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Optimal Inflation for the U.S.

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Abstract: What is the correctly measured inflation rate that monetary policy should aim for in the long-run? This paper characterizes the optimal inflation rate for the U.S. economy in a New Keynesian sticky-price model with an occasionally binding zero lower bound on the nominal interest rate. Real-rate and mark-up shocks jointly determine the optimal inflation rate to be positive but not large. Even allowing for the possibility of extreme model misspecification, the optimal inflation rate is robustly below 1 percent. The welfare costs of optimal inflation and the lower bound are limited.

Keywords: Commitment, liquidity trap, long-run trade-off, nonlinear, robust control, stationary distribution

JEL classification: C63, E31, E52

"...the optimal inflation rate is surely positive, perhaps as high as 2 or 3 percent." — Larry Summers (1991)

1 Introduction

Over the years, Vickrey (1954), Phelps (1972), Okun (1981), Summers (1991), Fischer (1996) and McCallum (2000), among others, noted the relevance of the lower bound on the nominal interest rate for the determination of the optimal inflation rate. Short-term nominal interest rates cannot be pushed below zero, so monetary policymakers should aim for a higher inflation rate to leave room for the conduct of stabilization policy.[1] However, sustaining higher inflation is costly to the economy. The 'optimal' inflation rate solves the trade-off between the benefits of a more flexible stabilization policy and the costs of higher inflation.

In recent years, the U.S. Federal Reserve and central banks of other major world economies have been generally successful in achieving low and stable inflation. This achievement, however, gives rise to concern about the effectiveness of monetary policy when interest rates are close to zero (see for example Bernanke, Reinhart and Sack (2004)). As a consequence, there is an ongoing debate on the correctly measured inflation rate that monetary policy should aim for in the long-run. So far discussions are limited to intuitions based on indirect evidence only, since a robust prescription for the optimal inflation rate is not available yet.

This paper directly addresses the issue of identifying the optimal inflation rate for the U.S. economy. The analysis is conducted in a stochastic version of the well-known New Keynesian sticky-price model as in Clarida, Galí and Gertler (1999) and Woodford (2003). In addition, the lower bound is explicitly taken into account as an occasionally binding constraint on policy. The optimal inflation rate is first evaluated under the standard assumption that there is no uncertainty surrounding the true model of the economy. The results show that the optimal inflation rate is positive but not large. Then, following the 'robust control' approach of Hansen and Sargent (2007), the implications of model misspecification for the determination of the optimal inflation rate are investigated. Even allowing for the possibility of extreme model misspecification, the optimal inflation rate is robustly below 1 percent.

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[1] In principle, achieving a negative nominal rate is feasible for example by giving up free convertibility of deposits and other financial assets into cash or by levying a tax on money holdings as in Buiter and Panigirtzoglou (2003) and Goodfriend (2000). However, a consensus on the applicability of such policy measures has to emerge yet.

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From a technical point of view, the monetary policy problem in this paper presents four specific challenges that significantly aggravate its solution. These difficulties help explain why the literature does not aim directly to provide an assessment of the optimal inflation rate. The four important features that this paper simultaneously addresses are: the forward-looking nature of private-sector expectations; the nonlinearity due to an occasionally binding constraint on policy; departure from the simplifying assumption that the future state of the economy is know with perfect foresight; and uncertainty about the true model of the economy. As a by-product, this paper provides a general solution strategy and an efficient nonlinear numerical procedure apt to solving models in a broad class of robust control problems with occasionally binding constraints.

A number of recent papers study how monetary policy should be conducted in the presence of the lower bound on the nominal interest rate. Several references are available for example in Svensson (2003). Closely related to this paper, one strand of literature examines the performance of simple monetary policy rules. Fuhrer and Madigan (1997), Coenen, Orphanides and Wieland (2004) and Reifschneider and Williams (2000) study dynamic rational expectations models using simulation methods. Wolman (2005) considers a dynamic general equilibrium framework. This set of papers shows that a targeted inflation rate close enough to zero can generate significant real distortions in the economy and a higher inflation target reduces the distortions. However, sustaining higher inflation is costly in welfare terms to the economy. These findings show the importance of investigating the 'optimal' inflation rate, which solves the trade-off between the gains of reducing the distortions and the costs of higher inflation.

Another strand of literature studies optimal monetary policy in models where private-sector expectations are backward-looking, that is they are formed over current and past knowledge of the state of the economy only. For example, Orphanides and Wieland (2000) and Kato and Nishiyama (2005) characterize the optimal conduct of monetary policy in a dynamic rational expectations model where expectations about future policy actions play no role. These papers show the relevance of 'preemptive' easing of interest-rate policy in the approach of a zero nominal rate, when the central bank cannot intervene in the economy through the expectational channel.

However, the 'management' of private-sector expectations is key to effective central-bank action. When policymakers credibly commit to a future course of policy, they have leverage on shaping expectations and thus can determine the condition of the economy also at a zero nominal rate. The economy can be stimulated by a lower real interest rate via an increase in expectations of inflation. Building on this insight of Krugman (1998), a further strand of the literature analyzes the purely forward-looking version of

the New Keynesian sticky-price model. In particular, Jung, Teranishi and Watanabe (2005) and Eggertsson and Woodford (2003) characterize optimal monetary policy in a perfect-foresight setup. In a series of two papers, Adam and Billi (2006, 2007) provide a treatment with uncertainty about the future state of the economy and assess the welfare gains of the policymaker's commitment to a policy plan.

This paper employs a more general version of the New Keynesian sticky-price model, which features a mark-up shock shifting the aggregate-supply relation and also price indexation to past inflation to match the inertial behavior of inflation observed in the data as in Christiano, Eichenbaum and Evans (2005). The dimension of the state space of this model is higher than in previous studies on optimal policy and the lower bound. Solving this model is a greater challenge. Yet, in order to provide a robust prescription for the goals of monetary policy in the long-run, such challenge cannot be avoided.

This paper first presents results for a stochastic rational expectations equilibrium assuming no uncertainty surrounds the true model of the economy and that all available information is used to rationally anticipate future variables. The results show that not only real-rate shocks but also mark-up shocks are relevant for the determination of the optimal inflation rate, which is about 0.17 percent per year under a baseline calibration.[2] The optimal inflation rate may be as high as 0.69 percent per year when the standard deviation of the innovations of both type of shocks are 50% larger than under the baseline. The welfare costs of optimal inflation and the lower bound are small, amounting to a permanent reduction in consumption for the representative agent of less than -0.01 percent per year under the baseline.

Next, model misspecification is introduced in the analysis. Following the robust control approach of Hansen and Sargent (2007), policy is designed to be robust to a worst-case scenario of model misspecification. Previous applications of robust control to the study of optimal policy design in the New Keynesian model, as in Giordani and Söderlind (2004) and Walsh (2004), ignore the lower bound constraint. Alternatively, Woodford (2005) investigates the robustness of optimal policy to a worst-case scenario in a setting where the monetary policymaker does not trust its model of how the private sector forms expectations about future policy. In a symmetric modeling framework, however, neither model misspecification nor the policymaker's limited ability to shape private-sector expectations are relevant for the determination of the optimal inflation rate. Increases and decreases in inflation are equally undesirable in a standard linear-

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[2] In the New Keynesian model, real-rate shocks shift the intertemporal Euler equation describing private-expenditure decisions of the representative household, and mark-up shocks shift the aggregate-supply relation summarizing the price-setting behavior of firms.

quadratic model abstracting from the asymmetric risks of the lower bound. Sims (2001) emphasizes that the lower bound is ignored in macroeconomic robust control exercises, and also in economic applications which consider structured forms of uncertainty as in Giannoni (2002) and Onatski and Stock (2002). Onatski and Williams (2003) neglect the role of the lower bound in a model that allows for both parametric and nonparametric misspecifications.

The results in this paper show that the optimal inflation rate is slightly higher when model misspecification is taken into account. The lowest detection error probability of a misspecification for which the algorithm can identify an equilibrium is about 29%. Corresponding to this extreme misspecification, the 'robustly' optimal inflation rate may be as high as 0.87 percent per year under a baseline calibration. The welfare costs of optimal inflation and the lower bound remain limited in the presence of model misspecification. The costs amount to a permanent reduction in consumption of less than -0.01 percent per year over the feasible range of detection error probabilities.

The remainder of this paper is structured as follows. Section 2 introduces the model, then section 3 illustrates the solution strategy and provides the equilibrium definition under the assumption of rational expectations. In section 4, the model is calibrated to the U.S. economy. The optimal inflation rate and the welfare costs of the lower bound are discussed in section 5. Section 6 assesses the robustly optimal inflation rate in a worst-case scenario of model misspecification. Section 7 briefly concludes. The numerical algorithm, derivation of welfare equivalent consumption losses, and calculation of detection error probabilities for the robust control exercise are described in the appendix.

2   Model

This setting adopts the well-known sticky-price version of the New Keynesian model, which is widely studied in the monetary policy literature as in Clarida, Galí and Gertler (1999) and Woodford (2003). The model consists of a representative consumer and firms in monopolistic competition facing restrictions on the frequency of price adjustments à la Calvo (1983). The monetary authority directly controls the nominal interest rate with the objective of maximizing welfare for the representative consumer. Standard linear-quadratic solution methods can be applied if the model abstracts from the lower bound on the nominal rate.

Once the lower bound is explicitly taken into account, however, finding the stochastic equilibrium requires highly intensive nonlinear numerical methods. It is crucial to economize in the number of state variables to cope with the curse of dimensionality.

The dimension of the state space can be contained by taking a log-linear approximation of the intertemporal equilibrium conditions around the first-best deterministic steady state associated with price stability. As discussed by Schmitt-Grohé and Uribe (2007), a higher-order approximation would require an additional state variable to keep track over time of relative price dispersion. Steady state price stability guarantees that the evolution of the measure of price dispersion has no real consequences up to first order.

The optimal monetary policy problem to be solved is the following:

$$\max_{\{\pi_t, x_t, i_t\}} - E_0 \sum_{t=0}^{\infty} \beta^t \left[(\pi_t - \gamma\pi_{t-1})^2 + \lambda x_t^2 \right] \tag{1}$$

s.t.

$$\pi_t - \gamma\pi_{t-1} = \beta E_t (\pi_{t+1} - \gamma\pi_t) + \kappa x_t + u_t \tag{2}$$

$$x_t = E_t x_{t+1} - \varphi (i_t - E_t \pi_{t+1} - r_t^n) \tag{3}$$

$$u_t = \rho_u u_{t-1} + \sigma_{\varepsilon u} \varepsilon_{ut} \tag{4}$$

$$r_t^n = (1 - \rho_r) r_{ss} + \rho_r r_{t-1}^n + \sigma_{\varepsilon r} \varepsilon_{rt} \tag{5}$$

$$i_t \geq 0 \tag{6}$$

where E_t denotes the mathematical expectations operator conditional on information available at time t. Expectations are assumed to be rational with no uncertainty surrounding the true model of the economy and all available information being used to rationally anticipate future variables (this assumption is relaxed in section 6). π_t denotes the quarterly inflation rate. y_t is the output gap that is the deviation of output from its 'natural' flexible-price equilibrium.[3] The quarterly nominal interest rate, i_t, is the instrument of monetary policy. Equations (1)-(6) describe a social planning problem. The monetary policymaker is implicitly allowed to select the equilibrium paths of inflation and output in addition to the nominal rate path, $\{\pi_t, x_t, i_t\}_{t=0}^{\infty}$.[4]

The welfare-theoretic objective function of the monetary policymaker in equation (1) is a quadratic (second-order Taylor series) approximation to the expected utility of the representative consumer.[5] The subjective discount factor is denoted by $\beta \in (0, 1)$. The

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[3] Output is assumed to be efficient at its deterministic steady state level for example thanks to an output subsidy that neutralizes the distortions from monopolistic competition as in chapter 6 of Woodford (2003).

[4] The implementation of interest rate policy in this nonlinear framework is a non trivial issue, which remains to be addressed in future research.

[5] The objective function in equation (1) arises under the assumption of backward-looking indexation of individual firm's prices to an aggregate price index as in chapter 6 of Woodford (2003).

relative welfare weight

$$\lambda \equiv \frac{\kappa}{\theta} > 0 \tag{7}$$

depends on the underlying structure of the economy, with $\theta > 1$ representing the price elasticity of demand substitution among alternative differentiated goods produced by firms operating in monopolistic competition. The model abstracts from transaction frictions and money demand distortions associated with positive nominal interest rates. The model can be interpreted as the 'cashless limit' of a model economy with money featuring steady state price stability.

Equation (2) is a log-linearized stochastic aggregate-supply relation, which describes the price-setting behavior of firms subject to staggered price setting à la Calvo (1983). The slope parameter

$$\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\frac{\varphi^{-1}+\omega}{1+\omega\theta} > 0 \tag{8}$$

is function of the underlying structure of the economy, with $\omega > 0$ being the elasticity of a firm's real marginal costs with respect to its own output level.[6] Each period, a share $\alpha \in (0,1)$ of randomly picked firms cannot adjust their individual prices, while the remaining $(1-\alpha)$ firms get to choose prices optimally. Substantial criticism is directed at the version of the New Keynesian model without price indexation, for its inability to capture the high persistence of inflation displayed in the data. Following Christiano, Eichenbaum and Evans (2005), $\gamma \in [0,1)$ denotes the degree of indexation of individual prices to the most recent aggregate price index.[7] The so-called 'cost-push' shock, u_t, shifts the aggregate-supply relation. The shifter is interpreted as a 'mark-up' shock that is variation over time in the degree of monopolistic competition between firms.

Equation (3) is a log-linearized stochastic intertemporal Euler equation describing the representative consumer's private expenditure decisions. $\varphi > 0$ denotes the intertemporal elasticity of substitution or, in other terms, the real-rate elasticity of output. Shifting the Euler equation is the 'natural' real-rate of interest shock, r_t^n. The real-rate shock summarizes all shocks that under flexible prices generate variation over time in the real interest rate; it captures the combined effects of preference shocks, productivity shocks, and exogenous changes in government expenditure.

Equations (4) and (5) describe the evolution of the exogenous mark-up shock (u_t)

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[6]For further details see chapter 3 of Woodford (2003).

[7]When $\gamma = 0$ equation (2) reduces to the purely forward-looking version of the Phillips curve. Lagged inflation (π_{t-1}) is an additional endogenous state variable of the model if $\gamma > 0$.

and the exogenous real-rate shock (r_t^n), respectively. These shocks are assumed to follow AR(1) stochastic processes with autoregressive coefficients denoted by $\rho_j \in (-1, 1)$ for $j = u, r$. The steady state real interest rate is given by $r_{ss} \equiv 1/\beta - 1$, such that $r_{ss} \in (0, +\infty)$. The innovations (ε_{jt} for $j = u, r$) are assumed independent both across time and cross-sectionally, and normally distributed with mean zero and standard deviations denoted by $\sigma_{\varepsilon j} \geq 0$ for $j = u, r$.

Finally, equation (6) represents an occasionally binding zero lower bound constraint on the nominal interest rate. Abstracting from the lower bound, the simpler optimal monetary policy problem (1)-(5) can be solved applying standard linear-quadratic methods. The lower bound renders the problem nonlinear and causes a break down of certainty equivalence. Therefore, the stationary distribution is not centered around the deterministic steady state around which the equilibrium conditions of the model are log-linearized.

3 Solving the Model

This section illustrates the strategy to solve the model introduced in the previous section and derives the stochastic rational expectations equilibrium. The policy problem is reduced to a nonlinear system of recursive equilibrium conditions, which define a fixed-point to be found numerically.

The infinite horizon Lagrangian for the optimal monetary policy problem (1)-(6) is easily obtained as

$$
\max_{\{\pi_t, x_t, i_t\}} \min_{\{m_{1t}, m_{2t}\}} \mathcal{L} \equiv E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ -\left(\pi_t - \gamma \pi_{t-1}\right)^2 - \lambda x_t^2 \tag{9}
$$
$$
+ m_{1t} \left[(1 + \beta\gamma)\, \pi_t - \gamma \pi_{t-1} - \kappa x_t - u_t \right] - m_{1t-1} \pi_t
$$
$$
+ m_{2t} \left[-x_t - \varphi \left(i_t - r_t^n \right) \right] + m_{2t-1} \beta^{-1} \left(x_t + \varphi \pi_t \right) \Big\}
$$
$$
\text{s.t.}
$$
$$
\text{Equations } (4)\text{-}(6)
$$

where m_{1t} and m_{2t} denote the Lagrange multipliers for the aggregate-supply relation (2) and the intertemporal Euler equation (3), respectively. This optimization problem is in recursive form.

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The Kuhn-Tucker conditions of the Lagrangian (9) are given by

$$\partial \mathcal{L}/\partial \pi_t = -2\left(\pi_t - \gamma \pi_{t-1}\right) + \left(1 + \beta\gamma\right) m_{1t} - m_{1t-1} + \beta^{-1}\varphi m_{2t-1} = 0 \qquad (10)$$

$$\partial \mathcal{L}/\partial x_t = -2\lambda x_t - \kappa m_{1t} - m_{2t} + \beta^{-1} m_{2t-1} = 0 \qquad (11)$$

$$\partial \mathcal{L}/\partial i_t \cdot i_t = -\varphi m_{2t} \cdot i_t = 0, \quad m_{2t} \geq 0, \quad i_t \geq 0 \qquad (12)$$

where either the Lagrange multiplier on the Euler equation $(m_{2t} \geq 0)$ or the Lagrange multiplier on the nominal rate $(i_t \geq 0)$ is zero, at all states and for each period.

Equations (2), (3) and (10)-(12) form a nonlinear system of five equations with five unknowns, which is satisfied by optimal policy in equilibrium. Solving the system delivers a five-dimensional nonlinear equilibrium response function

$$y\left(s_t\right) \equiv \left(\pi_t, x_t, i_t \geq 0, m_{1t}, m_{2t} \geq 0\right) \subset R^5$$

defined over a five-dimensional state space

$$s_t \equiv \left(u_t, r_t^n, \pi_{t-1}, m_{1t-1}, m_{2t-1} \geq 0\right) \subset R^5$$

Besides the three natural state variables, that is the exogenous shocks and lagged inflation (u_t, r_t^n, π_{t-1}), the state space contains two additional endogenous co-state variables given by the lagged values of the Lagrange multipliers $(m_{1t-1}, m_{2t-1} \geq 0)$.[8]

Associated with the equilibrium response function $y\left(s_t\right)$ is the equilibrium expectations function

$$E_t y_{t+1} = \int y\left(s_{t+1}\right) f\left(\varepsilon_{jt+1}\right) d\left(\varepsilon_{t+1}\right) \qquad (13)$$

where $f\left(\cdot\right)$ is the probability density function of the stochastic shock innovations, $\varepsilon_t \equiv \left(\varepsilon_{ut}, \varepsilon_{rt}\right) \in R^2$. The evolution of the exogenous shocks (u_t, r_t^n) is described by equations (4) and (5). The expectations function (13) is not integrated over a probability density function of shock innovations when agents are assumed to have perfect foresight $(\sigma_{\varepsilon j} \to 0$ for $j = u, r)$. The following definition of a stochastic rational expectations equilibrium is proposed.

Definition 1 (SREE) *Assume* $\sigma_{\varepsilon j} \geq 0$ *for* $j = u, r$. *A 'stochastic rational expectations equilibrium' of the optimal monetary policy problem (1)-(6) is a nonlinear pol-*

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[8] The co-state variables $(m_{1t-1}, m_{2t-1} \geq 0)$ can be interpreted as the social value costs of 'promises' kept from past policy commitments, which lead to deviations from purely forward-looking policy whenever their values differ from zero.

icy response function $y(s_t) \equiv (\pi_t, x_t, i_t \geq 0, m_{1t}, m_{2t} \geq 0)$ *specified over the state space* $s_t \equiv (u_t, r_t^n, \pi_{t-1}, m_{1t-1}, m_{2t-1} \geq 0)$, *such that the nonlinear system of equations (2), (3) and (10)-(12) is satisfied.*

Solving for a SREE requires finding a fixed-point in the space of nonlinear response functions. Given the high dimensionality of the state space, a highly efficient nonlinear numerical procedure is employed. Details of the algorithm are in appendix A.1. Independent of the lower bound constraint, the perfect-foresight limit ($\sigma_{\varepsilon j} \to 0$ for $j = u, r$) of a SREE converges to the deterministic steady state around which the equilibrium conditions of the model are log-linearized.

4 Calibration

In this section the model is calibrated to the U.S. economy. The time period is assumed to be one quarter. The baseline parameter values are summarized in table 1 and expressed in quarters, unless otherwise noted.

Parameter Definition	Assigned Value
Subjective discount factor	$\beta = 0.9913$
Real-rate elasticity of output	$\varphi = 6.25$
Share of firms keeping prices fixed	$\alpha = 0.66$
Price elasticity of demand	$\theta = 7.66$
Elasticity of firms' marginal costs	$\omega = 0.47$
Slope of the Phillips curve	$\kappa = 0.024$
Weight on output in the loss function	$\lambda = 0.003$
Degree of inflation indexation	$\gamma = 0.9$
Steady state real interest rate	$r_{ss} = 3.5\%$ annualized
s.d. real-rate shock innovation	$\sigma_{\varepsilon r} = 0.24\%$
s.d. mark-up shock innovation	$\sigma_{\varepsilon u} = 0.30\%$
AR(1)-coefficient of real-rate shock	$\rho_r = 0.8$
AR(1)-coefficient of mark-up shock	$\rho_u = 0$

Table 1: Baseline Calibration (Quarterly Model)

The values assigned to the main structural parameters, $(\varphi, \alpha, \theta, \omega)$ and the resulting (κ, λ), are taken from tables 5.1 and 6.1 in Woodford (2003), which are based on the results of Rotemberg and Woodford (1997). Endogenous inflation persistence is required in the New Keynesian sticky-price model with steady state price stability to match the inertial behavior of inflation observed in the data. The degree of indexation of individual prices to the most recent aggregate price index is set to $\gamma = 0.9$. This value is

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consistent with estimates under the assumption of rational expectations as in Giannoni and Woodford (2005) and Milani (2007).[9]

Following the approach of Adam and Billi (2006), the parameters describing the evolution of the stochastic shock processes and the discount factor are estimated over the period $1983{:}1-2002{:}4$. The estimated expectations of inflation and output are constructed from the predictions of an unconstrained VAR in the output gap, inflation and the fed funds rate. These estimates for expectations are plugged into the model's intertemporal equilibrium conditions, (2) and (3), along with actual data. Then, the historical shock processes, u_t and r_t^n, are identified with the equation residuals. Fitting AR(1) processes to the identified historical shocks justifies the estimates reported in the table.[10] The quarterly subjective discount factor is set to $\beta = (1+r_{ss})^{-1} \approx 0.9913$, as implied by the estimate for the steady state real interest rate, namely $r_{ss} = 3.5\%$ annualized.

5 Stochastic Rational Expectations Equilibrium

This section presents results for the stochastic rational expectations equilibrium of the quarterly model under the baseline calibration in table 1. For readability, all results are presented as annualized percentage values. Real-rate and mark-up shocks are shown to jointly determine the stationary distribution and optimal inflation. The optimal inflation rate is a low positive value and the welfare costs of the lower bound are moderate. These findings are robust to a wide range of alternative calibrations.

5.1 Stationary Distribution

This subsection describes the stationary distribution that emerges in the long-run. The results show that the optimal inflation rate and the optimal nominal interest rate are higher when the shocks buffeting the economy are more volatile.

Figure 1 presents the stationary distribution of inflation, the output gap, nominal and real interest rates $(\pi, x, i, i - E\pi_{+1})$ under the baseline calibration. The distribution is expressed in terms of probability density. In each panel, the full vertical line indicates

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[9]Christiano, Eichenbaum and Evans (2005) assume full inflation indexation ($\gamma = 1$) in a model that abstracts form the lower bound. It is easily verified that the model with lower bound is not well defined under full indexation, since the nonlinear system in definition 1 would not have a determinate steady state for inflation and the nominal rate.

[10]Adam and Billi (2006) estimate a simpler model over the same period $1983{:}1-2002{:}4$ assuming no inflation indexation ($\gamma = 0$). The standard deviation of the mark-up shock is almost double the size for the empirically relevant case with inflation indexation ($\gamma = 0.9$).

the mean of the stationary distribution of the corresponding endogenous variable.[11] The dashed-dotted vertical line shows the deterministic steady state of the variable.[12] Because of the zero lower bound constraint the stationary distribution of the nominal rate is asymmetric and spans only non negative values. The lower bound renders the model nonlinear and causes a break down of certainty equivalence. The mean of the stationary distribution does not necessarily coincide with the deterministic steady state. Only in the perfect-foresight limit does the stationary distribution converge to the deterministic steady state around which the equilibrium conditions of the model are log-linearized. The figure shows that the optimal average values of inflation and the nominal rate are higher than their respective deterministic steady state values. There does not appear to be any significant difference between the average and deterministic steady state values for the output gap and the real rate.

[Figure 1 about here]

Figure 2 illustrates the stationary distribution of the same endogenous variables as in the previous figure when the shocks buffeting the economy are more volatile. The standard deviations of the real-rate and mark-up shock innovations are 50% larger than under the baseline calibration. Other parameters take the baseline values. Comparing the various panels in figures 1 and 2 clarifies the quantitative implications of the break down of certainty equivalence in the model. Inflation and the nominal rate are indeed subject to a positive 'stochastic mean bias' that is their optimal average values are higher with more volatile shocks. Yet, the deterministic steady state values of inflation and the nominal rate remain unchanged. The average values of output and the real rate are not very sensitive to the level of uncertainty attached to the stochastic shock processes.

[Figure 2 about here]

The economic intuition for the positive stochastic mean bias in inflation and the nominal rate is the following. A higher optimal inflation rate in the economy allows the monetary policymaker to support a higher nominal rate. A higher average nominal rate

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[11] For a given stochastic rational expectations equilibrium, the distribution is computed by assembling 10^5 stochastic simulations at a specific time period. The simulations are initialized to the deterministic steady state. The mean and standard deviation of the endogenous variables are obtained as summary statistics of the distribution. Tracking the time-evolution of the summary statistics, it is ascertained that the distribution reached its stationary configuration.

[12] For a given perfect-foresight limit equilibrium, it is verified that the steady state reached by a deterministic policy path is (approximately) equal to the deterministic steady state around which the stochastic equilibrium conditions of the model are log-linearized. This test insures that the numerical procedure provides solutions that are consistent with the log-linearized stochastic equilibrium conditions.

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protects the economy against more frequent episodes of zero nominal rates when shocks are more volatile. This argument also reveals that there is a policy tension in the long-run between supporting a higher average inflation rate, which is costly in welfare terms to the economy, and a higher likelihood of the lower bound on the nominal rate being a binding constraint for policy. The next subsection investigates how optimal policy resolves this trade-off.

5.2 Optimal Inflation and Welfare Costs

This subsection shows that the real-rate shock and the mark-up shock are both relevant for the determination of optimal inflation. The welfare costs of the lower bound are small when policy is conducted optimally.

Table 2 reports the optimal average and standard deviation of inflation and the expected frequency of zero nominal rates with alternative levels of uncertainty in the stochastic shock processes. Other parameters take the baseline values. The table compares results for the baseline level of uncertainty with those when the standard deviation of the mark-up and real-rate shock innovation are 50% larger. The optimal inflation rate is about 0.17 percent per year under the baseline. Optimal inflation rises to about 0.47 percent per year with a more volatile mark-up shock, and to about 0.34 percent per year with a more volatile real-rate shock. Thus, the mark-up shock and the real-rate shock are both determinants of optimal inflation. The optimal inflation rate rises further to about 0.69 percent per year when both type of shocks are more volatile.

$E\left(\pi\right)$ s.d.(π) Freq$(i=0)$	$1\cdot\sigma_{\varepsilon u}$	$1.5\cdot\sigma_{\varepsilon u}$
$1\cdot\sigma_{\varepsilon r}$	0.17	0.47
	1.9	2.8
	27	60
$1.5\cdot\sigma_{\varepsilon r}$	0.34	0.69
	1.9	2.8
	43	62

Table 2: Optimal Inflation And Expected Frequency Of Zero Nominal Rates (Annualized Percentage Values)

Table 2 also shows that the volatility of optimal inflation and the expected frequency of zero nominal rates are higher with more volatile shocks. This result confirms the economic intuition provided in the previous subsection about a positive stochastic mean

bias in inflation. The mean bias stems from a long-run policy tension which is directly a function of the degree of uncertainty about the future state of the economy. On the one hand, supporting a higher average inflation rate restrains the likelihood that the lower bound on the nominal rate may constrain policy. Without a rise in average inflation, the volatility of inflation and the expected frequency of zero nominal rates would otherwise be even higher in a more uncertain economic environment.[13] On the other hand, higher average inflation is costly in welfare terms to the economy. The optimal inflation rate resolves this trade-off between the costs of a less flexible stabilization policy and the costs of higher inflation.

Table 3 illustrates the welfare losses of optimal inflation and the lower bound with alternative levels of uncertainty in the stochastic shock processes. Again, other parameters take the baseline values. The losses are represented in terms of their welfare equivalent permanent reduction in consumption. Consumption equivalents are derived via a transformation of the unconditional losses in the objective function (1), see appendix A.2 for details.[14] The table reports the losses, μ, for the model that takes directly into account the lower bound constraint. Losses, μ_{LQ}, are reported also for a standard linear-quadratic approximation, which abstracts from the lower bound. A measure of the welfare costs of the lower bound and optimal inflation is provided by the additional losses for the model with lower bound relative to the one without:

$$\Delta\left(\mu\right) \equiv \mu - \mu_{LQ} \leq 0$$

The table shows that the welfare costs of optimal inflation are small.[15] The costs are higher with more volatile mark-up and real-rate shocks. This finding is consistent with the results in the previous table, since both type of shocks are determinants of optimal inflation. The costs amount to a permanent reduction in consumption for the representative agent of only about -0.0036 percent per year under the baseline calibration. The costs increase sixfold to about -0.0223 percent per year when the standard deviation

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[13] For example, solving the model abstracting from the lower bound under the baseline calibration, the standard deviation of inflation would rise from 1.9 to 2.1 percent per year and the expected frequency of non positive nominal rates, Freq($i \leq 0$), would rise from 27 to 37 percent per year.

[14] Unconditional losses are computed as the average discounted losses across $5 \cdot 10^3$ stochastic simulations, each 10^3 periods long after discarding several pre-simulated periods. The simulations are initialized to the average state. The number of pre-simulated periods to discard is determined by tracking the time-evolution of the summary statistics to ascertain that the distribution reached its stationary configuration.

[15] Clearly, the welfare costs of a suboptimally high level of inflation are expected to be greater than the costs under optimal policy. Research measuring the welfare costs of suboptimally high inflation is surveyed for example in Lucas (2004).

of the shock process innovations are 50% larger. Therefore, the welfare costs remain moderate even in a much more uncertain economic environment.

Losses for model without LB: μ_{LQ} Losses for model with LB: μ Additional losses from LB: $\Delta(\mu)$	$1 \cdot \sigma_{\varepsilon u}$	$1.5 \cdot \sigma_{\varepsilon u}$
$1 \cdot \sigma_{\varepsilon r}$	-0.2752	-0.6193
	-0.2788	-0.6304
	-0.0036	-0.0110
$1.5 \cdot \sigma_{\varepsilon r}$	-0.2752	-0.6193
	-0.2871	-0.6416
	-0.0119	-0.0223

Table 3: Welfare Costs Of Optimal Inflation And The Lower Bound (Annualized Percentage Values)

5.3 Robustness of Results

This subsection confirms the robustness of the findings by analyzing a wide range of changes to each structural parameter of the model.

Table 4 reports the optimal average and standard deviation of inflation, expected frequency of zero nominal rates, and welfare costs of the lower bound under alternative calibrations. Each calibration assumes a change in only one deep structural parameter. All other parameters take the baseline values. Parameter changes effect the results via two main channels in the model. On the one hand, the change modifies the intertemporal equilibrium conditions describing the behavior of the economy. In particular, the change alters the 'slope' of the aggregate-supply relation in equation (8). On the other hand, the parameter change also modifies the welfare 'weight' assigned to output stability relative to price stability in equation (7). Results for the various exercises are described in turn.

The interest-sensitivity of output ($\varphi > 0$) determines the leverage of nominal interest rate policy on output and also the relative welfare weight on output stability. With a higher real-rate elasticity of output the leverage of the policymaker is stronger, so the frequency of zero nominal rates and the welfare costs of the lower bound go down. The slope of the aggregate-supply relation in equation (8) falls, making it optimal to tolerate a relatively higher volatility of inflation. Yet, the relative welfare weight on output stability in equation (7) also falls implying that a relatively lower volatility of inflation is optimal instead. Depending on which of these two opposing effects dominates, the volatility of inflation can turn out either higher or lower. Thus, it is not clear whether

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Alternative Calibrations	$E(\pi)$	s.d.(π)	Freq$(i=0)$	$\Delta(\mu)$
Baseline	0.17	1.9	27	-0.0036
Less interest-sensitivity of output $(\varphi = 1)$	0.00	1.5	73	-0.0107
More interest-sensitivity of output $(\varphi = 10)$	0.31	2.0	22	-0.0019
Almost flexible prices $(\alpha = 0.1)$	0.12	0.4	0	-0.0000
Very sticky prices $(\alpha = 0.9)$	0.11	2.6	77	-0.0265
Less competition $(\theta = 3)$	0.51	2.0	10	-0.0004
More competition $(\theta = 15)$	0.04	1.9	35	-0.0169
Less elastic marginal costs $(\omega = 0.1)$	0.17	1.9	27	-0.0011
More elastic marginal costs $(\omega = 10)$	0.17	0.8	28	-0.0479
Less inflation indexation $(\gamma = 0.85)$	0.09	1.7	22	-0.0013
More inflation indexation $(\gamma = 0.95)$	0.37	2.4	38	-0.0069
Lower steady state real rate $(r_{ss} = 2\%)$	0.61	1.9	69	-0.0127
Higher steady state real rate $(r_{ss} = 5\%)$	0.08	2.0	2	-0.0010

Table 4: Robustness Of Results (Annualized Percentage Values)

the optimal inflation rate should rise or fall. The exercise shows that for a significantly higher real-rate elasticity of output, namely φ equal to 10, the optimal inflation rate goes up to about 0.31 percent per year.

The optimal inflation rate goes down for both very low and very high degrees of price stickiness $(0 < \alpha < 1)$. The intuition for this finding is the following. If prices are almost flexible, the volatility of inflation turns out to be lower, the lower bound is seldom reached and its welfare costs fall substantially. So, there is little incentive to support a higher average inflation rate to limit the expected frequency of zero nominal rates. Conversely, if prices are very sticky, the volatility of inflation is higher, and the lower bound is expected to be reached more frequently causing larger welfare costs. Thus, it is more costly to support a higher average inflation rate. The exercise shows that for a very low or a very high degree of price stickiness, namely α equal to 0.1 or 0.9, the optimal inflation rate goes down to about 0.12 or 0.11 percent per year, respectively.

The price elasticity of demand substitution among alternative differentiated goods produced by firms $(\theta > 1)$ determines the degree of monopolistic competition in the model. There is a stronger incentive for policy to support higher average inflation if there is less competition among firms. Higher average inflation restrains the expected frequency of zero nominal rates and the welfare costs of the lower bound. Significantly less competition, namely θ equal to 3, implies that the optimal inflation rate goes up to about 0.51 percent per year.

The relevance of the elasticity of a firm's real marginal costs with respect to its own

output level ($\omega > 0$) is also evaluated. The incentive to counteract the likelihood that the lower bound may constrain interest rate policy is stronger if marginal costs are less elastic. Yet, the exercise shows that such incentive is not strong enough to generate any noticeable difference in the optimal inflation rate.

The degree of inflation indexation ($0 \leq \gamma < 1$) determines policy effectiveness via the expectational channel. Policy is less (more) effective when private-sector expectations are relatively less (more) forward-looking. The ability to credibly commit to a policy plan provides the monetary policymaker the leverage to steer private-sector expectations about the future state of the economy. The leverage through the expectational channel is stronger when the private sector is more forward-looking. Thus, optimal inflation, the expected frequency of zero nominal rates and the welfare costs of the lower bound are higher if there is more inflation indexation When inflation indexation is 5% more than the baseline, namely γ equal to 0.95, the optimal inflation rate goes up to about 0.37 percent per year.

The steady state real interest rate ($0 < r_{ss} < +\infty$) determines the steady state nominal interest rate in the model economy. The optimal inflation rate, the expected frequency of zero nominal rates and the welfare costs of the lower bound are higher if the steady state interest rate is lower. When the steady state real rate is down to half the value of the baseline, namely r_{ss} equal to 2% per year, the optimal inflation rate goes up to about 0.61 percent per year.

Therefore, the findings appear to be robust to a wide range of changes in the calibration of the deep structural parameters of the model. The optimal inflation rate is positive but not large. The calibration exercises are carried out modifying only one structural parameter in isolation. These exercises serve the purpose of identifying the extent that a particular parameter modification can play in terms of generating the results. Yet, joint uncertainty along multiple dimensions of the parameter space may have important implications for the robustness of the findings. This issue is investigated in the following section.

6   Model Misspecification

This section investigates the robustness of the results when policy is designed to be robust to a worst-case scenario of model misspecification. A robust control version of the optimal monetary policy problem is introduced. The solution strategy to solve this more general model is illustrated, then results for the stochastic robust control equilibrium are presented. The optimal inflation rate is slightly higher, and the welfare costs of the

lower bound remain limited.

6.1   Robust Control Model

This subsection develops a generalization of the model introduced in section 2. The assumption of rational expectations is relaxed by assuming that the true model of the economy is surrounded by unmeasurable uncertainty. Following the approach of Hansen and Sargent (2007), the policy problem is represented as a 'robust control' problem.

In the robust control approach, the model is viewed as an approximation of the true model of the economy. The true model is unknown, but is known to be in a neighborhood around its approximating model. The robust control problem is described as a game, played between the policymaker who aims to maximize its objective function, and a fictitious adversary agent (or nature) who instead attempts to minimize the objective function.

The robust control version of the optimal monetary policy problem is:

$$\max_{\{\pi_t, x_t, i_t\}} \min_{\{w_{1t}, w_{2t}\}} -\hat{E}_0 \sum_{t=0}^{\infty} \beta^t \left[(\pi_t - \gamma \pi_{t-1})^2 + \lambda x_t^2 - \Theta \left(w_{1t}^2 + w_{2t}^2 \right) \right] \quad (14)$$

s.t.

$$\pi_t - \gamma \pi_{t-1} = \beta \hat{E}_t \left(\pi_{t+1} - \gamma \pi_t \right) + \kappa x_t + u_t \quad (15)$$

$$x_t = \hat{E}_t x_{t+1} - \varphi \left(i_t - \hat{E}_t \pi_{t+1} - r_t^n \right) \quad (16)$$

$$u_t = \rho_u u_{t-1} + \sigma_{\varepsilon u} \left(\varepsilon_{ut} + w_{1t} \right) \quad (17)$$

$$r_t^n = (1 - \rho_r) r_{ss} + \rho_r r_{t-1}^n + \sigma_{\varepsilon r} \left(\varepsilon_{rt} + w_{2t} \right) \quad (18)$$

$$i_t \geq 0 \quad (19)$$

where $\hat{E}_t$ denotes the mathematical expectations operator conditional on information available at time t. The accent is added above the expectations operator to reflect that expectations are based on the approximating model underlying the robust control problem. As in the standard rational expectations framework, the policymaker and the private-sector are assumed to share the same model of the economy, since the policymaker seeks to maximize welfare for the representative consumer. The policymaker and the private-sector are also assumed to share the same concern for model misspecification. With both the policymaker and the private-sector facing model uncertainty, the adversary agent is able to exploit the policymaker's commitment to a policy plan and hamper its ability to shape private-sector expectations.

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The adversary agent seeks to minimize the objective function of the monetary policymaker in equation (14) by choosing the variables w_{1t} and w_{2t} to manipulate the evolution of the exogenous stochastic shock processes, (17) and (18), respectively. The variables w_{1t} and w_{2t} represent unmeasurable uncertainty surrounding the true model. The parameter $\Theta \geq 0$ in the objective function (14) measures the degree of potential misspecification between the approximating model and a worst-case scenario. Intuitively, when Θ is smaller a more severe modeling misspecification can arise. When $\Theta \rightarrow +\infty$ there is no concern for misspecification and the robust control problem simplifies to the standard rational expectations case, already analyzed in the previous sections.

Abstracting from the lower bound, the simpler optimal monetary policy problem (14)-(18) can be solved applying standard linear-quadratic methods. As in the standard rational expectations framework, the lower bound constraint (19) renders the problem nonlinear.

6.2   Solving the Robust Control Model

This subsection illustrates the strategy to solve the model introduced in the previous section and derives the stochastic robust control equilibrium. The policy problem is reduced to a nonlinear system of recursive equilibrium conditions, which defines a fixed-point to be found numerically.

The infinite horizon Lagrangian for a robust control version of the optimal monetary policy problem (14)-(19) is easily obtained as

$$
\max_{\{\pi_t, x_t, i_t\}} \min_{\{m_{1t}, m_{2t}, w_{1t}, w_{2t}\}} \hat{\mathcal{L}} \equiv \hat{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ -\left(\pi_t - \gamma \pi_{t-1}\right)^2 - \lambda x_t^2 + \Theta \left(w_{1t}^2 + w_{2t}^2\right) \right. \tag{20}
$$
$$
+ m_{1t} \left[(1 + \beta\gamma) \pi_t - \gamma \pi_{t-1} - \kappa x_t - u_t\right] - m_{1t-1}\pi_t
$$
$$
\left. + m_{2t} \left[-x_t - \varphi \left(i_t - r_t^n\right)\right] + m_{2t-1}\beta^{-1} \left(x_t + \varphi \pi_t\right)\right\}
$$
$$
\text{s.t.}
$$
$$
\text{Equations (17)-(19)}
$$

where m_{1t} and m_{2t} denote the Lagrange multipliers for the intertemporal equilibrium conditions (15) and (16), respectively. This optimization problem is in recursive form.

The Kuhn-Tucker conditions of the Lagrangian (20) are given by

$$\partial \hat{\mathcal{L}} / \partial \pi_t = -2 \left(\pi_t - \gamma \pi_{t-1} \right) + \left(1 + \beta \gamma \right) m_{1t} - m_{1t-1} + \beta^{-1} \varphi m_{2t-1} = 0 \qquad (21)$$

$$\partial \hat{\mathcal{L}} / \partial x_t = -2\lambda x_t - \kappa m_{1t} - m_{2t} + \beta^{-1} m_{2t-1} = 0 \qquad (22)$$

$$\partial \hat{\mathcal{L}} / \partial i_t \cdot i_t = -\varphi m_{2t} \cdot i_t = 0, \quad m_{2t} \geq 0, \quad i_t \geq 0 \qquad (23)$$

$$\partial \hat{\mathcal{L}} / \partial w_{1t} = 2\Theta w_{1t} - \sigma_{\varepsilon u} m_{1t} = 0 \qquad (24)$$

$$\partial \hat{\mathcal{L}} / \partial w_{2t} = 2\Theta w_{2t} + \sigma_{\varepsilon r} \varphi m_{2t} = 0 \qquad (25)$$

where either the Lagrange multiplier on the Euler equation ($m_{2t} \geq 0$) or the Lagrange multiplier on the nominal rate ($i_t \geq 0$) is zero, at all states and for each period.

Solving the nonlinear system of equations (15), (16) and (21)-(25) delivers a seven-dimensional nonlinear equilibrium response function

$$\hat{y} \left(s_t \right) \equiv \left(\pi_t, x_t, i_t \geq 0, m_{1t}, m_{2t} \geq 0, w_{1t}, w_{2t} \leq 0 \right) \subset R^7$$

defined over a five-dimensional state space

$$s_t \equiv \left(u_t, r_t^n, \pi_{t-1}, m_{1t-1}, m_{2t-1} \geq 0 \right) \subset R^5$$

The accent is added above the equilibrium response function to set it apart from the standard rational expectations case. There are now two additional control variables, ($w_{1t}, w_{2t} \leq 0$) representing model misspecification.[16]

Associated with the equilibrium response function $\hat{y} \left(s_t \right)$ is the equilibrium expectations function

$$\hat{E}_t \hat{y}_{t+1} = \int \hat{y} \left(s_{t+1} \right) f \left(\varepsilon_{jt+1} \right) d \left(\varepsilon_{t+1} \right) \qquad (26)$$

where $f \left(\cdot \right)$ is the probability density function of the stochastic shock innovations, $\varepsilon_t \equiv \left(\varepsilon_{ut}, \varepsilon_{rt} \right) \in R^2$. The evolution of the exogenous shocks $\left(u_t, r_t^n \right)$ is described by equations (17) and (18). As in the standard rational expectations framework, the expectations function (26) is not integrated over a probability density function of shock innovations when agents are assumed to have perfect foresight ($\sigma_{\varepsilon j} \to 0$ for $j = u, r$). The following definition of a stochastic robust control equilibrium is proposed.

Definition 2 (SRCE) *Assume $\sigma_{\varepsilon j} \geq 0$ for $j = u, r$ and $\Theta \geq 0$. A 'stochastic robust control equilibrium' of the optimal monetary policy problem (14)-(19) is a nonlinear*

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[16] Equation (25), $m_{2t} \geq 0$, $\sigma_{\varepsilon r} \geq 0$, $\varphi > 0$ and $\Theta \geq 0$ imply $w_{2t} \leq 0$.

policy response function $\hat{y}(s_t) \equiv (\pi_t, x_t, i_t \geq 0, m_{1t}, m_{2t} \geq 0, w_{1t}, w_{2t} \leq 0)$, *specified over the state space* $s_t \equiv (u_t, r_t^n, \pi_{t-1}, m_{1t-1}, m_{2t-1} \geq 0)$, *such that the nonlinear system of equations (15), (16) and (21)-(25) is satisfied.*

When $\Theta \to +\infty$ the model abstracts from potential misspecification and a robust control problem converges to the standard rational expectations case. Therefore, without a concern for model misspecification, a stochastic robust control equilibrium (SRCE) simplifies to a standard stochastic rational expectations equilibrium (SREE). This observation justifies the following corollary.

Corollary 3 *Assume* $\sigma_{\varepsilon j} \geq 0$ *for* $j = u, r$ *and* $\Theta \geq 0$. *Given* $\hat{y}(s_t)$ *solving a SRCE and* $y(s_t)$ *solving a SREE*, $\hat{y}(s_t) \to y(s_t)$ *for* $\Theta \to +\infty$.

Proof. When $\Theta \to +\infty$ equations (24) and (25) imply $w_{1t} = w_{2t} = 0$. And, the system of equations (15), (16) and (21)-(25) in definition 2 coincides with the system of equations (2), (3) and (10)-(12) in definition 1. ∎

Solving for a SRCE involves finding a fixed-point in the space of nonlinear response functions. Details of the nonlinear numerical procedure are in appendix A.1. Independent of the lower bound constraint, the perfect-foresight limit ($\sigma_{\varepsilon j} \to 0$ for $j = u, r$) of the SRCE converges to the deterministic steady state around which the equilibrium conditions of the model are log-linearized.

6.3 Robustly Optimal Inflation and Welfare Costs

This subsection presents results for the stochastic robust control equilibrium. The model is solved for the baseline quarterly calibration in table 1. For readability, all results are presented as annualized percentage values.

With respect to the standard rational expectations framework, there is one additional parameter to calibrate ($\Theta \geq 0$). Following Hansen and Sargent (2007), a statistical theory of model selection can be employed to determine a context-specific value for the degree of model misspecification Θ. This procedure allows to discriminate between the approximating model and a worst-case scenario associated with that particular value for Θ. Taking an agnostic view on whether observed equilibrium outcomes are generated by the approximating model or a worst-case model, one selects a reasonable probability of making a detection error, $p(\Theta)$. Intuitively, no concern for robustness ($\Theta \to +\infty$) corresponds to a detection error probability of 50%, as the robust control problem converges to the standard rational expectations case. When Θ is smaller a more severe modeling

misspecification can arise, which is more easily detected observing equilibrium outcomes. Details of the procedure are in appendix A.3.

Figure 3 presents the stationary distribution of inflation, the output gap, nominal and real interest rates $(\pi, x, i, i - E\pi_{+1})$. The detection error probability is about 29%, which is the most extreme model misspecification for which the algorithm can identify an equilibrium.[17] As in the previous figures, the distribution is expressed in terms of probability density. In each panel, the full vertical line indicates the mean of the stationary distribution of the corresponding endogenous variable. The dashed-dotted vertical line shows the deterministic steady state of the variable. As in the standard rational expectations framework, the lower bound renders the model nonlinear and causes a break down of certainty equivalence. The optimal average values of inflation and the nominal rate are higher than their respective deterministic steady state values. Comparing the various panels in figures 1 and 3, however, reveals that model misspecification increases the positive stochastic mean bias in inflation and the nominal rate. The optimal inflation rate and the optimal nominal rate are higher with uncertainty about the true model of the economy. There does not appear to be any significant difference between the average and deterministic steady state values for the output gap and the real rate.

Figure 4 depicts the optimal average and standard deviation of inflation and the expected frequency of non positive nominal rates over a range of detection error probabilities. Other parameters take the baseline values. A detection error probability of 50% reproduces results for the standard rational expectations case. As already mentioned, the lowest detection error probability for which the algorithm can find an equilibrium is about 29%. In each panel, the line with circles indicates values for the solution to the model that takes directly into account the lower bound constraint. The line with squares shows values obtained for a standard linear-quadratic approximation which abstracts from the lower bound. The top panel shows that the optimal inflation rate is increasing in a nonlinear fashion with the degree of model misspecification, although the deterministic steady state value of inflation is unaffected by the particular choice of detection error probability. The optimal inflation rate is about 0.17 percent per year under the assumption of rational expectations. When the detection error probability from about 30% is reduced just a little to 29%, the optimal inflation rate jumps up from about 0.53 to 0.87 percent per year.

The middle and bottom panels of figure 4 together reveal that model misspecification

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[17]Detection error probabilities are computed averaging across 10^4 stochastic simulations. The sample size is set to 80 periods, which is equal to number of observations over the estimation period $1983{:}1-2002{:}4$ used in section 4 to calibrate the quarterly model.

effects the long-run policy tension as a result of the lower bound. Misspecification worsens the trade-off between supporting a higher average inflation rate which is costly in welfare terms to the economy, and a higher likelihood that the lower bound may constrain interest rate policy. Without a rise in average inflation, the volatility of inflation and the expected frequency of zero nominal rates would otherwise increase by even more for any given detection error probability. Solving the model abstracting from the lower bound under the lowest admissible detection error probability, namely about 29%, the standard deviation of inflation would rise from 2.9 to 3.4 percent per year. And, the expected frequency of non positive nominal rates, Freq($i \leq 0$), would rise from 56 to 71 percent per year.

Figure 5 illustrates the welfare costs of optimal inflation and the lower bound over a range of detection error probabilities. Other parameters take the baseline values. The losses are represented in terms of a permanent reduction in consumption equivalents derived via a transformation of the unconditional losses in the objective function (14), as explained in appendix A.2. The line with circles shows the losses for the model that takes directly into account the lower bound. The line with squares are the losses for a standard linear-quadratic approximation. In both cases, model misspecification inflicts greater welfare losses to the economy. The distance between the two lines provides a measure of the welfare costs of optimal inflation and the lower bound. The costs amount to a permanent reduction in consumption of less than -0.01 percent per year over the feasible range of detection error probabilities. This result confirms that the welfare costs of the lower bound remain moderate when policy is conducted optimally.

7 Conclusions

The achievement of low and stable inflation gives rise to concern about the effectiveness of monetary policy when interest rates are close to zero. As a consequence, there is an ongoing debate on the correctly measured inflation rate that monetary policy should aim for in the long-run. So far discussions are limited to intuitions based on indirect evidence only, since a robust prescription for the optimal inflation rate is not available yet.

A direct assessment of the optimal inflation rate is provided in a New Keynesian sticky-price model with an occasionally binding zero lower bound on the nominal rate. The results show that not only real-rate shocks but also mark-up shocks are relevant for the determination of the optimal inflation rate, which is about 0.17 percent per year under a baseline calibration.

When monetary policy is designed to be robust to a worst-case scenario of model misspecification, the 'robustly' optimal inflation rate may be as high as 0.87 percent per year. The welfare costs of inflation and the lower bound are limited when policy is conducted optimally.

A   Appendix

A.1   Numerical Algorithm

This appendix illustrates the numerical procedure for solving a robust control version of the optimal monetary policy problem (14)-(19). This algorithm is of more general interest than the one employed in Billi (2005) and also in Adam and Billi (2006, 2007) to solve a standard rational expectations case.

As explained in subsection 6.2, solving for a stochastic robust control equilibrium (SRCE) involves finding the response function, $\hat{y}(s) \subset R^7$, satisfying the nonlinear system of equations in definition 2. The state space, $s \subset R^5$, is discretized into a set of N interpolation nodes, $\{s_n | n = 1, ..., N\}$ where $s_n \in s$. The response function is evaluated at intermediate values of the discretization grid by resorting to multilinear interpolation. The stochastic shock innovations, $\varepsilon \in R^2$, are assumed normally distributed. The associated expectations function (26) is evaluated at the interpolation nodes by an M-node Gaussian-Hermite quadrature scheme, $\{\varepsilon_m | m = 1, ..., M\}$ where $\varepsilon_m \in \varepsilon$ as in chapter 7 of Judd (1998). A quadrature scheme is not necessary and the numerical procedure is much less involved when solving the perfect-foresight limit ($\sigma_\varepsilon \to 0$).

The fixed-point of the nonlinear system of equations in definition 2 is found with an iterative update rule

$$\hat{y}^{k+1} \leftarrow \hat{y}^k + \iota \left(\hat{y}^{k+1} - \hat{y}^k \right), \text{ from step } k \text{ to } k+1 \tag{27}$$

where $\iota \in (0, 1]$ is the step size, which is chosen to guarantee algorithm stability as in chapter 4 of Bertsekas (1999).

The **Algorithm** proceeds as follows:

Step 1: Choose the degree of approximation, N and M, and assign the interpolation and quadrature nodes with an efficient method. Guess an initial value for the response function at the interpolation nodes, $\hat{y}^0(s_n)$.

Step 2: Evaluate the expectations function, $\hat{E}\hat{y}^k_{+1}$, consistent with the current guess for the response function, $\hat{y}^k$. Then derive a new guess, $\hat{y}^{k+1}$, by applying the iterative

update rule (27).

Step 3: Stop if $\max_{n=1,...,N} \left\| \hat{y}^{k+1} - \hat{y}^k \right\| < \tau$, where $\tau > 0$ is the convergence tolerance level. Otherwise repeat step 2.

Once convergence is achieved, the accuracy of the solution is checked. Compute the Euler equation residuals, $R(s_r)$, for the model's implementability constraints, (15) and (16), at an arbitrary set of residual nodes, $\{s_r | r = 1, ..., R\}$ where $s_r \in s$ as in Santos (2000). Then, examine the maximum approximation error, $e \equiv \max_{r=1,...,R} \left\| R(s_r) \right\|$.

Employing a sparse grid method is crucial when assigning the nodes, since for a stochastic nonlinear numerical procedure the dimension of the state space is very high. Achieving with a sparse grid an acceptable degree of approximation for the baseline requires setting $N \approx 3.6 \cdot 10^4$ and $M \approx 45$, as opposed to a linearly spaced grid which would require over 10^6 interpolation nodes. The support is chosen to cover ± 4 unconditional standard deviations for the exogenous shocks (u_t, r_t^n) and to avoid erroneous extrapolation of the endogenous state variables $(\pi_{t-1}, m_{1t-1}, m_{2t-1})$ when performing stochastic simulations.

For efficiency, an approximation refinement method is employed. The initial guess for the response function is set to the linearized solution around the deterministic steady state, and a coarse grid of interpolation and quadrature nodes $N^0 \times M^0$ is chosen. The nonlinear solution obtained for the coarse grid is interpolated over a finer grid and used as a new guess to resolve. The degree of approximation is progressively increased towards the final set of nodes, $\{N^0 \times M^0 < N^1 \times M^1 < ... < N \times M\}$.

The convergence tolerance level is set to the square root of machine precision, namely $\tau = 1.49 \cdot 10^{-8}$. The Euler equation residuals are verified both at the interpolation nodes and over a finer linearly spaced grid of $R = 10^5$ residual nodes. The maximum approximation error encountered is close to the square root of machine precision.

A.2 Consumption Equivalent Losses

This appendix shows how to derive welfare equivalent consumption losses for a robust control version of the optimal monetary policy problem (14)-(19). This procedure is a generalization of the one described in Adam and Billi (2007) for a standard rational expectations case.

This procedure extends the results in chapter 6 of Woodford (2003) by introducing model misspecification. The expected discounted sum of utility of the representative

household can be validly approximated up to second order by

$$\hat{E}_0 \sum_{t=0}^{\infty} \beta^t U_t = \frac{U_c \overline{Y}}{2} \frac{\alpha \theta \left(1 + \omega \theta\right)}{\left(1 - \alpha\right)\left(1 - \alpha\beta\right)} \hat{L} \tag{28}$$

where $\overline{Y}$ denotes steady state output, $U_c > 0$ is steady state marginal utility of consumption, and

$$\hat{L} \equiv -\hat{E}_0 \sum_{t=0}^{\infty} \beta^t \left[(\pi_t - \gamma\pi_{t-1})^2 + \lambda x_t^2 - \Theta \left(w_{1t}^2 + w_{2t}^2\right) \right] \leq 0$$

corresponds to the objective function in equation (14) which the monetary policymaker aims to maximize and nature attempts to minimize.

Assuming a permanent reduction in consumption, $\mu \leq 0$, the utility loss generated can be validly approximated up to second order by

$$\frac{1}{1 - \beta} \left(U_c \overline{Y} \mu + \frac{U_{cc}}{2} \left(\overline{Y}\mu\right)^2 \right) = \frac{U_c \overline{Y}}{1 - \beta} \left(\mu - \frac{1}{2\varphi}\mu^2 \right) \tag{29}$$

where $U_{cc} < 0$ is the second derivative of utility with respect to consumption evaluated at the steady state, and $\varphi \equiv -U_c / \left(\overline{Y} U_{cc}\right) > 0$ measures the intertemporal elasticity of substitution of aggregate expenditure.

Equating (28) and (29) delivers

$$\frac{1}{2\varphi}\mu^2 - \mu + \frac{1 - \beta}{2} \frac{\alpha \theta \left(1 + \omega \theta\right)}{\left(1 - \alpha\right)\left(1 - \alpha\beta\right)} \hat{L} = 0$$

The utility equivalent loss in permanent consumption is then given by

$$\mu = \varphi \left(1 - \sqrt{1 - \frac{1 - \beta}{\varphi} \frac{\alpha \theta \left(1 + \omega \theta\right)}{\left(1 - \alpha\right)\left(1 - \alpha\beta\right)} \hat{L}} \right) \tag{30}$$

A.3   Detection Error Probabilities

This appendix explains how to calculate detection error probabilities to calibrate the parameter $\Theta \geq 0$, which measures model misspecification in a robust control version of the optimal monetary policy problem (14)-(19).

Following Hansen and Sargent (2007), consider a fixed sample of observations of moderate size T. In a small sample, it is difficult to statistically detect the distance between an approximating model and a distorted model associated with the worst-case

perturbation. The uncertainty surrounding the approximating model tends to disappear as the sample size increases.

On the one hand, estimate the log-likelihood ratio under the assumption that the data is generated by an approximating model

$$r_A = \frac{1}{T} \sum_{t=0}^{T-1} \left[\frac{1}{2} w'_{At} w_{At} - w'_{At} \varepsilon_{At} \right]$$

where w_{At} is a vector of control variables distorting the evolution of the exogenous stochastic shock processes (17) and (18). ε_{At} is a vector of normally distributed innovations, independent both across time and cross-sectionally. The paths for the controls are obtained performing stochastic simulations under the assumption that the dynamics of the shocks are not distorted.

On the other hand, estimate the log-likelihood ratio under the assumption that the data generating process is the distorted model associated with the worst-case perturbation

$$r_B = \frac{1}{T} \sum_{t=0}^{T-1} \left[\frac{1}{2} w'_{Bt} w_{At} + w'_{Bt} \varepsilon_{Bt} \right]$$

where w_{Bt} is a vector of control variables distorting the evolution of the exogenous stochastic shock processes (17) and (18). ε_{Bt} is a vector of normally distributed innovations, independent both across time and cross-sectionally.

Assigning equal prior weights to the approximating model and the distorted model, the overall detection error probability is given by

$$p(\Theta) = \frac{1}{2} (p_A + p_B)$$

where $p_j = \text{Freq}(r_j \leq 0)$, for $j = A, B$.

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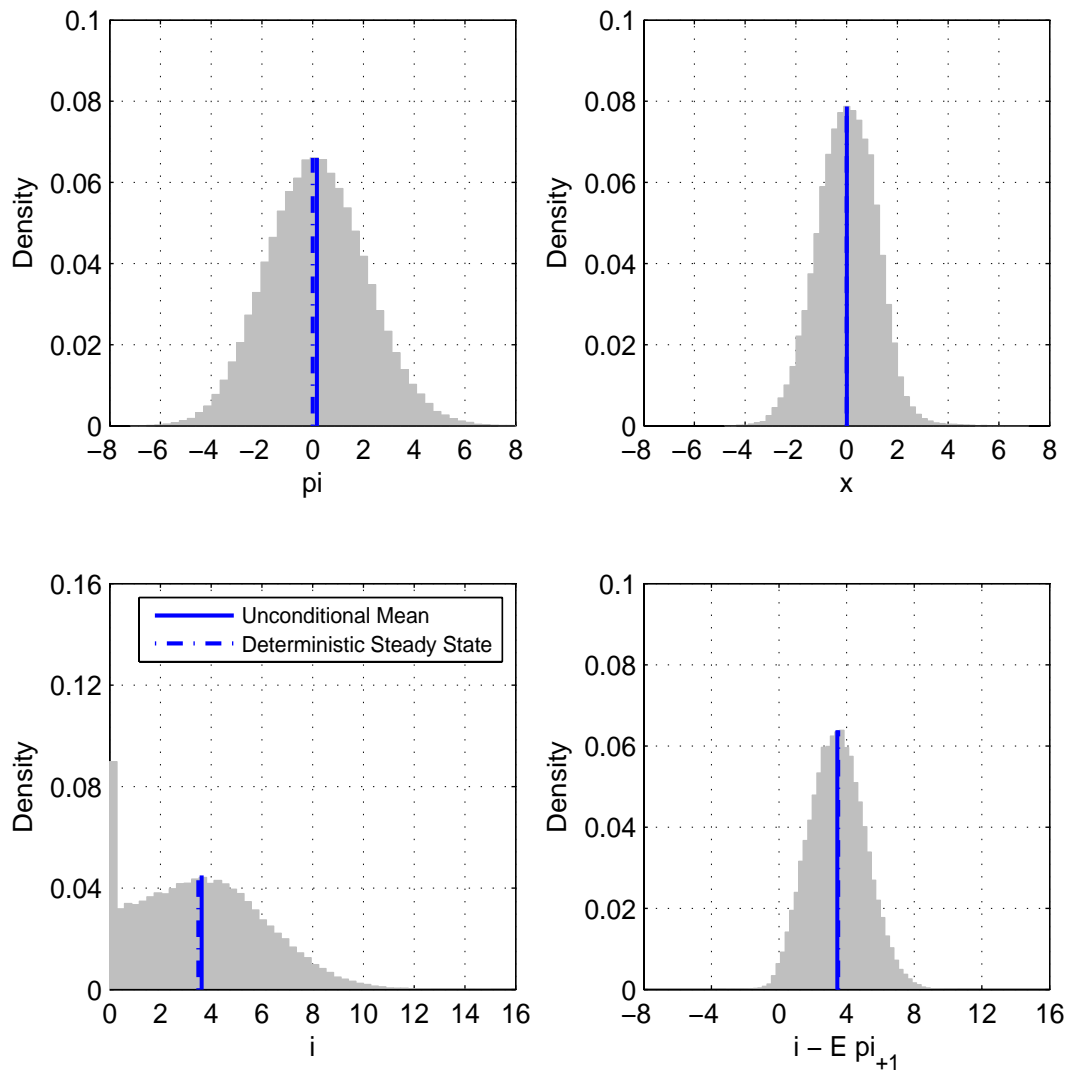
Figure 1: Stationary Distribution Of Inflation, Output Gap, Nominal And Real Interest Rates Under A Baseline Calibration (Annualized Percentage Values)
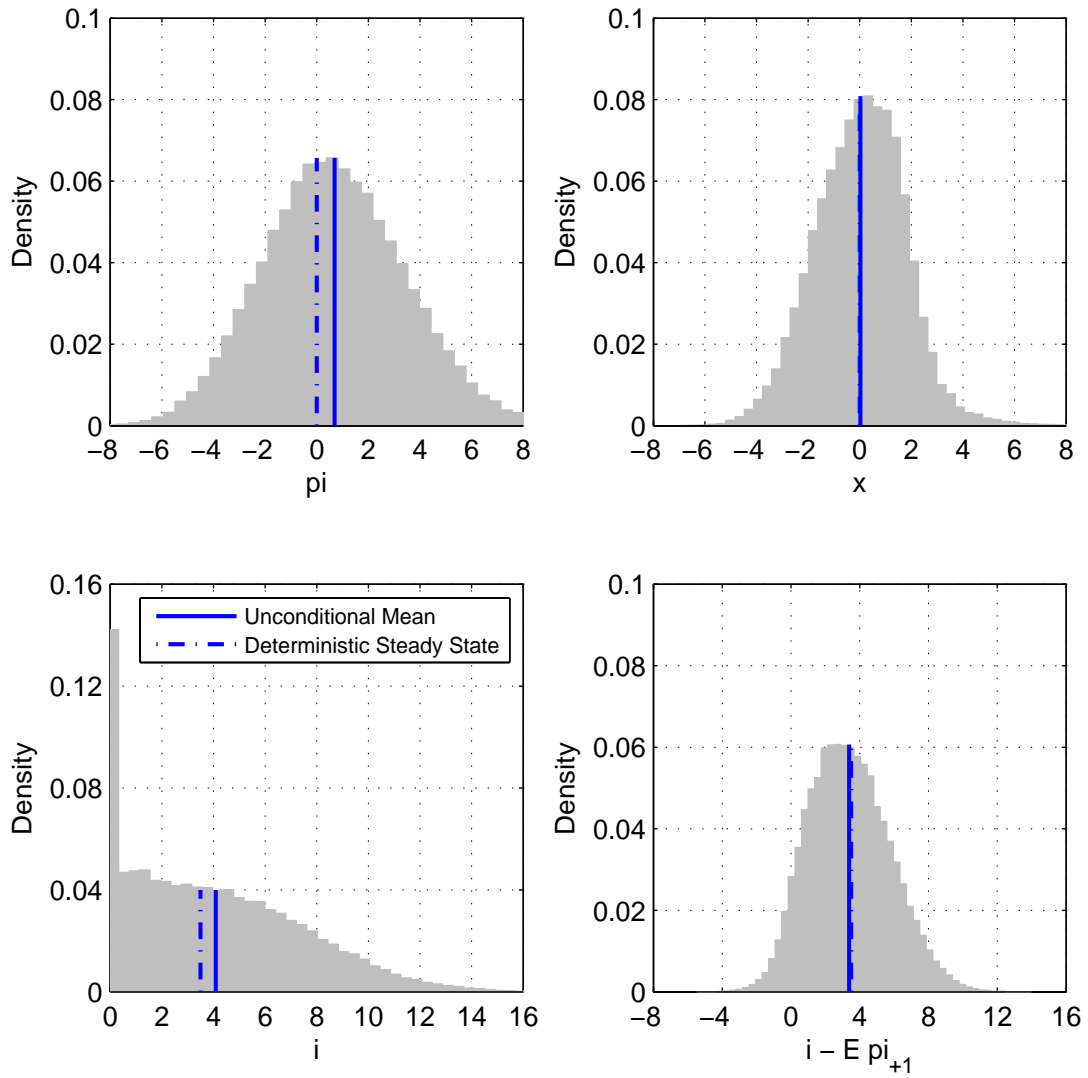
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Figure 2: Stationary Distribution In A More Uncertain Economic Environment, i.e., Standard Deviation Of Shock Innovations 50% Larger Than Under The Baseline (Annualized Percentage Values)
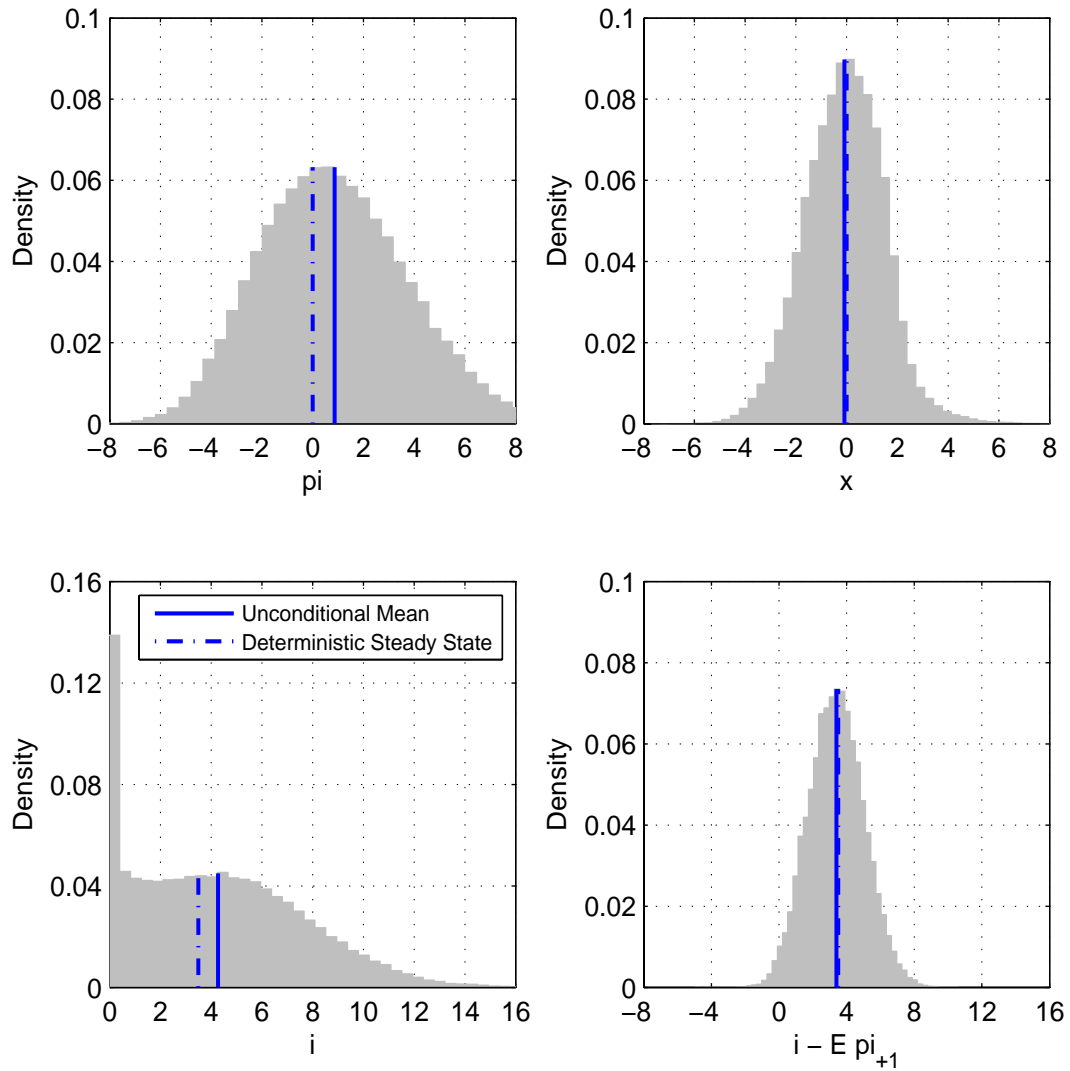
Figure 3: Stationary Distribution With Extreme Robustness To Model Misspecification, i.e., Lowest Detection Error Probability For Which A Solution Is Found (Annualized Percentage Values)
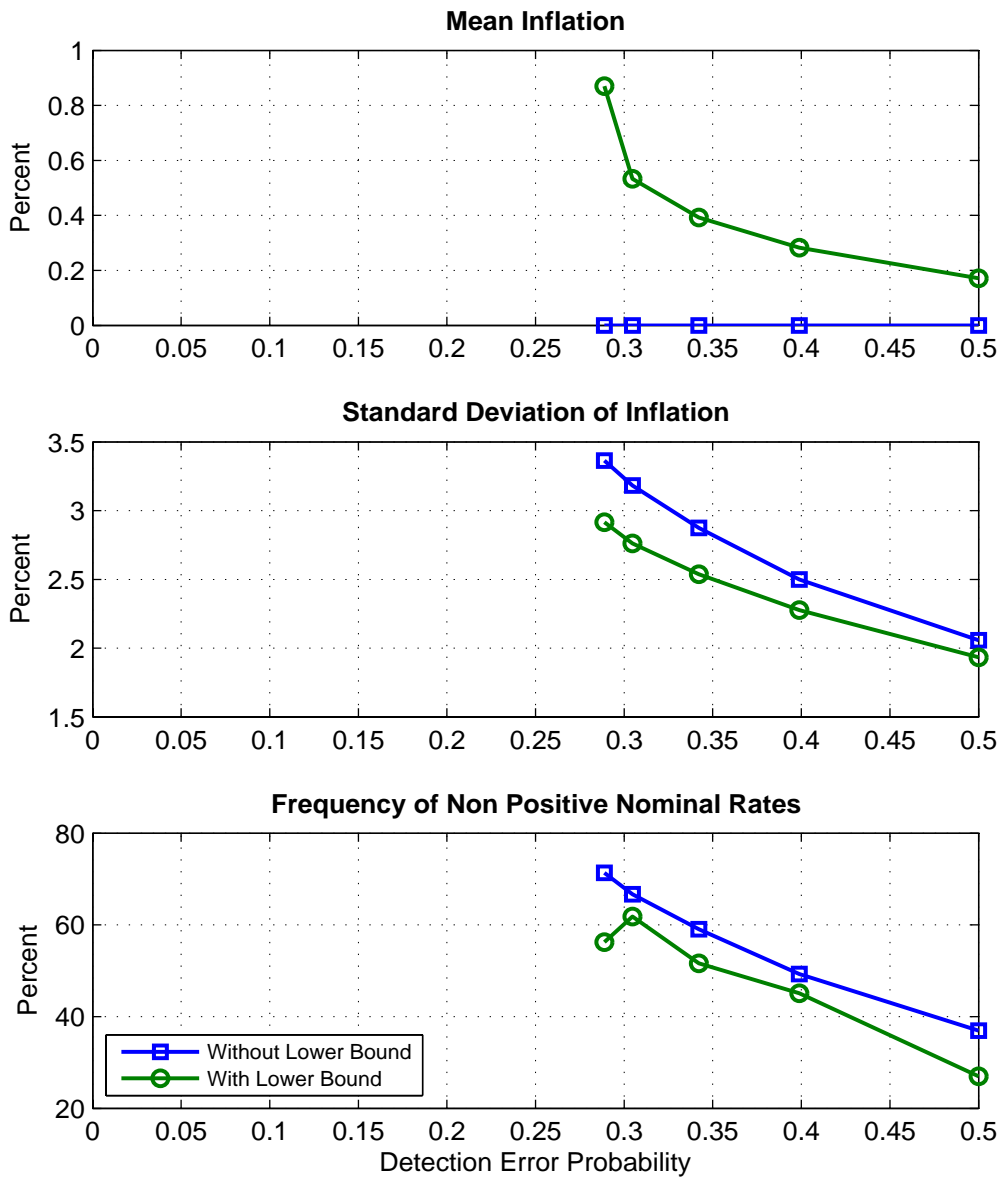
Mean Inflation

Standard Deviation of Inflation

Frequency of Non Positive Nominal Rates

Without Lower Bound
With Lower Bound

Detection Error Probability

Figure 4: Optimal Inflation And Robustness To Model Misspecification (Annualized Percentage Values)
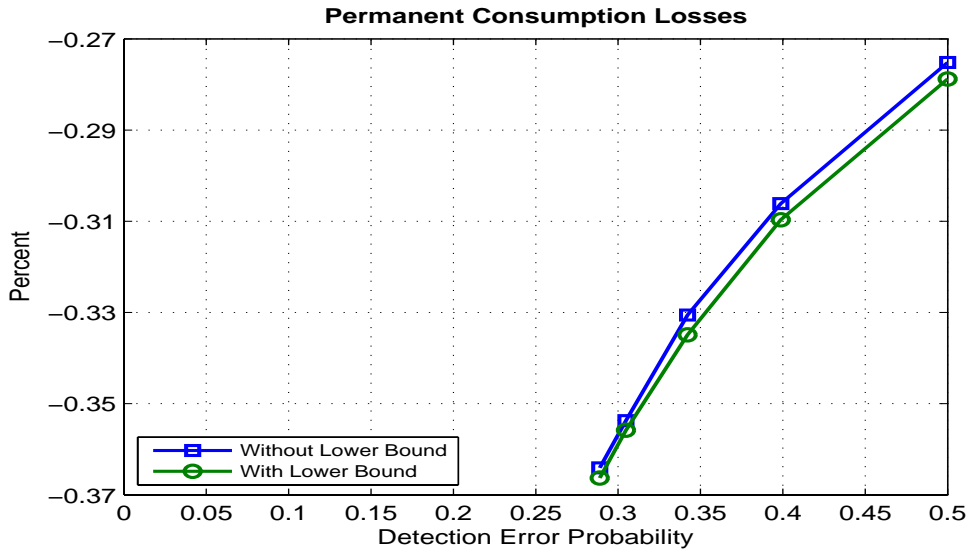
Figure 5: Welfare Costs Of Optimal Inflation And Robustness To Model Misspecification (Annualized Percentage Values)