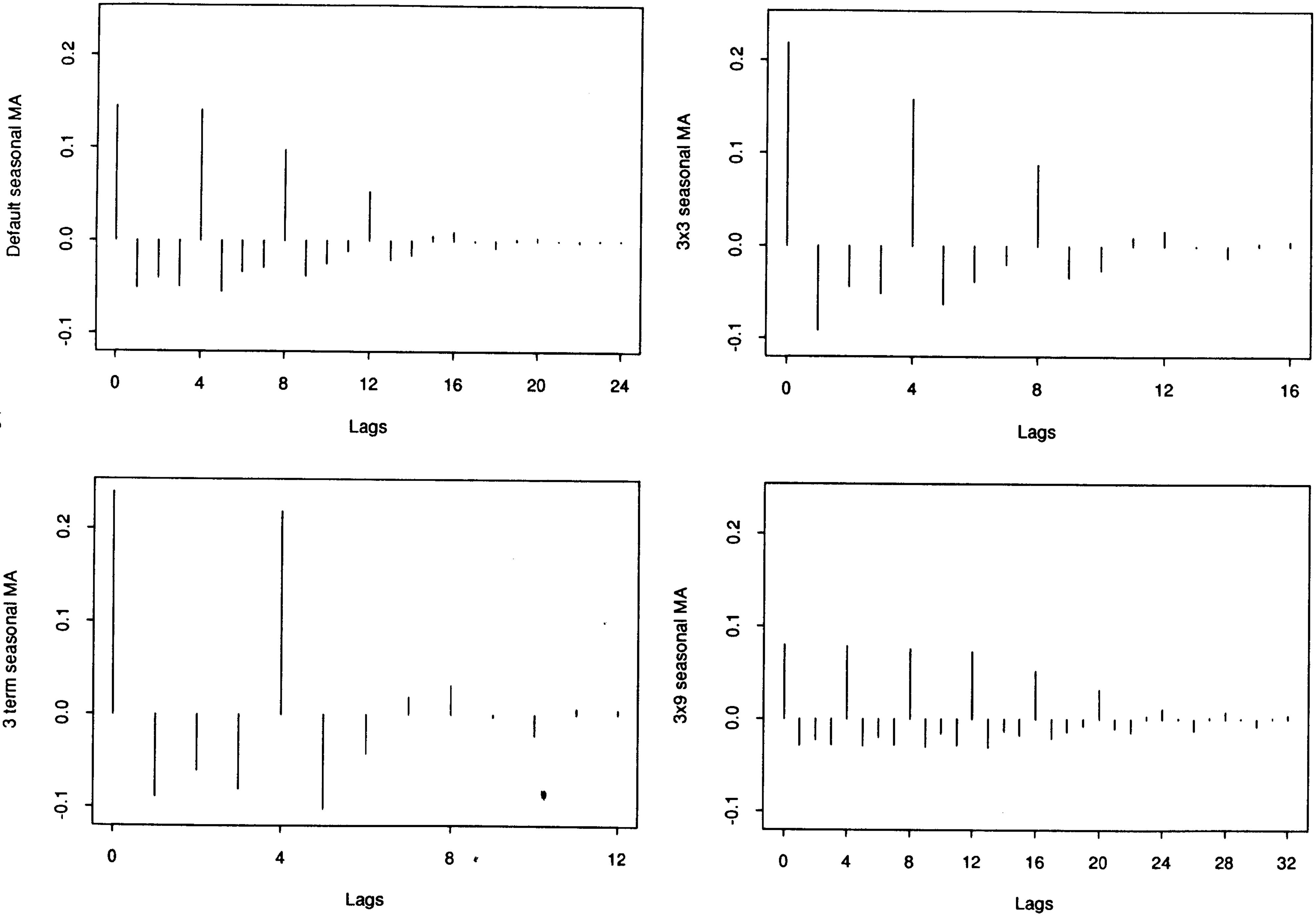


Quarterly seasonal filter weights, 5-term Henderson



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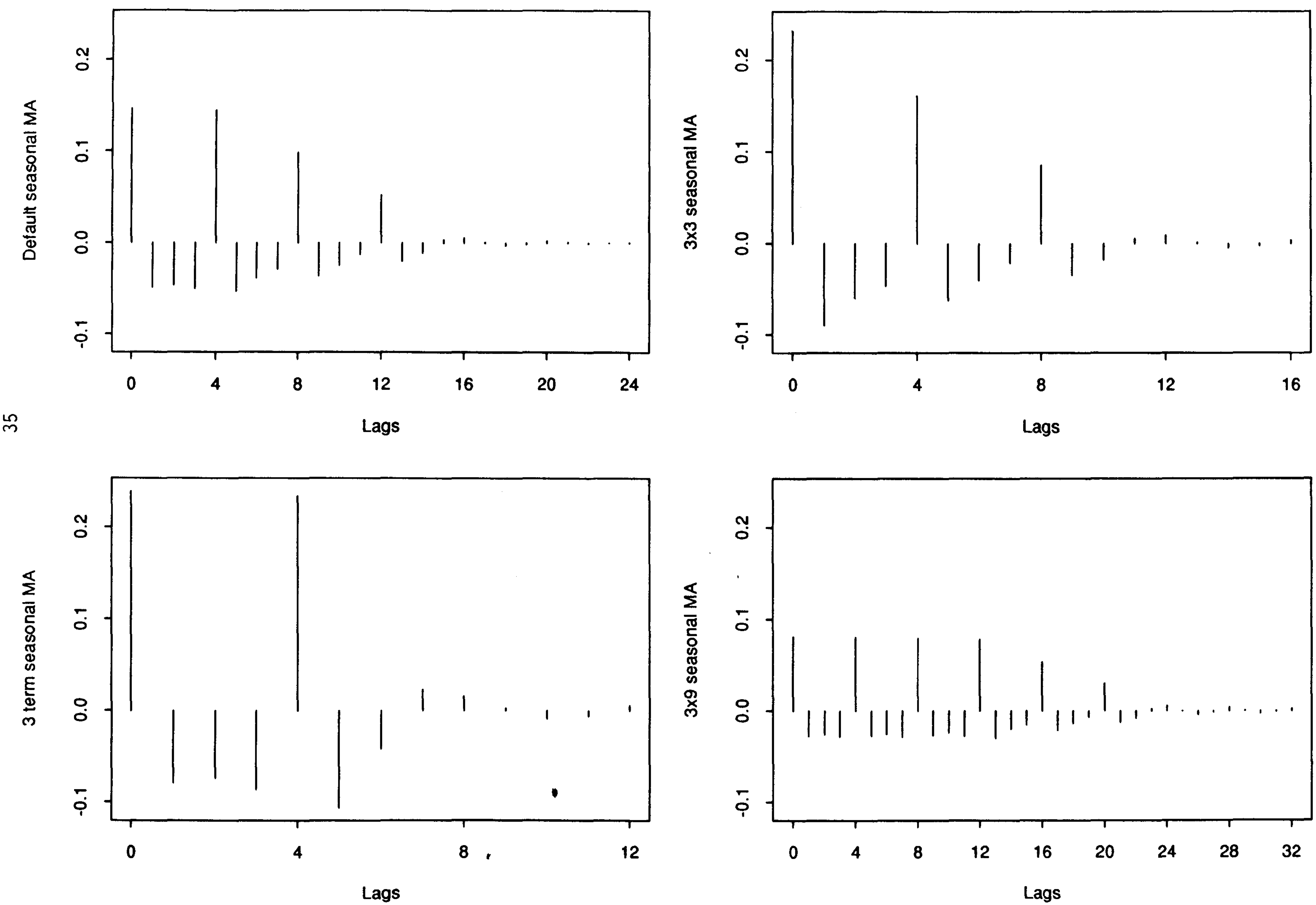

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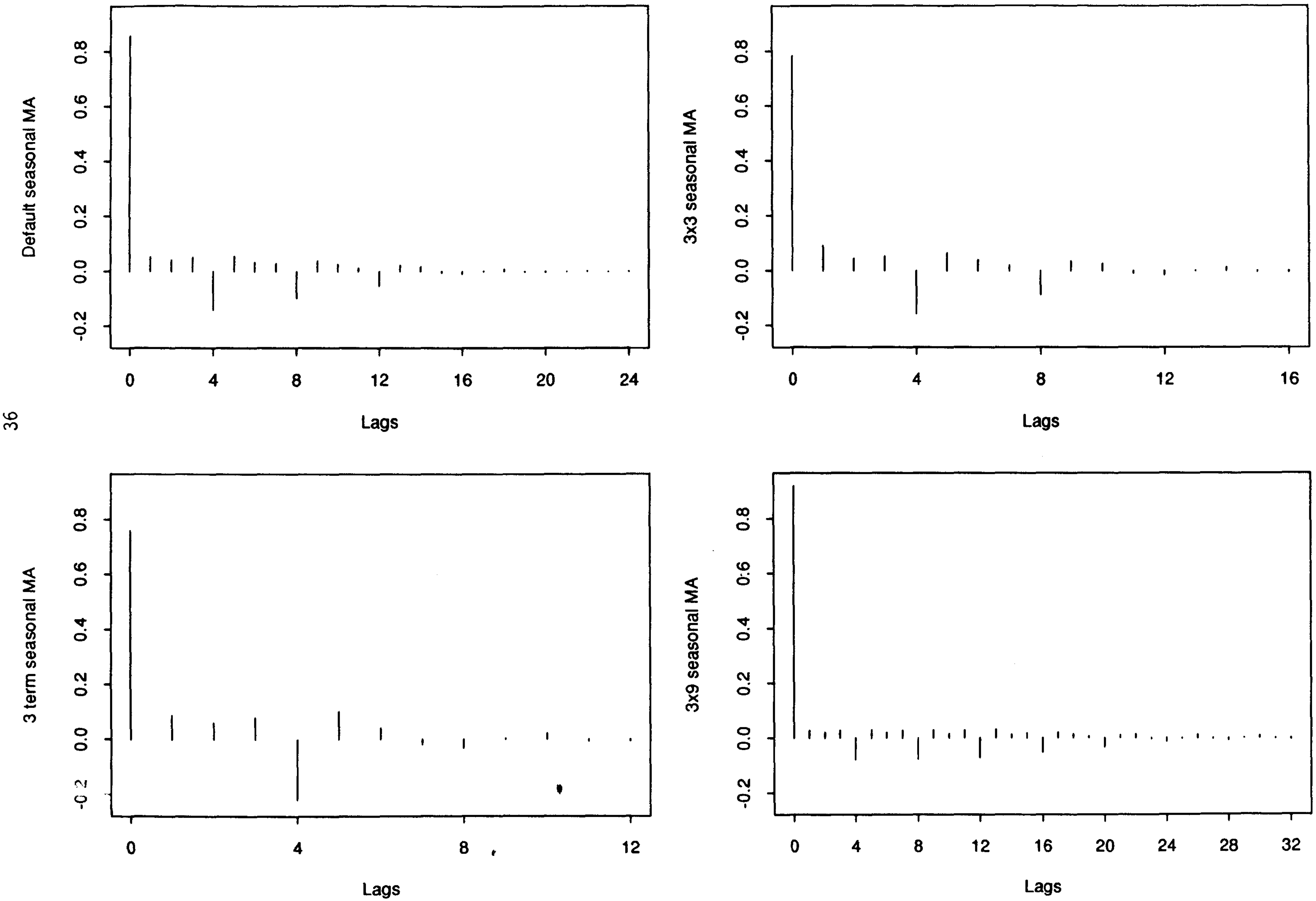

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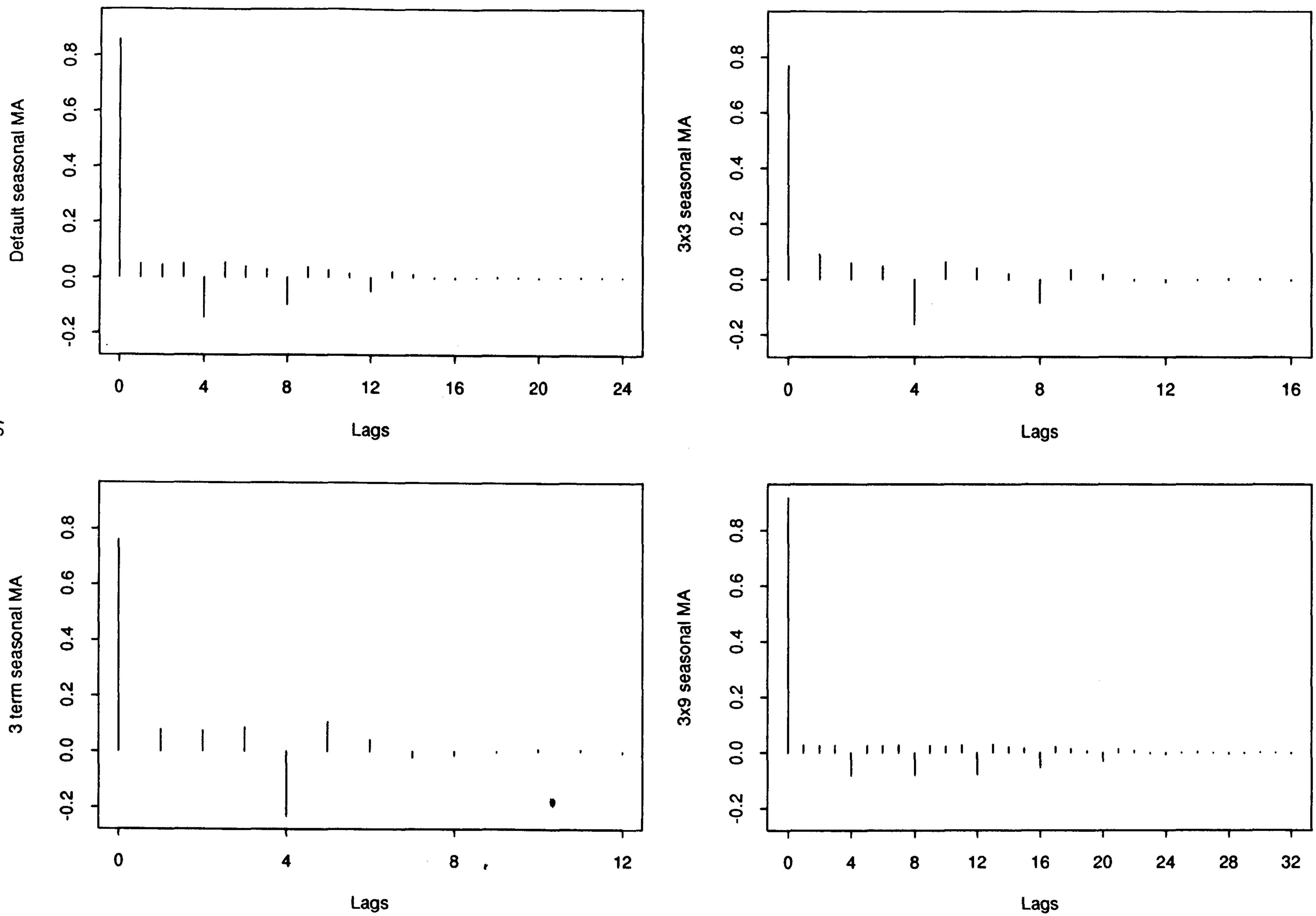

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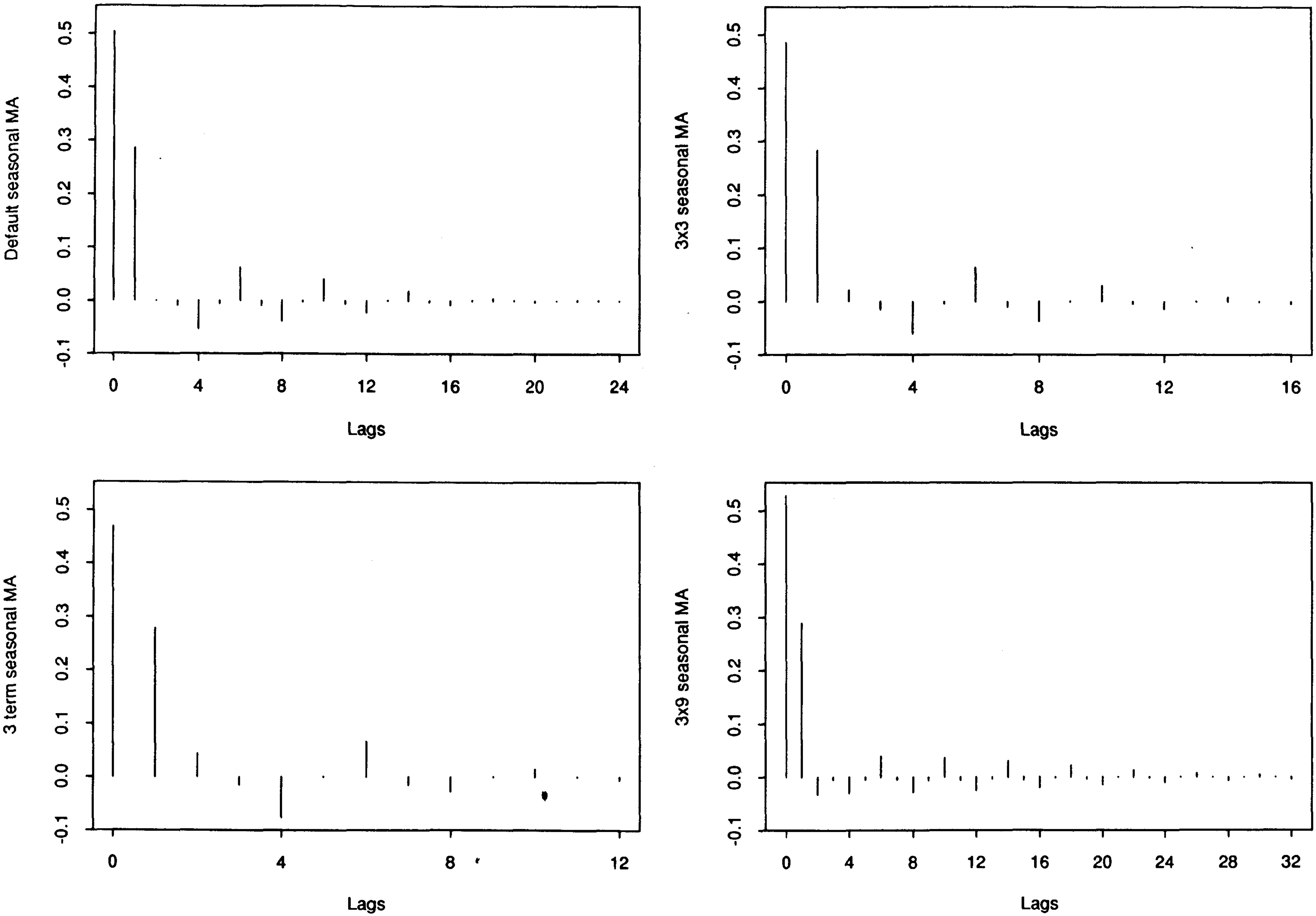

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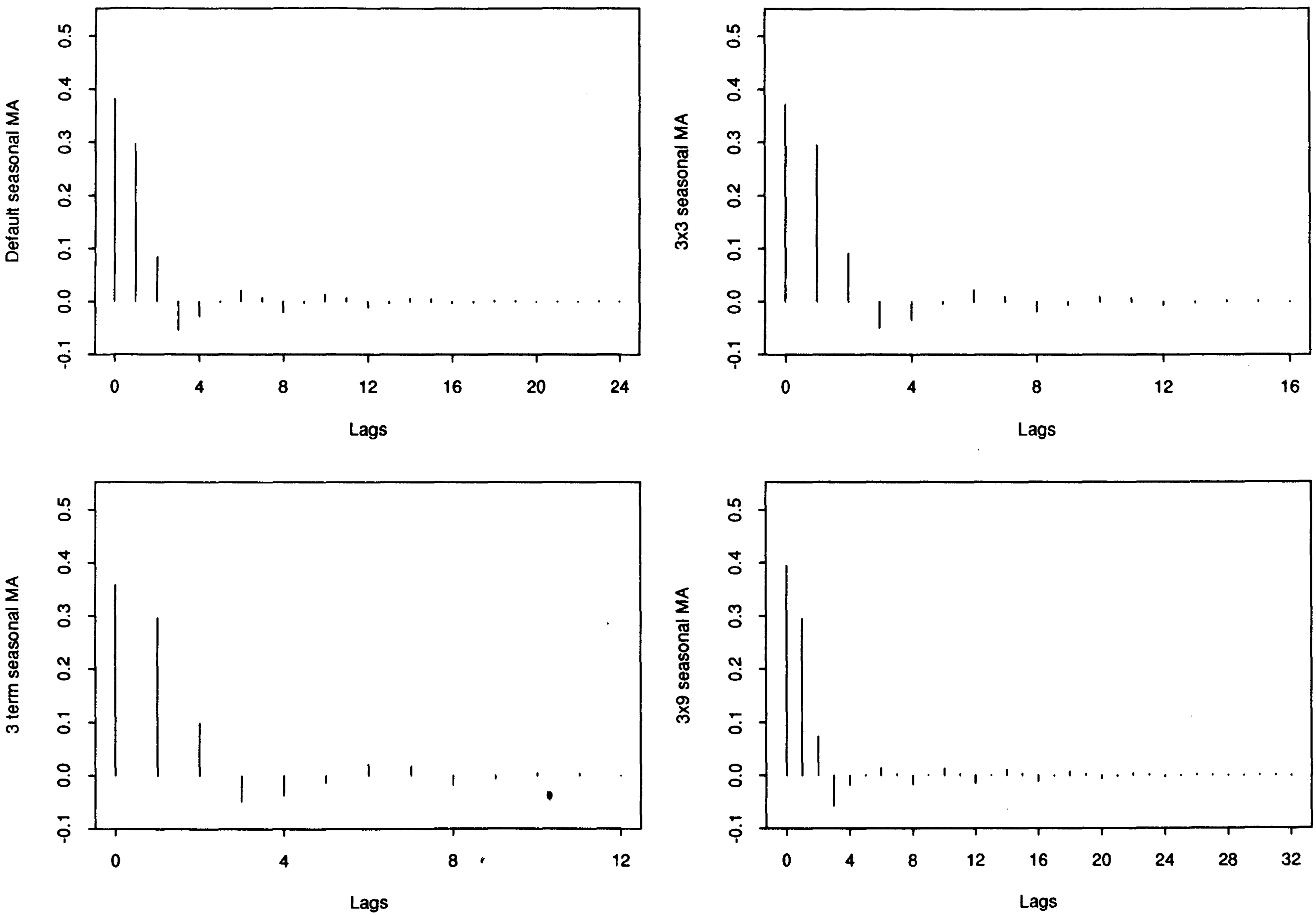

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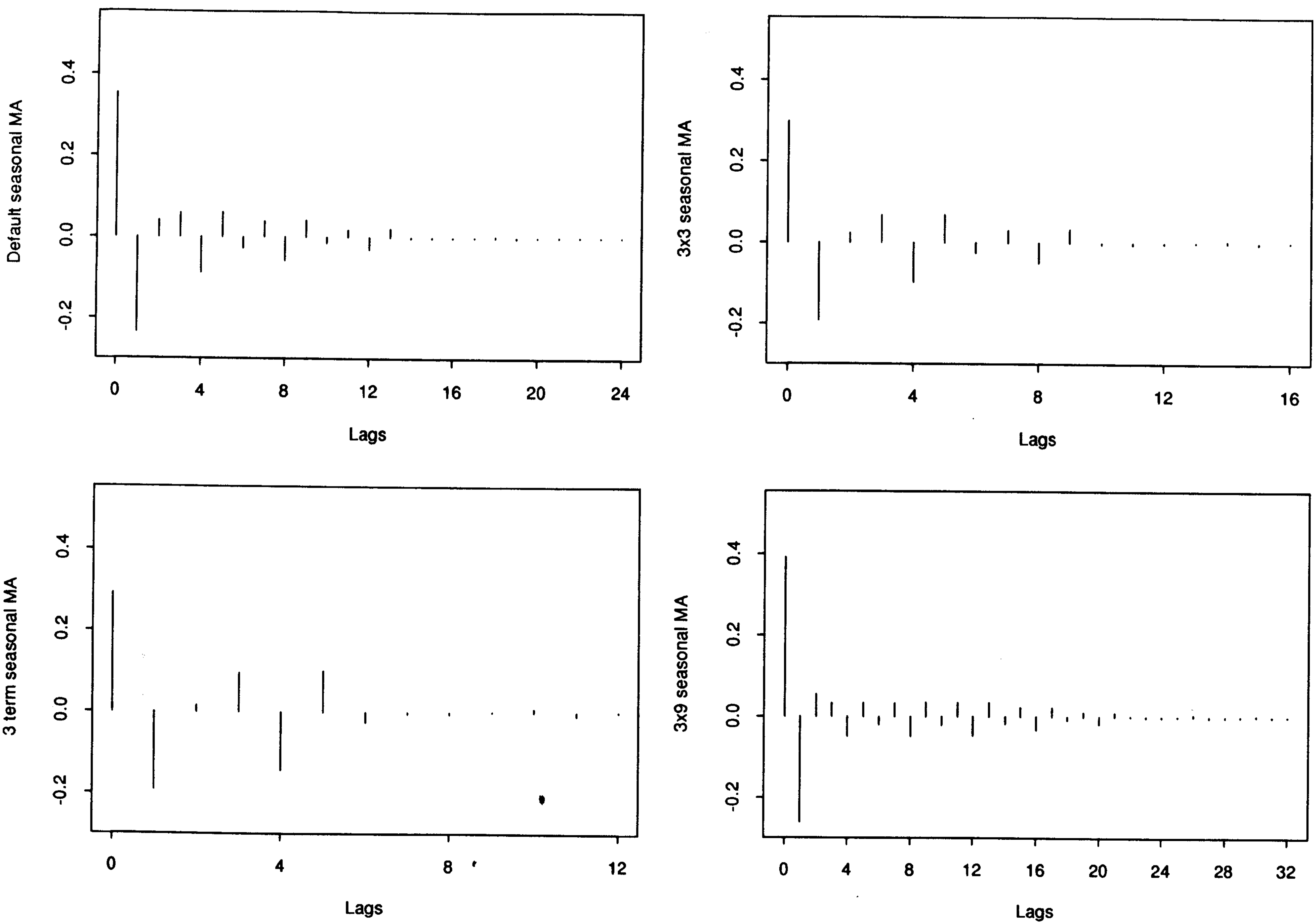
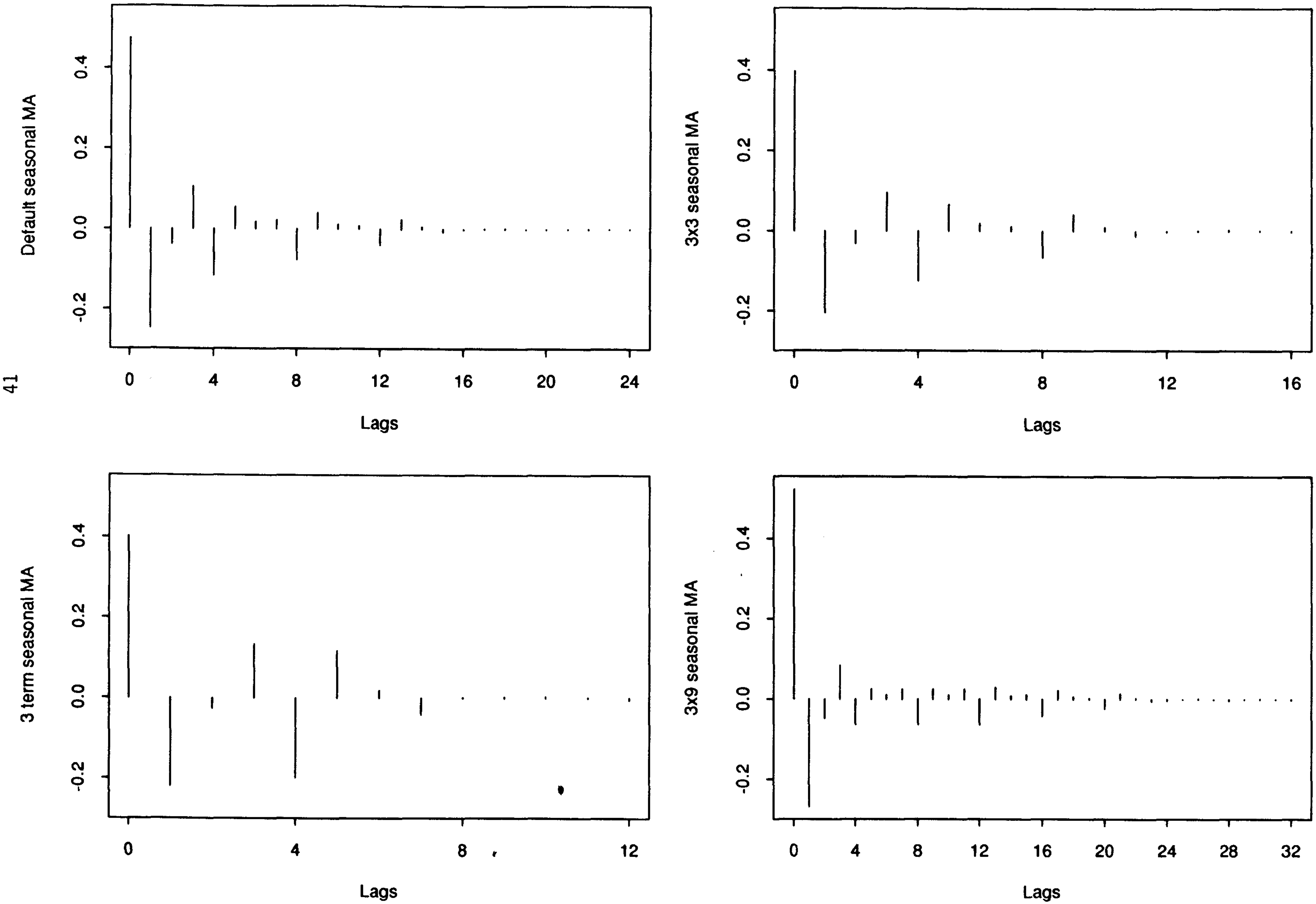
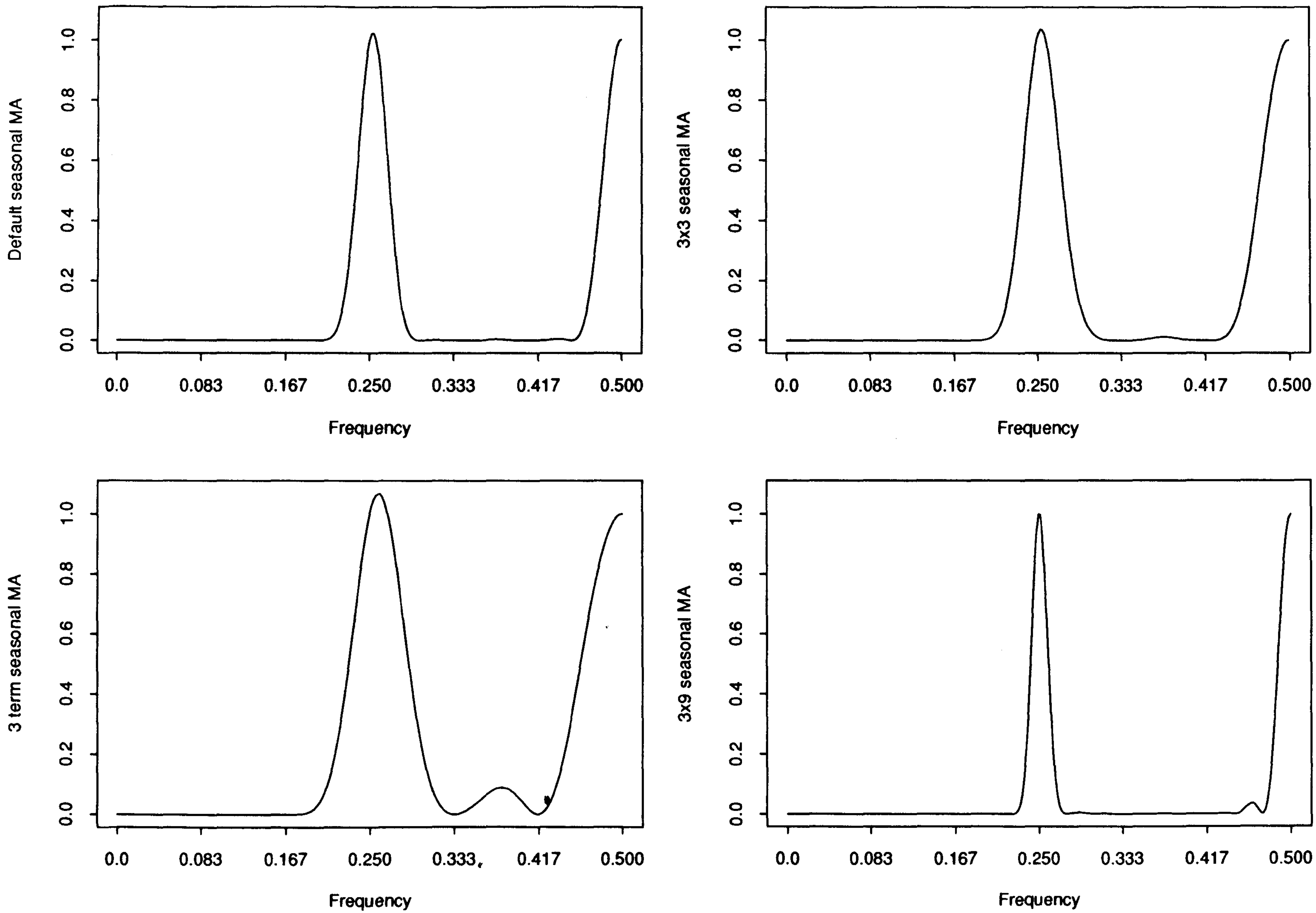


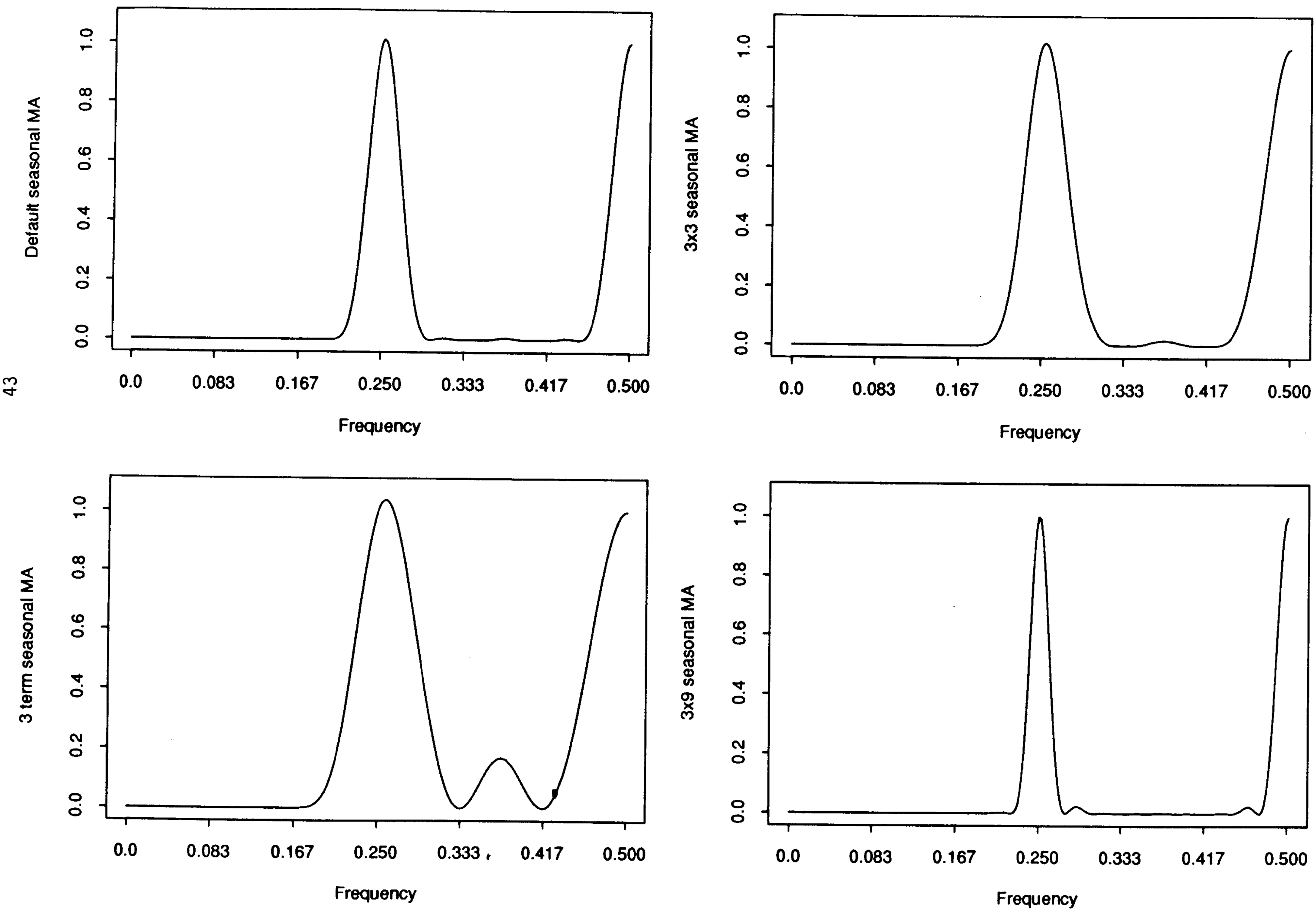
Quarterly irregular filter weights, 7-term Henderson



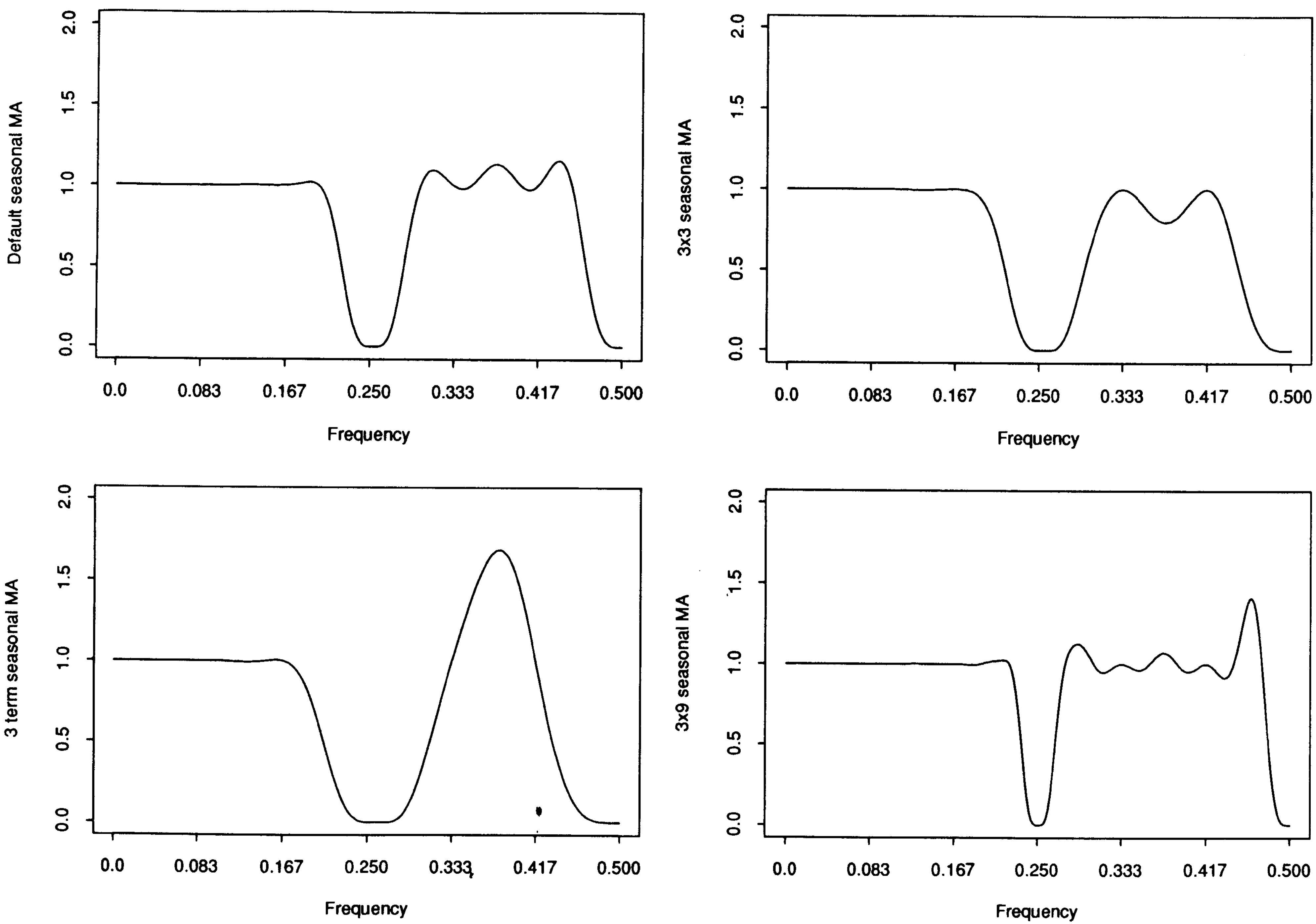
Quarterly seasonal filter squared gain functions, 5-term Henderson



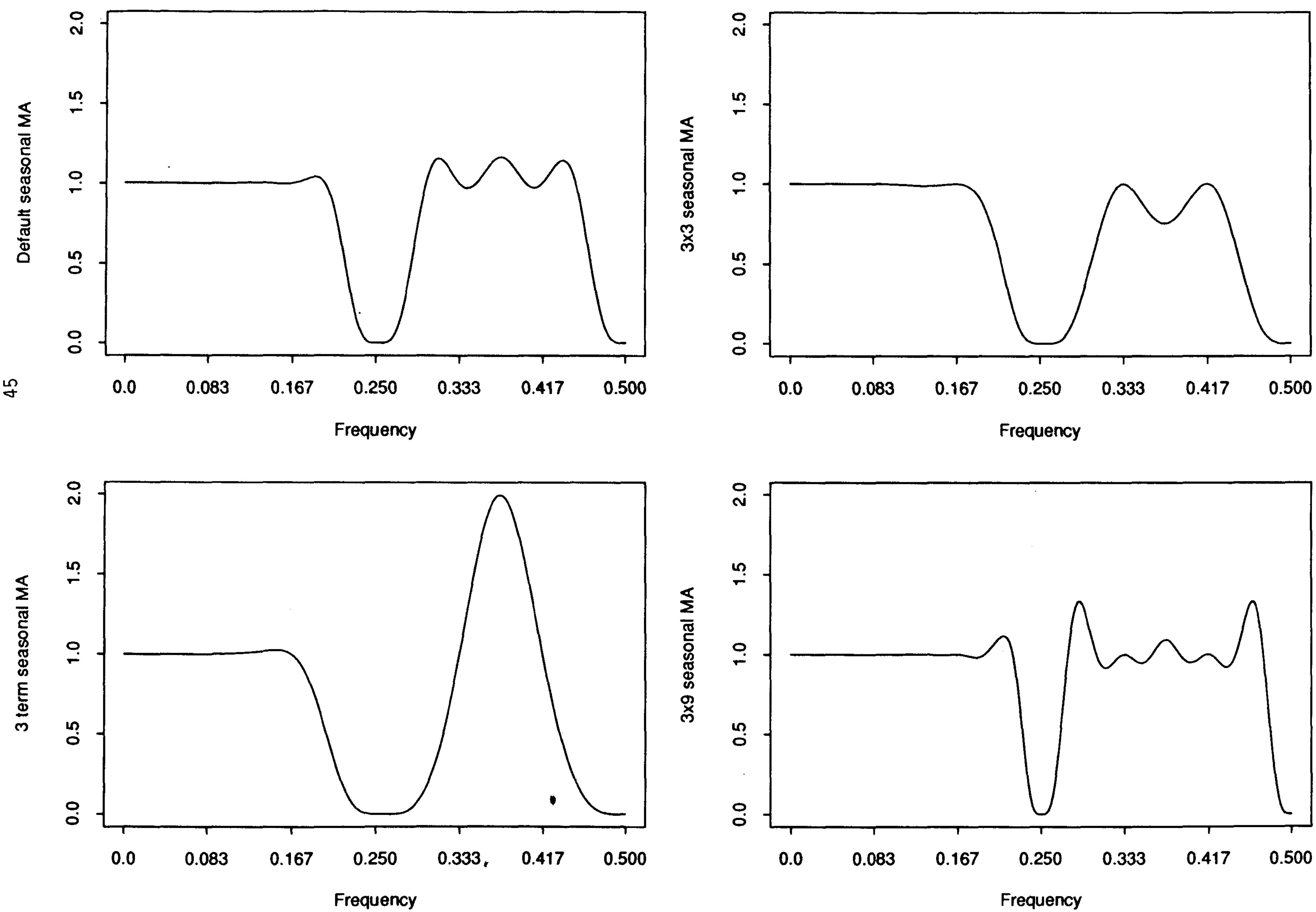
Quarterly seasonal filter squared gain functions, 7-term Henderson



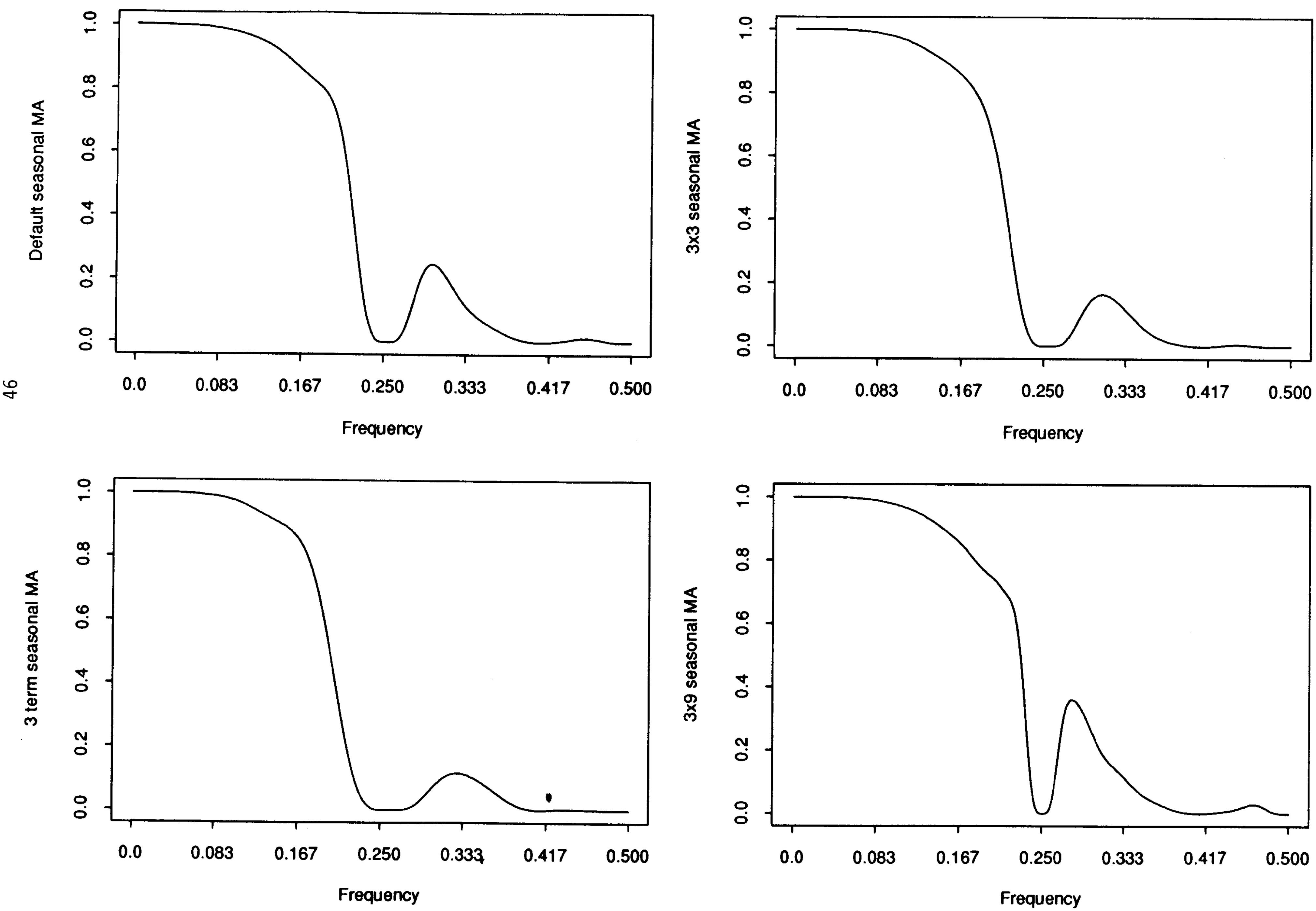
Quarterly adjustment filter squared gain functions, 5-term Henderson



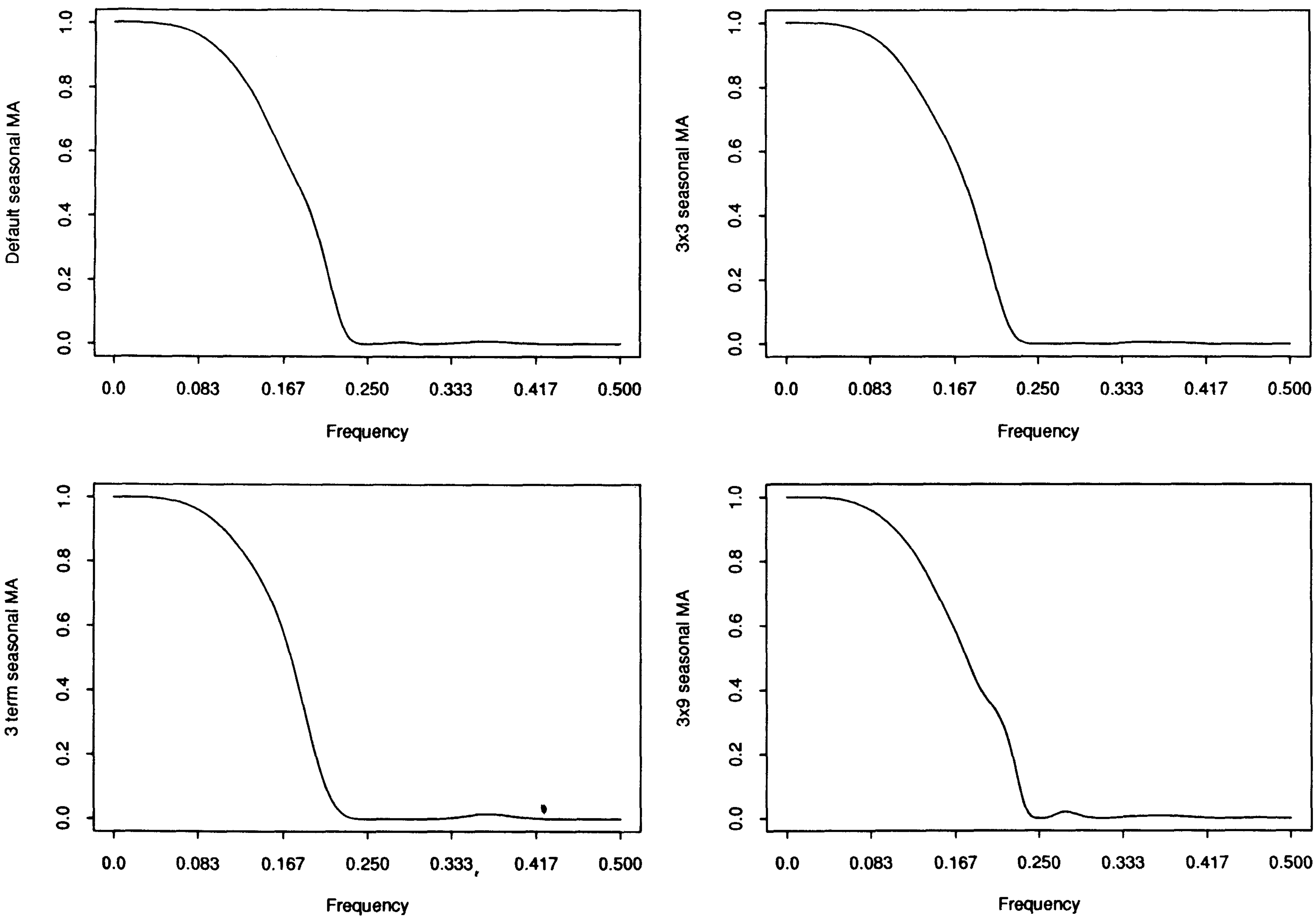
Quarterly adjustment filter squared gain functions, 7-term Henderson



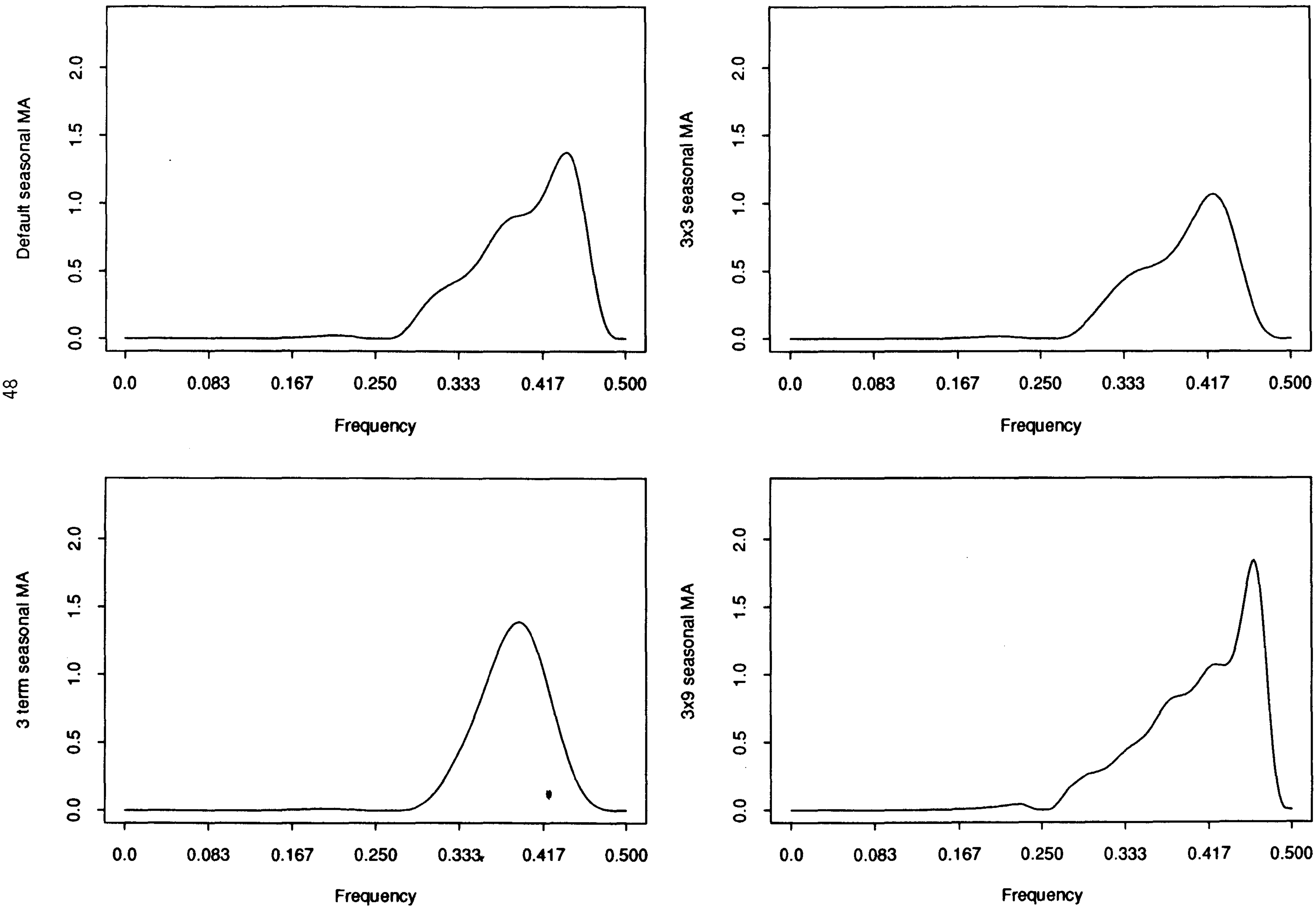
Quarterly trend filter squared gain functions, 5-term Henderson



Quarterly trend filter squared gain functions, 7-term Henderson



Quarterly irregular filter squared gain functions, 5-term Henderson



Quarterly irregular filter squared gain functions, 7-term Henderson