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In Noise We Trust?
Optimal Monetary Policy with Random Targets

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Abstract:
We show that a monetary policy in which the central bank commits to a randomized inflation target allows for potentially faster expectations convergence than with a fixed target. The randomized target achieves faster convergence in particular in transition environments: those demonstrating either particularly high or low inflation.

Keywords: monetary policy, asymmetric information, Bayesian rational expectations, commitment

JEL Codes: E52, E61, E42

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I. Introduction

This paper addresses the intersection of two lines of debate in monetary policy. The first of these is the degree to which central banks have the ability to influence the real economy in the short run. Perspectives on this topic are well-known. For discursive and illustrative purposes, and acknowledging recent advances, we use a particularly extreme position, complete rational expectations. Since our conclusions can be extended logically into less strict formulations, the use of the monetarist framework is appropriate. Under such conditions, Taylor's (1975) well known conclusion is that the monetary authority can impact the real economy only during an expectations formation period at the onset of a new policy regime. At all other times, monetary policy has well-understood limited effects on the real economy.

The second debate is the appropriate monetary policy framework or mechanism to use in transition environments. Such transitions are clearly important in both persistently high and persistently low inflation environments. Case examples of these include Latin America in the 1980s, Japan in the late 1990s, and Zimbabwe today. Current conclusions are that an inflation targeting regime can contribute to a relatively rapid inflation decline in conditions of hyperinflation (see, for example, Bernanke et al. 1999). The practice involves ever bolder declarations of inflation targets, and more highly skilled central bankers to implement the plans. In each of these cases, credibility plays a crucial role in determining the efficacy of the monetary policy regime—and arguably the appropriate method of monetary policy action. In high inflation environments, the population will be unlikely to believe announcements of low-inflation policy. Even under the assumption of fully rational expectations, if the central bank's promise is not credible, it will take time until the population's expectations converge to the actual central bank target. Similarly, given the Japanese populace's expectations of the central bank's low-inflation prejudice, they are unlikely to believe the bank's desire to raise inflation (see, for example, Krugman 1998). In practice, this "transitional expectations" period is a common occurrence in countries undergoing regime changes and/or in countries with "sticky" expectations (see Sachs 1990).

In our model, we find that randomization of monetary policy in transition cases provides a better response mechanism to the population's behavior, because it eliminates a portion of the information asymmetry problem. Through such a policy, the monetary authority can "encourage" rapid convergence without an increase in credibility or reputation. Consider the following example. Suppose a new government takes over monetary policy in a developing country with a "traditional" inflation rate of 40-50 percent a year. The new government "knows" that the optimal inflation level is 5 percent. However, since a 5 percent target would not be credible, the government would potentially have to announce a higher target or to implement a relatively slow convergence plan in an inflation targeting framework.[1] We show that if the government commits to a so-called "random target," the situation can change. The government would enact a policy whereby it commits to implement the targeting decisions of a computer run by some independent third party.[2] The computer in each period would generate a random number based on a target around 5 percent, with some small variance.[3] Notice that the government's commitment to a randomized policy is credible, because there is no *a priori* reason why the government is better off under the fixed target. Notice as well the distinction here from *inflation targeting* in that the stated commitment of the central bank is to a target that is itself moving, not to a band around a fixed target. Thus, the remaining credibility question is the ability of the bank to implement the computer-generated policy. We will show that this mechanism allows for faster expectations convergence than a fixed target.

Monetary policy randomization has potential application in at least three cases. Two of these cover the

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[1] We argue below that with non-zero discounting, the optimal inflation level is greater in an economy without government credibility.

[2] The third party is used to illustrate the feasibility of a trustworthy random draw. The computer could even be placed within the offices of a "trusted" institution outside the country.

3 In fact, the model will specify that the computer generate a random walk target, though for the sake of explanation, random is sufficient at this stage.

“high” inflation situation, hyperinflation and persistent inflation. In situations of hyperinflation, a faster convergence rate is clearly desirable, especially since most “losses” occur in the short run. The Japanese deflation-cum-liquidity trap case is also an example in which expectations are too sticky (this time downwards) and in which, for structural and credibility reasons a change in inflationary expectations is needed where traditional monetary policy has failed.

We have focused our discussion on such “extreme” cases because we believe lack of credibility to be an important limitation in the availability of monetary policy options for such countries. While credibility can be built over time, these countries may find themselves too often in a “transitional” period, in which credibility building is impractical. Our solution prescribing inflation targeting with a random target rate seems to provide at least some improvement in the efficacy of monetary policy in such cases.

The existing literature on gradual vs. “shock” approaches to transition draws either on credibility, or on trend or targeting approaches. Guillermo Calvo’s (1983) model of stochastic price adjustments suggests that immediate disinflation can be costless, and Laurence Ball (1994) suggests that a credible and rapid disinflation can even prompt a boom. Thomas Sargent (1994) has suggested that four historic cases of hyperinflation were halted through *credible* institutional change. While these go some distance in resolving new Keynesian concerns of wage and price friction in economic adjustment, they don’t capture the credibility constraints at hand in transition environments. The practical constraints implied in targeting approaches are briefly discussed above.

At the intersection of the two debates above, we discuss the interaction between central banks and the population under the framework of economic conditions of very high or very low inflation. As alluded to above, these cases are reasonably considered to have a set of characteristics dissimilar to “normal” times, and consequently a set of monetary policy actions that may reasonably be different as well. Of these, we point to two in particular. First, use of credibility as a monetary-policy enhancement tool can potentially result in the two cases mentioned above. In addition to being a successful mechanism for the direction of expectations, it can also fail (as in Latin America in the 1980s), or be too successful (Japan in the late 1990s). In each of these cases, the actual actions of the central bank take on the central role in the formation of expectations; thus, a monetarist style adjustment through intent announcement may be sub-optimal. Similarly, gradualist approaches that rely on perfect credibility will be constrained. Second, the use by the monetary authorities of a loss function that ignores the real cost of inflation can be inappropriate in some contexts.[4] See Woodford (1999) for a derivation of monetary policy loss functions from microeconomic foundations. See also Walsh (2003) and Clarida et al. (1999), where using the squared-deviation from optimal inflation as a determinant of social losses may not capture the importance of the expected inflation's deviation from the socially optimal level. This particular issue is not a new debate; see Rogoff (1985), for example, for discussion of anticipated and unanticipated inflation. This can be seen easily in the limiting case of hyperinflation, where the real costs to the economy of high inflation cannot be measured solely by the deviation from expected or target inflation. There is some real cost to the inflation itself.

With such a real cost, the government must optimize some social loss function that includes not only the cost of deviation from normal inflation, but also the cost of "sticky" inflationary expectations, that is, the cost of its own lack of credibility. It must do so in a manner that is consistent with the population's continued attempts to predict inflation. In what follows, we will construct a governmental loss function through which the government targets the population's inflationary expectations.[5] Such a social loss function seems more appropriate to situations of hyperinflation or chronic deflation than to "normal"

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4 The most common generalization of this is, $\Lambda_t = (n_t - n_t^*)^2 + \chi(\pi_t - \pi^*)^2$, where *n* is some macro-variable (real GDP, unemployment, etc.), n_t^* is the target level of the variable at time *t*, π is a measure of inflation, and π^* is the target level of inflation (Svensson, 1999).

5 By targeting public inflationary expectations, we simply mean that these expectations become part of the government's loss function in a way to be described in more detail below. We do not mean, as do Bernanke and Woodford (1997), that the government uses public expectations as a signal for some disturbance in the economy that the public alone can observe; in our framework the government has full information.

times. We then solve for the government's optimal inflation target in light of a population with rational expectations, but with incomplete information.

Two situations are discussed: a "classic" one in which the government commits to a particular inflation target; and our proposal for a "randomized" target, in which the government pre-commits to a particular randomization of its inflation target. In either case, the target inflation rate is unknown to the public. We cover in Section II the rationale behind the ability of the government to obtain real gains through monetary policy, even under rational expectations, and we address the effect on expectations formation of increased uncertainly in inflation policy. In order to achieve this increased uncertainty, we use a "random target." Specifically, this refers to a periodic change in the target inflation rate of the country's monetary authority. In each time period the target is moved up or down a "notch." These "notches" are normally distributed. The result is a random walk in the target inflation rate. Section III describes our model of monetary policy equilibrium in the case of a fixed monetary target. Subsequently, Section IV explains optimal monetary policy when the government chooses a random walk inflation target. The benefits of the randomized policy are discussed. We show simulated results in Section V and conclude in Section VI.

II. A transitional expectations framework

In a world of full information and fully rational expectations, monetary policy has limited effect. Indeed, economic agents, given full information, can rationally determine the inflation that the government wants to achieve, and adjust their prices and wages accordingly. This is clearly not the case in what Taylor (1975) describes as a transitional period towards rational expectations, a period during which the central bank's target inflation rate is either not known or not credible. Under such conditions of asymmetric information, Taylor shows that monetary policy has a significant impact in the short run, that is, until expectations fully adjust to reflect the actual chosen rate.

Taylor uses an output-inflation targeting loss function (that is, the loss is measured in squared deviations from some target output level and some target inflation rate, both of which are held constant across time). In particular, Taylor is concerned with the potential gains from generating some extra "unexpected" inflation. Indeed, suppose that for some reason (fiscal policy distortions, etc.) the current rate of inflation (which is fully known by the public and thus also expected by the public) is below the optimal rate. If the central bank unexpectedly moves to target the optimal rate, it will obtain some output gains in the transitional period until the population learns the new rate. The central bank can capitalize on creating "confusion" about the true inflation rate.

At the outset of this project, our hypothesis about the benefits of a random target was the opposite of our current conclusion. Consider the following logic: if an unexpected change in the inflation target generates some social gains (or reduce losses due to transition), then increasing the transition period should increase those gains (decrease losses). Indeed, if the central bank could "confuse" the public for a longer period, it would be better off than under the Taylor model. One might expect, as we did, that randomization would create this additional confusion. In fact, further randomization of the monetary rule led to an *increased* ability of the population to discover the target rate, and thus decreased the ability of the central bank to generate output gains during the period of transitional expectations.[6] Before illustrating this point, let us walk through Taylor's construction.

First, we explain, as Taylor has, the logic behind the inefficacy of monetary policy under rational expectations. Suppose the relationship between inflation and unemployment can be shown as[7]

$$\pi(t) = \phi(u(t)) + x(t)\,,\ \phi'(.) < 0\,,\ \phi(u^*) = 0\,,\quad u^* > 0. \tag{1}$$

6 The surprising result led to the formulation of the remainder of this paper: increased randomization can facilitate convergence to the government target rate—a particularly useful property in some situations.
7 For notational convenience and comparability purposes, we have generally adopted Taylor's notation.

Here $\pi(t)$ and $x(t)$ are the actual and expected rates of inflation, respectively, at each time t. Taylor further assumes that the monetary authority has direct control over the inflation rate through control of the money supply,[8] so that $\pi(t)$ is effectively a choice variable. Long-run effects on employment are not possible in (1), but short-run effects are. This is a well-known argument made by Phelps, who considers that inflationary expectations adapt according to the linear rule:

$$dx(t) = \beta[\pi(t) - x(t)]dt \,. \tag{2}$$

The government chooses an inflation target that maximizes:

$$\int_0^\infty e^{-\rho t} W(x,u)\, dt \,, \tag{3}$$

where $W(x,u)$ is the instantaneous social utility rate, u is the unemployment rate, and β and ρ are discount rates. The short-run effects mentioned above are obtained when $\pi(t)$ is systematically greater than $x(t)$. Rational expectations theory suggests that over a period of time, agents understand that their expectations based on (2) are biased and modify accordingly. Taylor explains that under rational expectations the formulation appears as:

$$x(t|t_0) = E[\pi(t)|I(t_0)] \,. \tag{4}$$

Here $I(t_0)$ represents information available at time t_0. This explains that under a deterministic policy with no uncertainty, $\phi[u(t)]=0$ and $u(t)=u^*$ for all t (Taylor 1975).

[8] We later relax this assumption in order to add reality and as the mechanism for further uncertainty.

III. Monetary policy commitment to a fixed target

From the above argument, it follows that some information asymmetry is necessary in order for the short-run Phillips curve to have an effect on output. We will thus model the monetary policy effects of central bank policies as an incomplete information game between the central bank and the public. The game is set in continuous time, so that stochastic random variables are used to model random disturbances.[9] The public does not have knowledge of the government's methods or desires, but is able to learn from the government's prior actions. This is an abstraction that seems appropriate in a situation of low government credibility. Essentially it assumes zero credibility – any announcement is equivalent to none at all. The public derives all new information from the actions themselves. Taylor illustrates such a system: during the time period when old beliefs mix with new information, the rational expectations hypothesis no longer holds, and monetary policy has real and significant effects.

THE MODEL

A. The Government (and Central Bank)

The government commits itself at time $t = 0$ to a target inflation rate for the economy, which we denote by μ. Once the target inflation rate is set, the economy generates an actual inflation rate $\pi(t)$. We assume that the government does not have perfect control over the actual value of the inflation rate that is generated. More precisely, $\pi(t)$ is modeled as a normally distributed random variable, where $E[\pi(t)] = \mu$ (thus, the government can control the average inflation rate it generates, but not the actual value).

[9] Taylor models the information asymmetry by having the government cum central bank follow an inflation target that is *unknown* to the public. We consider this construction to be a particularly realistic one.

Furthermore, we assume that $var[\pi(t)]=\sigma$, where σ is a known constant depending on the characteristics of the interaction between the central bank and the economy.[10]

To translate these ideas into stochastic calculus equations that can be solved, let us define price level at each time t as $p(t)$, and then let $z(t) = \log[p(t)]$. When $\sigma = 0$, the government has complete control over inflation, and by definition $\pi(t) = \mu(t)$. With $\sigma > 0$, inflation is generated through the diffusion process:

$$dz(t) = \mu dt + \sigma dv, \qquad (5)$$

where $v(t)$ is a standard Wiener process with a zero mean and unit variance (that is, at each time t, $dv(t)$ is randomly chosen from a normal distribution with mean zero and unit variance).

Once inflation is generated, the public observes it and (as discussed below) forms some inflationary expectations $x(t) = E_{public}[\pi(t)]$. We assume that the government knows the process through which these expectations are formed, so the government can infer $x(t)$.

The government also has a negative payoff function or loss function $L(t)$ (see Svensson 1999).[11] At this point, we will not specify L; suffice it to say that L depends on two arguments, actual inflation $\pi(t)$ and some other target variable (unemployment, or real GDP, etc.) which depends again on inflation and on public inflationary expectations, $x(t)$. Since by choosing $\mu(t)$ the government determines both $\pi(t)$ and $x(t)$, the government's goal is to choose a target μ that minimizes the expected value of the loss function. Indeed, both $\pi(t)$ and $x(t)$ are stochastic, and the government can control only their distribution, not the

10 A lower σ implies a more precise monetary policy. In particular, we assume that σ is the minimum possible variance; that is, the government cannot improve the precision with which it generates inflation beyond this point.
11 As discussed above, Taylor uses a social welfare function instead of a loss function. We use the loss function because the terminology is compatible with more recent papers on monetary policy targeting.

actual realized values. In a full commitment regime, the government chooses a value μ that minimizes the (discounted) expected future losses, $\int_0^\infty e^{-\delta t} E[L(t)]dt$.

B. The Public

At each time t, the public observes the inflation rate $\pi(t)$ generated in the economy. The public, however, does not have a full knowledge of the government's target inflation rate μ. More precisely, consider that at time zero the public expects the government's inflation target to be $\mu(0) = \mu_0$, different from μ. In other words, $E_{public}[\mu(0)] = x(0) = \mu_0$. Also, the public belief about the accuracy of this initial guess is such that $\text{var}_{\text{public}}(\mu(0)) = \sigma_0$. We assume throughout that the government knows both μ_0 and σ_0. The fact that the initial public expectations are different from the actual government target may result, as Taylor emphasizes, from the fact that the initial basis for expectations of inflation setting is a combination of expectations from a new regime and information from an old regime.[12] Moreover, the same situation can result from a lack of credibility of the "new" government. Such a construction captures the reality that an emerging market government's open commitment to a particularly "low" inflation level, μ, may not be entirely credible.

The public is assumed (as in Bernanke and Woodford 1997) to have a quadratic loss function, depending on the square of the difference between its inflationary expectations $x(t)$ and the true inflation $\pi(t)$. It follows that the public's goal is to generate at each time $t > 0$ the most appropriate inflationary expectations. The public observes the actual inflation generated at each time t, and based on it and on its current inflationary expectations "infers" the most likely inflation target chosen by the government and

12 Under complete rational expectations, the population would be able to calculate the government's initial action perfectly.

adjusts its own expectations. Note that this method assumes that the public knows σ, but not μ.[13] Under these conditions, it can be shown that the population's inflationary expectations are given by the following stochastic differential equation:

$$dx(t) = \frac{1}{t + \sigma^2 / \sigma_0^2}[dz(t) - x(t)dt]. \tag{6}$$

Note that since the above is a stochastic differential equation, the solution will also be stochastic. This implies that even under perfect information, the government will not be able to have deterministic knowledge of the expectations path *x(t)*. However, as it will become apparent shortly, the government only needs to know the mean and the variance of the populations' expectations. [14]

If we define:

$$\begin{aligned} \bar{x}(t) &= E_{government}[x(t)] \\ \tilde{x}(t) &= \text{var}_{government}[x(t)] = E_{government}[x(t)^2] - \bar{x}(t)^2, \end{aligned} \tag{7}$$

then, from the theory of linear stochastic differential equations (Arnold 1974) it follows that these deterministic variables are given by the following deterministic differential equations:

$$d\bar{x}(t) = [-\frac{1}{t + \sigma^2 / \sigma_0^2}\bar{x}(t) + \frac{\mu}{t + \sigma^2 / \sigma_0^2}]dt \text{ , with } \bar{x}(0) = \mu_0 \text{ , and} \tag{8}$$

$$d\tilde{x}(t) = [-\frac{2}{t + \sigma^2 / \sigma_0^2}\tilde{x}(t) + \frac{\sigma^2}{(t + \sigma^2 / \sigma_0^2)^2}]dt \text{ , with } \tilde{x}(0) = \sigma_0^2 . \tag{9}$$

Solving the above equations for the mean and variance of the inflation expectations we get:

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13 In particular, it follows that σ0 ≥ σ, because σ imposes a natural limit on the precision of the public's "guess."

14 This is the case because the loss functions that are currently used are quadratic forms.

$$\bar{x}(t) = \mu + \frac{\mu_0 - \mu}{t + \sigma^2 / \sigma_0^2} \text{, and} \tag{10}$$

$$\tilde{x}(t) = \frac{\sigma^2}{t + \sigma^2 / \sigma_0^2} . \tag{11}$$

From this knowledge of the expected path of people's expectations, the government can set μ appropriately to minimize some given loss function (which we outline below). The population, as implied above, simply attempts to have full understanding of the inflation rate in order to set prices and wages to match.

C. The loss function and the resulting Bayesian equilibrium

The loss function suggested by Taylor in order to derive the government's optimal policy is a traditional one (See Kydland and Prescott (1977) and Barro and Gordon (1983)), associated with a target output level (as well as inflation):

$$L_t = (\pi_t - \pi^*)^2 + \lambda(y_t - y^*)^2 . \tag{12}$$

However, such a function does not necessarily suggest that rapid transition is optimal; on the contrary, the government can benefit if the transition is delayed. To the extent that unanticipated inflation impacts unemployment, and thus output, the benefits of a rapid convergence to π^* depend on the magnitude of output loss and λ.

Such logic, however, seems to reflect poorly the conditions of emerging markets with systemically high inflation or the case of Japanese deflation. Clearly the output losses sustained by continued deflation in

Japan are not captured by the squared deviation term in the loss function. More precisely, by following the Rogoff (1985) reasoning, we use a loss function given by:

$$L(t) = A[x(t) - \mu]^2 + B[\pi(t) - \pi^*]^2. \tag{13}$$

Indeed, the Rogoff loss function is constructed as follows. The first term of Rogoff's loss function is the squared deviation from optimal unemployment. This deviation enters the function as a result of the loss created from "wrong" wage setting (this affects unemployment and thus output). Firms in his model set wages to equal expected inflation in the next period (in our model this corresponds to *x(t)*). However, the ideal wage that would generate the ideal unemployment is the wage firms would choose if they had perfect knowledge about future inflation. But in our model, if firms had perfect information they would choose a wage μ, as this is the true expected inflation. Since the firms do not know (or do not believe) μ, they wrongly set wages to *x(t)* instead. We thus suggest the introduction in the loss function of a term that measures the cost of "sticky" expectations. This term is weighted by a constant factor, *A*. See Williams (1999) and Rudebusch and Svensson (1999) for further discussion.[15] Rogoff's second term is the deviation from an "optimal" inflation level, π^*. We have included this term as well, weighted by a constant *B*.

Thus, the expected loss, *EL(t)* is given by:

$$EL(t) = A[E(x(t)) - \mu]^2 + B[E(\pi(t)) - \pi^*]^2 + A\,\mathrm{var}(x(t)) + B\,\mathrm{var}(\pi(t)), \tag{14}$$

but $E(x(t)) = \bar{x}(t)$; $E(\pi(t)) = \mu$; $\mathrm{var}(x(t)) = \tilde{x}(t)$ and $\mathrm{var}(\pi(t)) = \sigma^2$. Thus, the expected loss function can be re-written as:

15 Under the general targeting framework described by Svensson (1999), such a loss function corresponds to a combination of inflation targeting and inflationary expectations targeting, where the target inflation rate is π^*, and the target expectation is the actual policy choice μ.

$$EL(t) = A[\bar{x}(t) - \mu]^2 + B[\mu - \pi^*]^2 + A\tilde{x}(t) + B\sigma^2 . \quad (15)$$

In equilibrium, the government thus chooses μ so that $(\partial TL / \partial \mu) = 0$, where $TL = \int_0^\infty e^{\delta t} EL(t) dt$.

Solving for the optimal μ, we get:

$$\mu^* = \frac{\mu_0 (AI) + \pi^* (B / \delta)}{AI + B / \delta} \text{ , where } I = \int_0^\infty \frac{e^{-\delta t}}{(t + \sigma^2 / \sigma_0^2)^2} dt . \quad (16)$$

A Bayesian Nash Equilibrium exists in which the government chooses a target inflation rate as described above. The population does not know this rate, but its inflationary expectations converge to the above rate. Note that with discounting, the equilibrium target rate is not the socially optimal inflation rate. In particular, in a hyperinflationary situation, the government will choose a target rate that is higher than optimal (a lower rate would not be credible and would generate losses due to sticky expectations). With no discounting, the optimal policy is to choose the ideal inflation rate, and this is the same result as Taylor's similar model.

IV. Monetary policy commitment to a random target

We have seen from the above discussion that the government is constrained to choosing a sub-optimal target inflation rate in a situation in which its commitment is not credible. However, if we could manage to accelerate the rate at which the public changes its expectations, we would be able to draw closer to the socially optimal level of inflation. We model such a possibility below.

A. The Government (and Central Bank)

As before, the government chooses at time t = 0 some target inflation rate μ. However, at each time, the government "randomizes" its target around μ. More precisely, the government chooses the target rate $\mu(t)$ to follow a random walk with drift μ,[16] and with a variance ω^2. Now, inflation is generated according to the same rule as before, but it is more volatile, because both the government and the state of the economy add uncertainty regarding the final observed inflation rate level. We now have:

$$d\mu(t) = \omega dw \text{, and } \mu(0) = \mu \text{, and} \tag{17}$$

$$dz(t) = \mu(t)dt + \sigma dv \, . \tag{18}$$

Here, $v(t)$ and $w(t)$ are standard Wiener processes with a zero mean and unit variance, independent of each other.

B. The public

The public again attempts to "guess" the full-information expected inflation rate μ, based on its initial assumptions and on the observed inflation rate, which is an imperfect signal of the true target rate. The public's information set at time t consists of the actual inflation rate, and of the parameters σ and ω.

One can show that the population's inflationary expectations are given by the following stochastic differential equation:

[16] Note that the inclusion of a drift parameter here does not alter the key conclusion that convergence is faster under the random targeting framework. It however provides a potentially more reasonable parameterization of a government policy.

$$dx(t) = \alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1}[dz(t) - x(t)dt], \text{ where } \alpha = \omega / \sigma \text{ and } C = \frac{\omega\sigma + \sigma_0^2}{\omega\sigma - \sigma_0^2}. \quad (19)$$

Again, since this equation will generate stochastic solutions, we look for the (deterministic) paths of the mean and variance of the inflationary expectations:

$$d\overline{x}(t) = \left[-\alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1}\overline{x}(t) + \mu\alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1} \right] dt, \text{ with } \overline{x}(0) = \mu_0, \text{ and} \quad (20)$$

$$d\widetilde{x}(t) = \left[-2\alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1}\widetilde{x}(t) + \alpha^2 \left(\frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1} \right)^2 (\sigma^2 + \omega^2) \right] dt, \text{ with } \widetilde{x}(0) = \sigma_0^2. \quad (21)$$

Solving the above equations for the mean and variance of the inflation expectations we get:

$$\overline{x}(t) = \mu + (\mu_0 - \mu)\frac{2\alpha}{(\alpha + \beta)e^{\alpha t} + (\alpha - \beta)e^{-\alpha t}}, \text{ where } \alpha = \frac{\omega}{\sigma} \text{ and } \beta = \frac{\sigma_0^2}{\sigma^2}; \text{ and} \quad (22)$$

$$\widetilde{x}(t) = \frac{\alpha}{2}(1 + \alpha^2) + \frac{2\alpha^2(1 - \alpha^2)\beta - \alpha(1 + \alpha^2)(\alpha^2 - \beta^2)(1 + 2\alpha t) - \alpha(1 + \alpha^2)(\alpha - \beta)^2 e^{-2\alpha t}}{[(\alpha + \beta)e^{\alpha t} + (\alpha - \beta)e^{-\alpha t}]^2}. \quad (23)$$

C. The loss function and the Bayesian equilibrium

Again, intuitively, a fast convergence without the benefit of the monetary authority's full credibility is particularly useful in emerging market transition situations. The loss function is described by (13): $L(t) = A[x(t) - \mu]^2 + B[\pi(t) - \pi^*]^2$, and the expected loss function by (14): $EL(t) = A[E(x(t)) - \mu]^2 + B[E(\pi(t)) - \pi^*]^2 + A\,\mathrm{var}(x(t)) + B\,\mathrm{var}(\pi(t))$

Here, $E(x(t)) = \bar{x}(t)$; $E(\pi(t)) = \mu$; $\text{var}(x(t)) = \tilde{x}(t)$ and $\text{var}(\pi(t)) = \omega^2 + \sigma^2$.[17]

It follows that the expected loss function can be re-written as:

$$EL(t) = A[\bar{x}(t) - \mu]^2 + B[\mu - \pi^*]^2 + A\tilde{x}(t) + B(\omega^2 + \sigma^2) . \qquad (24)$$

Now, both μ and ω are choice variables, and the government thus chooses them so that

$$\frac{\partial TL}{\partial \mu} = \frac{\partial TL}{\partial \omega} = 0 \text{ , where } TL = \int_0^\infty e^{-\delta t} EL(t) dt . \qquad (25)$$

For simplicity, let's denote the ratio of ω/σ by α.. Since σ is a constant, choosing ω is the same as choosing α. Solving (25) for μ while holding ω (and thus α.) temporarily constant we get:

$$\mu^* = \frac{\mu_0(AJ) + \pi^*(B/\delta)}{AJ + B/\delta} \text{ , where } J \equiv \int_0^\infty \frac{4\alpha^2 e^{-\delta t}}{[(\alpha+\beta)e^{\alpha t} + (\alpha - \beta)e^{-\alpha t}]^2} dt,\ \alpha = \frac{\omega}{\sigma} \text{ and } \beta = \frac{\sigma_0^2}{\sigma^2}. \qquad (26)$$

At this point, μ is a function of α. To solve the problem completely, we would need to determine the value of α > 0 for which the value of the total loss function is minimized. Unfortunately, the equation is too complex to allow for a closed solution; instead some intuitive properties of the optimal policy will be derived.

First, let us note that expectations converge faster under the randomized target policy (it is apparent by inspection of equations (10) and (22) that the convergence under the fixed target is hyperbolic, while under the random target it is exponential). This result is apparently counterintuitive: by randomizing monetary policy, that is, creating further uncertainty for the public, the central bank actually makes the

[17] It has been suggested that a more appropriate formulation would have $\text{var}(\pi(t)) = \omega^2 t + \sigma^2$. The authors have tested this formulation and the findings are consistent under both.

public converge faster to its true target rate. The key to this apparent contradiction is that by randomizing the target, the central bank conveys *more* rather then *less* information about its true mean target rate.

Our best understanding of this effect is that the public is more capable of adapting to a band around some unknown mean than to the unknown mean itself. Intuitively, in the long-run, the public can assume that the mean of any distribution lies at the "center" of accumulated observations, a known band size, despite adding to uncertainty of the realization of the mean itself, allows observers to converge more quickly. Imagine giving the public knowledge that a band of width X = 3σ must capture over 99 percent of all observations. Observations that are separated by some distance close to X will permit close approximation of the mean. Under a system where a fixed mean has to be derived, two such observations allow less information.

We have thus shown that convergence is faster in the random target case, indicating that the method may be useful in a context in which the central bank needs to overcome sticky expectations. On the same grounds, it can be shown that $I(\sigma, \sigma_0, \delta) \geq J(\omega, \sigma, \sigma_0, \delta), \forall\, \omega > 0.$ If we now define the optimal target described in (26) as $\mu^*(\alpha)$ and the optimal commitment target defined in (16) by μ^*_F, we notice that both $\mu^*(\alpha)$ and μ^*_F are weighted averages of μ_0 and π^*, with J and I as weights on the side of μ_0. But since J is lower than I, it follows that under random targeting a lower weight is put on the public's initial inflation guess. Consequently, the resulting optimal target is closer to the socially optimal π^* under random targeting than under fixed targeting.[18]

The final question is whether or not the total loss associated with implementing the optimal random target

18 Note, incidentally, that this conclusion does not apply if there is no discounting of the future. If $\delta = 0$, both policies imply setting the target inflation rate to the optimal level π^*. It is easy to see why this is so: with no discounting of the future, the speed with which the convergence takes place is irrelevant compared with the level at which expectations converge. Thus, in order to prevent losses from some point *t* to infinity, expectations have to become arbitrarily close to π^*. This can only be achieved if the target inflation rate is set to π^* as well in both the fixed and the random scenarios.

policy is lower than the loss under a fixed target, that is, whether or not $TL_{Random}(\mu^*(\omega^*)) \leq TL_{Fixed}(\mu_F^*)$. Since we cannot solve directly for ω^*, we cannot check directly whether this inequality holds in general (or for particular values of the parameters *A*, *B*, σ, σ_0, δ). However, the inequality seems intuitively possible. Indeed, consider that we start with a fixed target policy μ^*_F. If at this point we introduce some randomization of the policy (say a small ω), we can reduce the target rate to $\mu^*(\omega)$. *W*hat are the effects of this small change on the total loss function? First, there is the negative effect of generating more volatile inflation; and this effect is weighted by the factor *B*. There exist, however, two positive effects. On the one hand, we have brought the inflationary policy closer to the social optimum, which decreases the value of the loss (again scaled by the factor *B*). On the other hand, because randomization leads to faster convergence of the inflationary expectations, we have a second improvement in the loss function, scaled by the factor *A*. Whether the two positive effects dominate the negative effect or not will most likely depend on the parameters of the problem, but it is conceivable that in the case of "mild" randomization (when $\alpha = \omega / \sigma \cong 0$), the negative effect on the volatility of inflation will not be dominant.

Indeed, it is possible to show mathematically that this is the case under fairly un-restrictive conditions on the parameters of the problem. If the initial guess of the public is not accurate, so that $\sigma_0 > \sigma$ (with strict inequality), it can be shown that there exists a continuous range of small values of α (and thus ω) for which the total loss associated with an optimal randomized policy is *strictly* lower than the total loss associated with an optimal fixed target. This finding suggests that for situations characterized by a high degree of uncertainty, randomized targeting both decreases social losses and moves the economy closer to the optimal inflation rate.[19]

19 The case $\sigma_0 = \sigma$ is more difficult to treat mathematically. For this case, we have constructed total loss curves, graphing the total loss in the random targeting scenario as a function of the choice of α. The optimal α should be chosen as the point where these curves bottom; however, the shape of the curves depends highly on the other parameters of the problem. As a result, depending on the particular values of the parameters, there may or may not exist values of α for which the total loss in random targeting is lower than the total loss in fixed targeting.

V. Some simulations and results

In this section we provide some basic illustrations of the theoretical results contained above. We simulate a policy game as described above with a 200 period path of inflation. In each period, given a set of initial parameters, we calculate expected inflation, mean squared error and total loss from both the Taylor baseline and the random policy. Our parameters are set as follows. Recall that σ is the variance of the inflation rate, σ_0 is the public's initial expectation of inflation variance, σ^* is the government's choice of a variance rate, μ^* is defined in equation (26), and μ_0 is the public's initial guess of the government inflation target. We choose parameters as follows:

TABLE I. Parameter set and their values.

Parameter	Value
σ^*	0.1
σ	0.1
σ_0	0.01
μ^*	3%
μ_0	15%

Chart 1, below, shows a sample case that might be relevant for a high inflation country. In this circumstance, a change in government policy or other structural shift might lead the population to believe that the government has a target of about 15 percent. Note that the exercise is not sensitive to this choice. The chart shows a sample time path of inflation under the two cases, with a target inflation rate of 3 percent and an initial public guess of the target of 15 percent. One can see that under this parameterization, the random policy leads to a more rapid convergence to the target rate than the Taylor version. Chart 2 repeats the exercise given a government in a deflationary environment. Consider a public that expects an optimal inflation rate of -1 percent.

We can also consider the squared error and mean squared error for each of these cases. Charts 3 and 4 show the squared error and mean squared error respectively for the high inflation case. Charts 5 and 6 show the same for the deflationary case.

VI. Policy implications and conclusions

Under rational expectations theory, central banks automatically achieve rapid convergence of public inflation beliefs to the actual rate of inflation. In many cases, this is not plausible monetary policy, given credibility constraints. If the public is ignorant of the parameters of the government's optimization problem, then the announcement of a new target will not be automatically credible. This may be particularly true in developing countries where, because of political instability, the central banks and/or government may minimize a private loss function that does not reflect the societal optimization. We find that in such situations, a government can achieve faster expectations convergence to an economy's optimal inflation rate through the use of a randomized inflation target.

References:

Arnold, L., 1974, Stochastic Differential Equations: Theory and Applications (John Wiley and Sons, New York).

Ball, L., 1984, Credible Disinflation with Staggered Price Setting, The American Economic Review 84, 282-289.

Barro, R. and D. Gordon, A Positive Theory of Monetary Policy in a Natural Rate Model, The Journal of Political Economy 91, 589-610.

Bernanke, B., T. Laubach, F. Mishkin, and A. Posen, 1999, Inflation Targeting: Lessons from the International Experience, (Princeton University Press, Princeton).

Bernanke, B. and M. Woodford, 1997, Inflation Forecasts and Monetary Policy, Journal of Money, Credit and Banking 29, 653-685.

Calvo, G., 1983, Staggered Prices in a Utility-Maximizing Framework, Journal of Monetary Economics 12, 393-98.

Clarida, R., J. Gali and M. Gertler, 1999, The Science of Monetary Policy: A New Keynesian Perspective, Journal of Economic Literature 37, 1661-1707.

Fleming, W., and R. Rishel, 1975, Deterministic and Stochastic Optimal Control. (Springer-Verlag, New York).

Krugman, P., 1998, It's Baaack, Japan's Slump and the Return of the Liquidity Trap, Brookings Papers on Economic Activity.

Kydland, F., and E. Prescott., Rules Rather than Discretion: The Inconsistency of Optimal Plans, The Journal of Political Economy 85, 473-492.

Rogoff, K., 1985, The Optimal Degree of Commitment to and Intermediate Monetary Target, Quarterly Journal of Economics, 1169-1189.

Rudebusch, G., and L. Svensson., 1999, Policy Rules for Inflation Targeting, in: J. B. Taylor, ed., Monetary Policy Rules, (University of Chicago Press, Chicago).

Sachs, J., Ed., 1990, Developing Country Debt and Economic Performance: Country Studies—Argentina, Bolivia, Brazil, Mexico. (University of Chicago Press, Chicago).

Sargent, T., 1994, The Ends of Four Big Inflations, in: M. Parkin, ed., The Theory of Inflation, (Aldershot: UK) 433-89.

Svensson, L., 1999, Inflation Targeting as a Monetary Policy Rule, Journal of Monetary Economics 43, 607-654.

Taylor, J., 1975, Monetary Policy during a Transition to Rational Expectations, The Journal of Political Economy 83, 1009-1022.

Walsh, C., 2003, Monetary Theory and Policy. (MIT Press, Cambridge).

Williams, J., 1999, Simple Rules for Monetary Policy, Finance and Economics Discussion Series Paper no. 12, Federal Reserve Board.

Woodford, M. 2001, Inflation Stabilization and Welfare, NBER Working Paper Series, Working Paper 8071.

Appendix I:

Determination of the public's expected inflation path under a fixed inflation target

As indicated, the public wants to determine the true government inflation target μ. The public cannot observe μ directly, but instead observes a noisy signal (inflation) that is based on μ. The equation of the signal is: $d[\log(p)] = dz(t) = \mu dt + \sigma dv$.

The public thus observes *z(t)*, and must infer the value of μ. This is known as a filtering problem; it was solved in continuous time by Kalman and Bucy (Fleming and Rishel 1975, pp. 133-140). The public is assumed to start with an initial guess μ_0, and then to develop conditional expectations based on current knowledge and the observed signal. The expectations value for μ, which we denote by *x(t)* follows the stochastic differential equation:

$$dx(t) = F(t)[dz(t) - x(t)dt], \text{ where } x(0) = E_{public}[\mu(0)] = \mu_0, \qquad \text{(A1)}$$

and $F(t) = R(t)/\sigma^2$, where *R(t)* follows the deterministic differential equation (known as the Ricatti equation):

$$dR(t) = -\frac{R(t)^2}{\sigma^2}dt \text{ with initial condition } R(0) = \sigma_0^2. \qquad \text{(A2)}$$

Solving for *R*, we get: $R(t) = 1/(t/\sigma^2 + 1/\sigma^2)$, and thus $F(t) = 1/(t + \sigma^2/\sigma_0^2)$.

This implies that the public's expectations follow the stochastic differential equation:

$$dx(t) = \frac{1}{t + \sigma^2/\sigma_0^2}[dz(t) - x(t)dt]. \qquad \text{(A3)}$$

Solving for *x* would generate a solution with a deterministic part and a stochastic part, because *x* must be a stochastic variable. Such a solution would not be particularly interesting, because the government is mostly interested in the expected mean and variance of the population's expectations. Let us define:

$$\bar{x}(t) = E_{government}[x(t)]$$
$$\tilde{x}(t) = \text{var}_{government}[x(t)] = E_{government}[x(t)^2] - \bar{x}(t)^2. \qquad \text{(A4)}$$

The government also knows μ, so it can rewrite (A3) as:

$$dx(t) = \frac{1}{t + \sigma^2 / \sigma_0^2}[\mu dt + \sigma dv - x(t)dt], \text{ or:} \qquad \text{(A5)}$$

$$dx(t) = \left[-\frac{1}{t + \sigma^2 / \sigma_0^2} x(t) + \frac{\mu}{t + \sigma^2 / \sigma_0^2} \right] dt + \frac{\sigma}{t + \sigma^2 / \sigma_0^2} dv. \qquad \text{(A6)}$$

Based on Arnold (1973, pp. 130), we know that the mean and variance of *x* follow the deterministic differential equations:

$$d\bar{x}(t) = \left[-\frac{1}{t + \sigma^2 / \sigma_0^2} \bar{x}(t) + \frac{\mu}{t + \sigma^2 / \sigma_0^2} \right] dt, \text{ with } \bar{x}(0) = \mu_0, \text{ and} \qquad \text{(A7)}$$

$$d\tilde{x}(t) = \left[-\frac{2}{t + \sigma^2 / \sigma_0^2} \tilde{x}(t) + \frac{\sigma^2}{(t + \sigma^2 / \sigma_0^2)^2} \right] dt, \text{ with } \tilde{x}(0) = \sigma_0^2. \qquad \text{(A8)}$$

Solving the above equations for the mean and variance of the inflation expectations we get:

$$\bar{x}(t) = \mu + \frac{\mu_0 - \mu}{t + \sigma^2 / \sigma_0^2} \text{ : the mean inflationary expectations; and} \qquad \text{(A9)}$$

$$\tilde{x}(t) = \frac{\sigma^2}{t + \sigma^2 / \sigma_0^2} \text{ : the variance of the inflationary expectations.} \qquad \text{(A10)}$$

In equilibrium, the population forms rational expectations based on a Bayesian rule. These expectations are described by (A3). The government knows (A3), and thus derives the deterministic paths of the expectations' mean and variance, (A9) and (A10), respectively. Based on these paths, and on its own loss function, the government finally chooses the optimal commitment level μ^*.

Appendix II:

Determination of the optimal government commitment with a fixed target

As justified in the text, the expected loss associated with monetary policy is given by:

$$EL(t) = A[\bar{x}(t) - \mu]^2 + B[\mu - \pi^*]^2 + A\tilde{x}(t) + B\sigma^2 . \qquad \text{(A11)}$$

The total loss is the integral of the (discounted) losses from the introduction of the policy to infinity:

$$TL = A(\mu - \mu_0)^2 \int_0^\infty \frac{e^{-\delta t}}{(t + \sigma^2/\sigma_0^2)^2} dt + B[\mu - \pi^*]^2 \int_0^\infty e^{-\delta t} dt + A\sigma^2 \int_0^\infty \frac{e^{-\delta t}}{t + \sigma^2/\sigma_0^2} dt + B\sigma^2 \int_0^\infty e^{-\delta t} dt . \text{(A12)}$$

The government then chooses μ to minimize the total loss, by setting the first derivative of the total loss with respect to μ to zero. (It is clear from the below expression that the second derivative is positive, so this is indeed a minimum).

$$\frac{\partial TL}{\partial \mu} = 2A(\mu - \mu_0)\int_0^\infty \frac{e^{-\delta t}}{(t + \sigma^2/\sigma_0^2)^2} dt + 2B[\mu - \pi^*]/\delta . \qquad \text{(A13)}$$

Let us define: $I = \int_0^\infty [e^{-\delta t}/(t + \sigma^2/\sigma_0^2)]dt$, which is a function of δ, σ, and σ_0, but not μ. Now, setting the first derivative of the total loss with respect to the target rate to zero, we get:

$\partial TL/\partial \mu = 2A(\mu - \mu_0)I + 2B[\mu - \pi^*]/\delta = 0 \rightarrow \mu(AI + B/\delta) = \mu_0 AI + \pi^* B/\delta$.

Finally, we obtain:

$$\mu^* = \frac{\mu_0(AI) + \pi^*(B/\delta)}{AI + B/\delta} \text{ , where } I = \int_0^\infty \frac{e^{-\delta t}}{(t + \sigma^2/\sigma_0^2)^2} dt . \qquad \text{(A14)}$$

Appendix III:

Determination of the public's expected inflation path under random targeting

As indicated in the text, the public wants to determine the true government inflation target μ. The public knows that μ is generated randomly so that $d\mu(t) = \omega dw$, but cannot observe μ directly. The public instead observes a noisy signal (inflation), which is based on μ. The equation of the signal is:

$$d[\log(p)] = dz(t) = \mu(t)dt + \sigma dv \ . \qquad \text{(A15)}$$

The public thus observes *z(t)*, and must infer the value of $\mu(t)$. The Kalman-Bucy solution for the filtering problem is described by the stochastic differential equation:

$$dx(t) = F(t)[dz(t) - x(t)dt], \text{ where } x(0) = E_{public}[\mu(0)] = \mu_0 \ . \qquad \text{(A16)}$$

As before, $F(t) = R(t)/\sigma^2$, and *R(t)* follows the deterministic Ricatti equation:

$$dR(t) = \left[-\frac{R(t)^2}{\sigma^2} + \omega^2\right]dt \text{ with initial condition } R(0) = \sigma_0^2 \ . \qquad \text{(A17)}$$

Solving for *R*, we get:

$$R(t) = \sigma\omega\frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1}\ , \text{ where } \alpha = \omega/\sigma \text{ and } C = \frac{\omega\sigma + \sigma_0^2}{\omega\sigma - \sigma_0^2}\ , \qquad \text{(A18)}$$

and thus:

$$F(t) = \alpha\frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1}\ , \text{ where } \alpha = \omega/\sigma \text{ and } C = \frac{\omega\sigma + \sigma_0^2}{\omega\sigma - \sigma_0^2}\ . \qquad \text{(A19)}$$

This implies that the public's expectations follow the stochastic differential equation:

$$dx(t) = \alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1}[dz(t) - x(t)dt]. \tag{A20}$$

Let us define as before:

$$\begin{aligned} \bar{x}(t) &= E_{government}[x(t)] \\ \tilde{x}(t) &= \text{var}_{government}[x(t)] = E_{government}[x(t)^2 - \bar{x}(t)] \end{aligned} . \tag{A21}$$

Rewriting (A20) based on (A15) and the definition of $\mu(t)$, we obtain:

$$dx(t) = \alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1}[\mu dt + \sigma dv - x(t)dt] \text{, or:} \tag{A22}$$

$$dx(t) = \left[-\alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1} x(t) + \mu \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1} \right] dt + \sigma\alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1} dv \tag{A23}$$

Based on Arnold (1973) the mean and variance of *x* follow the deterministic differential equations:

$$d\bar{x}(t) = \left[-\alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1} \bar{x}(t) + \mu\alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1} \right] dt \text{, with } \bar{x}(0) = \mu_0 \text{, and} \tag{A24}$$

$$d\tilde{x}(t) = \left[-2\alpha \frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1} \tilde{x}(t) + \alpha^2 \left(\frac{Ce^{2\alpha t} - 1}{Ce^{2\alpha t} + 1} \right)^2 (\sigma^2 + \omega^2) \right] dt \text{, with } \tilde{x}(0) = \sigma_0^2 . \tag{A25}$$

Solving the above equations for the mean and variance of the inflation expectations we get:

$$\bar{x}(t) = \mu + (\mu_0 - \mu) \frac{2\alpha}{(\alpha + \beta)e^{\alpha t} + (\alpha - \beta)e^{-\alpha t}} \text{, where } \alpha = \frac{\omega}{\sigma} \text{ and } \beta = \frac{\sigma_0^2}{\sigma^2} \text{, and} \tag{A26}$$

$$\tilde{x}(t) = \frac{\alpha}{2}(1+\alpha^2) + \frac{2\alpha^2(1-\alpha^2)\beta - \alpha(1+\alpha^2)(\alpha^2-\beta^2)(1+2\alpha t) - \alpha(1+\alpha^2)(\alpha-\beta)^2 e^{-2\alpha t}}{[(\alpha+\beta)e^{\alpha t} + (\alpha-\beta)e^{-\alpha t}]^2} \text{ . (A27)}$$

Appendix IV:

Determination of the optimal government commitment with a random target

As before, the expected loss at each time *t* is given by:

$$EL(t) = A[\bar{x}(t) - \mu]^2 + B[\mu - \pi^*]^2 + A\tilde{x}(t) + B(\sigma^2 + \omega^2) \text{ .} \qquad \text{(A28)}$$

It follows that the total loss is

$$TL = A(\mu-\mu_0)^2 \int_0^\infty \frac{4\sigma^2\omega^2 e^{-\delta t}}{[(\sigma\omega+\sigma_0^2)e^{\alpha t} + (\sigma\omega-\sigma_0^2)e^{-\alpha t}]^2} dt + B[\mu-\pi^*]^2 \int_0^\infty e^{-\delta t} dt + A\int_0^\infty e^{-\delta t}\tilde{x}(t) dt + B(\sigma^2+\omega^2)\int_0^\infty e^{-\delta t} dt \text{ .} \quad \text{(A29)}$$

To optimize governmental policy, let us first determine the best monetary target under the temporary assumption that ω is constant (that is, not a choice variable). This is achieved by setting the derivative of the total loss function with respect to the target rate to zero:

$$\frac{\partial TL}{\partial \mu} = 2A(\mu-\mu_0)\int_0^\infty \frac{(4\sigma^2\omega^2)e^{-\delta t}}{[(\sigma\omega+\sigma_0^2)e^{\alpha t} + (\sigma\omega-\sigma_0^2)e^{-\alpha t}]^2} dt + 2B[\mu-\pi^*]/\delta \text{ .} \qquad \text{(A30)}$$

Let us define: $J(\omega) \equiv \int_0^\infty [(4\sigma^2\omega^2)e^{-\delta t}]/[(\sigma\omega+\sigma_0^2)e^{\alpha t} + (\sigma\omega-\sigma_0^2)e^{\alpha t}]^2 dt$, which again does not depend on μ. Now, setting the first derivative of the total loss with respect to the target rate to zero, we get:

$\partial TL/\partial\mu = 2A(\mu-\mu_0)J + 2B[\mu-\pi^*]/\delta = 0 \rightarrow \mu(AJ + B/\delta) = \mu_0 AJ + \pi^* B/\delta$.

Finally, we obtain

$$\mu^*(\omega) = \frac{\mu_0(AJ(\omega)) + \pi^*(B/\delta)}{AJ(\omega) + B/\delta} \quad \text{,where} \tag{A31}$$

$$J \equiv \int_0^\infty \frac{4\alpha^2 e^{-\delta t}}{[(\alpha+\beta)e^{\alpha t} + (\alpha-\beta)e^{-\alpha t}]^2} dt,\ \alpha = \frac{\omega}{\sigma} \text{ and } \beta = \frac{\sigma_0^2}{\sigma^2}.$$

In equilibrium, the government should choose a target inflation rate described by (A31) for any particular (predetermined) level of randomization ω. Subsequently, the government should choose the particular level of randomization ω^* that minimizes the total loss function (A29). Due to the complexities of the equations involved, a closed solution for ω^* is not available.

Appendix V

Comparison of expectations convergence speed under fixed and random targets

We attempt to show that the (expected) convergence of the expectations towards the true target rate is faster under random targeting (A26) than under fixed targeting (A9).

Let $f(t) = e^{\alpha t}, t \in \Re$ be an exponential function, with $\alpha = \omega/\sigma$ as above. Since f is convex, we have from Jensen's inequality:

$$\begin{aligned} &s \cdot f(t_1) + (1-s)\cdot f(t_2) \geq f[s\cdot t_1 + (1-s)\cdot t_2], \\ &\forall\, t_1, t_2, s \in \Re \end{aligned} \tag{A32}$$

In particular, let us set: $t_1 = t$; $t_2 = -t$, and $s = (\varpi\omega + \sigma_0^2)/2\sigma\omega$.

From the above inequality (A32), we get:

$$\frac{\sigma\omega + \sigma_0^2}{2\sigma\omega} e^{\alpha t} + \frac{\sigma\omega - \sigma_0^2}{2\sigma\omega} e^{-\alpha t} \geq e^{\alpha \frac{\sigma_0^2}{\sigma\omega} t} = e^{\frac{\sigma_0^2}{\sigma^2} t}. \tag{A33}$$

But for any value q, we know that $e^q \geq 1 + q$. In particular:

$$e^{\frac{\sigma_0^2}{\sigma^2}t} \geq 1 + \frac{\sigma_0^2}{\sigma^2}t = \frac{\sigma_0^2}{\sigma^2}(t + \frac{\sigma^2}{\sigma_0^2}) . \tag{A34}$$

Based on the fact that σ corresponds to the maximum inflation generation precision, it follows that:

$$\sigma \leq \sigma_0, \text{ and so we also have: } \frac{\sigma_0^2}{\sigma^2} \geq 1. \tag{A35}$$

Combining (A33), (A34) and (A35), we get:

$$\frac{\sigma\omega + \sigma_0^2}{2\sigma\omega}e^{\alpha t} + \frac{\sigma\omega - \sigma_0^2}{2\sigma\omega}e^{-\alpha t} \geq t + \frac{\sigma^2}{\sigma_0^2} . \tag{A36}$$

Inverting, we obtain:

$$\left(\frac{\sigma\omega + \sigma_0^2}{2\sigma\omega}e^{\alpha t} + \frac{\sigma\omega - \sigma_0^2}{2\sigma\omega}e^{-\alpha t}\right)^{-1} \leq \left(t + \frac{\sigma^2}{\sigma_0^2}\right)^{-1} . \tag{A37}$$

Multiplying by μ_0 - μ we obtain:

$$\frac{2\sigma\omega(\mu_0 - \mu)}{(\sigma\omega + \sigma_0^2)e^{\alpha t} + (\sigma\omega - \sigma_0^2)e^{-\alpha t}} \leq \frac{\mu_0 - \mu}{t + \frac{\sigma^2}{\sigma_0^2}}, \text{ iff } \mu < \mu_0. \text{ And,} \tag{A38a}$$

$$\frac{2\sigma\omega(\mu_0 - \mu)}{(\sigma\omega + \sigma_0^2)e^{\alpha t} + (\sigma\omega - \sigma_0^2)e^{-\alpha t}} \geq \frac{\mu_0 - \mu}{t + \frac{\sigma^2}{\sigma_0^2}}, \text{ iff } \mu > \mu_0. \tag{A38b}$$

The above equations, together with the equations (A26) and (A9), show that the convergence of $\bar{x}$ towards μ is always faster under random targeting than under fixed rate targeting.

Moreover, if we compare (A14) and (A31), with (A37), we find that the weighting factor I that appears in the fixed rate optimization problem is always higher than the factor J that appears in the random rate optimization. Indeed, squaring (A37) and using the definitions of I and J, we obtain that $I \geq J(\omega), \forall\, \omega > 0$. In particular, this implies that the optimal target under randomization will be always closer to the socially desired level π^* than will the optimal rate under a fixed target policy.

Appendix VI

Comparison of total losses under fixed and random targets

We have shown that optimality in the fixed and randomized targeting cases requires the government to choose μ_F^* and α^*, $\mu_R^*(\alpha^*)$ respectively. These choices generate respective values of the loss function of $TL^*_F = TL(\mu_F^*)$ and $TL^*_R = TL(\mu_R^*(\alpha^*))$. Below, we establish a sufficient (but not necessary) condition for TL^*_R to be lower than TL^*_F.

For simplicity, calculations below are carried out using the symbols $\alpha = \omega / \sigma$ and $\beta = \sigma_0^2 / \sigma^2$.

Lemma 1: Under random targeting, $\lim_{\alpha \to 0} \tilde{x}(t) = \sigma^2[\beta(1+t))/(1+\beta t)^2], \forall t > 0$.

Proof of Lemma 1*:* Starting with the formula for $\tilde{x}(t)$ in (A27), we apply the L'Hospital rule twice, since α is the only variable factor. The fact that the resulting limit exists and is finite proves the lemma (for clarity of exposition, actual calculations are skipped here).

Lemma 2*:* Let μ_F^* be the optimal inflation target chosen under fixed targeting, and $TL^*_F = TL(\mu_F^*)$ the total loss associated with it. Also, for a given $\alpha > 0$, let $\mu_R^*(\alpha)$ be the optimal inflation target chosen under random targeting, and $TL_R(\alpha) = TL(\mu_R^*(\alpha))$. Then, the following relation holds: $\lim_{\alpha \to 0} TL_R(\mu_R^*(\alpha)) \leq TL_F^*$, with equality iff $\beta = 1$.

Proof of Lemma 2*:* We know that

$TL_F - TL_R(\mu_R^*(\alpha)) =$

$$A(\mu-\mu_0)^2\int_0^\infty \frac{e^{-\delta t}}{(t+\sigma^2/\sigma_0^2)^2}dt + B[\mu-\pi^*]^2\int_0^\infty e^{-\delta t}dt + A\sigma^2\int_0^\infty \frac{e^{-\delta t}}{t+\sigma^2/\sigma_0^2}dt + B\sigma^2\int_0^\infty e^{-\delta t}dt -$$

$$- A(\mu-\mu_0)^2\int_0^\infty \frac{4\sigma^2\omega^2 e^{-\delta t}}{[(\sigma\omega+\sigma_0^2)e^{\alpha t}+(\sigma\omega-\sigma_0^2)e^{-\alpha t}]^2}dt - B[\mu-\pi^*]^2\int_0^\infty e^{-\delta t}dt - A\int_0^\infty e^{-\delta t}\tilde{x}(t)dt - B(\sigma^2+\omega^2)\int_0^\infty e^{-\delta t}dt \cdot \tag{A39}$$

In the above equality, let us take $\alpha \to 0$, which also implies $\omega \to 0$. First of all, from the derivation in Appendix V (or simply applying the l'Hospital rule twice) we have that:

$$\lim_{\alpha\to 0} A(\mu-\mu_0)^2\int_0^\infty \frac{4\sigma^2\omega^2 e^{-\delta t}}{[(\sigma\omega+\sigma_0^2)e^{\alpha t}+(\sigma\omega-\sigma_0^2)e^{-\alpha t}]^2}dt = A(\mu-\mu_0)^2\int_0^\infty [\lim_{\alpha\to 0}\frac{4\sigma^2\omega^2 e^{-\delta t}}{[(\sigma\omega+\sigma_0^2)e^{\alpha t}+(\sigma\omega-\sigma_0^2)e^{-\alpha t}]^2}]dt =$$

$$= A(\mu-\mu_0)^2\int_0^\infty \frac{e^{-\delta t}}{(t+\sigma^2/\sigma_0^2)^2}dt. \tag{A39a}$$

Second, it is trivial to see that:

$$\lim_{\alpha\to 0} B(\sigma^2+\omega^2)\int_0^\infty e^{-\delta t}dt = B(\sigma^2+\lim_{\alpha\to 0}\omega^2)\int_0^\infty e^{-\delta t}dt = B\sigma^2\int_0^\infty e^{-\delta t}dt. \tag{A40}$$

Using the above (A39) and (A40) results, we obtain:

$$TL_F - \lim_{\alpha\to 0} TL_R(\mu_R^*(\alpha)) = A\sigma^2\int_0^\infty \frac{e^{-\delta t}}{t+\sigma^2/\sigma_0^2}dt - A\int_0^\infty e^{-\delta t}[\lim_{\alpha\to 0}\tilde{x}(t)]dt\,. \tag{A41}$$

Based on Lemma 1, we can simplify further:

$$TL_F - \lim_{\alpha\to 0} TL_R(\mu_R^*(\alpha)) = A\sigma^2\int_0^\infty e^{-\delta t}\left[\frac{1}{t+1/\beta} - \frac{\beta(1+t)}{(1+\beta t)^2}\right]dt = A\sigma^2\int_0^\infty e^{-\delta t}\left[\frac{\beta}{1+\beta t} - \frac{\beta(1+t)}{(1+\beta t)^2}\right]dt =$$

$$= A\sigma^2\int_0^\infty e^{-\delta t}\left[\frac{\beta+\beta^2 t-\beta-\beta t}{(1+\beta t)^2}\right]dt = = A\sigma^2\int_0^\infty e^{-\delta t}\frac{\beta t(\beta-1)}{(1+\beta t)^2}dt\,. \tag{A42}$$

Since by construction $\beta \geq 1$, it follows that $A\sigma^2 \int_0^\infty e^{-\delta t}[(\beta t(\beta - 1))/(1+\beta t)^2 dt \geq 0$, with equality iff $\beta = 1$.

Combining this with (A42), we reach the desired conclusion, namely, that:

$$\lim_{\alpha \to 0} TL_R(\mu_R^*(\alpha)) \leq TL_F^*, \text{ with equality iff } \beta = 1. \tag{A43}$$

Lemma 3*:* With the same notation as above, $\lim_{\alpha \to \infty} TL_R(\mu_R^*(\alpha)) = \infty$.

Proof of Lemma 3*:* The result is derived trivially by passing α to infinity in (A28) and noticing that the first three terms are all positive and that the last one is infinite.

Theorem 1*:* If $\sigma_0 > \sigma$, that is, if $\beta > 1$, then $\exists \overline{\omega} > 0$ (and thus $\overline{\alpha} = \overline{\omega} / \sigma$) , such that

$$TL_R(\mu_R^*(\alpha)) < TL_F \forall \alpha \in (0, \overline{\alpha}) .$$

Proof of Theorem 1*:* Since $\beta > 1$, we know from Lemma 2 that $\lim_{\alpha \to 0} TL_R(\mu_R^*(\alpha)) < TL_F^*$, This implies that there exists a neighborhood of zero, say, $N = (0, 2\varepsilon)$ with $\varepsilon > 0$, such that:

$$TL_R(\mu_R^*(\alpha)) < TL_F^*, \forall \varepsilon \in N = (0, 2\varepsilon). \tag{A44}$$

In particular, it follows that:

$$\exists \varepsilon > 0 \therefore TL_R(\mu_R^*(\varepsilon)) < TL_F^*. \tag{A44a}$$

From Lemma 3, we know that $\lim_{\alpha \to \infty} TL_R(\mu_R^*(\alpha)) = \infty$. This implies that there exists a neighborhood of infinity, say $N' = (\delta/2, \infty)$ with $\delta > \varepsilon > 0$, such that:

$$TL_R(\mu_R^*(\alpha)) < TL_F^*, \forall \varepsilon \in N' = (\delta / 2, \infty). \tag{A45}$$

In particular, it follows that:

$$\exists \delta > \varepsilon \therefore TL_R(\mu_R^*(\delta)) > TL_F^* \tag{A45a}$$

The expressions (A44a) and (A45a) show that the function $TL_R(\mu_R^*(\cdot))$ takes values both below and above TL^*_F. But $TL_R(\mu_R^*(\alpha))$ is a continuous function of α, and thus (A44a) and (A45a) guarantee that:

$$\exists \hat{\alpha} \in (\varepsilon, \delta) \therefore TL_R(\mu_R^*(\hat{\alpha})) = TL_F. \tag{A46}$$

Let us now define the set A as follows: $A = \{x > 0 \mid TL_R(\mu_R^*(x)) = TL_F^*$. Expression (A46) guarantees that the set A is non-empty. Let $\overline{\alpha} = \inf_x A$. We can easily see that $\overline{\alpha} > 0$. Indeed, if this were not the case, there would exist points x arbitrarily close to zero such that $TL_R(\mu_R^*(x)) = TL_F$. But this contradicts (A45), which says that there exists an entire neighborhood of zero with points for which the relation above is not true.

We contend now that the $\overline{\alpha} > 0$ as defined above has the desired property: $TL_R(\mu_R^*(\alpha)) < TL_F^* \forall \varepsilon \in (0, \overline{\alpha})$.

Indeed, suppose by contradiction that this were not the case, that is,

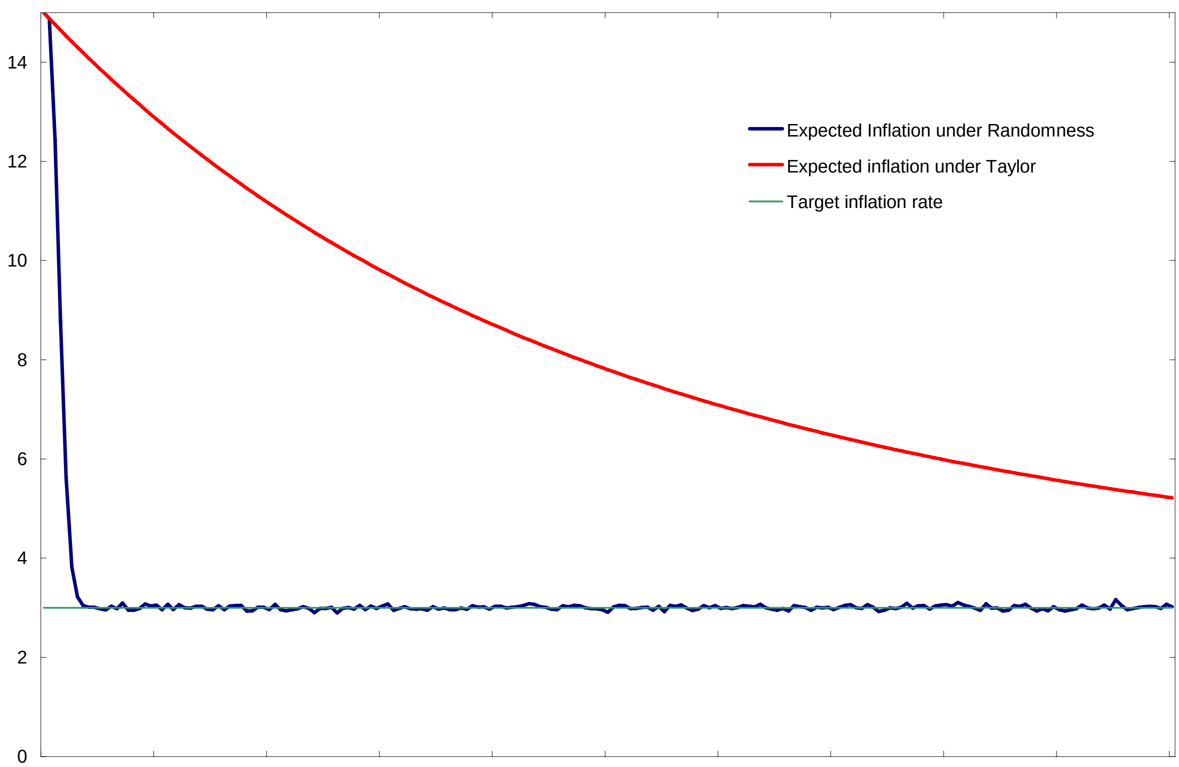
$$\exists \theta \in (0, \overline{\alpha}) \therefore TL_R(\mu_R^*(\theta)) > TL_F \tag{A47}$$

From (A45) we know that $TL_R(\mu_R^*(\cdot))$ must take a value lower than TL^*_F in $(0, \overline{\alpha})$. If (A47) were true, then the function would also take a value greater than TL^*_F. By reason of continuity of the function $TL_R(\mu_R^*(\cdot))$, we conclude that $\exists \gamma \in (0, \overline{\alpha}) \therefore TL_R(\mu_R^*(\gamma)) = TL_F^*$. It follows that $\gamma \in A$ and thus, by construction, $\gamma \geq \overline{\alpha} = \inf_x A$. But this contradicts the fact that by construction $\gamma \in (0, \overline{\alpha})$. It follows that (A47) cannot be true, and thus we conclude that:

$$\exists\overline{\alpha} > 0 \therefore TL_R(\mu_R^*(\alpha)) < TL_F^* \forall \in (0, \overline{\alpha}) \quad \text{(A48).}$$

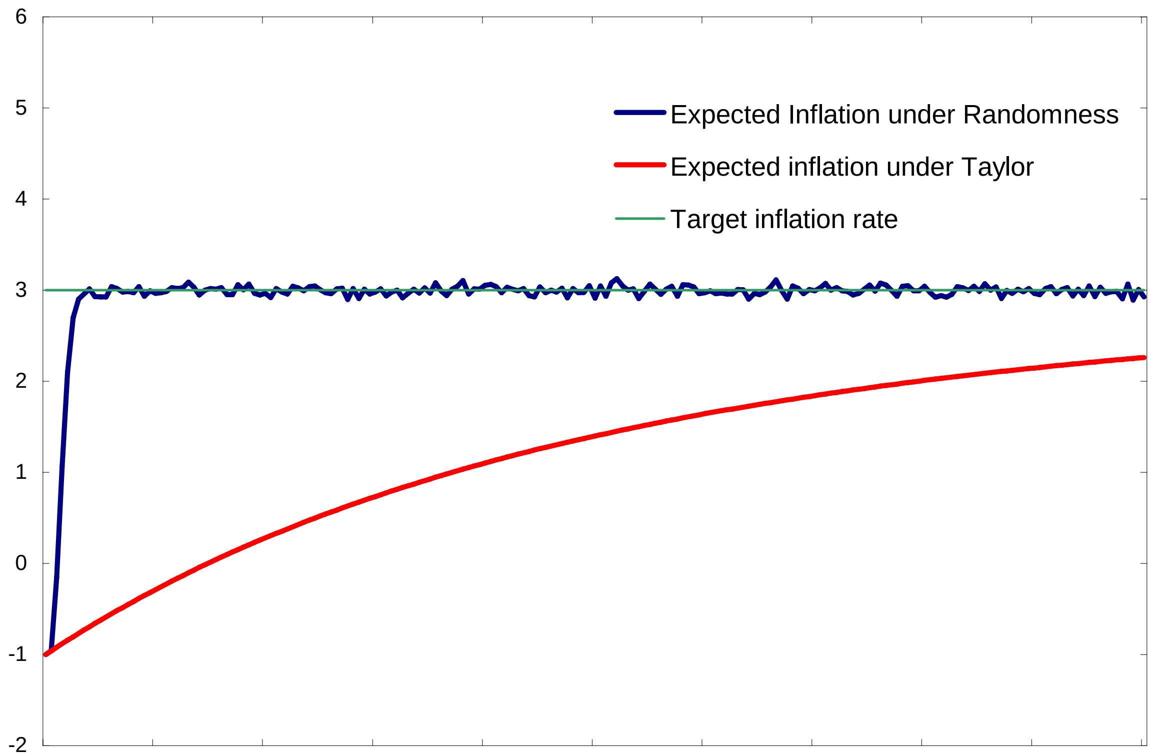
Figures:

Figure 1: Inflation Vs. Time: High Inflation Case



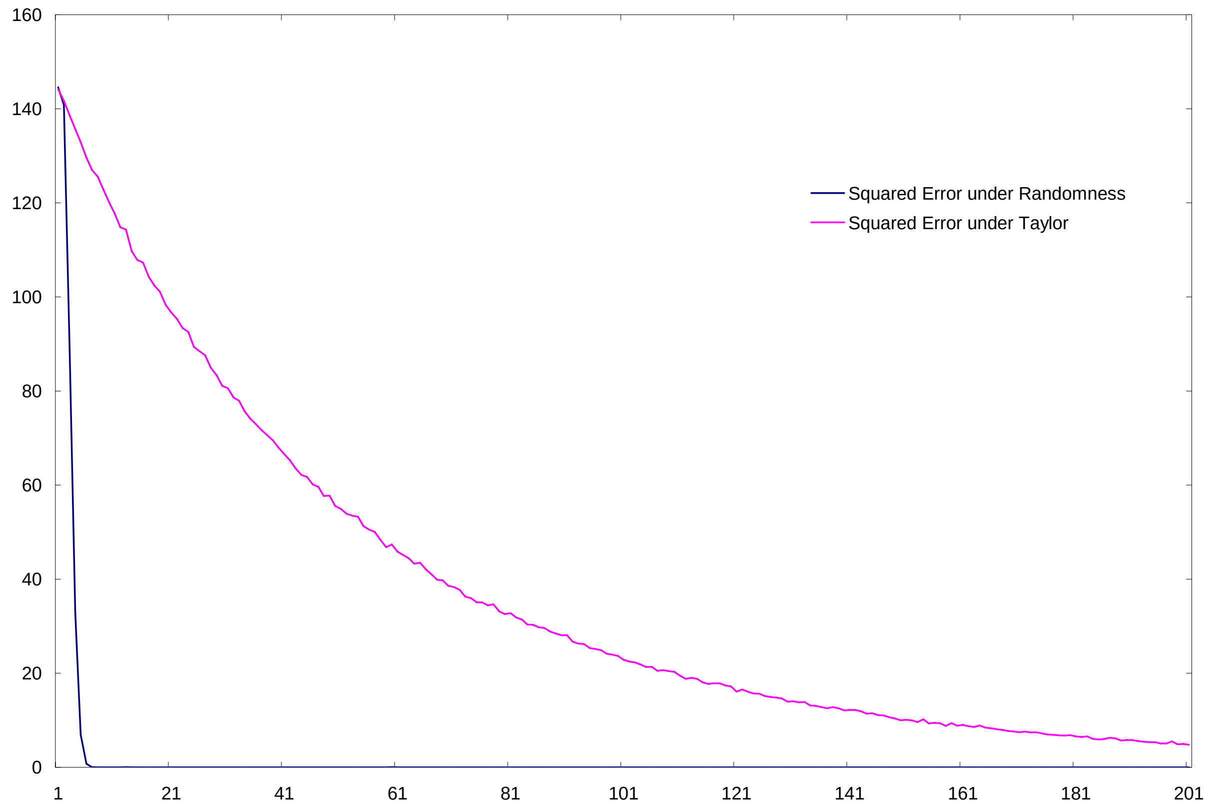
Plot of inflation vs. time with a government target inflation rate of 3 percent and a public guess of the target at 15 percent. The parameters for the calculation are as follows: σ^* = 0.1, σ = 0.1, σ_0 = 0.01, μ^* = 3, and μ_0 = 15.

Figure 2: Inflation Vs. Time: Deflation Case



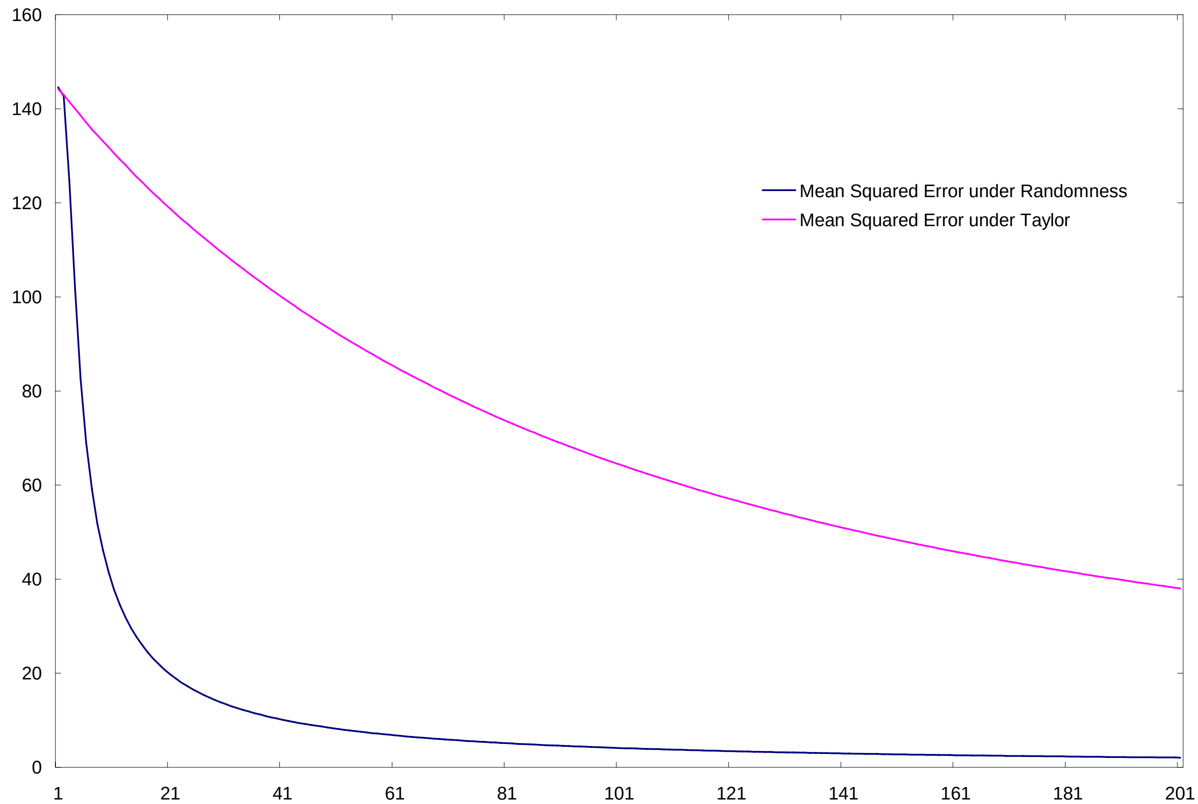
Plot of inflation vs. time for the deflationary case of a government target inflation rate of 3 percent and a public expectation of the optimal inflation rate at -1 percent. The parameters for the calculation are as follows: $\sigma^* = 0.1$, $\sigma = 0.1$, $\sigma_0 = 0.01$, $\mu^* = 3$, and $\mu_0 = -1$.

Figure 3: Squared Error for High Inflation Case.



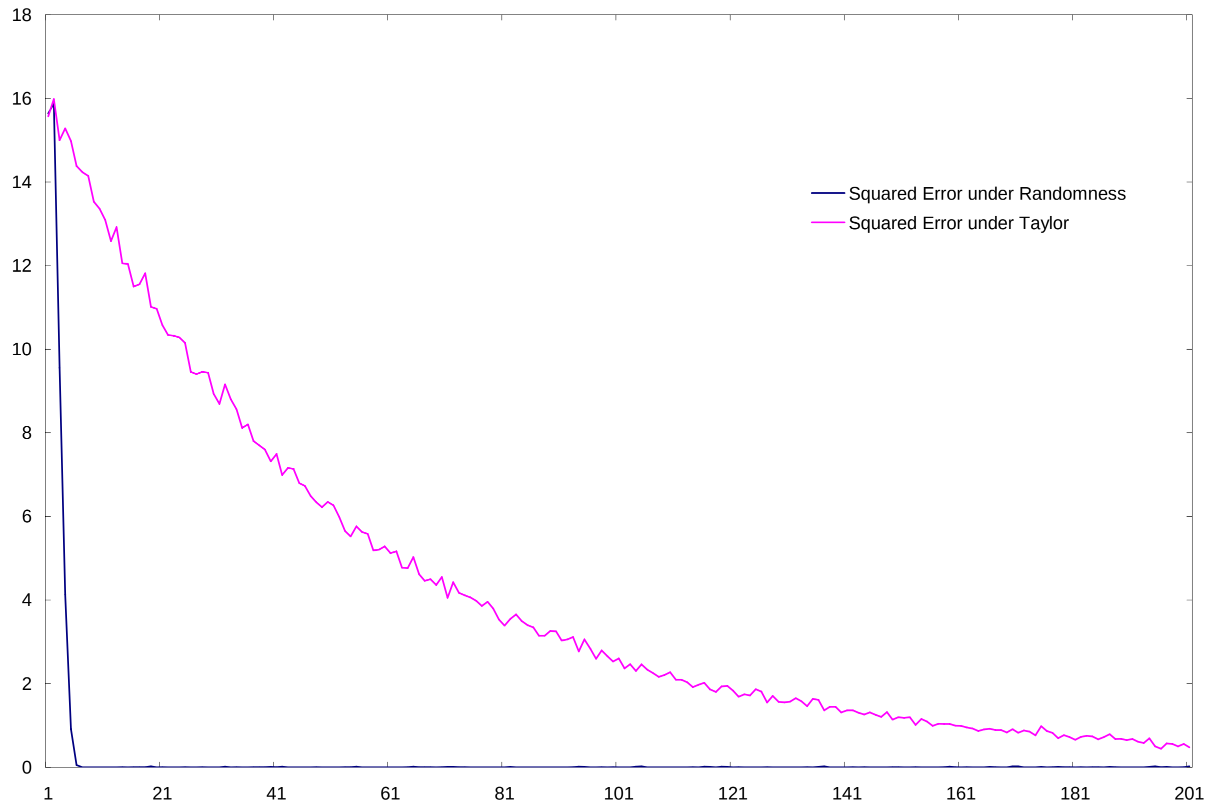
Plot of the squared error for Figure 1, with a government target inflation rate of 3 percent and a public guess of the target at 15 percent.

Figure 4: Mean Squared Error for High Inflation Case.



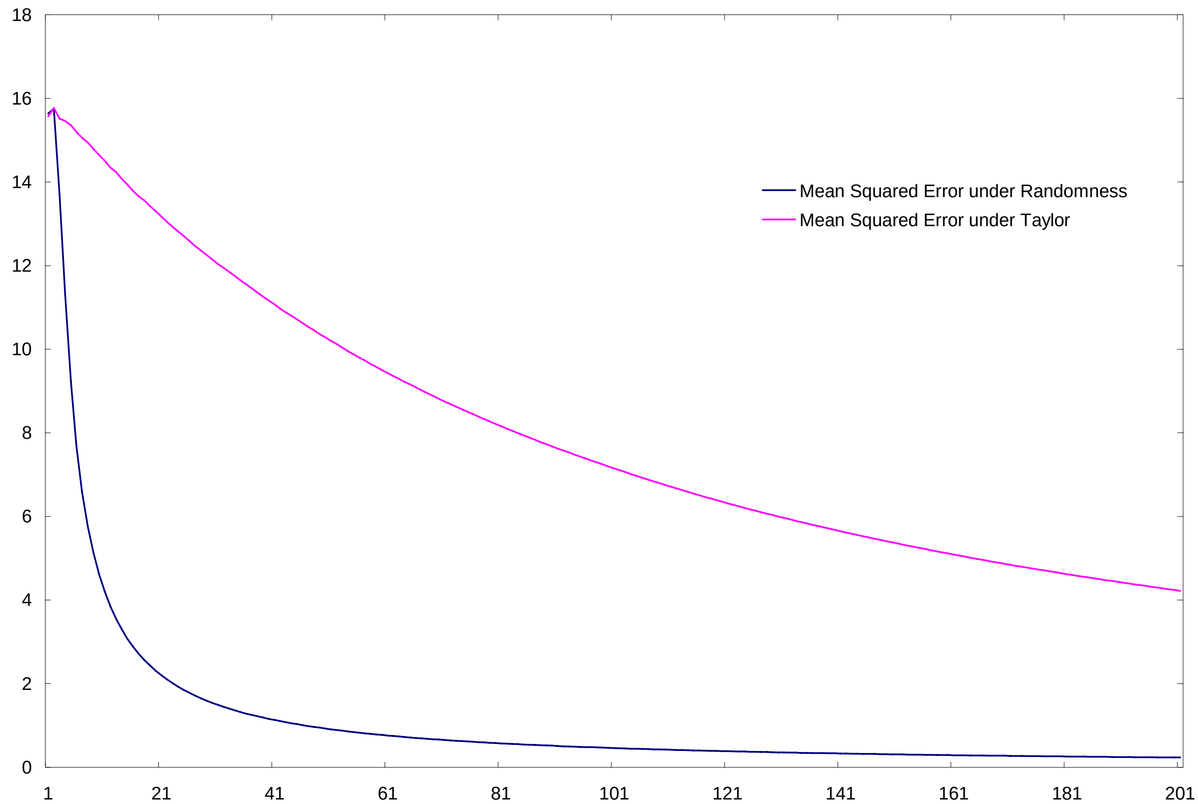
Plot of the mean squared error for Figure 1, with a government target inflation rate of 3 percent and a public guess of the target at 15 percent.

Figure 5: Squared Error for Deflation Case.



Plot of the squared error for Figure 2, with a government target inflation rate of 3 percent and a public guess of the target at -1 percent.

Figure 6: Mean Squared Error for Deflation Case.



Plot of the mean squared error for Figure 2, with a government target inflation rate of 3 percent and a public guess of the target at -1 percent.