FDIC Center for Financial Research Working Paper

No. 2008-03

Time Changed Markov Processes in Unified Credit-Equity Modeling*

December 2007

FDIC CFR

Federal Deposit Insurance Corporation • Center for Financial Research

Time Changed Markov Processes in Unified Credit-Equity Modeling*

Peter Carr † **Vadim Linetsky**‡ **Rafael Mendoza**§

December 23, 2007

Abstract

This paper develops a novel class of hybrid credit-equity models with state-dependent jumps, local-stochastic volatility and default intensity based on time changes of Markov processes with killing. We model the defaultable stock price process as a time changed Markov diffusion process with state-dependent local volatility and killing rate (default intensity). When the time change is a Lévy subordinator, the stock price process exhibits jumps with state-dependent Lévy measure. When the time change is a time integral of an activity rate process, the stock price process has local-stochastic volatility and default intensity. When the time change process is a Lévy subordinator in turn time changed with a time integral of an activity rate process, the stock price process has state-dependent jumps, local-stochastic volatility and default intensity. We develop two analytical approaches to the pricing of credit and equity derivatives in this class of models. The two approaches are based on the Laplace transform inversion and the spectral expansion approach, respectively. If the resolvent (the Laplace transform of the transition semigroup) of the Markov process and the Laplace transform of the time change are both available in closed form, the expectation operator of the time changed process is expressed in closed form as a single integral in the complex plane. If the payoff is square-integrable, the complex integral is further reduced to a spectral expansion. To illustrate our general framework, we time change the jump-to-default extended CEV model (JDCEV) of Carr and Linetsky (2006) and obtain a rich class of analytically tractable models with jumps, local-stochastic volatility and default intensity. These models can be used to jointly price and hedge equity and credit derivatives.

*This research was supported in part by the grants from the Federal Deposit Insurance Corporation, Moody's Investor Services, and the National Science Foundation under grant DMI-0200429.

†New York University, Courant Institute, 251 Mercer Street, New York, NY 10012, Phone: (212) 617-5056, E-mail: pcarr@nyc.rr.com.

‡Department of Industrial Engineering and Management Sciences, McCormick School of Engineering and Applied Sciences, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208, Phone: (847) 491 2084, E-mail: linetsky@iems.northwestern.edu, Web: http://users.iems.northwestern.edu/∼linetsky.

§Department of Industrial Engineering and Management Sciences, McCormick School of Engineering and Applied Sciences, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208, E-mail: r-mendoza2@northwestern.edu.

1 Introduction

The volume in credit derivatives continues to grow. In its 2007 Mid-Year Market Survey ISDA reported that the notional amount outstanding of credit derivatives grew by seventy five percent in the past year to \$45 trillion. Innovation also continues unabated in the equity derivatives market. ISDA Survey reported that the notional amount outstanding of equity derivatives grew by fifty seven percent in the past year to \$10 trillion. The joint growth in the equity and credit derivatives markets has lead to renewed interest in the modeling of asset prices in the presence of default risk.

Until recently equity derivatives pricing models and credit derivatives pricing models have developed more or less independently of each other. Equity derivatives models largely concentrated on modeling the implied volatility smile by introducing jumps and/or stochastic volatility into the stock price process (see Gatheral (2006) for a survey), and ignored the possibility of default of the firm underlying the option contract. At the same time, credit models focused on modeling the default event and ignored the information available in the equity derivatives market (see Bielecki and Rutkowski (2002), Duffie and Singleton (2003), and Lando (2004) for surveys of credit risk models). Recently, market practitioners have realized that equity derivatives markets and credit markets are closely related, and a variety of cross-market trading and hedging strategies have emerged in the industry under such names as equity-to-credit and credit-to-equity. Indeed, a deep-out-of-the-money put on a firm's stock that has little chance to be exercised unless the firm goes bankrupt and its stock price drops to zero or near zero, is effectively a credit derivative that pays the strike price in the event of bankruptcy. Indeed, over the past several years, every time the credit markets become seriously concerned about the possibility of bankruptcy of a firm, the open interest, daily volume of trading, and the implied volatility of deep-out-of-the-money puts on the firm's stock explode many times over their historical average. In late 2005 and early 2006 the credit markets were concerned about the possibility of General Motors bankruptcy. While the GM stock traded between \$18 and \$22 in the December 2005 — January 2006 period, January 2007 puts with strikes of \$10, \$7.50, \$5, and even \$2.50 all had very substantial open interest, large daily trading volumes, and implied volatilities of between 100% and 140%. In August and September of 2007 a similar story took place with deep-out-of-the-money puts on Countrywide Financial on Countrywide's bankruptcy concerns due to its substantial exposure to subprime mortgages.

In this paper we propose a flexible analytically tractable modeling framework which unifies the valuation of all credit derivatives and equity derivatives related to a given firm. We model the firm's stock price as the fundamental state variable that is assumed to follow a *time changed Markov process with killing.* Our model architecture is to start with an analytically tractable Markov process with killing (e.g., a one-dimensional diffusion with killing) and subject it to a stochastic time change (clock) with the analytically tractable Laplace transform. If the resolvent (the Laplace transform of the transition semigroup) of the Markov process and the Laplace transform of the time change are both known in closed form, then the expectation operator of the time changed process and, hence, the corresponding pricing operator, can be recovered via the Laplace transform inversion. Moreover, if the spectral representation of the transition semigroup is known in closed form, then the Laplace inversion for the time changed process can also be accomplished in closed form, leading to analytical pricing of credit and equity derivatives.

Many properties of the clock are inherited by the time changed process, allowing us to produce desired behavior in the stock price process modeled as a time changed Markov process.

To introduce jumps, we add a jump component into the clock. To introduce stochastic volatility, we add an absolutely continuous component into the clock. By composing the two types of time changes we construct models that exhibit both state-dependent jumps and stochastic volatility. The time changed process also inherits many properties of the original process. If the original process is a Markov process with killing, then the time changed process also has killing with the state-dependent killing rate, leading to models with the default intensity dependent on the stock price. *Thus, our modeling framework incorporates duffusive dynamics, state-dependent jumps, stochastic volatility, and state-dependent default intensity in an analytically-tractable way.*

Our modeling framework can parsimoniously capture many fundamental empirical observations in equity and credit markets, including the well-known positive relationship between credit default swap (henceforth CDS) spreads and corporate bond yields and implied volatilities of equity options, the leverage effect (the negative relationship between the realized volatility of a stock and its price level), the volatility skew/smile effects, and jumps in the stock price process. As such, the class of models we propose is very general, nesting many of the models already in the credit and equity derivatives literatures as special cases corresponding to a particular choice of the Markov process and the time change.

The class of models developed in the present paper can be thought of as a far-reaching generalization of the hybrid credit-equity models that describe the stock price dynamics as a one-dimensional diffusion with the local volatility and default intensity specified to be some functions of the stock price. In this class of models, in the event of default the stock price is assumed to drop to zero. Along these lines, Linetsky (2006) recently solved in closed form an extension of the Black-Scholes-Merton (BSM) model with bankruptcy, where the hazard rate of bankruptcy (default intensity) is a negative power of the stock price. The limitation of this model is that, while the default intensity is a function of the stock price, the local volatility of the diffusive stock price dynamics is constant, as in the original BSM model. To relax this restriction, Carr and Linetsky (2006) proposed and solved in closed form a *jump-to-default extended constant elasticity of variance* model (*JDCEV* for short). This model introduces stock-dependent default intensity into Cox's CEV model. This model features state-dependent local volatility and default intensity. Moreover, the default intensity is specified to be a linear function of the local variance. This specification provides a direct link between the stock price volatility and default intensity. However, the JDCEV model is still a one-dimensional diffusion model, with all the attendant limitations. In particular, the stock price volatility does not have an independent stochastic component, and there are no jumps in the stock price process. By appropriately time changing one-dimensional diffusions with killing, such as the Brownian motion with killing in Linetsky (2006) and the JDCEV diffusion in Carr and Linetsky (2006), we obtain models with jumps, stochastic volatility, and default.

The class of models developed in the present paper can also be thought of as a far-reaching generalization of the framework of *time changed Lévy processes with stochastic volatility* of Carr et al. (2003). Clark (1973) introduced into finance the notion of stochastic time changes, in which the observed price process is assumed to arise by running a time-homogeneous process on a second process called a clock. A clock is an increasing process which is normalized to start at zero and which can have a stochastic component. The requirement that time increases precludes the modeling of the clock as a diffusion, although it is frequently modeled as a time integral of a positive diffusion. Alternatively, the clock is often modeled as a *Lévy subordinator*, a Lévy process with positive jumps and non-negative drift. Time changing (subordinating) with Lévy subordinators goes back to the pioneering work of Bochner (1948), (1955) and is often called

Bochner's subordination. It is well-known that if we subordinate a Lévy process, we obtain another Lévy process (see Sato (1998)). In fact, many Lévy processes popular in finance can be represented as subordinate Brownian motions with drift with appropriately chosen subordinators (see Geman et al. (2001) for a survey). The variance gamma (VG) model of Madan and Milne (1991), Madan and Seneta (1990), and Madan et al. (1998), the normal inverse Gaussian (NIG) model of Barndorff-Nielsen (1998), and the Carr et al. (2002) model (CGMY) can all be represented as subordinate Brownian motions (for the latter see Madan and Yor (2006)). On the other hand, if one time changes Brownian motion with a time change that is a time integral of a CIR diffusion, one obtains Heston's (1998) stochastic volatility model. Building on this idea, Carr et al. (2003) time change general Lévy processes with time changes that are time integrals of other positive processes (e.g., CIR processes) and introduce a class of models termed *Lévy processes with stochastic volatility*. If the time change is an integral of another process, called the *activity rate* process, then the Lévy measure of the time changed process scales with the activity rate process. Thus, the activity rate speeds up or slows down jumps in the time changed process, in addition to speeding up or slowing down diffusive dynamics when time changing a Brownian motion (see also Barndorff-Nielsen et al. (2002) for related work on time changes and stochastic volatility).

However, there are two significant limitations in the framework of Carr et al. (2003). First, the process to be time changed is a space-homogeneous Lévy process with state-independent Lévy measure and constant volatility. Through the time change, both the volatility and the Lévy measure scale with the activity rate process, but there is no explicit dependence of the volatility and the Lévy measure on the stock price. This space homogeneity contradicts the accumulated empirical evidence. In the context of pure diffusion models, the so-called *local-stochastic volatility* models take the volatility process to be a product of a function of the stock price (such as the power function in the CEV model) and the stochastic volatility component (see Hagan et al. (2004), Lipton (2002), and Lipton and McGhee (2002)). These models generalize stochastic volatility models such as Heston's to introduce explicit stock price dependence into the local volatility. In the context of jump models, we would like the Lévy measure to include both some explicit state dependence on the stock price as well as on the stochastic volatility. This is not addressed in the framework of Carr et al. (2003). The second limitation of Carr et al. (2003) is that they do not include default in their models. The original process is a Lévy process with infinite lifetime. As a result, the time changed Lévy process with stochastic volatility also has infinite lifetime. Thus, these are pure equity derivatives models that do not capture the possibility of default of the firm. Several interesting recent papers also exploit time changes in derivatives pricing. Albanese and Kuznetsov (2004) apply time changes to construct equity derivatives pricing models with stochastic volatility and jumps, Boyarchenko and Levendorskiy (2007) apply time changes to construct interest rate models with jumps, and Ding et al. (2006) apply time changes to birth processes to generate multiple defaults processes for multi-name credit derivatives. However, in contrast to the focus of the present paper, neither of these references model equity derivatives and credit derivatives in a unified fashion.

The present paper develops the next generation of *hybrid credit-equity models with state-dependent jumps, local-stochastic volatility and default intensity* based on time changes of Markov processes with killing. The class of models proposed here remedy a number of limitations of the previous generations of models. By starting from a one-dimensional diffusion with killing and time changing it with a composite time change that can be represented as a subordinator in turn time changed with a time integral of another process (a *subordinator with stochastic volatility*), we construct processes with state-dependent jumps, local-stochastic volatility, and state-dependent default intensity. Moreover, due to special properties of one-dimensional diffusions, we retain analytical tractability in this general framework. This is in contrast with the previous generations of analytically tractable jump-diffusion and pure jump models based on Lévy processes with space homogeneous jumps (Merton (1976), Kou (2002), Kou and Wang (2004), Barndorff-Nielsen (1998), Eberlein et al. (1998), Madan et al. (1998), Carr et al. (2002)). The state dependence of the Lévy measure in our approach is inherited from the state dependence of the local volatility of the original diffusion subject to time change. At the same time, many existing models, including local volatility models (e.g., CEV), stochastic volatility models (e.g., Heston), local-stochastic volatility models (e.g., SABR), Lévy processes with stochastic volatility, and diffusion models with state-dependent default intensity are all nested as special cases in our general framework. Advantages of our hybrid credit-equity modeling framework include the ability to consistently price and cross hedge the entire book of credit as well as equity derivatives, in addition to the ability to incorporate a rich assortment of empirically relevant features.

The rest of this paper is organized as follows. In section 2, we present our model architecture. We define the defaultable stock price process as a time changed Markov diffusion process with killing. In section 3 we described the three major classes of time changes studied in this paper: subordinators, absolutely continuous time changes (time integrals of an activity rate process), and composite time changes (subordinators with stochastic volatility). In section 4 we prove a series of key theorems on the martingale and Markov properties of our time changed stock price processes. In section 5 we apply our defaultable stock model to set-up the general framework for the unified valuation of credit derivatives and equity derivatives. In section 6 we present our Laplace transform approach to the valuation of contingent claims on time changed Markov processes with the known resolvent (Laplace transform of the transition semigroup) and the known Laplace transform of the time change. In section 7 we present our spectral expansion approach that works in the special case of symmetric Markov processes and contingent claims with square-integrable payoffs. In this case the Laplace transform inversion is accomplished in closed form and results in a spectral expansion for the contingent claim value function. To illustrate our general approach, in section 8 we present a detailed study of time changing the jump-to-default extended CEV process of Carr and Linetsky (2006). Section 8.1 presents explicit expressions for the resolvent kernel, the spectral expansion of the transition probability density, the survival probability for the JDCEV process, and the spectral expansion for put options under the JDCEV process (call options are obtained via the call-put parity). In section 8.2 we introduce jumps and stochastic volatility into the JDCEV process and construct and numerically illustrate the time changed JDCEV model by calculating default probabilities, term structures of credit spreads, and implied volatility skews in a JDCEV model time changed with an Inverse Gaussian subordinator in turn time changed with a time integral of a CIR process (subordinator with stochastic volatility). The resulting stock price process is a pure jump process with state-dependent Lévy measure, stochastic volatility, and default intensity dependent both on the stock price and on the stochastic volatility. The computations are done by applying our analytical methods based on the Laplace transform and on the spectral expansion. Section 9 summarizes our results, discusses avenues for further research and applications, and concludes the paper. Appendix A contains the proofs. Appendix B collects various results on special functions used in the development of the time changed JDCEV process.

2 Model Architecture

We assume frictionless markets, no arbitrage, and take an equivalent martingale measure (EMM) $\mathbb{Q}$ chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{Q})$ as given. All stochastic processes defined in the following live on this probability space, and all expectations are with respect to $\mathbb{Q}$ unless stated otherwise. We model the stock price dynamics under the EMM as a stochastic process $\{S_t, t \geq 0\}$ defined by:

$$S_t = \mathbf{1}_{\{t<\tau_d\}} e^{\rho t} X_{T_t} \equiv \begin{cases} e^{\rho t} X_{T_t}, & t < \tau_d \\ 0, & t \geq \tau_d \end{cases} . \tag{2.1}$$

We now describe the ingredients in our model.

(i) **Background Markov Process X.** $\{X_t, t \geq 0\}$ is a time-homogeneous Markov diffusion process starting from a positive value $X_0 = x > 0$ and solving a stochastic differential equation (SDE)

$$dX_t = [\mu + h(X_t)]X_t\, dt + \sigma(X_t)X_t\, dB_t, \tag{2.2}$$

where $\sigma(x)$ and $\mu + h(x)$ are the state-dependent instantaneous volatility and drift rate, $\mu \in \mathbb{R}$ is a constant parameter, and $\{B_t, t \geq 0\}$ is a standard Brownian motion. We assume that $\sigma(x)$ and $h(x)$ are Lipschitz continuous on $[\epsilon, \infty)$ for each $\epsilon > 0$, $\sigma(x) > 0$ on $(0, \infty)$, $h(x) \geq 0$ on $(0, \infty)$, and $\sigma(x)$ and $h(x)$ remain bounded as $x \to \infty$. We do not assume that $\sigma(x)$ and $h(x)$ remain bounded as $x \to 0$. Under these assumptions the process X does not explode to infinity (infinity is a *natural boundary* for the diffusion process; see Borodin and Salminen (2002), p.14 for boundary classification of diffusion processes), but, in general, may reach zero, depending on the behavior of $\sigma(x)$ and $h(x)$ as $x \to 0$. The SDE (2.2) has a unique solution up to the first hitting time of zero,

$$H_0 = \inf\{t \geq 0 : X_t = 0\}.$$

If the process can reach zero, we kill it at H_0 and send it to an solated state Δ called the *cemetery state* in the terminology of Markov processes (see Borodin and Salminen (2002), p.4), where it remains for all $t \geq H_0$ (zero is a *killing boundary*). If the process cannot reach zero (zero is an inaccessible boundary), we set $H_0 = \infty$ by convention. We call the process X the *background Markov process*. We could have included jumps in the process X, thus starting from a jump-diffusion process, rather than a pure diffusion as is done here. Instead, we start from a diffusion process and introduce jumps through time changing the diffusion with a Lévy subordinator. By introducing jumps via time changes we gain some important analytical tractability as will be seen later. After the jump-inducing time change, we have a Markov jump-diffusion process, which we can again time change to introduce stochastic volatility.

(ii) **Time Change Process T.** The process $\{T_t, t \geq 0\}$ is a random time change (called a *directing process*) assumed independent of X. It is a right-continuous with left limits (RCLL) increasing process starting at zero, $T_0 = 0$. We also assume that $\mathbb{E}[T_t] < \infty$ for every $t > 0$. In this paper we focus on two important classes of time changes: Lévy subordinators (Lévy processes with positive jumps and non-negative drift) that are employed to introduce jumps, and absolutely continuous time changes

$$T_t = \int_0^t V_u\, du$$

with a positive rate process $\{V_t, t \geq 0\}$ called *activity rate* that are employed to introduce stochastic volatility. We also consider composite time changes of the form

$$T_t = T^1_{T^2_t},$$

where T^1_t is a Lévy subordinator and T^2_t is an absolutely continuous time change with some activity rate process V. This can be thought of as first time changing the diffusion process X with the Lévy subordinator T^1 to introduce jumps, and then time changing the resulting Markov jump-diffusion process with the absolutely continuous time change T^2 to introduce stochastic volatility. Alternatively, the process T can be understood as a subordinator with stochastic volatility along the lines of time changed Lévy processes of Carr et al. (2003). We describe these classes of time changes in detail in section 3.

(iii) **Default Time τ_d.** The stopping time τ_d models the time of default of the firm on its debt. We assume that in default strict priority rules are followed, so that while debt holders receive some recovery, the stock becomes worthless (stock price is equal to zero in default). The time of default τ_d is constructed as follows. Let H_0 be the first time the diffusion process X reaches zero as defined previously. Let $\mathcal{E}$ be an exponential random variable with unit mean, $\mathcal{E} \sim \text{Exp}(1)$, and independent of X and T. Define

$$\zeta := \inf\{t \in [0, H_0] : \int_0^t h(X_u)du \geq \mathcal{E}\}, \tag{2.3}$$

where $h(x)$ is the function appearing in the drift of X (in Eq.(2.3) we assume that $\inf\{\emptyset\} = H_0$ by convention). It can be interpreted as the first jump time of a doubly-stochastic Poisson process with the state-dependent intensity (hazard rate) $h(X_t)$ if it jumps before time H_0, or H_0 if there is no jump in $[0, H_0]$. At time ζ we kill the process X and send it to the cemetery state Δ, where it remains for all $t \geq \zeta$. We note that, in general, the process X may be killed either at time H_0 via diffusion to zero if $\zeta = H_0$ or at the first jump time ζ of the doubly stochastic Poisson process with intensity h if $\zeta < H_0$ (according to our definition, $\zeta \leq H_0$). In the latter case, the process is killed from a positive value $X_{\zeta-} > 0$. The process X is thus a Markov process with killing with *lifetime* ζ.[1]

The drift in (2.1) includes the hazard rate h to make the process $1_{\{t<\zeta\}}X_t$ with $\mu = 0$ into a martingale. The inclusion of the hazard rate in the drift compensates for the possibility of killing the process from a positive state, i.e., a jump of the process X_t from a positive value $X_{\zeta-} > 0$ to the cemetery state Δ and, correspondingly, a jump of the process $1_{\{t<\zeta\}}X_t$ from a positive value $X_{\zeta-} > 0$ to zero. This compensation of the jump to zero makes the process $1_{\{t<\zeta\}}X_t$ with $\mu = 0$ into a martingale (our assumptions on $\sigma(x)$ and $h(x)$ ensure that this process is a true martingale and not just a local martingale).

After applying the time change T to the process X with lifetime ζ, the lifetime of the time changed process X_{T_t} is:

$$\tau_d := \inf\{t \geq 0 : T_t \geq \zeta\}. \tag{2.4}$$

While the process X_t is in the cemetery state for all $t \geq \zeta$, the time changed process X_{T_t} is in the cemetery state for all times t such that $T_t \geq \zeta$ or, equivalently, $t \geq \tau_d$ with τ_d defined by

[1]The process killed at $\zeta \leq H_0$ is a subprocess of the process killed at H_0. We could have used different notation for the process killed at ζ to distinguish it from the process killed at H_0. To simplify notation, we denote both processes by X. It should not lead to any confusion as it should be clear from the context whether we are working with the process killed at H_0 or its subprocess killed at $\zeta \leq H_0$.

Eq.(2.4). That is, τ_d defined by Eq.(2.4) is the first time the time changed process X_{T_t} is in the cemetery state. We take τ_d to be the time of default. Since we assume that the stock becomes worthless in default, we set $S_t = 0$ for all $t \geq \tau_d$, so that $S_t = 1_{\{t<\tau_d\}} e^{\rho t} X_{T_t}$.

(iv) **Scaling Factor $e^{\rho t}$.** To gain some additional modeling flexibility, we also include a scaling factor $e^{\rho t}$ with some constant $\rho \in \mathbb{R}$ in our definition of the stock price process (2.1).

(v) **The Martingale Condition.** For the model (2.1) to be well defined, the functions $\sigma(x)$, $h(x)$, the time change process T, and the constant parameters μ and ρ must be such that the discounted stock price process with the dividends reinvested is a non-negative martingale under the EMM $\mathbb{Q}$, i.e.,

$$\mathbb{E}[S_t] < \infty \text{ for every } t \tag{2.5}$$

and

$$\mathbb{E}[S_{t_2}|\mathcal{F}_{t_1}] = e^{(r-q)(t_2-t_1)} S_{t_1} \text{ for every } t_1 < t_2, \tag{2.6}$$

where $r \geq 0$ is the risk-free interest rate and $q \geq 0$ is the dividend yield (in this paper we assume r and q are constant). The martingale condition (2.5-6) imposes important restrictions on the model parameters. In section 3 we describe the classes of time changes we work with, and in section 4 prove key theorems that give the necessary and sufficient conditions for the martingale condition (2.5-6) to hold.

3 Time Change Processes

3.1 Lévy Subordinators

Let $\{T_t, t \geq 0\}$ be a Lévy *subordinator*, i.e., a non-decreasing Lévy process with positive jumps and non-negative drift with the Laplace transform

$$\mathbb{E}[e^{-\lambda T_t}] = e^{-t\phi(\lambda)} \tag{3.1}$$

with the Laplace exponent given by the Lévy-Khintchine formula

$$\phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} (1 - e^{-\lambda s})\nu(ds) \tag{3.2}$$

with the Lévy measure $\nu(ds)$ satisfying

$$\int_{(0,\infty)} (s \wedge 1)\nu(ds) < \infty, \tag{3.3}$$

non-negative drift $\gamma \geq 0$, and the transition probability $\mathbb{Q}(T_t \in ds) = \pi_t(ds)$,

$$\int_{[0,\infty)} e^{-\lambda s}\pi_t(ds) = e^{-t\phi(\lambda)}. \tag{3.4}$$

The standard references on subordinators include Bertoin (1996), (1999) and Sato (1999) (see also Geman et al. (2001) for finance applications). A subordinator starts at zero, T_0, drifts at the constant non-negative drift rate γ, and experiences positive jumps controlled by the Lévy measure $\nu(ds)$ (we exclude the trivial case of constant time changes with $\nu = 0$ and $\gamma > 0$). The Lévy measure ν describes the arrival rates of jumps so that jumps of sizes in some Borel set A

bounded away from zero occur according to a Poisson process with intensity $\nu(A) = \int_A \nu(ds)$. If $\int_{\mathbb{R}^+} \nu(ds) < \infty$, the subordinator is of compound Poisson type with the Poisson arrival rate $\alpha = \int_{\mathbb{R}^+} \nu(ds)$ and the jump size probability distribution $\alpha^{-1}\nu$. If the integral $\int_{\mathbb{R}^+} \nu(ds)$ is infinite, the subordinator is of infinite activity. All subordinators are processes of finite variation and, hence, the truncation of small jumps is not necessary in the Lévy-Khintchine formula (3.2) in this case.

Consider an exponential moment $\mathbb{E}[e^{\mu T_t}]$ of a subordinator T with Lévy measure ν. When $\mu < 0$, it is always finite and is given by the Lévy-Khintchine formula with $\lambda = -\mu$. We will also need to consider the case $\mu \geq 0$. Generally, we are interested in the set $\mathcal{I}_\nu$ of all $\mu \in \mathbb{R}$ such that $\mathbb{E}[e^{\mu T_t}] < \infty$. As a corollary of Theorem 25.17 of Sato (1999), $\mathbb{E}[e^{\mu T_t}] < \infty$ for all $t \geq 0$ if and only if

$$\int_{[1,\infty)} e^{\mu s} \nu(ds) < \infty. \tag{3.5}$$

For a given subordinator with Lévy measure ν, the set $\mathcal{I}_\nu$ of all μ such that (3.5) holds is an interval $(-\infty, \bar{\mu})$ or $(-\infty, \bar{\mu}]$. The right endpoint $\bar{\mu} \geq 0$ may be finite or infinite and, if it is finite, may or may not belong to the set $\mathcal{I}_\nu$. It is also possible that $\bar{\mu} = 0$. For all $\mu \in \mathcal{I}_\nu$ we have:

$$\mathbb{E}[e^{\mu T_t}] = e^{-t\phi(-\mu)}. \tag{3.6}$$

A simple example of a finite activity subordinator is a compound Poisson process with jump arrival rate $\alpha > 0$ and exponentially distributed jumps with mean $1/\eta > 0$ with the Lévy measure:

$$\nu(ds) = \alpha\eta e^{-\eta s} ds.$$

The Laplace exponent (3.2) of a subordinator with this Lévy measure and drift $\gamma > 0$ is:

$$\phi(\lambda) = \gamma\lambda + \frac{\alpha\lambda}{\lambda + \eta}$$

and $\mathcal{I}_\nu = (-\infty, \eta)$. The transition probability measure can be written in closed form:

$$\pi_t(ds) = e^{-\alpha t}\delta_{\{\gamma t\}}(ds) + \sum_{n=1}^{\infty} e^{-\alpha t} \frac{(\alpha t \eta)^n}{n!(n-1)!}(s + \gamma t)^{n-1} e^{-\eta(s+\gamma t)} ds,$$

where $\delta_{\{\gamma t\}}(ds)$ is the Dirac measure with unit mass at $s = \gamma t$ ($e^{-\alpha t}$ is the probability of no jumps by time t).

A more general compound Poisson Lévy measure is of the form

$$\nu(ds) = \alpha F(ds),$$

where $F(ds)$ is a probability measure on $\mathbb{R}^+$ so that positive jumps arrive according to a Poisson process with intensity α and are distributed according to F. The Laplace exponent (3.2) for the compound Poisson subordinator simplifies to:

$$\phi(\lambda) = \gamma\lambda + \alpha[1 - \mathcal{L}(F)(\lambda)],$$

where $\mathcal{L}(F)(s)$ is the Laplace transform of the probability measure F,

$$\mathcal{L}(F)(\lambda) = \int_0^\infty e^{-\lambda s} F(ds).$$

An important family of Lévy subordinators is defined by the following three-parameter family of Lévy measures

$$\nu(ds) = Cs^{-Y-1}e^{-\eta s}ds$$

with $C > 0$, $\eta > 0$, and $Y < 1$. For $Y \in (0,1)$ these are the so-called *tempered stable subordinators* (exponentially dampened counterparts of stable subordinators with $\nu(ds) = Cs^{-Y-1}ds$). The special case $Y = 1/2$ is known as the *inverse Gaussian process* (Barndorff-Nielsen (1998)). The limiting case $Y = 0$ is the *gamma process* (see Madan et al. (1998)). The processes with $Y \in [0,1)$ are infinite activity processes. For $Y < 0$ these are compound Poisson processes with gamma distributed jump sizes. The previously discussed compound Poisson process with exponential jumps is a special case with $Y = -1$ (and $C = \alpha\eta$). For $Y \neq 0$ the Laplace exponent (3.2) is given by:

$$\phi(\lambda) = \gamma\lambda - C\Gamma(-Y)[(\lambda+\eta)^Y - \eta^Y],$$

where $\Gamma(x)$ is the gamma function. For the gamma process with $Y = 0$ and drift $\gamma \geq 0$ the Laplace exponent (3.2) is given by:

$$\phi(\lambda) = \gamma\lambda + C\ln(1+\lambda/\eta).$$

For $Y \in [0,1)$ the transition measures $\pi_t(ds)$ are known in closed form only for the two special cases with $Y = 0$ (gamma process) and $Y = 1/2$ (inverse Gaussian process) and are given by:

$$\pi_t^G(ds) = \frac{\eta^{Ct}}{\Gamma(Ct)}(s+\gamma t)^{Ct-1}e^{-\eta(s+\gamma t)}ds$$

and

$$\pi_t^{IG}(ds) = \frac{Ct}{(s+\gamma t)^{3/2}}\exp\left(2Ct\sqrt{\pi\eta} - \eta(s+\gamma t) - \pi C^2t^2/(s+\gamma t)\right)ds,$$

respectively. For $Y < 0$, the transition measure of the compound Poisson process (CPP) with gamma distributed jumps is (the CPP with exponential jumps discussed above is a special case with $Y = -1$):

$$\pi_t(ds) = e^{-\alpha t}\delta_{\{\gamma t\}}(ds) + \sum_{n=1}^{\infty} e^{-\alpha t}\frac{(\alpha t\eta^{|Y|})^n}{n!\Gamma(n|Y|)}(s+\gamma t)^{n|Y|-1}e^{-\eta(s+\gamma t)}ds.$$

The interval $\mathcal{I}_\nu = (-\infty, \eta]$ for $Y \in (0,1)$ and $\mathcal{I}_\nu = (-\infty, \eta)$ for $Y \leq 0$. For general $Y \in (0,1)$ the transition measure is not known in closed form and has to be computed numerically by inverting the Laplace transform (3.4). Further information on subordinators can be found in Applebaum (2004), Bertoin (1996), (1999), Sato (1999). For applications in finance see Geman et al. (2001), Boyarchenko and Levendorskiy (2002), Cont and Tankov (2004), and Schoutens (2003).

3.2 Absolutely Continuous Time Change Processes

Let $\{Z_t, t \geq 0\}$ be a conservative n-dimensional Markov process independent of X (Z can have a diffusion component and a jump component, but no killing, so that Z has infinite lifetime). Consider an integral process:

$$T_t = \int_0^t V(Z_u)du, \tag{3.7}$$

where $V(z)$ is some positive function from the state space $D \subset \mathbb{R}^n$ of the process Z into $(0, \infty)$ so that the *activity rate* process $\{V_t := V(Z_t), t \geq 0\}$ is positive (we exclude the trivial case of constant time changes with constant $V > 0$). The process T_t is strictly increasing and starts at the origin. We are interested in such Markov processes Z and such functions $V(z)$ that the Laplace transform

$$\mathcal{L}_z(t, \lambda) = \mathbb{E}_z[e^{-\lambda \int_0^t V(Z_u)du}] \tag{3.8}$$

is known in closed form (the subscript z signifies that the Laplace transform $\mathcal{L}_z(t, \lambda)$ explicitly depends on the initial state $Z_0 = z$ of the Markov process Z).

A key example is given by the CIR activity rate process (in this case $V(z) = z$ so that $V_t = Z_t$):

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t,$$

where the standard Brownian motion W is independent of the Brownian motion B driving the SDE (2.2), the activity rate process starts at some positive value $V_0 = v > 0$, $\kappa > 0$ is the rate of mean reversion, $\theta > 0$ is the long-run activity rate level, $\sigma_V > 0$ is the activity rate volatility, and it is assumed that the Feller condition is satisfied $2\kappa\theta \geq \sigma_V^2$ to ensure that the process never hits zero (zero is an inaccessible boundary for the CIR process when the Feller condition is satisfied). Due to the Cox, Ingersoll and Ross (1985) result giving the closed form solution for the zero-coupon bond in the CIR interest rate model (note that the Laplace transform (3.8) can be interpreted as the price of a unit face value zero-coupon bond with maturity at time t when the short rate process is $r_t = \lambda V_t$), we have:

$$\mathcal{L}_v(t, \lambda) = A(t, \lambda) e^{-B(t,\lambda)v},$$

where $V_0 = v$ is the initial value of the activity rate process and

$$A(t, \lambda) = \left(\frac{2\varpi e^{(\varpi + \kappa)t/2}}{(\varpi + \kappa)(e^{\varpi t} - 1) + 2\varpi} \right)^{\frac{2\kappa\theta}{\sigma_V^2}}, \quad B(t, \lambda) = \frac{2\lambda(e^{\varpi t} - 1)}{(\varpi + \kappa)(e^{\varpi t} - 1) + 2\varpi},$$

where

$$\varpi = \sqrt{2\sigma_V^2 \lambda + \kappa^2}.$$

Heston's stochastic volatility model is based on Brownian motion time changed with the integral of the CIR process. The CIR activity rate process has been used more generally in Carr et al. (2003) to time change Lévy processes to introduce stochastic volatility in the popular Lévy models, such as VG, NIG, CGMY, etc.

More generally, there are several known classes of Markov processes that yield closed form expressions for the Laplace transform (3.8). The first class are affine jump-diffusion processes with the affine function $V(z)$ (Duffie et al.(2000), (2003)). In this class the Laplace transform of the time change is the exponential of an affine function of the initial state $Z_0 = z$ of the Markov process Z driving the activity rate process. The CIR example is a particular representative of the affine class. The second class are the so-called quadratic models (Leippold and Wu (2002)), where the function $V(z)$ is quadratic in the state vector, and the state vector follows an n-dimensional Gaussian Markov process (an n-dimensional Ornstein-Uhlenbeck process). In this case the Laplace transform of the time change is the exponential of a quadratic function of the initial state $Z_0 = z$ of the Markov process. The third class are Ornstein-Uhlenbeck processes driven by Lévy processes used by Carr et al. (2003) to time change Lévy processes. Explicit expressions for the Laplace transforms of these time changes can be found in this reference. Carr and Wu (2004) use all three of these classes of absolutely continuous time changes to time change Lévy processes (a listing of closed form expressions for Laplace transforms of these time changes can be found in Tables 1 and 2 in this reference). Here we use them to time change Markov processes. We note that, while the Laplace transforms are known in closed form for these three classes of absolutely continuous time changes, in general the transition probability distributions $\mathbb{Q}_z(T_t \in ds) = \pi_t(z, ds)$ can only be obtained numerically by Laplace transform inversion (note that they explicitly depend on the initial state $Z_0 = z$ of the Markov process Z driving the activity rate V).

3.3 Composite Time Changes

Furthermore, we can compose the two types of time changes and consider a composite time change process:

$$T_t = T^1_{T^2_t}, \tag{3.9}$$

where T^1_t is a subordinator with Laplace exponent ϕ and T^2_t is an integral of some positive function of a Markov process with analytically tractable Laplace transform $\mathcal{L}_z(t, \lambda)$. That is, the process T is obtained by time changing a Lévy subordinator T^1 with an absolutely continuous time change T^2. The process T is in the class of Lévy processes time-changed with an integral of an activity rate process studied by Carr et al. (2003). By conditioning on T^2, the Laplace transform of the composite time change is:

$$\mathbb{E}[e^{-\lambda T_t}] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\lambda T^1_{T^2_t}\right) | T^2_t\right]\right] = \mathbb{E}[e^{-T^2_t \phi(\lambda)}] = \mathcal{L}_z(t, \phi(\lambda)). \tag{3.10}$$

We note that after we have done the absolutely continuous time change T^2_t, further time changes will no longer have analytically tractable Laplace transforms, since, in contrast to subordinators with the Laplace transform $e^{-t\phi(\lambda)}$ that depends on time exponentially, the Laplace transform $\mathcal{L}_z(t, \lambda)$ may have a complicated general dependence on time.

4 Martingale and Markov Properties of the Defaultable Stock Model

We now prove key theorems that establish when our stock price model (2.1) satisfies the martingale condition (2.5)–(2.6) and when it is a Markov process.

4.1 Time Changing with Lévy Subordinators

Theorem 4.1 *Let X be a background diffusion process as described in section 2(i) with $\mu \in \mathbb{R}$ and $h(x)$ and $\sigma(x)$ satisfying the assumptions listed there, let T be a Lévy subordinator with drift $\gamma \geq 0$ and Lévy measure ν with the characteristic exponent $\phi(\lambda)$ and with the interval $\mathcal{I}_\nu$ as described in section 3.1, and let τ_d be the default time as described in section 2(iii). Then the stock price process (2.1) satisfies the martingale condition (2.5)–(2.6) if and only if*

$$\mu \in \mathcal{I}_\nu \tag{4.1}$$

and

$$\rho = r - q + \phi(-\mu). \tag{4.2}$$

Proof. The proof is by conditioning on the time change T that is independent of X and using Eq.(3.6) to compute the expectation and is given in Appendix A. $\square$

Thus, when the time change T is a Lévy subordinator, our model (2.1) is characterized by the local volatility function $\sigma(x)$, hazard rate $h(x)$, Lévy measure ν and drift $\gamma \geq 0$ of the Lévy subordinator, and a constant $\mu \in \mathcal{I}_\nu$. Depending on the Lévy measure, it may or may not be possible to select $\mu \in \mathcal{I}_\nu$ so that

$$\rho = r - q + \phi(-\mu) = 0. \tag{4.3}$$

¿From (3.2) we see that $-\phi(-\mu)$ is a strictly increasing function on $\mathcal{I}_\nu$. Thus, the equation (4.3) has at most one solution in $\mathcal{I}_\nu$. If it exists, we denote it μ_0 and call the corresponding model (2.1) with $\mu = \mu_0$ and $\rho = 0$ the *zero-ρ model.* If the equation (4.3) has no solution in $\mathcal{I}_\nu$, one possible choice is to set $\mu = 0$ so that $\rho = r - q$. We call this choice the *zero-μ model.* For this choice the process $\mathbf{1}_{\{t<\zeta\}} X_t$ and the time changed process $\mathbf{1}_{\{t<\tau_d\}} X_{T_t}$ are both martingales, and the desired mean for the stock price process $S_t = \mathbf{1}_{\{t<\tau_d\}} e^{(r-q)t} X_{T_t}$ is achieved by including the factor $e^{\rho t} = e^{(r-q)t}$. We now establish when Eq.(4.3) has a solution.

Theorem 4.2 *Eq.(4.3) has at most one solution in $\mathcal{I}_\nu$. If $r < q$, then Eq.(4.3) has no solution in $\mathcal{I}_\nu$ if and only if $\gamma = 0$ and the subordinator is of finite activity with finite Lévy measure with Poisson intensity $\alpha = \int_{(0,\infty)} \nu(ds)$ such that $-\alpha > r - q$. If $r > q$, then Eq.(4.3) has no solution in $\mathcal{I}_\nu$ if and only if $\bar{\mu}$ is included in $\mathcal{I}_\nu$ (i.e., $\int_{[1,\infty)} e^{\bar{\mu}s}\nu(ds) < \infty$) and $r - q > -\phi(-\bar{\mu})$. If $r = q$, Eq.(4.3) has a unique solution $\mu = 0$ in $\mathcal{I}_\nu$.*

Proof. The proof follows from the analysis of Eq.(3.2) and is given in Appendix A.$\square$

We now turn to the question of whether the model (2.1) is Markovian. It turns out that when T is a Lévy subordinator, the time changed process X_{T_t} is again a Markov process.

Theorem 4.3 *Let X be a background diffusion process with lifetime ζ as described in section 2(i) with assumptions listed there, and let T be a Lévy subordinator with drift $\gamma \geq 0$ and Lévy measure $\nu(ds)$ as described in section 3.1. Then the time changed process (the superscript ϕ refers to the subordinate quantities with the subordinator with the Laplace exponent ϕ)*

$$X_t^\phi := X_{T_t} = \begin{cases} X_{T_t}, & T_t < \zeta \\ \Delta, & T_t \geq \zeta \end{cases} \equiv \begin{cases} X_{T_t}, & t < \tau_d \\ \Delta, & t \geq \tau_d \end{cases} \tag{4.4}$$

is a Markov jump-diffusion process with lifetime τ_d and with the Lévy-type *infinitesimal generator $\mathcal{G}^\phi$ that for any twice continuously differentiable function with compact support $f \in C_c^2((0,\infty))$ can be written in the form:*

$$\mathcal{G}^\phi f(x) = \frac{1}{2}\gamma^2\sigma^2(x)x^2\frac{d^2 f}{dx^2}(x) + b(x)\frac{df}{dx}(x) - k(x)f(x) \tag{4.5}$$

$$+ \int_{(0,\infty)} \left(f(y) - f(x) - \mathbf{1}_{\{|y-x|\leq 1\}}(y-x)\frac{df}{dx}(x) \right) \Pi(x, dy)$$

*with the jump measure (*state-dependent Lévy measure*)*

$$\Pi(x, dy) = \pi(x, y)dy \tag{4.6}$$

with the density defined for all $x, y > 0$, $x \neq y$, by

$$\pi(x,y) = \int_{(0,\infty)} p(s;x,y)\nu(ds), \tag{4.7}$$

killing rate

$$k(x) = \gamma h(x) + \int_{(0,\infty)} P_s(x,\{\Delta\})\nu(ds), \tag{4.8}$$

and drift with respect to the truncation function $\mathbf{1}_{\{|y-x|\leq 1\}}$

$$b(x) = \gamma[\mu + h(x)]x + \int_{(0,\infty)} \left(\int_{\{y>0:|y-x|\leq 1\}} (y-x)p(s;x,y)dy \right) \nu(ds). \tag{4.9}$$

Here $p(t;x,y)$ is the transition probability density of the background Markov process X with lifetime ζ, so that the probability to find the process in a Borel set $A \subset (0,\infty)$ at time t if the process starts at $X_0 = x$ at time zero is $P_t(x,A) = \int_A p(t;x,y)dy$, and

$$P_t(x,\{\Delta\}) = 1 - \int_{(0,\infty)} p(t;x,y)dy \tag{4.10}$$

is the transition probability of the background process X with lifetime ζ from the state $x > 0$ to the cemetery state Δ by time t.

The transition density $p^{\phi}(t;x,y)$ of the time changed Markov process X^{ϕ} with lifetime τ_d is given by:

$$p^{\phi}(t;x,y) = \int_{[0,\infty)} p(s;x,y)\pi_t(ds), \tag{4.11}$$

where $p(s;x,y)$ is the transition density of the background Markov process X with lifetime ζ and $\pi_t(ds)$ is the transition measure of the subordinator T. The transition probability of the process X^{ϕ} with lifetime τ_d from the state $x > 0$ to the cemetery state Δ by time t is given by:

$$P_t^{\phi}(x,\{\Delta\}) = 1 - \int_{(0,\infty)} p^{\phi}(t;x,y)dy = \int_{[0,\infty)} P_s(x,\{\Delta\})\pi_t(ds). \tag{4.12}$$

Proof. The proof relies on R.S. Phillips' theorem on subordination of Markov semigroups and is given in Appendix A. □

The theorem asserts that when the background process is Markov and the time change is a Lévy subordinator, the time-changed process is again Markov and gives explicitly its local characteristics (volatility, drift with respect to the truncation function, killing rate, and jump measure). Intuitively, for any $x > 0$ and a Borel set $A \subset (0,\infty)\backslash\{x\}$ bounded away from x, the Lévy measure $\Pi(x,A)$ gives the arrival rate of jumps from the state x into the set A, i.e., the transition probability from the state x into the set A bounded away from x has the following asymptotics:

$$P_t(x,A) \sim \Pi(x,A)t \text{ as } t \to 0.$$

The truncation function in the integral in (4.5) is only needed when jumps are of infinite variation. When

$$\int_{\{y>0:|x-y|\leq 1\}} |y-x|\Pi(x,dy) < \infty \tag{4.13}$$

for all $x > 0$, jumps of the time changed process are of finite variation, the truncation is not needed, and the infinitesimal generator (4.5) of the time changed Markov process simplifies to:

$$\mathcal{G}^\phi f(x) = \frac{1}{2}\gamma^2\sigma^2(x)x^2\frac{d^2 f}{dx^2}(x) + \gamma[\mu + h(x)]x\frac{df}{dx}(x) - k(x)f(x) + \int_{(0,\infty)} (f(y) - f(x))\,\Pi(x, dy). \tag{4.14}$$

If Π is a finite measure with $\lambda(x) := \Pi(x, (0, \infty)) < \infty$ for every $x > 0$, then the process has a finite number of jumps in any finite time interval, and $\lambda(x)$ is the (state-dependent) jump arrival rate. The subordinated process X^ϕ has finite activity jumps if and only if the subordinator T has finite activity jumps. Note that, while subordinators are jump processes of finite variation, the subordinated processes X^ϕ may have jumps of either finite or infinite variation, depending on whether the Lévy measure (4.6-7) satisfies the integrability condition (4.13).

¿From Eqs.(4.5-8) we see that time changing the process X with a Lévy subordinator with drift $\gamma \geq 0$ and Lévy measure ν scales volatility and drift with γ, introduces jumps with state-dependent Lévy measure with Lévy density $\pi(x, y) = \int_{(0,\infty)} p(s; x, y)\nu(ds)$ determined by the Lévy measure of the subordinator and the transition density of the diffusion process X, and modifies the killing rate by scaling the original killing rate with γ and adding the term $\int_{(0,\infty)} P_s(x, \{\Delta\})\nu(ds)$ determined by the Lévy measure of the subordinator and the killing probability of the Markov process X. If $\gamma > 0$, we can set $\gamma = 1$ without loss of generality. Then the effect of the time change is to introduce jumps into the original diffusion process, so that the resulting process is a jump-diffusion with the same diffusion as the original process X plus jumps induced by the time change, and to modify the killing rate. Thus, the subordination procedure allows us to introduce jumps into any diffusion process. If $\gamma = 0$, then the time changed process has no diffusion component and is a pure jump process with killing.

Thus, we have a complete characterization of the time changed process X_t^ϕ as a Markov process with killing. The stock price process (2.1) can be written as $S_t = \mathbf{1}_{\{t<\tau_d\}}e^{\rho t}X_t^\phi$. The stock price process stays positive prior to the default time τ_d (lifetime of X_t^ϕ) and jumps into zero at τ_d. We call this *jump-to-default*. It is thus a Markov jump-diffusion process with zero specified as an absorbing state.

4.2 Absolutely Continuous Time Changes

We now turn to absolutely continuous time changes.

Theorem 4.4 *Let X be a background diffusion process as described in section 2(i) with $\mu \in \mathbb{R}$ and $h(x)$ and $\sigma(x)$ satisfying the assumptions listed there, let T be an absolutely continuous time change with a positive activity rate process V_t as described in section 3.2, and let τ_d be the default time as described in section 2(iii). Then the stock price process (2.1) satisfies the martingale condition (2.5)–(2.6) if and only if*

$$\mu = 0, \ \ \rho = r - q. \tag{4.15}$$

Proof. The proof is given in Appendix A. □

Since the time change process $\{T_t, t \geq 0\}$ is continuous and strictly increasing (we assume the activity rate process V is strictly positive), the *inverse process* $\{A_t, t \geq 0\}$ defined by $T_{A_t} = t$ is also continuous and strictly increasing and $A_{T_t} = t$. To understand the effect of the absolutely continuous time change on the process X, we write for $T_t < \zeta$ (equivalently $t < \tau_d$)

$$\begin{aligned} X_{T_t} &= x + \int_0^{T_t} h(X_u)X_u du + \int_0^{T_t} \sigma(X_u)X_u dB_u \\ &= x + \int_0^t h(X_{T_s})X_{T_s} dT_s + \int_0^t \sigma(X_{T_s})X_{T_s} dB_{T_s} \\ &= x + \int_0^t h(X_{T_s})X_{T_s} V(Z_s)ds + \int_0^t \sigma(X_{T_s})X_{T_s}\sqrt{V(Z_s)}d\tilde{B}_s. \end{aligned} \tag{4.16}$$

In the first equality we did a change of variable in the integral, $u = T_s$ (with the inverse $s = A_u$). In the second equality we observed that $dT_s = V_s ds$ and $dB_{T_s} = \sqrt{V_s}d\tilde{B}_s$, where $\tilde{B}_t = \int_0^t \frac{dB_{T_s}}{\sqrt{V_s}}$ is a standard Brownian motion (it is a continuous martingale with quadratic variation t and, hence, is a standard Brownian motion by Lévy's theorem). The process X_t is killed at time $\zeta = \inf\{t \in [0, H_0] : \int_0^t h(X_u)du \geq \mathcal{E}\}$. Then the time changed process X_{T_t} is killed at time

$$\tau_d = \inf\{t \in [0, A_{H_0}] : \int_0^{T_t} h(X_u)du \geq \mathcal{E}\} = \inf\{t \in [0, A_{H_0}] : \int_0^t h(X_{T_s})V(Z_s)ds \geq \mathcal{E}\}, \tag{4.17}$$

where we did a change of variable $u = T_s$ in the integral. From Eqs.(4.16) and (4.17) we observe that the time changed process $Y_t = X_{T_t}$ has the local volatility

$$\sigma(x, z) = \sqrt{V(z)}\sigma(x) \tag{4.18}$$

and killing rate

$$k(x, z) = V(z)h(x) \tag{4.19}$$

so that for $t < \tau_d$ the process Y solves the SDE:

$$dY_t = V(Z_t)h(Y_t)Y_t dt + \sqrt{V(Z_t)}\sigma(Y_t)Y_t d\tilde{B}_t. \tag{4.20}$$

Thus, the time change scales the volatility with the square root of the activity rate and scales the killing rate with the activity rate. The activity rate plays a role of stochastic volatility that both drives the instantaneous volatility of the time changed process and the killing rate (default intensity). Thus, by construction, this class of models possesses a natural built-in connection between the stock price volatility and the firm's default intensity. This manifests itself in the connection between the implied volatility skew in the stock options market and the credit spreads in the credit markets. The linkages between credit spreads and equity volatility (both realized and implied in options prices) have been widely documented in the empirical literature (see the discussion and the references in the introduction of Carr and Linetsky (2006)). Our class of models based on time changing a diffusion with killing with an integral of an activity rate (stochastic volatility) process is ideally suited to the task of modeling the linkages between equity volatility and credit spreads, as the activity rate drives both the local-stochastic volatility of the stock price and the default intensity. See Carr and Wu (2006) for the empirical support of the linkage between the volatility and default intensity in the framework of affine models.

We thus conclude that the time changed process Y is no longer a one-dimensional Markov process. However, the process (Y, Z) is an $(n + 1)$-dimensional Markov process with lifetime τ_d

and with the infinitesimal generator $\mathcal{G}$ that for any twice continuously differentiable function with compact support $f \in C_c^2((0,\infty) \times D)$ (where $D \subset \mathbb{R}^n$ is the state space of the process Z) can be written in the form:

$$\mathcal{G}f(x,z) = V(z)\mathcal{G}^X f(x,z) + \mathcal{G}^Z f(x,z), \tag{4.21}$$

where $\mathcal{G}^X$ is the infinitesimal generator of the background process X with lifetime ζ,

$$\mathcal{G}^X f(x) = \frac{1}{2}\sigma^2(x)x^2\frac{\partial^2 f}{\partial x^2}(x) + h(x)x\frac{\partial f}{\partial x}(x) - h(x)f(x) \tag{4.22}$$

and $\mathcal{G}^Z$ is the infinitesimal generator of the n-dimensional Markov process Z driving the activity rate $V_t = V(Z_t)$.

The fact that, in general, the time changed process is not Markovian is illustrated by the Heston model. If we start with Brownian motion and do a time change with the time change process taken to be an integral of an independent CIR process, the resulting time changed process is no longer a one-dimensional Markov process because of the second source of uncertainty (stochastic volatility) entering through the time change. The Markov property is restored in an enlarged two-dimensional state space with both the stock price and its instantaneous volatility as two state variables.

4.3 Composite Time Changes

We now turn to composite time changes where we first time change the diffusion process X with a Lévy subordinator to introduce jumps, and then time change the resulting Markov jump-diffusion process with an absolutely continuous time change to introduce stochastic volatility as described in section 3.3. Equivalently, we can think of it as a single time change, where the process T_t is a time-changed Lévy process with stochastic volatility as in Carr et al. (2003).

Theorem 4.5 *Let X be a background diffusion process as described in section 2(i) with $\mu \in \mathbb{R}$ and $h(x)$ and $\sigma(x)$ satisfying the assumptions listed there, let T_t be a composite time change (3.9), where T^1 is a Lévy subordinator with drift $\gamma \geq 0$ and Lévy measure ν and T^2 is an absolutely continuous time change with a positive activity rate process $V_t = V(Z_t)$ as described in sections 3.2 and 3.3, and let τ_d be the default time as described in section 2(iii). Then the stock price process (2.1) satisfies the martingale condition (2.5)–(2.6) if and only if*

$$\mu = 0, \;\; \rho = r - q.$$

Proof. The proof is given in Appendix A. □

Recalling Theorem 4.2 and arguing as in section 4.2, we conclude that the process (Y, Z), where $Y_t = X_{T_t} = X_{T^1_{T^2_t}}$, is an $(n+1)$-dimensional Markov jump-diffusion process with the infinitesimal generator $\mathcal{G}$ that for any twice continuously differentiable function with compact support $f \in C_c^2((0,\infty) \times D)$ (where $D \subset \mathbb{R}^n$ is the state space of the process Z) can be written in the form (we set $\mu = 0$ in the drift of X according to Theorem 4.5; here $\mathcal{G}^\phi$ is the infinitesimal generator (4.5) after the first time change with the Lévy subordinator and $\mathcal{G}^Z$ is the infinitesimal generator of the n-dimensional Markov process Z):

$$\mathcal{G}f(x,z) = V(z)\mathcal{G}^\phi f(x,z) + \mathcal{G}^Z f(x,z) \tag{4.23}$$

$$= \frac{1}{2}\gamma^2 V(z)\sigma^2(x)x^2\frac{\partial^2 f}{\partial x^2}(x,z) + b(x,z)\frac{\partial f}{\partial x}(x,z) - k(x,z)f(x,z)$$
$$+ \int_{(0,\infty)} \left(f(y,z) - f(x,z) - \mathbf{1}_{\{|y-x|\le 1\}}(y-x)\frac{\partial f}{\partial x}(x,z)\right)\Pi(x,z;dy) + \mathcal{G}^Z f(x,z)$$

with the jump measure (state-dependent Lévy measure)

$$\Pi(x,z;dy) = \pi(x,z;y)dy \tag{4.24}$$

with the density defined for all $x, y > 0$, $x \neq y$, $z \in D$ by

$$\pi(x,z;y) = V(z)\int_{(0,\infty)} p(s;x,y)\nu(ds), \tag{4.25}$$

killing rate

$$k(x,z) = V(z)\left(\gamma h(x) + \int_{(0,\infty)} P_s(x,\{\Delta\})\nu(ds)\right), \tag{4.26}$$

and drift with respect to the truncation function $\mathbf{1}_{\{|y-x|\le 1\}}$

$$b(x) = V(z)\left[\gamma h(x)x + \int_{(0,\infty)}\left(\int_{\{y>0:|y-x|\le 1\}}(y-x)p(s;x,y)dy\right)\nu(ds)\right]. \tag{4.27}$$

Here $p(t;x,y)$ is the transition probability density of the process X with lifetime ζ and $P_t(x,\{\Delta\})$ is the transition probability of the process X from the state $x > 0$ to the cemetery state Δ by time t given by Eq.(4.10).

The first time change T^1 scales the volatility with γ, introduces jumps with the Lévy measure (4.6–7), and modifies the killing rate by scaling the old killing rate h with γ and adding the term to it as in (4.8). The second time change introduces stochastic volatility by scaling the volatility with $\sqrt{V(z)}$, and scaling the Lévy measure (4.25) and the killing rate (4.27) with $V(z)$.

5 Unified Valuation of Corporate Debt, Credit Derivatives, and Equity Derivatives

We assume that the stock price follows the process (2.1). We view the stock price as the fundamental observable state variable and, within the framework of our reduced-form model (2.1), view all securities related to a given firm, such as corporate debt, credit derivatives, and equity derivatives, as contingent claims written on the stock price process (2.1). Before proceeding with the valuation of contingent claims, we first consider the calculation of the (risk-neutral) *survival probability* — the probability of no default up to time $t > 0$. Conditioning on the time change, we have:

$$\mathbb{Q}(\tau_d > t) = \mathbb{Q}(\zeta > T_t) = \int_0^\infty \mathbb{Q}(\zeta > s)\pi_t(ds) = \int_0^\infty P_s(x,(0,\infty))\pi_t(ds), \tag{5.1}$$

where $P_t(x,(0,\infty)) = \mathbb{Q}(\zeta > t)$ is the survival probability for the Markov process X with lifetime ζ (transition probability for the Markov process X with lifetime ζ from the state $x > 0$ into $(0,\infty)$, $P_t(x,(0,\infty)) = 1 - P_t(x,\{\Delta\})$) and $\pi_t(ds)$ is the probability distribution of the time change T_t. If the survival probability for the process X and the probability distribution of the time change $\pi_t(ds)$ are known in closed form, then we can obtain the survival probability for the stock price process (2.1) by integration (5.1).

Next, consider a European-style contingent claim with the payoff $\Psi(S_t)$ at maturity $t > 0$ given no default by time t, and constant recovery payment $R > 0$ if default occurs by t. Separating the claim into two building blocks, a claim with the payoff Ψ and no recovery and the recovery payment, the valuation is done by conditioning on the time change similar to the calculation of the survival probability (5.1). For the European claim with the payoff $\Psi(S_t)$ given no default by time t and with no recovery if default occurs by t we have:

$$e^{-rt}\mathbb{E}[\mathbf{1}_{\{\tau_d>t\}}\Psi(S_t)] = e^{-rt}\mathbb{E}[\mathbf{1}_{\{\zeta>T_t\}}\Psi(e^{\rho t}X_{T_t})] = e^{-rt}\int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{\zeta>s\}}\Psi(e^{\rho t}X_s)\right]\pi_t(ds); \quad (5.2)$$

For the fixed recovery R paid at time t if default occurs by t we have:

$$Re^{-rt}[1-\mathbb{Q}(\tau_d > t)], \quad (5.3)$$

where the survival probability is given by Eq.(5.1).

¿From Eqs.(5.1)-(5.3) we observe that, by conditioning on the time change, the calculation of the survival probability and the valuation of contingent claims reduce to computing expressions of the form:

$$\mathbb{E}[\mathbf{1}_{\{\tau_d>t\}}f(X_{T_t})] = \mathbb{E}[\mathbf{1}_{\{\zeta>T_t\}}f(X_{T_t})] = \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{\zeta>s\}}f(X_s)\right]\pi_t(ds) \quad (5.4)$$

for some function f (to compute the survival probability set $f = 1$). This involves first computing the expectation $\mathbb{E}\left[\mathbf{1}_{\{\zeta>s\}}f(X_s)\right]$ for the background diffusion process X and then integrating the result in time against the probability distribution of the time change T_t, if the probability distribution of the time change is known in closed form (e.g., the closed form expressions for compound Poisson, Gamma and inverse Gaussian subordinators given in section 3.1). In general, if the closed form expression for the distribution of the time change is not available, it can be recovered by inverting the Laplace transform numerically, which involves numerical integration in the complex plane by means of the Bromwich Laplace inversion formula. The second step is to compute the integral from zero to infinity in Eqs.(5.1) and (5.2). Thus, if we can determine the expectation $\mathbb{E}\left[\mathbf{1}_{\{\zeta>s\}}f(X_s)\right]$ for the original Markov process X in closed form, we still need to perform double numerical integration in order to compute (5.4) for the time changed process. Fortunately, when the function f satisfies an additional integrability condition, there is an alternative approach that avoids any need for Laplace transform inversion to recover $\pi_t(ds)$ and for numerical integration in s in (5.4). In the next section we will present a remarkably powerful Laplace transform approach that will effectively evaluate both of these integrals in closed form.

The two building blocks (5.2) and (5.3) can be used to value corporate debt, credit derivatives, and equity derivatives. In particular, a *defaultable zero-coupon bond* with unit face value, maturity $t > 0$, and recovery $R \in [0, 1]$ can be represented as the European claim with $\Psi(S_t) = 1$ and valued at time zero by:

$$B_R(x,t) = e^{-rt}\mathbb{Q}(\tau_d > t) + Re^{-rt}[1-\mathbb{Q}(\tau_d > t)] = e^{-rt}R + e^{-rt}(1-R)\mathbb{Q}(\tau_d > t), \quad (5.5)$$

where we indicate explicitly the dependence of the bond value function on the initial stock price $S_0 = X_0 = x$. Our recovery assumption corresponds to the *fractional recovery of treasury* assumption (see, e.g., Lando (2004), p.120). Defaultable bonds with coupons can be valued as portfolios of defaultable zeros.

A *European call option* with strike $K > 0$ with the payoff $(S_t - K)^+$ at expiration t has no recovery if the firm defaults. A *European put option* with strike $K > 0$ with the payoff $(K - S_t)^+$ can be decomposed into two parts: the put payoff $(K - S_t)^+ \mathbf{1}_{\{\tau_d > t\}}$, given no default by time t, and a recovery payment equal to the strike K at expiration in the event of default $\tau_d \leq t$. The pricing formulas for European-style call and put options take the form:

$$C(x; K, t) = e^{-rt}\mathbb{E}[(e^{\rho t}X_{T_t} - K)^+ \mathbf{1}_{\{\tau_d > t\}}] = e^{-rt}\int_0^\infty \mathbb{E}[(e^{\rho t}X_s - K)^+ \mathbf{1}_{\{\zeta > s\}}]\pi_t(ds), \qquad (5.6)$$

and

$$P(x; K, t) = P_0(x; K, t) + P_D(x; K, t), \qquad (5.7)$$

where

$$P_0(x; K, t) = e^{-rt}\int_0^\infty \mathbb{E}[(K - e^{\rho t}X_s)^+ \mathbf{1}_{\{\zeta > s\}}]\pi_t(ds) \qquad (5.8)$$

and

$$P_D(x; K, t) = Ke^{-rt}[1 - \mathbb{Q}(\tau_d > t)], \qquad (5.9)$$

respectively. One notes that the put pricing formula (5.7) consists of two parts: the present value $P_0(x; K, t)$ of the put payoff conditional on no default given by Eq.(5.8) (this can be interpreted as the down-and-out put with the down-and-out barrier at zero), as well as the present value $P_D(x; K, t)$ of the cash payment equal to the strike K in the event of default given by Eq.(5.9). This recovery part of the put is a *European-style default claim*, a credit derivative that pays a fixed cash amount K at maturity t if and only if the underlying firm has defaulted by time t. Thus, the put option contains an embedded credit derivative. Generally, we emphasize that in our model, corporate debt, credit derivatives, and equity options are all valued in an unified framework as contingent claims written on the defaultable stock.

While we will now focus on deriving explicit closed-form expressions for European-style securities by probabilistic methods, the framework of this section can be extended to the valuation of American-style options and more complicated securities with American features, such as convertible bonds. The standard results imply that the value function solves the appropriate partial integro-differential equation (PIDE) with the integro-differential operator $\mathcal{G}$ (the infinitesimal generator of the time changed Markov process; one-dimensional in the case of time changes by Lévy subordinators or $(n + 1)$-dimensional in the case of absolutely continuous or composite time changes) on the appropriate domain and subject to appropriate terminal and boundary conditions. The solution can be derived via numerical methods.

6 Valuation of Contingent Claims on Time Changed Markov Processes: A Laplace Transform Approach

We now present a powerful method to compute expectations of the form (5.4) needed to value contingent claims in our model. We will tackle it in two steps. First, we show how to use the Laplace transform to compute the expectation operator

$$\mathcal{P}_t f(x) = \mathbb{E}_x \left[\mathbf{1}_{\{\zeta > t\}} f(X_t)\right], \qquad (6.1)$$

where X is a one-dimensional diffusion process with lifetime ζ started at x at time zero and the function f satisfies some integrability conditions to be specified below. Second, we show how the time change can be accomplished so that the integral with respect to the time variable in the expectation (5.4) is evaluated in closed form from the knowledge of the Laplace transform representation for the expectation (6.1) for the process X and the Laplace transform of the time change T, without any need to recover the probability distribution of the time change.

We need some preliminary material on one-dimensional diffusions to proceed (see Borodin and Salminen (2002), Chapter II for more details). Let $\{X_t, t \geq 0\}$ be a one-dimensional, time-homogeneous diffusion on the interval I with endpoints ℓ and r, $-\infty \leq \ell < r \leq \infty$, and with the infinitesimal generator $\mathcal{G}$ acting on twice continuously differentiable functions on (ℓ, r) by:

$$\mathcal{G}f(x) = \frac{1}{2}a^2(x)\frac{d^2 f}{dx^2}(x) + b(x)\frac{df}{dx}(x) - c(x)f(x). \tag{6.2}$$

We assume that the diffusion coefficient $a(x)$ is continuous and strictly positive on the open interval (ℓ, r), drift $b(x)$ is continuous on (ℓ, r), and killing rate $c(x)$ is continuous and non-negative on (ℓ, r) (the continuity assumptions are not necessary, but will simplify our discussion). We assume that if a boundary point ℓ or r is accessible, the process is killed at the first hitting time of the boundary and is sent to the cemetery state Δ. Thus, the lifetime ζ of X is either the first hitting time of the boundary or the first jump time of a Poisson process with intensity $c(X_t)$ if it occurs before the process hits the boundary. In the context of our credit-equity model (2.1), we have $a(x) = \sigma(x)x$, $b(x) = [\mu + h(x)]x$ and $c(x) = h(x)$, $\ell = 0$ is either a killing boundary if it is accessible or a natural boundary if it is inaccessible, and $r = \infty$ is a natural boundary. In this section we work with a general diffusion process with the infinitesimal generator (6.2), as the results are general and are applicable to other diffusion-based financial models, such as interest rate models.

Under our assumptions, the diffusion process X with lifetime ζ has a positive transition density $p(t; x, y)$, so that

$$\mathcal{P}_t f(x) = \mathbb{E}_x\left[\mathbf{1}_{\{\zeta > t\}} f(X_t)\right] = \int_\ell^r f(y)p(t; x, y)dy \tag{6.3}$$

for any f for which the integral exists. Moreover, the density $p(t; x, y)$ is continuous in all of its variables and is known to satisfy the following symmetry property

$$p(t; x, y)\mathfrak{m}(x) = p(t; y, x)\mathfrak{m}(y), \tag{6.4}$$

where the function $\mathfrak{m}(x)$ is the so-called *speed density* of the diffusion process X and is constructed from the diffusion and drift coefficients as follows (see Borodin and Salminen (2002), p.17):

$$\mathfrak{m}(x) = \frac{2}{a^2(x)\mathfrak{s}(x)}, \text{ where } \mathfrak{s}(x) = \exp\left(-\int_{x_0}^x \frac{2b(y)}{a^2(y)}dy\right), \tag{6.5}$$

where $x_0 \in (\ell, r)$ is an arbitrary point in the state space. The function $\mathfrak{s}(x)$ is called the *scale density* of the diffusion process X.

Let $H_z := \inf\{t \geq 0 : X_t = z\}$ be the first hitting time of $z \in (\ell, r)$. Then the non-negative random variable H_z has the Laplace transform:

$$\mathbb{E}_x[e^{-sH_z}] = \begin{cases} \frac{\psi_s(x)}{\psi_s(z)}, & x \leq z \\ \frac{\phi_s(x)}{\phi_s(z)}, & x \geq z \end{cases}, \tag{6.6}$$

where $\psi_s(x)$ and $\phi_s(x)$ are continuous solutions of the second-order ordinary differential equation (the so-called *Sturm-Liouville equation*):

$$\mathcal{G}u(x) = \frac{1}{2}a^2(x)\frac{d^2u}{dx^2}(x) + b(x)\frac{du}{dx}(x) - c(x)u(x) = su(x). \tag{6.7}$$

For $s > 0$, the functions $\psi_s(x)$ and $\phi_s(x)$ can be characterized as the unique (up to a multiplicative factor independent of x) solutions of (6.7) by firstly demanding that $\psi_s(x)$ is increasing in x and $\phi_s(x)$ is decreasing, and secondly posing boundary conditions at accessible boundary points. For $\psi_s(x)$ the boundary condition is only imposed at ℓ if ℓ is an accessible boundary. Since in this paper we assume that accessible boundaries are specified as killing boundaries, we have a Dirichlet boundary condition at ℓ, $\psi_s(\ell) = 0$. For $\phi_s(x)$ we have, similarly, $\phi_s(r) = 0$ if r is an accessible boundary specified as a killing boundary. The functions $\psi_s(x)$ and $\phi_s(x)$ are called fundamental solutions of the Sturm-Liouville equation (6.7). They are linearly independent and all solutions can be expressed as their linear combinations. Moreover, the so-called *Wronskian* (where $s(x)$ is the scale density defined in Eq.(6.5))

$$w_s = \frac{1}{\mathfrak{s}(x)}(\psi_s'(x)\phi_s(x) - \psi_s(x)\phi_s'(x)) \tag{6.8}$$

is independent of x.

Introduce the *Green's function*, the Laplace transform of the transition density with respect to time:

$$G_s(x,y) = \int_0^\infty e^{-st}p(t;x,y)dt. \tag{6.9}$$

The Green's function admits an explicit representation in terms of the fundamental solutions ϕ_s and ψ_s(Borodin and Salminen (2002), p.19; note that we define the Green's function with respect to the Lebesgue measure, while Borodin and Salminen define it with respect to the speed measure $\mathfrak{m}(y)dy$, where $\mathfrak{m}(y)$ is the speed density, and so $\mathfrak{s}(y)$ does not appear in their expression):

$$G_s(x,y) = \frac{\mathfrak{m}(y)}{w_s}\begin{cases} \psi_s(x)\phi_s(y), & x \le y \\ \psi_s(y)\phi_s(x), & y \le x \end{cases}. \tag{6.10}$$

Therefore, the transition density of a one-dimensional diffusion can be found by, firstly, determining the increasing and decreasing solutions $\psi_s(x)$ and $\phi_s(x)$ of the Sturm-Liouville equation (6.7) and, secondly, inverting the Laplace transform (6.9):

$$p(t;x,y) = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{st}G_s(x,y)ds. \tag{6.11}$$

In this *Bromwich Laplace transform inversion formula* the integration is along the contour in the right half plane parallel to the imaginary axes $s = \varepsilon + iw$ with $\varepsilon > 0$ and $w \in \mathbb{R}$.

Now consider the computation of the expectation (6.1). Taking the Laplace transform in time, we define the *resolvent operator* $\mathcal{R}_s$ (the Laplace transform of the expectation operator; see Ethier and Kurtz (1986)):

$$\mathcal{R}_s f(x) := \int_0^\infty e^{-st}\mathcal{P}_t f(x)dt = \int_0^\infty e^{-st}\mathbb{E}_x\left[\mathbf{1}_{\{\zeta > t\}}f(X_t)\right]dt = \int_\ell^r f(y)G_s(x,y)dy$$

$$= \frac{\phi_s(x)}{w_s} \int_\ell^x f(y)\psi_s(y)\mathfrak{m}(y)dy + \frac{\psi_s(x)}{w_s} \int_x^r f(y)\phi_s(y)\mathfrak{m}(y)dy, \tag{6.12}$$

where we interchanged the Laplace transform integral in t and the expectation integral in y. This interchange is allowed by Fubini's theorem if and only if the function f is such that $\int_\ell^r |f(y)G_s(x,y)|dy < \infty$ or

$$\int_\ell^x |f(y)|\psi_s(y)\mathfrak{m}(y)dy < \infty \text{ and } \int_x^r |f(y)|\phi_s(y)\mathfrak{m}(y)dy < \infty \;\; \forall \;\; x \in (\ell, r), \;\; s > 0. \tag{6.13}$$

For f satisfying this integrability condition, we can then recover the expectation (6.1) by first computing the resolvent operator (6.12) and then inverting the Laplace transform via the Bromwich Laplace transform inversion formula (see Pazy (1983) for the Laplace inversion formula for operator semigroups):

$$\mathcal{P}_t f(x) = \mathbb{E}_x \left[\mathbf{1}_{\{\zeta > t\}} f(X_t)\right] = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{st} \mathcal{R}_s f(x) ds. \tag{6.14}$$

A crucial observation is that in the representation (6.14) time only enters through the exponential e^{st} (the temporal and spatial variables are separated). We can thus write:

$$\mathbb{E}[\mathbf{1}_{\{\zeta > T_t\}} f(X_{T_t})] = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \mathbb{E}[e^{sT_t}] \mathcal{R}_s f(x) ds = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \mathcal{L}(t, -s) \mathcal{R}_s f(x) ds, \tag{6.15}$$

where $\mathcal{L}(t, \lambda) = \mathbb{E}[e^{-\lambda T_t}]$ is the Laplace transform of the time change (here we require that $\mathbb{E}[e^{\varepsilon T_t}] = \mathcal{L}(t, -\varepsilon) < \infty$). This result has two significant advantages over the expression (5.4). First, it does not require the knowledge of the transition probability measure of the time change, and only requires the knowledge of the Laplace transform of the time change. Second, it does not require the knowledge of the expectation $\mathbb{E}[\mathbf{1}_{\{\zeta > t\}} f(X_t)]$ for the original process, and only requires the knowledge of the resolvent $\mathcal{R}_s f(x)$ given by Eq.(6.12).

The Laplace transform inversion in (6.15) can be performed by appealing to the Cauchy Residue Theorem to calculate the Bromwich integral in the complex plane. In order to do this, we need to analyze singularities of the function $\mathcal{R}_s f(x)$ in the complex plane $s \in \mathbb{C}$ (due to our assumption $\mathbb{E}[e^{\varepsilon T_t}] = \mathcal{L}(t, -\varepsilon) < \infty$, the Laplace transform of the time change $\mathcal{L}(t, -s)$ is analytic in the half-plane to the left of the integration contour in (6.15)).

Remark 6.1. If the background process X is a Lévy process (in particular, Brownian motion with drift), then the Laplace transform approach in this section can be shown to be equivalent to the Fourier transform approach of Carr et al. (2003). In this case, we do not need to work with the resolvent and can work with the characteristic functions instead as is done in Carr et al. (2003), leading to the Fourier inversion by the FFT as in Carr and Madan (). For Lévy processes, the characteristic function/Fourier transform approach is more straightforward to use in application. However, the Laplace transform approach in this section is much more general, as it can be applied to time changing any Markov process, not just a Lévy process.

Remark 6.2. Carr et al. (2003) work with Lévy processes without killing. We note that it is possible to introduce killing/default into the framework of time changed Lévy processes in Carr et al. (2003) as follows. Start with a Lévy process with killing. Recall that the killing rate k has to be constant in order for the killed process to be a Lévy process. That is, the Lévy process is killed at an independent exponential time. On time changing the Lévy process with an integral of an activity rate process V_t (such as the CIR), the time changed process acquires a stochastic default intensity kV_t. That is, the default intensity is the old constant killing rate scaled with the stochastic activity rate process that introduces stochastic volatility. To price contingent claims in this class of models based on Lévy processes with stochastic volatility and killing, one can directly follow the Fourier approach of Carr et al. (2003). However, the method developed in the present paper is much more general and is applicable to any Markov process with killing.

Remark 6.3. If the background Markov process X is a one-dimensional diffusion and the time change process is a Lévy subordinator with the exponential Lévy measure $\nu(ds) = \alpha\eta e^{-\eta s}ds$ as discussed in section 3, then we note that the state-dependent Lévy density (4.7) of the time changed process is the Green's function (6.10) of the diffusion X evaluated at $s = \eta$ and scaled with $\alpha\eta$. Indeed, from Eq.(4.7), we have:

$$\pi(x, y) = \alpha\eta \int_0^\infty p(s; x, y)e^{-\eta s}ds = \alpha\eta G_\eta(x, y). \tag{6.16}$$

7 Valuation of Contingent Claims on Time Changed Markov Processes: A Spectral Expansion Approach

Studying the Green's function as a function of the complex variable s, one can invert the Laplace transform (6.9) and obtain a *spectral representation of the transition density* for one-dimensional diffusions originally due to McKean (1956) (see also Ito and McKean (1974, Section 4.11), Wong (1964), and Karlin and Taylor (1981)). Indeed, considered as a linear operator in the Hilbert space of functions square-integrable with the speed density $\mathfrak{m}(x)$, the expectation operator $\mathcal{P}_t$ (6.3) is *self-adjoint*. Namely, define the inner product

$$(f, g) := \int_\ell^r f(x)g(x)\mathfrak{m}(x)dx \tag{7.1}$$

and let $L^2((\ell, r), \mathfrak{m})$ be the Hilbert space of functions on (ℓ, r) square-integrable with the speed density, i.e., with $\|f\| < \infty$, where $\|f\|^2 = (f, f)$. Then the semigroup $\{\mathcal{P}_t, t \geq 0\}$ of expectation operators (6.3) indexed by time is self-adjoint in $L^2((\ell, r), \mathfrak{m})$, i.e.,

$$(\mathcal{P}_t f, g) = (f, \mathcal{P}_t g)$$

for every $f, g \in L^2((\ell, r), \mathfrak{m})$ and $t \geq 0$. This follows from the symmetry property (6.4) of the density (note that this symmetry property is apparent from the structure of the Green's function (6.10)). The infinitesimal generator $\mathcal{G}$ of a self-adjoint semigroup, as well as the resolvent operators $\mathcal{R}_s$, are also self-adjoint, and we can appeal to the Spectral Theorem for self-adjoint operators in Hilbert space to obtain their spectral representations. One-dimensional diffusions are examples of *symmetric Markov processes* with symmetric transition semigroups and self-adjoint infinitesimal generators (the standard reference is Fukushima et al. (1994).

In the important special case when the spectrum of $\mathcal{G}$ in $L^2((\ell, r), m)$ is purely discrete, the spectral representation has the following form. Let $\{\lambda_n\}_{n=1}^\infty$, $0 \leq \lambda_1 < \lambda_2 < ...$, $\lim_{n\uparrow\infty} \lambda_n = \infty$, be the eigenvalues of $-\mathcal{G}$ and let $\{\varphi_n\}_{n=1}^{\infty}$ be the corresponding eigenfunctions normalized so that $\|\varphi_n\|^2 = 1$. That is, (λ_n, φ_n) solve the *Sturm-Liouville eigenvalue-eigenfunction problem* for the (negative of the) differential operator (6.2):

$$-\mathcal{G}\varphi_n = \lambda_n \varphi_n \tag{7.2}$$

(Dirichlet boundary condition is imposed at an endpoint if it is a killing boundary). Then the spectral representations for the transition density $p(t; x, y)$, the Green's function $G_s(x, y)$, the resolvent operator $\mathcal{R}_s f(x)$, and the expectation operator $\mathcal{P}_t f(x)$ for $f \in L^2((\ell, r), \mathfrak{m})$ take the form of *eigenfunction expansions* (for $t > 0$ the eigenfunction expansion (7.3) converges uniformly on compact squares in $(\ell, r) \times (\ell, r)$):

$$p(t; x, y) = \mathfrak{m}(y) \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y), \tag{7.3}$$

$$G_s(x, y) = \mathfrak{m}(y) \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(y)}{s + \lambda_n}, \tag{7.4}$$

$$\mathcal{R}_s f(x) = \sum_{n=1}^{\infty} \frac{c_n \varphi_n(x)}{s + \lambda_n}, \tag{7.5}$$

$$\mathcal{P}_t f(x) = \mathbb{E}_x \left[\mathbf{1}_{\{\zeta > t\}} f(X_t) \right] = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \varphi_n(x) \tag{7.6}$$

with the expansion coefficients

$$c_n = (f, \varphi_n) \tag{7.7}$$

satisfying the Parseval equality $\|f\|^2 = \sum_{n=1}^{\infty} c_n^2 < \infty$. The eigenfunctions $\{\varphi_n(x)\}_{n=1}^{\infty}$ form a complete orthonormal basis in the Hilbert space $L^2((\ell, r), \mathfrak{m})$, i.e., $(\varphi_n, \varphi_n) = 1$ and $(\varphi_n, \varphi_m) = 0$ for $n \neq m$. They are also eigenfunctions of the expectation operator:

$$\mathcal{P}_t \varphi_n(x) = \mathbb{E}_x \left[\mathbf{1}_{\{\zeta > t\}} \varphi_n(X_t) \right] = e^{-\lambda_n t} \varphi_n(x) \tag{7.8}$$

with eigenvalues $e^{-\lambda_n t}$, and of the resolvent operator:

$$\mathcal{R}_s \varphi_n(x) = \frac{\varphi_n(x)}{s + \lambda_n} \tag{7.9}$$

with eigenvalues $1/(s + \lambda_n)$.

More generally, the spectrum of the infinitesimal generator $\mathcal{G}$ in $L^2((\ell, r), m)$ may be continuous, in which case the sums in (7.3-6) are replaced with the integrals. We do not reproduce general results on spectral expansions with continuous spectrum here and instead refer the reader to the literature. For further details on the spectral representation for one-dimensional diffusions and their applications in asset pricing we refer the reader to Davydov and Linetsky (2003), Lewis (1998), (2000), and Linetsky (2004a), (2004b), (2004c), (2007). We also refer the reader to Amrein et al. (2005) for a detailed mathematical treatment of the Sturm-Liouville theory and numerous references.

We now comment on the relationship of the spectral representation (7.3) and the Laplace transform representation (6.11). The spectral representation (7.3) can be obtained from (6.11) as follows. If the spectrum of the infinitesimal generator $\mathcal{G}$ is purely discrete, then the only singularities of the Green's function $G_s(x,y)$ in the complex plane $\{s \in \mathbb{C}\}$ are simple poles at $s = -\lambda_n$, where $-\lambda_n$ are eigenvalues of $\mathcal{G}$. Calculating the residues at these poles produces expansion terms $\mathfrak{m}(y)e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)$ with the eigenfunctions corresponding to these eigenvalues, and the eigenfunction expansion (7.3) of the transition density arises as the sum over all poles in the Cauchy Residue Theorem. The eigenfunction expansion (7.6) is then obtained by substituting (7.3) into (6.3) and integrating term-by-term.

A key feature of the spectral representation is that it separates the temporal and spatial variables. Moreover, time enters the expression (7.6) only through the exponentials $e^{-\lambda_n t}$, thus setting the stage for time changes. We now turn to computing expectations of the form (5.4). Let $f \in L^2((\ell, r), \mathfrak{m})$. Substituting the eigenfunction expansion (7.6) into (5.4), we have:

$$\mathbb{E}[\mathbf{1}_{\{\zeta > T_t\}} f(X_{T_t})] = \sum_{n=1}^{\infty} c_n \mathbb{E}[e^{-\lambda_n T_t}]\varphi_n(x) = \sum_{n=1}^{\infty} c_n \mathcal{L}(t, \lambda_n)\varphi_n(x), \tag{7.10}$$

where $\mathcal{L}(t,\lambda)$ is the Laplace transform of the time change. In particular, for the eigenfunctions we have:

$$\mathbb{E}_x\left[\mathbf{1}_{\{\zeta > T_t\}}\varphi_n(X_{T_t})\right] = \mathcal{L}(t, \lambda_n)\varphi_n(x). \tag{7.11}$$

Due to the fact that time enters the spectral expansion only through the exponentials $e^{-\lambda_n s}$, integrating this exponential against the distribution of the time change $\pi_t(ds)$, the integral in s in (5.4) reduces to the Laplace transform of the time change, $\int_{[0,\infty)} e^{-\lambda_n s}\pi_t(ds) = \mathcal{L}(t, \lambda_n)$. Thus, in one shot, we both compute the integral in s in (5.4) and get rid of the necessity to invert the Laplace transform to recover the distribution of the time change. In effect, the spectral expansion approach reduces the total required number of integrations by two. In general, the spectral expansion approach is tailor-made for time changes due to the exponential dependence on time (see also Chen and Song (2005), (2007) and Linetsky (2007) for related results).

Remark 7.1. We stress that the spectral expansions (7.5-6) are only valid for functions f that are square-integrable with the speed density $\mathfrak{m}$. For those functions that are not in $L^2((\ell, r), \mathfrak{m})$ but satisfy the integrability conditions (6.13) one needs to apply the Cauchy Residue Theorem directly to the expression (6.15) since the resolvent $\mathcal{R}_s f(x)$ may have singularities that do not coincide with the singularities of the Green's function $G_s(x,y)$, and the evaluation of (6.15) has to be done case-by-case for each non-square-integrable f.

Remark 7.2. If the process X is a Lévy process (e.g., Brownian motion with drift), then the result of the spectral method can be shown to be equivalent to the Fourier transform method based on the characteristic function. The Fourier method is more straightforward in this case. However, the spectral method is much more general, as it is applicable to any symmetric Markov process (and to any one-dimensional diffusion in particular).

8 Time Changing the Jump-to-Default Extended CEV Process

8.1 The Jump-to-Default Extended CEV Process

Carr and Linetsky (2006) recently proposed the following extension of the classical *constant elasticity of variance (CEV) model* of Cox (1975). Recall that to be consistent with the leverage effect and the implied volatility skew, the instantaneous volatility in the CEV model is specified as a power function (see Cox (1975), Schroder (1989), Davydov and Linetsky (2001), (2003), Linetsky (2004b), and Jeanblanc et al. (2007, Chapter 6) for background on the CEV process):

$$\sigma(x) = ax^{\beta}, \tag{8.1}$$

where $\beta < 0$ is the volatility elasticity parameter and $a > 0$ is the volatility scale parameter. The limiting case with $\beta = 0$ corresponds to the constant volatility assumption in the Black-Scholes-Merton model. To be consistent with the empirical evidence linking corporate bond yields and CDS spreads to equity volatility, Carr and Linetsky (2006) propose to specify the default intensity as an affine function of the instantaneous variance of the underlying stock price process:

$$h(x) = b + c\,\sigma^2(x) = b + c\,a^2x^{2\beta}, \tag{8.2}$$

where $b \geq 0$ is a constant parameter governing the state-independent part of the intensity and $c \geq 0$ is a constant parameter governing the sensitivity of the intensity to σ^2. In Carr and Linetsky (2006) a and b are taken to be deterministic functions of time. In the present paper we assume that a and b are constant. The infinitesimal generator (6.2) of this diffusion process on $(0, \infty)$ with killing at the rate (8.2) has the form:

$$\mathcal{G}f(x) = \frac{1}{2}a^2x^{2\beta+2}\frac{d^2f}{dx^2}(x) + (\mu + b + c\,a^2x^{2\beta})x\frac{df}{dx}(x) - (b + c\,a^2x^{2\beta})f(x). \tag{8.3}$$

This model specification introduces the possibility of a jump to default from a positive value for the CEV process and is referred to as the *jump to default extended CEV* process, or *JDCEV* for short. This model nests the standard CEV model as a limiting case with vanishing default intensity $b = c = 0$. In the standard CEV model default can only occur when the stock price hits zero through diffusion. When $c = 0$, the intensity is independent of the stock price, and the model is that of the CEV process killed at an independent exponential time with mean $1/b$ (the first jump time of a Poisson process with constant intensity b). In this case default can occur either through hitting zero by diffusion or through a jump to zero from a positive stock price value. When $b = 0$, the intensity does not have a state-independent term and is entirely governed by the stock price process. In the general specification $b > 0$ and $c > 0$ the intensity has two parts — a state-independent part and a state-dependent part. When $c > 0$, default can only occur through a jump from a positive value, since the default intensity increases so fast as the stock falls, that the jump to default will almost surely arrive prior to the diffusion process hitting zero.

In this section we use the general theory developed in the previous sections to construct far reaching extensions of the original Carr-Linetsky JDCEV model. By assuming that the process X in (2.1) follows a JDCEV process and time changing it as described in Section 3, we introduce jumps and stochastic volatility into the JDCEV model. In order to be able to value contingent claims in time changed JDCEV models, we need to be able to compute expectations of the form (6.1) for the JDCEV process as described in Sections 6 and 7. The scale and speed densities (6.5) of the JDCEV process are:

$$\mathfrak{m}(x) = \frac{2}{a^2}x^{2c-2-2\beta}e^{Ax^{-2\beta}}, \;\; \mathfrak{s}(x) = x^{-2c}e^{-Ax^{-2\beta}}, \;\text{ where }\; A := \frac{\mu + b}{a^2|\beta|}. \tag{8.4}$$

The following theorem presents the fundamental solutions $\psi_s(x)$ and $\phi_s(x)$ entering the expression for the Green's function (6.10) and their Wronskian w_s (6.8) for the JDCEV process.

Without loss of generality we assume that $\mu + b \geq 0$.[2] There are two distinct cases: $\mu + b > 0$ and $\mu + b = 0$.

Theorem 8.1 *(i) For a JDCEV diffusion process with the infinitesimal generator (8.3) with parameters $\beta < 0$, $a > 0$, $b \geq 0$, $c \geq 0$ and such that $\mu + b > 0$, the increasing and decreasing fundamental solutions $\psi_s(x)$ and $\phi_s(x)$ are:*

$$\psi_s(x) = x^{\frac{1}{2}+\beta-c} e^{-\frac{1}{2}Ax^{-2\beta}} M_{\varkappa(s),\frac{\nu}{2}}(Ax^{-2\beta}), \tag{8.5}$$

$$\phi_s(x) = x^{\frac{1}{2}+\beta-c} e^{-\frac{1}{2}Ax^{-2\beta}} W_{\varkappa(s),\frac{\nu}{2}}(Ax^{-2\beta}), \tag{8.6}$$

where $M_{k,m}(z)$ and $W_{k,m}(z)$ are the first and second Whittaker functions (see Appendix B) with indexes

$$\nu = \frac{1+2c}{2|\beta|}, \quad \varkappa(s) = \frac{\nu - 1}{2} - \frac{s+\xi}{\omega}, \quad \text{where } \omega = 2|\beta|(\mu + b), \quad \xi = 2c(\mu + b) + b, \tag{8.7}$$

and the constant A is defined in Eq.(8.4). The Wronskian w_s defined by Eq.(6.8) reads:

$$w_s = \frac{2(\mu + b)\Gamma(1+\nu)}{a^2\Gamma(\nu/2 + 1/2 - \varkappa(s))}. \tag{8.8}$$

(ii) For a JDCEV diffusion process with the infinitesimal generator (8.3) with parameters $\beta < 0$, $a > 0$, $b \geq 0$, $c \geq 0$ and such that $\mu + b = 0$, the increasing and decreasing fundamental solutions $\psi_s(x)$ and $\phi_s(x)$ are:

$$\psi_s(x) = x^{\frac{1}{2}-c} I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right), \quad \phi_s(x) = x^{\frac{1}{2}-c} K_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right), \tag{8.9}$$

where $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions (see Appendix B) with index ν given in Eq.(8.7). The Wronskian w_s defined by Eq.(6.8) reads:

$$w_s = |\beta|. \tag{8.10}$$

Proof. The proof is by reduction of the Sturm-Liouville equation (6.7) for the JDCEV operator (8.3) to the Whittaker equation when $\mu + b > 0$ and to the Bessel equation when $\mu + b = 0$. See Appendix A. □

Theorem 8.1 generalizes Proposition 5 in Davydov and Linetsky (2001) that gives the fundamental solutions for the standard CEV model. Their results are a special case of our Proposition 8.1 for vanishing default intensity with $b = c = 0$. The Green's function is given by Eq.(6.10). Inverting the Laplace transform (6.11) leads to the spectral representation of the transition density.

[2]For absolutely continuous and composite time changes, $\mu = 0$ by Theorems 4.4 and 4.5, while $b \geq 0$. For Lévy subordinators, by Theorem 4.1 $\mu \in \mathcal{I}_\nu$ can always be selected so that $\mu + b \geq 0$. Thus, we do not consider the case $\mu + b < 0$.

Theorem 8.2 *(i) When $\mu + b > 0$, the spectrum of the negative of the infinitesimal generator (8.3) is purely discrete with the eigenvalues and eigenfunctions:*

$$\lambda_n = \omega n + \xi, \;\; \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!(\mu+b)}{\Gamma(\nu+n)}} x e^{-Ax^{-2\beta}} L_{n-1}^{(\nu)}(Ax^{-2\beta}), \;\; n = 1, 2, ..., \tag{8.11}$$

where $L_n^{(\nu)}(x)$ are the generalized Laguerre polynomials (see Appendix B) and ξ and ω are defined in (8.7). The spectral representation (eigenfunction expansion) of the JDCEV transition density is given by Eq.(7.3) with these eigenvalues and eigenfunctions and the speed density (8.4).

(ii) When $\mu + b = 0$, the spectrum of the infinitesimal generator (8.3) is purely absolutely continuous and the spectral representation for the transition density reads:

$$p(t; x, y) = \frac{1}{2|\beta|} \mathfrak{m}(y) \int_0^\infty e^{-(\lambda+b)t} (xy)^{1/2-c} J_\nu \left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|} \right) J_\nu \left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|} \right) d\lambda, \tag{8.12}$$

where $J_\nu(x)$ is the Bessel function of the first kind with index ν given in (8.7)

Proof. The proof is based on applying the Cauchy Residue Theorem to calculate the Bromwich Laplace inversion integral. See Appendix A. □

Theorem 8.2 generalizes Proposition 8(i) in Davydov and Linetsky (2003) that gives the eigenvalues and eigenfunctions for the standard CEV model. Their results are a special case of our Proposition 8.2 for vanishing default intensity with $b = c = 0$.

Carr and Linetsky (2006) present closed form solutions for the survival probability and call and put options in the JDCEV model (Proposition 5.5, pp. 319-320). However, those expressions are not suitable for time changes since they depend on time in a complicated fashion. Here, based on the theory in sections 6 and 7 and Theorems 8.1 and 8.2, we obtain alternative closed-form expressions for the survival probability and call and put options in the JDCEV model with time entering only through exponentials. We first present the result for the survival probability.

Theorem 8.3 *(i) For a JDCEV diffusion process with the infinitesimal generator (8.3) with parameters $\beta < 0$, $a > 0$, $b \geq 0$, $c \geq 0$, $\mu + b > 0$ and started at $x > 0$, the survival probability $\mathbb{Q}(\zeta > t)$ is given by:*

$$\mathbb{Q}(\zeta > t) = \sum_{n=0}^{\infty} e^{-(b+\omega n)t} \frac{\Gamma(1 + \frac{c}{|\beta|}) \left(\frac{1}{2|\beta|}\right)_n}{\Gamma(\nu+1) n!} A^{\frac{1}{2|\beta|}} x e^{-Ax^{-2\beta}} \; {}_1F_1 \left(1 - n + \frac{c}{|\beta|}; \nu + 1; Ax^{-2\beta} \right), \tag{8.13}$$

where ${}_1F_1(a; b; x)$ is the confluent hypergeometric function (see Appendix B), $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)...(a+n-1)$ is the Pochhammer symbol, and the constants A, ν, and ω are as defined in Theorem 8.1.

(ii) For $\mu + b = 0$ the JDCEV survival probability $\mathbb{Q}(\zeta > t)$ is given by:

$$\mathbb{Q}(\zeta > t) = x^{1/2-c} (\sqrt{2}a|\beta|)^{\frac{2c-1}{2|\beta|}} \frac{\Gamma(1 + \frac{c}{|\beta|})}{\Gamma(\frac{1}{2|\beta|})} \int_0^\infty e^{-(b+\lambda)t} \lambda^{-\frac{2c-1}{4|\beta|}-1} J_\nu \left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|} \right) d\lambda, \tag{8.14}$$

where $J_\nu(x)$ is the Bessel Function of the first kind.

Proof. The proof is based on first computing the resolvent (6.12) with $f(x) = 1$ and then inverting the Laplace transform (6.14) analytically. Since constants are not square-integrable on $(0,\infty)$ with the speed density (8.4), we cannot use the spectral expansion approach of section 7 and instead follow the Laplace transform approach of section 6. See Appendix A. □.

We now present the result for the put option. The put option price in our model (2.1) is given by Eqs.(5.7)-(5.9). In particular, in order to compute the price of the put payoff conditional on no default before expiration, $P_0(x; K, t)$, we need to compute the expectation $\mathbb{E}[(K - e^{\rho t}X_s)^+\mathbf{1}_{\{\zeta>s\}}] = e^{\rho t}\mathbb{E}[(e^{-\rho t}K - X_s)^+\mathbf{1}_{\{\zeta>s\}}]$ for the JDCEV process (8.3). The survival probability entering the put pricing formula is already computed in Theorem 8.3. The pricing formula for the call option is obtained via the put-call parity.

Theorem 8.4 *(i) For a JDCEV diffusion process with the infinitesimal generator (8.3) with parameters $\beta < 0$, $a > 0$, $b \geq 0$, $c \geq 0$ and such that $\mu + b > 0$, the expectation $\mathbb{E}[(k - X_t)^+\mathbf{1}_{\{\zeta>t\}}]$ is given by the eigenfunction expansion (7.6) with the eigenvalues λ_n and eigenfunctions $\varphi_n(x)$ given in Theorem 8.2 and expansion coefficients:*

$$c_n = \frac{A^{\nu/2+1}k^{2c+1-2\beta}\sqrt{\Gamma(\nu+n)}}{\Gamma(\nu+1)\sqrt{(\mu+b)(n-1)!}}$$

$$\times\left\{\frac{|\beta|}{c+|\beta|}\,{}_2F_2\left(\begin{array}{cc} 1-n, & \frac{c}{|\beta|}+1 \\ \nu+1, & \frac{c}{|\beta|}+2\end{array}; Ak^{-2\beta}\right) - \frac{\Gamma(\nu+1)(n-1)!}{\Gamma(\nu+n+1)}L_{n-1}^{(\nu+1)}\left(Ak^{-2\beta}\right)\right\}, \qquad (8.15)$$

where ${}_2F_2$ is the generalized hypergeometric function (see Appendix B).

(ii) For $\mu + b = 0$ the expectation has a spectral expansion with absolutely continuous spectrum:

$$\mathbb{E}[(k - X_t)^+\mathbf{1}_{\{\zeta>t\}}] = \int_0^\infty e^{-(\lambda+b)t}c(\lambda)x^{1/2-c}J_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)d\lambda, \qquad (8.16)$$

with the expansion coefficients:

$$c(\lambda) = \frac{\lambda^{\nu/2}k^{2c+1-2\beta}}{2^{\nu/2+1}\Gamma(\nu+1)(c+|\beta|)|\beta|^{\nu+1}a^{\nu+2}}\,{}_1F_2\left(\begin{array}{cc} \frac{c}{|\beta|}+1, & \\ \nu+1, & \frac{c}{|\beta|}+2\end{array}; -\frac{k^{-2\beta}\lambda}{2a^2|\beta|^2}\right)$$

$$-\frac{k^{c+1/2-\beta}}{\sqrt{2\lambda}|\beta|a}J_{\nu+1}\left(\frac{k^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right), \qquad (8.17)$$

where ${}_1F_2$ is the generalized hypergeometric function (see Appendix B).

Proof. The put payoff $f(x) = (k - x)^+$ is in the Hilbert space $L^2((0,\infty), \mathfrak{m})$ of functions square-integrable with the speed density (8.3) and, hence, the expectation has a spectral expansion. The proof follows by applying the spectral expansion approach. See Appendix A. □

Remark 8.1. When $b = c = 0$, all results in this section reduce to the corresponding results for the standard CEV model (without jump to default) in Davydov and Linetsky (2001), (2003). In the standard CEV model default can only occur through the stock price hitting zero via diffusion. In this case the survival probability in Theorem 8.3 is equal to the probability of the CEV diffusion not hitting zero by time t.

Remark 8.2. The series representation (8.14) for the survival probability is equivalent to the expression (5.14) in Carr and Linetsky (2006). To prove this one needs to apply the multiplication identity for the Whittaker functions given in Eq.(B.10) in Appendix B. Due to this identity, the series of hypergeometric functions in (8.14) collapses to the closed-form expression (5.14) in Carr and Linetsky (2006). For $\mu + b = 0$, one needs to use the integral (B.13) in Appendix B. Similarly, the eigenfunction expansion for the put in Theorem 8.4 is equivalent to the closed-form expression (5.18) in Carr and Linetsky (2006). To prove this one needs to apply the Hille-Hardy formula for Laguerre polynomials (Erdelyi (1953), p.189; valid for all $|t| < 1$, $\nu > -1$, $a, b > 0$):

$$\sum_{n=0}^{\infty} \frac{t^n n!}{\Gamma(n+\nu+1)} L_n^{\nu}(a)\, L_n^{\nu}(b) = \frac{(abt)^{-\nu/2}}{1-t} \exp\left\{-\frac{(a+b)\,t}{1-t}\right\} I_{\nu}\left(\frac{2\sqrt{abt}}{1-t}\right). \tag{8.18}$$

The closed form formulas in Carr and Linetsky (2006) are more suitable for pricing under the original JDCEV model without time changes than the series expansions developed in the present paper, as they are easier to compute. However, they are generally not suitable for time changed models since they have complicated functional dependence on time. In contrast, the expansions in this paper explicitly depend on time only through the exponentials and are, thus, ideally suited for time changes with known Laplace transforms.

8.2 Introducing Jumps and Stochastic Volatility into the JDCEV Process via Time Changes: Numerical Examples

In this section we illustrate our approach with numerical examples. We take the background diffusion process X to be a JDCEV process with $\mu = 0$ and time change it with the composite time change process $T_t = T^1_{T^2_t}$, where T^1 is the Inverse Gaussian (IG) subordinator with the Lévy measure $\nu(ds) = Cs^{-3/2}e^{-\eta s}$ and the Laplace exponent $\phi(s) = \gamma s + 2C\sqrt{\pi}(\sqrt{s+\eta} - \sqrt{\eta})$ and T^2 is the time integral of the activity rate following the CIR process. That is, the time change process is an IG process with stochastic volatility in the terminology of Carr et al. (2003). In order to satisfy the martingale condition, according to Theorem 4.5 we set $\mu = 0$ and $\rho = r - q$. The time changed process $Y_t := X_{T_t}$ is a martingale and the process (Y_t, V_t) is a two-dimensional Markov process with the infinitesimal generator

$$\begin{aligned}\mathcal{G}f(x,v) &= \frac{1}{2}\gamma^2 v a^2 x^{2\beta+2}\frac{\partial^2 f}{\partial x^2}(x,v) + \gamma v(b + c\,a^2x^{2\beta})x\frac{\partial f}{\partial x}(x,v) - k\,(x,v)\,f(x,v) + \\ &+ \int_{(0,\infty)} (f(y,v) - f(x,v))\, v\pi(x,y)dy + \frac{\sigma_V^2}{2}v\frac{\partial^2 f}{\partial v^2}(x,v) + \kappa(\theta - v)\frac{\partial f}{\partial v}(x,v),\end{aligned} \tag{8.19}$$

where the killing rate $k(x)$ and the state-dependent Lévy density $\pi(x,y)$ are:

$$k\,(x,v) = \gamma v(b + c\,a^2x^{2\beta}) +$$

$$+vC\int_{(0,\infty)} \left(1 - \frac{\Gamma\left(\frac{c}{|\beta|}+1\right)(\tau(s))^{\frac{1}{2|\beta|}}\,e^{-\tau(s)-b\,s}}{\Gamma(\nu+1)}\,{}_1F_1\left(\begin{array}{c}\frac{c}{|\beta|}+1\\ \nu+1\end{array};\tau(s)\right)\right) s^{-3/2}e^{-\eta\,s}ds, \tag{8.20}$$

where

$$\tau(s) := \frac{\omega\, x^{-2\beta}}{2|\beta|^2 a^2 \left(1 - e^{-\omega\, s}\right)}, \tag{8.21}$$

and

$$\pi\,(x,y) = 2|\beta| AC \left(\frac{y}{x}\right)^{c-\frac{1}{2}} y^{-(2\beta+1)}$$
$$\times \int_{(0,\infty)} \frac{s^{-3/2} e^{\left(\frac{\omega\nu}{2}-\xi-\eta\right)s}}{e^{\omega s}-1} \exp\left\{-A\left(\frac{x^{-2\beta}e^{\omega s} + y^{-2\beta}}{e^{\omega s}-1}\right)\right\} I_\nu\left(\frac{A\,(xy)^{-\beta}}{\sinh\left(\omega s/2\right)}\right) ds. \tag{8.22}$$

The stock price process in this model is a pure jump process with a jump-to-default that sends the process to zero, an absorbing state.

Remark 8.3. In Eqs.(8.20) and (8.22) it is convenient to use the closed form expressions for the survival probability and the transition density of the JDCEV process obtained in Carr and Linetsky (2006). The spectral expansion of the JDCEV transition probability of the form (7.3) with the eigenfunctions and eigenvalues given in Theorem 8.2 collapses to the closed-form expression in terms of the Bessel function on applying the Hille-Hardy formula (8.18).

The parameter values in our numerical example are listed in Table 1. The JDCEV pro-

JDCEV	S	50	*CIR*	V	1
	a	10		θ	1
	β	-1		σ_V	1
	c	0.5		κ	4
	b	0.01	*IG*	γ	0
	r	0.05		η	8
	q	0		C	$2\sqrt{2/\pi}$

Table 1: *Parameter values.*

cess parameter a entering into the local volatility function $\sigma(x) = a\, x^\beta$ is selected to that the local volatility is equal to twenty percent when the stock price is equal to fifty dollars, i.e., $a = 0.2 * 50^{-\beta} = 10$ for the case of $\beta = -1$ considered here. In this example we select $\gamma = 0$, so the time changed process is a pure jump process with no diffusion component (recall that the diffusion component vanishes for time changes with $\gamma = 0$). For this particular choice of parameters of the IG time change and the CIR activity rate process the time change has the mean and variance $E[T_1] = 1$ and $Var[T_1] = 1/16$ at $t = 1$. If we replace the background JDCEV process with Brownian motion with drift, then the time changed process is a Normal Inverse Gaussian (NIG) process with stochastic volatility following the CIR process as in Carr et al. (2003). Our model extends Carr et al. (2003) in two important respects. By taking the background process to be a diffusion process with state-dependent volatility and drift, the resulting Lévy density after time change is state-dependent, in contrast to the space homogeneous Lévy jumps. Secondly, the time changed process has a state-dependent killing rate (default intensity) in contrast to the absence of default in Carr et al. (2003). By extending the framework of Carr et al. (2003) to state-dependent jumps and default intensity, we gain the flexibility of being able to calibrate the model jointly to options prices and CDS spreads. Moreover, the state dependence of jumps allows for more flexibility in fitting implied volatility surfaces observed in the equity options market than is available under space homogeneous Lévy models.

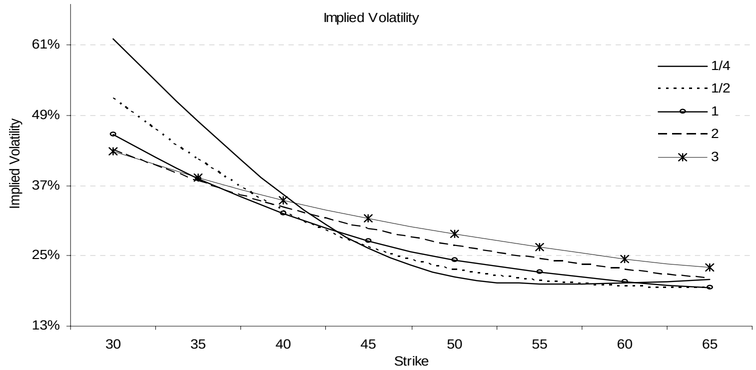

Figure 1: *Implied volatility smile/skew curves as functions of the strike price. Current stock price level is 50.*

Figure 1 plots the implied volatility smile/skew curves of options priced under this model for different maturities. The implied volatility values are shown in Table 2. We compute options prices in this model using Theorem 8.4 and then compute implied volatilities of these options by inverting the Black-Scholes formula. We observe that in this model shorter maturity skews are steeper, and flatten out as maturity increases, consistent with empirical observations in options markets. We also observe that the short maturity skew exhibits a true volatility smile with the increase in implied volatilities both to the right and to the left of the at-the-money strike. This behavior cannot be captured in the pure diffusion JDCEV model. In JDCEV the implied volatility skew results from the leverage effect (the local volatility is a decreasing function of stock price) and the possibility of default (the default intensity is a decreasing function of stock price). The resulting implied volatility skew is a decreasing function of strike. After the time change with jumps, the resulting jump process has both positive and negative jumps. This results in the implied volatility smile pattern. Table 3 presents sample put prices for several strike and maturity combinations. The prices are computed to the accuracy of 10^{-4} (all of the decimals presented in the table are correct) by computing the corresponding eigenfunction expansions.

Time/Strike	*30*	*35*	*40*	*45*	*50*	*55*	*60*	*65*
1/4	62.04	47.94	35.52	26.19	21.41	20.09	20.28	20.88
1/2	51.94	41.47	32.72	26.39	22.64	20.72	19.84	19.46
1	45.74	38.24	32.14	27.53	24.30	22.12	20.65	19.64
2	43.03	37.68	33.23	29.61	26.72	24.45	22.66	21.25
3	42.80	38.34	34.55	31.34	28.64	26.39	24.52	22.96

Table 2: *Implied volatilities (in %) for different strike prices and times to maturity (years). Current stock price level is 50.*

Figure 2 plots the default probability and the credit spread (assuming zero recovery in default) as functions of time to maturity for different levels of the stock price. As the stock price decreases, the credit spreads of all maturities increase, but the shorter and intermediate

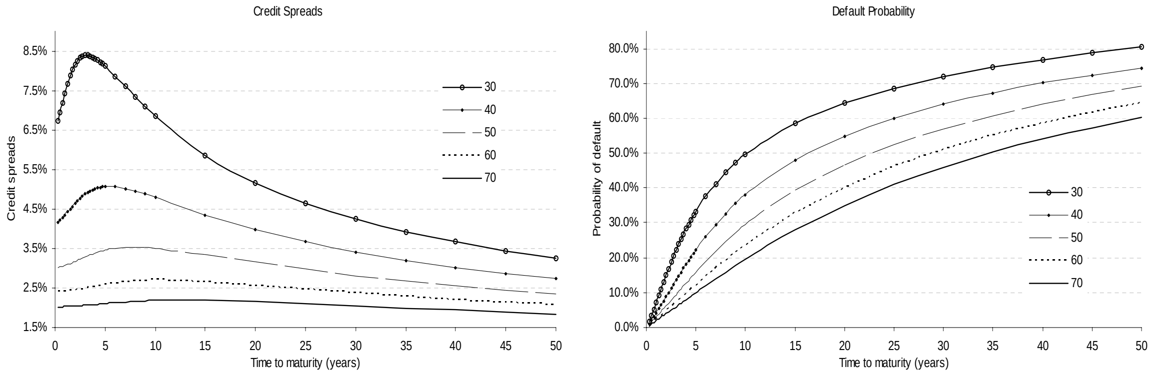

Figure 2: *Credit spreads and default probabilities as functions of time to maturity for current stock price levels* $S = 30,\ 40,\ 50,\ 60,\ 70$.

maturities increase the fastest. In particular intermediate maturities of between three and six years increase the fastest. This results in a pronounced hump in the term structure of credit spreads around four to five year maturities for lower stock prices. This increase in credit spreads with the decrease in the stock price is accounted for both the leverage effect through the increase in the local volatility of the original diffusion and, hence, more jump volatility for the jump process after the time change, as well as the increase in the default intensity of both the original diffusion process and the jump process after the time change. Figure 3 plots the default intensity (killing rate) in this model after the time change as a function of the stock price given by Eq.(8.20). The default intensity is a decreasing function of the stock price. The stock price process in this model is a pure jump process with a jump-to-default that sends the process to zero, an absorbing state.

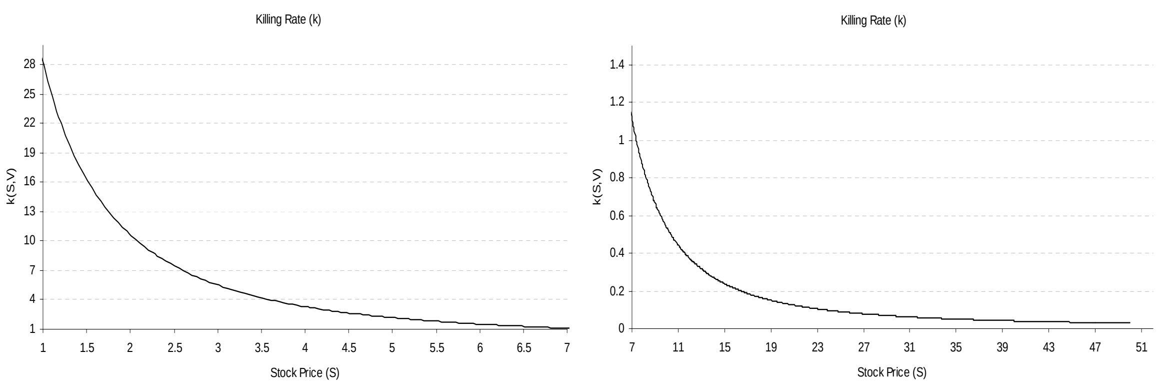

Figure 3: *Killing rate (default intensity) k(S,V) as a function of the stock price S when the activity rate is fixed at* $V = 1$*. The solid line is the default intensity in the model obtained by time changing the JDCEV model. The dashed line is the default intensity obtained by time changing the standard CEV model with* $b = c = 0$ *without the jump to default (see Remark 8.3).*

Remark 8.3. We note the following interesting feature of our model. If we take the standard Cox's CEV diffusion process (without the jump to default introduce in Carr and Linetsky (2006)) to serve as the background Markov process and time change it with a Lévy subordinator, the resulting process acquires a default intensity, even though the original CEV process does not have any killing rate. Indeed, the default event in the CEV process can only occur via hitting the origin by diffusing down to the zero stock price. That is, the default time in the original CEV process is predictable with an announcing sequence of hitting times of stock price levels decreasing towards zero. However, after a pure jump time change, the default time in the time changed jump process becomes totally inaccessible with the intensity given by the integral of the default probability of the original CEV process with the Lévy measure of the subordinator in Eq.(8.20) (in this case $b = 0$ since the standard CEV process does not have any default intensity). This default intensity is plotted in Figure 3 as a dashed line. Intuitively, one can understand this as follows. Suppose one observes a sample path of a diffusion process that hits zero. Since the process is continuous, one observes the announcing sequence of the default event. When the diffusion is subjected to a pure jump time change, one can no longer observe the announcing sequence, as the hitting times of the intermediate stock price levels are left unobservable when the time jumps through those times. As a result, the default event in the time changed process looks like an unpredictable jump to default from a positive value. We plan to further explore this mechanism of inducing a default intensity via a pure jump time change in our future work, in particular in the context of structural models.

Strike/Time	*1/4*	*1/2*	*1*	*2*	*3*	*4*	*5*
30	(0.2227, 0.0006) 0.2233	(0.4405, 0.0059) 0.4464	(0.8633, 0.039) 0.9023	(1.6669, 0.1334) 1.8003	(2.42, 0.1855) 2.6055	(3.1062, 0.189) 3.2952	(3.7063, 0.1667) 3.8730
35	(0.2598, 0.0053) 0.2650	(0.5139, 0.0317) 0.5457	(1.0072, 0.1293) 1.1365	(1.9447, 0.3088) 2.2535	(2.8233, 0.3779) 3.2012	(3.6239, 0.3646) 3.9885	(4.3241, 0.3135) 4.6376
40	(0.2969, 0.0416) 0.3385	(0.5873, 0.1494) 0.7368	(1.1511, 0.3786) 1.5297	(2.2226, 0.6489) 2.8715	(3.2266, 0.7056) 3.9323	(4.1416, 0.6463) 4.7879	(4.9418, 0.5423) 5.4841
45	(0.334, 0.2852) 0.6192	(0.6607, 0.5886) 1.2493	(1.295, 0.9697) 2.2647	(2.5004, 1.2505) 3.7509	(3.63, 1.2261) 4.8560	(4.6593, 1.0714) 5.7308	(5.5595, 0.8786) 6.4381
50	(0.3711, 1.4541) 1.8252	(0.7342, 1.8376) 2.5718	(1.4389, 2.159) 3.5978	(2.7782, 2.2248) 5.0030	(4.0333, 2.0021) 6.0354	(5.177, 1.6801) 6.8571	(6.1772, 1.3499) 7.5271
55	(0.4082, 4.5108) 4.9190	(0.8076, 4.3813) 5.1889	(1.5827, 4.1757) 5.7584	(3.056, 3.6744) 6.7304	(4.4366, 3.0944) 7.5310	(5.6947, 2.5121) 8.2068	(6.795, 1.9837) 8.7787
60	(0.4453, 8.9164) 9.3618	(0.881, 8.1355) 9.0165	(1.7266, 7.0824) 8.8090	(3.3338, 5.6661) 9.0000	(4.84, 4.552) 9.3919	(6.2125, 3.6029) 9.8153	(7.4127, 2.8062) 10.2189
65	(0.4824, 13.7265) 14.2090	(0.9544, 12.5737) 13.5281	(1.8705, 10.7341) 12.6047	(3.6117, 8.2104) 11.8220	(5.2433, 6.4037) 11.6470	(6.7302, 4.9796) 11.7098	(8.0304, 3.8396) 11.8700

Table 3: *Put prices. For each combination of strike and time to maturity two values are given in parenthesis. The first value is the price P_D of the default claim Eq.(5.9). The second value is the price P_0 of the put payoff paid only if there is no default Eq.(5.8). The third value below is the put option price equal to $P_D + P_0$.*

9 Conclusion

This paper develops a novel class of hybrid credit-equity models with state-dependent jumps, local-stochastic volatility and default intensity based on time changes of Markov processes with killing. We model the defaultable stock price process as a time changed Markov diffusion process with state-dependent local volatility and killing rate (default intensity). When the time change is a Lévy subordinator, the stock price process exhibits jumps with state-dependent Lévy measure. When the time change is a time integral of an activity rate process, the stock price process has local-stochastic volatility and default intensity. When the time change process is a Lévy subordinator in turn time changed with a time integral of an activity rate process, the stock price process has state-dependent jumps, local-stochastic volatility and default intensity. This framework offers far reaching extensions of the framework of time changed Lévy processes with stochastic volatility of Carr et al. (2003). By time changing Markov processes we relax the space homogeneity assumption inherent in Lévy models. Moreover, the mechanism of killing a Markov process at a state-dependent rate is well suited to modeling the default event.

This paper develops two analytical approaches to the pricing of credit and equity derivatives in this class of models. The two approaches are based on the Laplace transform inversion and the spectral expansion approach, respectively. If the resolvent (the Laplace transform of the transition semigroup) of the diffusion process and the Laplace transform of the time change are both available in closed form, the expectation operator of the time changed process is expressed in closed form as a single integral in the complex plane. If the payoff is square-integrable, the complex integral is further reduced to a spectral expansion. To illustrate our general framework, we time change the jump-to-default extended CEV model (JDCEV) of Carr and Linetsky (2006) and obtain a rich class of analytically tractable models with jumps, local-stochastic volatility and default intensity. These models can be used to jointly price and hedge equity and credit derivatives. In particular, we compute implied volatility surfaces, default probabilities, and credit spreads under the JDCEV process subject to the time change that is an inverse Gaussian subordinator that is itself subject to a time change with a CIR activity rate process. This process is a pure jump process with state dependent jumps and killing (jump to default) in contrast to the pure diffusion JDCEV model of Carr and Linetsky (2006).

The contribution of this paper is in the development of a flexible modeling framework, as well as in the development of the analytical methods to solve this class of models. A wide range of models can be constructed within this model architecture by pairing background diffusion processes with different time changes. We hope that this paper will stimulate empirical research into the joint credit-equity dynamics and the interplay between credit and equity derivatives markets.

A Proofs

A.1 Proof of Theorem 4.1

Since, by Theorem 4.3, the process $e^{-\rho t}S_t = \mathbf{1}_{\{t<\tau_d\}}X_{T_t}$ is a time-homogeneous Markov process, it is enough to prove that

$$\mathbb{E}[\mathbf{1}_{\{t<\tau_d\}}X_{T_t}] = e^{(r-q-\rho)t}x \text{ for all } t>0, \tag{A.1}$$

where $S_0 = X_0 = x > 0$. Let $\mathcal{F}_t^X = \sigma\{X_s, s \le t\}$ and $\mathcal{F}_t^T = \sigma\{T_s, s \le t\}$ be the filtrations generated by the background diffusion process X and the time change T. Observing that

$$\mathbf{1}_{\{t<\tau_d\}} = \mathbf{1}_{\{T_t<\zeta\}} = \mathbf{1}_{\{T_t<H_0\}}\mathbf{1}_{\{T_t<\zeta\}}$$

and

$$\mathbb{E}[\mathbf{1}_{\{t<H_0\}}\mathbf{1}_{\{t<\zeta\}}|\mathcal{F}_t^X] = \mathbf{1}_{\{t<H_0\}}e^{-\int_0^t h(X_u)du},$$

we can write

$$\mathbb{E}[\mathbf{1}_{\{t<\tau_d\}}X_{T_t}] = \mathbb{E}[\mathbf{1}_{\{T_t<H_0\}}e^{-\int_0^{T_t} h(X_u)du}X_{T_t}]$$

$$= x\mathbb{E}[\mathbf{1}_{\{T_t<H_0\}}e^{-\int_0^{T_t} h(X_u)du}e^{\int_0^{T_t}[\mu+h(X_u)]du+\int_0^{T_t}\sigma(X_u)dB_u-\frac{1}{2}\int_0^{T_t}\sigma^2(X_u)du}]$$

$$= x\mathbb{E}[e^{\mu T_t}\mathbf{1}_{\{T_t<H_0\}}e^{\int_0^{T_t}\sigma(X_u)dB_u-\frac{1}{2}\int_0^{T_t}\sigma^2(X_u)du}],$$

where in the second equality we used the SDE (2.2). Since the volatility $\sigma(x)$ remains bounded as $x \to \infty$, the process $\mathbf{1}_{\{t<H_0\}}e^{\int_0^t\sigma(X_u)dB_u-\frac{1}{2}\int_0^t\sigma^2(X_u)du}$ stopped at H_0 is an exponential martingale starting at one (and not just a local martingale; see, e.g., Delbaen and Shirakawa (2002)).

Now suppose $\mu \in \mathcal{I}_\nu$. Then, conditioning on the time change, we have:

$$x\mathbb{E}[e^{\mu T_t}\mathbb{E}[\mathbf{1}_{\{T_t<H_0\}}e^{\int_0^{T_t}\sigma(X_u)dB_u-\frac{1}{2}\int_0^{T_t}\sigma^2(X_u)du}|\mathcal{F}_t^T]] = x\mathbb{E}[e^{\mu T_t}] = xe^{-t\phi(-\mu)}.$$

Comparing the right hand side with that of Eq.(A.1), we conclude that Eq.(A.1) holds if and only if $\rho = r - q + \phi(-\mu)$. If $\mu \notin \mathcal{I}_\nu$, then $\mathbb{E}[e^{\mu T_t}]$ is infinite and (A.1) cannot be satisfied and, hence, the process (2.1) does not satisfy the martingale condition (2.5–6). □

A.2 Proof of Theorem 4.2

Define $f(\mu) := -\phi(-\mu)$. If $\gamma > 0$ or $\gamma = 0$ *and* the subordinator is of infinite activity ($\int_{(0,\infty)}\nu(ds) = \infty$), then $f(\mu)$ tends to $-\infty$ as $\mu \to -\infty$. If $\gamma = 0$ and the subordinator is of finite activity ($\int_{(0,\infty)}\nu(ds) = \alpha < \infty$), then $f(\mu)$ tends to $-\alpha$. If $\bar{\mu}$ is not included in $\mathcal{I}_\nu$, then $f(\mu)$ tends to $+\infty$ as $\mu \to \bar{\mu}$. If $\bar{\mu}$ is included in $\mathcal{I}_\nu$, then $f(\bar{\mu}) = \gamma\bar{\mu} + \int_{(0,\infty)}(e^{\bar{\mu}s} - 1)\nu(ds) < \infty$. We thus have the following alternatives for the existence of solutions of the equation $f(\mu) = r-q$. If $r < q$, then there is a unique solution μ_0 for all subordinators except for subordinators with zero drift and finite activity Lévy measure with Poisson intensity α such that $-\alpha > r - q$. If $r > q$, then there is a unique solution if either $\bar{\mu}$ is not included in $\mathcal{I}_\nu$ or $\bar{\mu}$ is included in $\mathcal{I}_\nu$ and $r - q \le f(\bar{\mu})$. Otherwise, if $\bar{\mu}$ is included in $\mathcal{I}_\nu$ and $r - q > f(\bar{\mu})$, Eq.(4.3) has no solution in $\mathcal{I}_\nu$. The statement for the case $r = q$ is immediate from the fact that $\phi(0) = 0$. □

A.3 Proof of Theorem 4.3

The idea of time changing a Markov process with a Lévy subordinator is originally due to Bochner (1948), (1956). The following fundamental theorem due to R.S. Phillips (1952) (see Sato (1999), Theorem 32.1, p.212) characterizes the time-changed transition semigroup and its infinitesimal generator.

Theorem A.1 (Phillip's Theorem; Sato (1999), p.212) *Let $\{T_t, t \ge 0\}$ be a subordinator with Lévy measure ν, drift γ, Laplace exponent $\phi(\lambda)$, and transition kernel $\pi_t(ds)$. Let $\{\mathcal{P}_t, t \ge 0\}$ be a strongly continuous contraction semigroup of linear operators in the Banach space* **B** *with* *infinitesimal generator $\mathcal{G}$. Define (the superscript ϕ refers to the subordinated quantities with the subordinator with the Laplace exponent ϕ):*

$$\mathcal{P}_t^\phi f = \int_{[0,\infty)} (\mathcal{P}_s f)\pi_t(ds), \quad f \in \mathbf{B}. \tag{A.2}$$

Then $\{\mathcal{P}_t^\phi, t \geq 0\}$ is a strongly continuous contraction semigroup of linear operators on $\mathbf{B}$. Denote its infinitesimal generator by $\mathcal{G}^\phi$. Then $\mathrm{Dom}(\mathcal{G}) \subset \mathrm{Dom}(\mathcal{G}^\phi)$, $\mathrm{Dom}(\mathcal{G})$ is a core of $\mathcal{G}^\phi$, and

$$\mathcal{G}^\phi f = \gamma \mathcal{G} f + \int_{(0,\infty)} (\mathcal{P}_s f - f)\nu(ds), \quad f \in \mathrm{Dom}(\mathcal{G}). \tag{A.3}$$

In our case the Banach space $\mathbf{B}$ is $C_0((0,\infty))$ (the space of continuous bounded functions on $(0,\infty)$ vanishing at infinity) and the semigroup $\{\mathcal{P}_t, t \geq 0\}$ is the Feller transition semigroup of the diffusion process X with lifetime ζ. Given our assumptions, the one-dimensional diffusion X always has a transition density $p(t; x, y)$ with respect to the Lebesgue measure, so that

$$\mathcal{P}_t f(x) = \mathbb{E}_x[\mathbf{1}_{\{t<\zeta\}} f(X_t)] = \int_{(0,\infty)} f(y) p(t; s, y) dy,$$

and, moreover, $p(t; x, y)$ is continuous in all its variables. Then from Eq.(A.2) we obtain the density (4.14) of the subordinate process $X^\phi = X_{T_t}$.

¿From Eq.(A.3) we can identify the infinitesimal generator of X^ϕ. (for mathematical references on the Lévy characteristics of subordinate Markov processes see Okura (2002, Theorem 2.1) and Chen and Song (2005), Section 2). For $f \in C_c^2((0,\infty))$ we have for the integral term in (A.3):

$$\int_{(0,\infty)} (\mathcal{P}_s f(x) - f(x))\nu(ds) = \int_{(0,\infty)} \left(\int_{(0,\infty)} p(s; x, y) f(y) dy - f(x) \right) \nu(ds)$$

$$= \int_{(0,\infty)} \left\{ \int_{(0,\infty)} p(s; x, y) \left[\left(f(y) - f(x) - \mathbf{1}_{\{|y-x|\leq 1\}}(y-x)\frac{df}{dx}(x) \right) \right.\right.$$

$$\left.\left. + f(x) + \mathbf{1}_{\{|y-x|\leq 1\}}(y-x)\frac{df}{dx}(x) \right] dy - f(x) \right\} \nu(ds)$$

$$= \int_{(0,\infty)} \left(f(y) - f(x) - \mathbf{1}_{\{|y-x|\leq 1\}}(y-x)\frac{df}{dx}(x) \right) \left(\int_{(0,\infty)} p(s; x, y)\nu(ds) \right) dy$$

$$- \left(\int_{(0,\infty)} \left(1 - \int_{(0,\infty)} p(s; x, y) dy \right) \nu(ds) \right) f(x)$$

$$+ \left(\int_{(0,\infty)} \left(\int_{\{y>0: |y-x|\leq 1\}} (y-x) p(s; x, y) dy \right) \nu(ds) \right) \frac{df}{dx}(x)$$

$$= \int_{(0,\infty)} \left(f(y) - f(x) - \mathbf{1}_{\{|y-x|\leq 1\}}(y-x)\frac{df}{dx}(x) \right) \Pi(x, dy)$$

$$- \left(\int_{(0,\infty)} P_s(x, \{\Delta\}) \nu(ds) \right) f(x)$$

$$+ \left(\int_{(0,\infty)} \left(\int_{\{y>0:|y-x|\leq 1\}} (y-x) p(s; x, y) dy \right) \nu(ds) \right) \frac{df}{dx}(x).$$

Substituting this result in (A.3), we arrive at Eqs.(4.5)-(4.9). □

A.4 Proof of Theorem 4.4

The proof is similar to the proof of Theorem 4.1. Since the process $(e^{-\rho t} S_t, Z_t) = (\mathbf{1}_{\{t<\tau_d\}} X_{T_t}, Z_t)$ is an $(n+1)$-dimensional time-homogeneous Markov process, it is enough to prove that Eq.(A.1) holds. Suppose $\mu \in \mathbb{R}$ is such that

$$\mathbb{E}[e^{\mu T_t}] = \mathcal{L}(t, -\mu) < \infty. \tag{A.4}$$

Proceeding as in the proof of Theorem 4.1 and conditioning on the time change, the left hand side of Eq.(A.1) reduces to:

$$x\mathbb{E}[e^{\mu T_t} \mathbb{E}[\mathbf{1}_{\{T_t < H_0\}} e^{\int_0^{T_t} \sigma(X_u) dB_u - \frac{1}{2} \int_0^{T_t} \sigma^2(X_u) du} | \mathcal{F}_t^T]] = x\mathbb{E}[e^{\mu T_t}] = x\mathcal{L}(t, -\mu).$$

We conclude that Eq.(A.1) holds if and only if

$$\mathcal{L}(t, -\mu) = e^{(r-q-\rho)t}. \tag{A.5}$$

However, for $\mu \neq 0$, the Laplace transform (A.4) is an exponential function of time if and only if the time change process has stationary and independent increments, i.e., is a Lévy subordinator. The only absolutely continuous time change that is a Lévy subordinator is a trivial time change with constant activity rate $V_t = \gamma$ so that $T_t = \gamma t$. Hence we conclude that Eq.(A.5) cannot hold for any $\mu \neq 0$ for any non-trivial absolutely continuous time change. For $\mu = 0$ we have that $\mathcal{L}(t, 0) = 1$ and Eq.(A.5) is satisfied if and only if $\rho = r - q$. □

A.5 Proof of Theorem 4.5

The proof is completely analogous to that of Theorem 4.4. Suppose that $\mu \in \mathcal{I}_\nu$ and such that

$$\mathbb{E}[e^{\mu T_t}] = \mathcal{L}(t, \phi(-\mu)) < \infty. \tag{A.6}$$

Then arguing as in the proof of Theorem 4.4 we arrive at the following necessary and sufficient condition for the process S to satisfy the martingale condition (2.5)–(2.6):

$$\mathcal{L}(t, \phi(-\mu)) = e^{(r-q-\rho)t}. \tag{A.7}$$

The only solution for a composite time change (3.9) with T^2 having a non-constant activity rate process V is $\mu = 0$ and $\rho = r - q$.□

A.6 Proof of Theorem 8.1

(i) Consider the Sturm-Liouville equation (6.7) with the operator (8.3) with $\mu+b>0$. Substitute $u(x)=x^{\frac{1}{2}-c+\beta}e^{-\frac{A}{2}x^{-2\beta}}v(y)$ with $y=Ax^{-2\beta}$, where A is defined in (8.4). The Sturm-Liouville equation for the function $u(x)$ reduces to the Whittaker equation for the function $v(y)$ (see Appendix B):

$$\frac{d^2v}{dy^2}(y)+\left(-\frac{1}{4}+\frac{\varkappa(s)}{y}+\frac{1-\nu^2}{4y^2}\right)v(y)=0, \tag{A.8}$$

with ν, $\varkappa(s)$, ξ, and ω as defined in (8.7). The increasing and decreasing solutions of the Whittaker equation are given by the Whittaker functions $v_1(y)=M_{\varkappa(s),\frac{\nu}{2}}(y)$ and $v_2(y)=W_{\varkappa(s),\frac{\nu}{2}}(y)$, respectively. Their Wronskian is given by

$$\mathfrak{W}\left(v_1,v_2\right)(y):=v_1(y)v_2'(y)-v_1'(y)v_2(y)=-\frac{\Gamma(1+\nu)}{\Gamma\left(\frac{1+\nu}{2}-\varkappa(s)\right)}.$$

Thus, the increasing and decreasing solutions of the original Sturm-Liouville equation are given by (8.5) and (8.6), and the Wronskian w_s with respect to the scale density is given by (8.8).

(ii) When $\mu+b=0$, the substitution $u(x)=x^{\frac{1}{2}-c}v(y)$ with $y=\frac{x^{-\beta}}{a|\beta|}$ reduces the Sturm-Liouville equation (6.7) with the operator (8.3) to the modified Bessel equation of order ν (with ν as in (8.7)):

$$y^2\frac{d^2v}{dy^2}(y)+y\frac{dv}{dy}(y)-\left(\nu^2+2\left(s+b\right)y^2\right)v(y)=0. \tag{A.9}$$

The increasing and decreasing solutions of the modified Bessel equation are given by the modified Bessel functions $v_1(y)=I_\nu\left(y\sqrt{2\left(s+b\right)}\right)$ and $v_2(y)=K_\nu\left(y\sqrt{2\left(s+b\right)}\right)$, respectively. Their Wronskian is given by

$$\mathfrak{W}\left(v_1,v_2\right)(y)=-\frac{1}{y}.$$

Thus, the increasing and decreasing solutions of the original Sturm-Liouville equation are given by (8.9), and the Wronskian w_s with respect to the scale density is given by (8.10). □

A.7 Proof of Theorem 8.2

(i) We present the proof of the spectral expansion by directly inverting the Laplace transform (6.11) for the transition density by applying the Cauchy Residue Theorem. When $\mu+b>0$, the Green's function $G_s(x,y)$ Eq.(6.10) is given by:

$$G_s(x,y)=\frac{\Gamma\left(\frac{1+\nu}{2}-\varkappa(s)\right)}{(\mu+b)\Gamma(1+\nu)}x^{\frac{1}{2}-c+\beta}y^{c-\frac{3}{2}-\beta}e^{-\frac{A}{2}\left(x^{-2\beta}-y^{-2\beta}\right)}$$

$$\times\begin{cases} M_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right)W_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right), & x\le y\\ W_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right)M_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right), & y\le x\end{cases}.$$

The only singularities of the Green's function are simple poles of the Gamma function $\Gamma(\nu/2+1/2-\varkappa(s))$ at $\nu/2+1/2-\varkappa(s)=-n+1$, $n=1,2,...$, i.e., at $s=-\lambda_n$ with $\lambda_n=\omega n+\xi$ for

$n = 1, 2, ...$ Applying the Cauchy Residue Theorem, in this case the Laplace inversion integral is equal to the sum of residues at the poles:

$$p(t;x,y) = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{st} G_s(x,y)ds = \sum_{n=1}^{\infty} \mathrm{Res}_{s=-\lambda_n}\left(e^{st}G_s(x,y)\right).$$

The residues are:

$$\mathrm{Res}_{s=-\lambda_n}\left(e^{st}G_s(x,y)\right) = \left.\frac{\omega(-1)^{n+1}e^{st}G_s(x,y)}{(n-1)!\Gamma\left(\frac{1+\nu}{2} - \varkappa(s)\right)}\right|_{s=-\omega n-\xi}.$$

Substituting this into the sum, we have:

$$p(t;x,y) = \sum_{n=1}^{\infty} e^{-(\omega n+\xi)t}\frac{\omega(-1)^{n-1}}{(n-1)!(\mu+b)\Gamma(1+\nu)}x^{\frac{1}{2}-c+\beta}y^{c-\frac{3}{2}-\beta}e^{-\frac{A}{2}\left(x^{-2\beta}-y^{-2\beta}\right)}$$

$$\times\begin{cases} M_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}\left(Ax^{-2\beta}\right)W_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}\left(Ay^{-2\beta}\right), & x\le y\\ W_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}\left(Ax^{-2\beta}\right)M_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}\left(Ay^{-2\beta}\right), & y\le x\end{cases}.$$

When $\varkappa = \frac{\nu}{2}+n-\frac{1}{2}$, $n = 1, 2, ...$, the Whittaker functions $M_{\varkappa,\frac{\nu}{2}}(x)$ and $W_{\varkappa,\frac{\nu}{2}}(x)$ become linearly dependent and reduce to the generalized Laguerre polynomials (see Buchholz (1969), p.214):

$$M_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}(x) = \frac{(n-1)!\Gamma(1+\nu)}{\Gamma(\nu+n)}e^{-x/2}x^{\frac{\nu+1}{2}}L_{n-1}^{(\nu)}(x),$$

$$W_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}(x) = (n-1)!(-1)^{n-1}e^{-x/2}x^{\frac{\nu+1}{2}}L_{n-1}^{(\nu)}(x).$$

Substituting this result in the sum we obtain the spectral representation of the transition probability density:

$$p(t;x,y) = \mathfrak{m}(y)\sum_{n=1}^{\infty} e^{-(\omega n+\xi)t}\frac{A^{\nu}(\mu+b)(n-1)!}{\Gamma(\nu+n)}xye^{-A\left(x^{-2\beta}+y^{-2\beta}\right)}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)L_{n-1}^{(\nu)}\left(Ax^{-2\beta}\right)$$

$$= \mathfrak{m}(y)\sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)$$

with the eigenvalues and eigenfunctions (8.11).

(ii) When $\mu + b = 0$, the Green's function $G_s(x,y)$ is given by:

$$G_s(x,y) = \frac{2}{a^2|\beta|}x^{\frac{1}{2}-c}y^{c-3/2-2\beta}\begin{cases} I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)K_\nu\left(\frac{y^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right), & x\le y\\ K_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)I_\nu\left(\frac{y^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right), & y\le x\end{cases}.$$

The only singularity of the Green's function is a branching point at $s = -b$. We place the branch cut along the negative real line from $-\infty$ to $-b$. We now use the Cauchy's Theorem to calculate the Laplace inversion integral. We consider a closed contour in Figure 4. Since

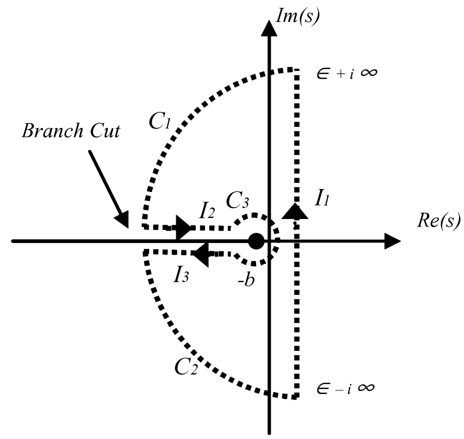

Figure 4: *The Bromwich Laplace inversion is over the vertical line I_1. C_1 and C_2 are the integration arcs at infinity, C_3 is the arc around the branching point. I_2 and I_3 are the lines of integration at each side of the branch cut.*

the function is analytic inside the contour, the integral along the closed contour vanishes. On the other hand, the integral is equal to the sum of the integral along the line parallel to the imaginary axes in the Bromwich Laplace inversion integral, the integrals along the two arcs at infinity, the integrals along each side of the branch cut, and the integral along the arc around the branching point $s = -b$. We now show that the integrals along the arcs at infinity and along the arc around the branching point vanish. We do this by considering the asymptotics of the Green's function.

Let $s = \rho e^{i\theta} - b$ and $ds = i\rho e^{i\theta} d\theta$. The asymptotic value of the Bessel functions' products in the Green's function vanish when either $\rho \to 0$ or when $\rho \to \infty$ and $\theta \in (\pi/2, \pi) \cup (-\pi, -\pi/2)$:

$$e^{t\rho e^{i\theta}} I_\nu\left(\frac{x^{-\beta}\sqrt{2\rho}e^{i\theta/2}}{a|\beta|}\right) K_\nu\left(\frac{y^{-\beta}\sqrt{2\rho}e^{i\theta/2}}{a|\beta|}\right)\rho e^{i\theta} \longrightarrow 0 \text{ as } \rho \to 0.$$

To show this, note that asymptotically for $\rho \to 0$ we have:

$$I_\nu\left(a\sqrt{\rho}\right) \approx \left(\frac{a}{2}\right)^\nu \frac{\rho^{\nu/2}}{\Gamma(\nu+1)} + \left(\frac{a}{2}\right)^{\nu+2} \frac{\rho^{\nu/2+1}}{\Gamma(\nu+2)} + \rho^{\nu/2} O\left(\rho^2\right) \text{ as } \rho \to 0,$$

$$K_\nu\left(b\sqrt{\rho}\right) \approx \frac{1}{2}\left(\frac{b}{2}\right)^\nu \Gamma(-\nu)\rho^{\nu/2} + \frac{1}{2}\left(\frac{b}{2}\right)^{\nu+2} \frac{\Gamma(-\nu)\rho^{\nu/2+1}}{1+\nu}$$
$$+\frac{1}{2}\left(\frac{b}{2}\right)^{-\nu}\Gamma(\nu)\rho^{-\nu/2} + \frac{1}{2}\left(\frac{b}{2}\right)^{2-\nu}\frac{\Gamma(\nu)\rho^{1-\nu/2}}{1-\nu} + \left(\rho^{\nu/2} + \rho^{-\nu/2}\right) O\left(\rho^2\right) \text{ as } \rho \to 0,$$

and, hence,

$$\rho I_\nu\left(a\sqrt{\rho}\right) K_\nu\left(b\sqrt{\rho}\right) \approx \left(\frac{a}{b}\right)^\nu \frac{\rho}{2\nu} + \frac{1}{2}\left(\frac{ab}{4}\right)^\nu \frac{\Gamma(-\nu)}{\Gamma(1+\nu)}\rho^{\nu+1} + (1+\rho^\nu) O\left(\rho^2\right) \text{ as } \rho \to 0.$$

Likewise we have:

$$e^{t\rho e^{i\theta}} I_\nu\left(\frac{x^{-\beta}\sqrt{2\rho}e^{i\theta/2}}{a|\beta|}\right) K_\nu\left(\frac{y^{-\beta}\sqrt{2\rho}e^{i\theta/2}}{a|\beta|}\right)\rho e^{i\theta} \longrightarrow 0 \text{ as } \rho \to \infty.$$

To show this, note that asymptotically as $\rho \to \infty$ we have:

$$I_\nu\left(a\sqrt{\rho}e^{\frac{\theta}{2}i}\right) \approx \frac{1}{\sqrt{2\pi}} e^{a\sqrt{\rho}e^{\frac{\theta}{2}i}} a^{-\frac{1}{2}}\rho^{-\frac{1}{4}}e^{-\frac{\theta}{4}i} \text{ as } \rho \to \infty,$$

$$K_\nu\left(b\sqrt{\rho}e^{\frac{\theta}{2}i}\right) \approx \frac{1}{\sqrt{2}}\sqrt{\pi}e^{-b\sqrt{\rho}e^{\frac{\theta}{2}i}} b^{-\frac{1}{2}}\rho^{-\frac{1}{4}}e^{-\frac{\theta}{4}i} \text{ as } \rho \to \infty,$$

and, hence,

$$e^{t\rho e^{i\theta}} I_\nu\left(a\sqrt{\rho}e^{\frac{\theta}{2}i}\right) K_\nu\left(b\sqrt{\rho}e^{\frac{\theta}{2}i}\right) \approx \frac{1}{2} e^{\sqrt{\rho}e^{\frac{\theta}{2}i}(a-b)+\rho e^{i\theta}t}(ab)^{-\frac{1}{2}}\rho^{-\frac{1}{2}}e^{-\frac{\theta}{2}i} \text{ as } \rho \to \infty.$$

According to Eq.(6.10) for the Green's function, the argument of I_ν is at most as large as the argument of K_ν (i.e., $a \le b$ for both cases $x < y$ and $x > y$), and since for $\theta \in (\pi/2, \pi) \cup (-\pi, -\pi/2)$ we have that $\Re\left(e^{i\theta}\right) = \cos(\theta) < 0$ and $\Re\left(e^{i\theta/2}(a-b)\right) = (a-b)\cos(\theta/2) \le 0$, the product vanishes as $\rho \to \infty$.

Thus, the integrals along the arcs at infinity and along the arc around the branching point vanish, and the Laplace inversion integral reduces to the integral of the jump across the branch cut (where we changed the integration variable according to $s = \left(\lambda e^{\pm\pi i} - b\right)$):

$$p(t;x,y) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{st}G_s(x,y)ds = \frac{1}{2\pi i}\int_0^\infty e^{-(\lambda+b)t}\left(G_{\lambda e^{-\pi i}-b}(x,y) - G_{\lambda e^{\pi i}-b}(x,y)\right)d\lambda.$$

Recall the following identities for Bessel functions:

$$I_\nu\left(e^{\pm\frac{\pi}{2}i}a\right) = e^{\pm\frac{\nu\pi}{2}}J_\nu(a),$$

$$K_\nu\left(e^{\pm\frac{\pi}{2}i}b\right) = \mp\frac{\pi i}{2}e^{\mp\frac{\nu\pi}{2}}\left(J_\nu(b) \mp iY_\nu(b)\right),$$

and

$$I_\nu\left(e^{-\frac{\pi}{2}i}a\right)K_\nu\left(e^{-\frac{\pi}{2}i}b\right) - I_\nu\left(e^{\frac{\pi}{2}i}a\right)K_\nu\left(e^{\frac{\pi}{2}i}b\right)$$
$$= \frac{\pi i}{2}J_\nu(a)\left(J_\nu(b) + iY_\nu(b)\right) + \frac{\pi i}{2}J_\nu(a)\left(J_\nu(b) - iY_\nu(b)\right) = \pi i J_\nu(a)J_\nu(b).$$

Substituting the expression for the Green's function into the integral across the branch cut and using this identity, we arrive at the spectral expansion for the transition density:

$$p(t;x,y) = \frac{y^{2c-2-2\beta}}{a^2|\beta|}(yx)^{\frac{1}{2}-c}\int_0^\infty e^{-(\lambda+b)t}J_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)d\lambda.$$

This completes the proof. □

A.8 Proof of Theorem 8.3

To compute the survival probability, we need to compute $\mathcal{P}_t f(x)$ for $f(x) = 1$ (note that the semigroup is non-conservative due to killing (default), and so $(\mathcal{P}_t 1)(x) \le 1$). Since in this case the constants are not square-integrable with the speed density, we cannot apply the spectral theory. Instead, we first compute the resolvent $(\mathcal{R}_s 1)(x)$ by integrating $f(x) = 1$ with the Green's function, and then invert the Laplace transform, as outlined in section 6.

(i) For $\mu + b > 0$ we have:

$$(\mathcal{R}_s 1)(x) = \frac{\Gamma\left(\frac{1+\nu}{2} - \varkappa(s)\right)}{(\mu+b)\Gamma(1+\nu)} x^{\frac{1}{2}-c+\beta} e^{-\frac{A}{2}x^{-2\beta}}$$

$$\times \left\{ M_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right) \int_x^\infty y^{c-\frac{3}{2}-\beta} e^{\frac{A}{2}y^{-2\beta}} W_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right) dy \right.$$

$$\left. + W_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right) \int_0^x y^{c-\frac{3}{2}-\beta} e^{\frac{A}{2}y^{-2\beta}} M_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right) dy \right\}.$$

Using the integrals (B.11) and (B.12) for the Whittaker functions, we calculate the integrals in closed form:

$$\int_x^\infty y^{c-\frac{3}{2}-\beta} e^{\frac{A}{2}y^{-2\beta}} W_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right) dy = \frac{A^{\frac{1-2c}{4|\beta|}-\frac{1}{2}}}{2|\beta|} \frac{\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(1+\frac{c}{|\beta|}\right)\Gamma\left(\frac{s+\xi}{\omega}-\frac{c}{|\beta|}\right)}{\Gamma\left(\frac{s+\xi}{\omega}+1\right)\Gamma\left(\frac{s+\xi}{\omega}+1-\nu\right)}$$

$$-\frac{A^{\frac{1-\nu}{2}} x^{-2\beta-1}\Gamma(\nu)}{(2|\beta|-1)\Gamma\left(\frac{s+\xi}{\omega}+1\right)} {}_2F_2\left(\begin{array}{cc} 1-\frac{1}{2|\beta|}, & 1+\frac{s+\xi}{\omega}-\nu \\ 2-\frac{1}{2|\beta|}, & 1-\nu \end{array}; Ax^{-2\beta}\right)$$

$$-\frac{A^{\frac{1+\nu}{2}} x^{2c-2\beta}\Gamma(-\nu)}{(2|\beta|+2c)\Gamma\left(\frac{s+\xi}{\omega}+1-\nu\right)} {}_2F_2\left(\begin{array}{cc} 1+\frac{c}{|\beta|}, & 1+\frac{s+\xi}{\omega} \\ 2+\frac{c}{|\beta|}, & 1+\nu \end{array}; Ax^{-2\beta}\right),$$

$$\int_x^\infty y^{c-\frac{3}{2}-\beta} e^{\frac{A}{2}y^{-2\beta}} M_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right) dy = \frac{A^{\frac{1+\nu}{2}} x^{2c-2\beta}}{(2|\beta|+2c)} {}_2F_2\left(\begin{array}{cc} 1+\frac{c}{|\beta|}, & 1+\frac{s+\xi}{\omega} \\ 2+\frac{c}{|\beta|}, & 1+\nu \end{array}; Ax^{-2\beta}\right).$$

Using the identity

$$\frac{\pi}{\sin(\pi\nu)} = -\Gamma(-\nu)\Gamma(1+\nu)$$

and Eq.(B.9), we obtain the resolvent $(\mathcal{R}_s 1)(x)$:

$$(\mathcal{R}_s 1)(x) = \frac{x^{\frac{1}{2}-c+\beta} e^{-\frac{A}{2}x^{-2\beta}}}{(\mu+b)}$$

$$\times \left\{ M_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right) \left[\frac{A^{\frac{1-2c}{4|\beta|}-\frac{1}{2}}}{2|\beta|} \frac{\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(1+\frac{c}{|\beta|}\right)\Gamma\left(\frac{s+\xi}{\omega}-\frac{c}{|\beta|}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{s+\xi}{\omega}+1-\nu\right)} \right. \right.$$

$$\left. -\frac{A^{\frac{1-\nu}{2}} x^{-2\beta-1}}{\nu(2|\beta|-1)} {}_2F_2\left(\begin{array}{cc} 1-\frac{1}{2|\beta|}, & 1+\frac{s+\xi}{\omega}-\nu \\ 2-\frac{1}{2|\beta|}, & 1-\nu \end{array}; Ax^{-2\beta}\right)\right]$$

$$\left. -M_{\varkappa(s),-\frac{\nu}{2}}\left(Ax^{-2\beta}\right) \frac{A^{\frac{1+\nu}{2}} x^{2c-2\beta}\Gamma(-\nu)}{\Gamma(1-\nu)(2|\beta|+2c)} {}_2F_2\left(\begin{array}{cc} 1+\frac{c}{|\beta|}, & 1+\frac{s+\xi}{\omega} \\ 2+\frac{c}{|\beta|}, & 1+\nu \end{array}; Ax^{-2\beta}\right)\right\}$$

Now we can proceed by inverting the Laplace transform Eq.(6.14) by means of the Cauchy Residue Theorem similar to the proof of Theorem 8.2.

The only singularities of the resolvent $(\mathcal{R}_s 1)(x)$ are simple poles of the Gamma function $\Gamma\left(\frac{s+\xi}{\omega}-\frac{c}{|\beta|}\right)$ in the first term at $\left(\frac{s+\xi}{\omega}-\frac{c}{|\beta|}\right) = -n$ for $n = 0, 1, 2, \ldots$, i.e., at $s = -(\omega n + b)$ for

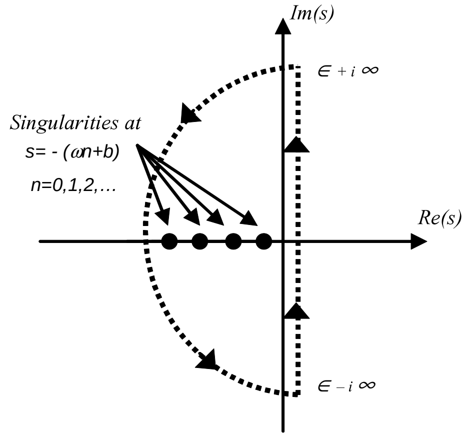

Figure 5: *Singularities enclosed by the integration contour are located at* $s = -(\omega n + b)$ *for* $n = 0, 1, 2...$

$n = 0, 1, 2, ...,$ this is shown in Figure 5. The last two terms have no singularities and thus do not contribute to the Laplace inversion integral. By applying the Cauchy Residue Theorem, we reduce the Laplace transform inversion integral for the survival probability to the sum over the residues at the poles:

$$\mathbb{Q}(\zeta > t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{st}(\mathcal{R}_s 1)(x)ds = \frac{\omega A^{\frac{1-2c}{4|\beta|}-\frac{1}{2}} x^{\frac{1}{2}-c+\beta} e^{-\frac{A}{2}x^{-2\beta}}}{2|\beta|(\mu+b)} \frac{\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(1+\frac{c}{|\beta|}\right)}{\Gamma\left(\nu+1\right)}$$

$$\times \sum_{n=0}^{\infty} \frac{e^{-(b+\omega n)t}\left(-1\right)^{-n}}{\Gamma\left(1-\frac{1}{2|\beta|}-n\right)n!} M_{n+\left(\frac{1-2(c+|\beta|)}{4|\beta|}\right),\frac{\nu}{2}}\left(Ax^{-2\beta}\right).$$

Finally, using the identity

$$(a)_n = (-1)^n \Gamma(1-a)/\Gamma(1-a-n)$$

and writing the Whittaker function in terms of the confluent hypergeometric function ${}_1F_1$ we obtain the explicit result (8.13) for the survival probability.

(ii) For $\mu + b = 0$ the resolvent is:

$$(\mathcal{R}_s 1)(x) = \frac{2}{a^2|\beta|} x^{\frac{1}{2}-c}\left\{K_\nu\left(\frac{x^{-\beta}\sqrt{2\left(s+b\right)}}{a|\beta|}\right)\int_0^x y^{c-3/2-2\beta} I_\nu\left(\frac{y^{-\beta}\sqrt{2\left(s+b\right)}}{a|\beta|}\right)dy\right.$$

$$\left. + I_\nu\left(\frac{x^{-\beta}\sqrt{2\left(s+b\right)}}{a|\beta|}\right)\int_x^\infty y^{c-3/2-2\beta} K_\nu\left(\frac{y^{-\beta}\sqrt{2\left(s+b\right)}}{a|\beta|}\right)dy\right\}.$$

Using the integrals (B.15) and (B.16), we calculate the integrals:

$$\int_x^\infty y^{c-3/2-2\beta} K_\nu\left(\frac{y^{-\beta}\sqrt{2\left(s+b\right)}}{a|\beta|}\right)dy = \left(\frac{a^2|\beta|}{2}\right)\frac{\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}}}{\left(s+b\right)^{1+\frac{2c-1}{4|\beta|}}}\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(\frac{c}{|\beta|}+1\right)$$

$$+\left(\sqrt{2}a|\beta|\right)^{\nu} x^{-2\beta-1}(s+b)^{-\nu/2}\frac{\Gamma(\nu)\Gamma\left(\frac{1}{2|\beta|}-1\right)}{4|\beta|\Gamma\left(\frac{1}{2|\beta|}\right)}\,{}_1F_2\left(\begin{array}{cc} 1-\frac{1}{2|\beta|} & \\ 1-\nu, & 2-\frac{1}{2|\beta|}\end{array};\frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right)$$

$$+\left(\sqrt{2}a|\beta|\right)^{-\nu} x^{2c-2\beta}(s+b)^{\nu/2}\frac{\Gamma(-\nu)\Gamma\left(-\frac{c}{|\beta|}-1\right)}{4|\beta|\Gamma\left(-\frac{c}{|\beta|}\right)}\,{}_1F_2\left(\begin{array}{cc} \frac{c}{|\beta|}+1 & \\ 1+\nu, & \frac{c}{|\beta|}+2\end{array};\frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right),$$

$$\int_x^\infty y^{c-3/2-2\beta} I_\nu\left(\frac{y^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)dy$$

$$=\frac{\left(\sqrt{2}a|\beta|\right)^{-\nu} x^{2c-2\beta}(s+b)^{\nu/2}}{2(c+|\beta|)\Gamma(\nu+1)}\,{}_1F_2\left(\begin{array}{cc} \frac{c}{|\beta|}+1 & \\ 1+\nu, & \frac{c}{|\beta|}+2\end{array};\frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right).$$

Using the identity

$$K_\nu(x)=\pi\left(I_{-\nu}(x)-I_\nu(x)\right)/(2\sin(\nu\pi))$$

together with $\pi/\sin(\pi\nu)=-\Gamma(-\nu)\Gamma(1+\nu)$ and $\Gamma(-(a+1))/\Gamma(-a)=-1/(a+1)$, we obtain the resolvent $(\mathcal{R}_s 1)(x)$:

$$(\mathcal{R}_s 1)(x)=\frac{2}{a^2|\beta|}x^{\frac{1}{2}-c}$$

$$\times\left\{-I_{-\nu}\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)\frac{\left(\sqrt{2}a|\beta|\right)^{-\nu} x^{2c-2\beta}(s+b)^{\nu/2}\Gamma(-\nu)}{4(c+|\beta|)}\,{}_1F_2\left(\begin{array}{cc} \frac{c}{|\beta|}+1 & \\ 1+\nu, & \frac{c}{|\beta|}+2\end{array};\frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right)\right.$$

$$+I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)\left[\left(\frac{a^2|\beta|}{2}\right)\frac{\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}}}{(s+b)^{1+\frac{2c-1}{4|\beta|}}}\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(\frac{c}{|\beta|}+1\right)+\right.$$

$$\left.\left.+\left(\sqrt{2}a|\beta|\right)^{\nu} x^{-2\beta-1}(s+b)^{-\nu/2}\frac{\Gamma(\nu)\Gamma\left(\frac{1}{2|\beta|}-1\right)}{4|\beta|\Gamma\left(\frac{1}{2|\beta|}\right)}\,{}_1F_2\left(\begin{array}{cc} 1-\frac{1}{2|\beta|} & \\ 1-\nu, & 2-\frac{1}{2|\beta|}\end{array};\frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right)\right]\right\}.$$

The resolvent $(\mathcal{R}_s 1)(x)$ has no poles except possibly at $s=-b$ when $2|\beta|>1$ due to the second term in the parenthesis:

$$\frac{I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)}{(s+b)^{1+\frac{2c-1}{4|\beta|}}}\approx\left(\left(\frac{\sqrt{2}x^{-\beta}}{2a|\beta|}\right)^{\nu}\frac{(s+b)^{\nu/2}}{\Gamma(\nu+1)}+\left(\frac{\sqrt{2}x^{-\beta}}{2a|\beta|}\right)^{\nu+2}\frac{(s+b)^{\nu/2+1}}{\Gamma(\nu+2)}\right)(s+b)^{-\left(1+\frac{2c-1}{4|\beta|}\right)}$$

$$=\left(\frac{\sqrt{2}x^{-\beta}}{2a|\beta|}\right)^{\nu}\frac{(s+b)^{\frac{1}{2|\beta|}-1}}{\Gamma(\nu+1)}+\left(\frac{\sqrt{2}x^{-\beta}}{2a|\beta|}\right)^{\nu+2}\frac{(s+b)^{\frac{1}{2|\beta|}}}{\Gamma(\nu+2)}.$$

However, this term does not contribute to the Laplace inversion since the corresponding residue vanishes:

$$\lim_{s\to-b}(s+b)\frac{I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)}{(s+b)^{1+\frac{2c-1}{4|\beta|}}}=0.$$

Thus, the only singularity of the resolvent is a branching point at $s = -b$. We place the branch cut along the negative real line from $-\infty$ to $-b$. Similar to part (ii) of the proof of Theorem 8.2, we use the Cauchy's Theorem to calculate the Laplace inversion integral. We consider a closed contour in Figure 4. Since the function is analytic inside the contour, the integral along the closed contour vanishes. On the other hand, the integral is equal to the sum of the integral along the line parallel to the imaginary axes in the Bromwich Laplace inversion integral, the integrals along the two arcs at infinity, the integrals along each side of the branch cut, and the integral along the arc around the branching point $s = -b$. Considering the asymptotics of the resolvent, it can be shown that the integrals along the arcs at infinity and along the arc around the branching point vanish. The analysis is similar to the one in the proof of Theorem 8.2(ii) and we omit it to save space.

Applying the Cauchy Theorem as in part (ii) of the proof of Theorem 8.2, the Laplace inversion integral reduces to the integral of the jump of the function across the branch cut (after a change of variable $s = \left(\lambda e^{\pm \pi i} - b\right)$):

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{st}(\mathcal{R}_s 1)(x)ds = \frac{1}{2\pi i}\int_0^\infty e^{-(\lambda+b)t}\left((\mathcal{R}_{\lambda e^{-\pi i}-b}1)(x) - (\mathcal{R}_{\lambda e^{\pi i}-b}1)(x)\right)d\lambda.$$

More explicitly, it is given by:

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{st}(\mathcal{R}_s 1)(x)ds = \frac{1}{a^2|\beta|\pi i}x^{\frac{1}{2}-c}$$

$$\times\Bigg\{-\frac{\left(\sqrt{2}a|\beta|\right)^{-\nu}x^{2c-2\beta}\Gamma(-\nu)}{4\left(c+|\beta|\right)}\int_0^\infty e^{-(\lambda+b)t}\lambda^{\nu/2}\,{}_1F_2\left(\begin{matrix}\frac{c}{|\beta|}+1\\ 1+\nu, \quad \frac{c}{|\beta|}+2\end{matrix};-\frac{x^{-2\beta}\lambda}{\left(\sqrt{2}a|\beta|\right)^2}\right)\times$$

$$\times\left[e^{-\frac{\nu\pi}{2}i}I_{-\nu}\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{-\frac{\pi}{2}i}}{a|\beta|}\right) - e^{\frac{\nu\pi}{2}i}I_{-\nu}\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{\frac{\pi}{2}i}}{a|\beta|}\right)\right]d\lambda$$

$$+\left(\frac{a^2|\beta|}{2}\right)\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}}\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(\frac{c}{|\beta|}+1\right)\int_0^\infty e^{-(\lambda+b)t}\lambda^{-\left(1+\frac{2c-1}{4|\beta|}\right)}$$

$$\times\left[e^{\left(1+\frac{2c-1}{4|\beta|}\right)\pi i}I_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{-\frac{\pi}{2}i}}{a|\beta|}\right) - e^{-\left(1+\frac{2c-1}{4|\beta|}\right)\pi i}I_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{\frac{\pi}{2}i}}{a|\beta|}\right)\right]d\lambda$$

$$+\left(\sqrt{2}a|\beta|\right)^{\nu}x^{-2\beta-1}\frac{\Gamma(\nu)\Gamma\left(\frac{1}{2|\beta|}-1\right)}{4|\beta|\Gamma\left(\frac{1}{2|\beta|}\right)}\int_0^\infty e^{-(\lambda+b)t}\lambda^{-\nu/2}\,{}_1F_2\left(\begin{matrix}1-\frac{1}{2|\beta|}\\ 1-\nu, \quad 2-\frac{1}{2|\beta|}\end{matrix};-\frac{x^{-2\beta}\lambda}{\left(\sqrt{2}a|\beta|\right)^2}\right)\times$$

$$\times\left[e^{\frac{\nu\pi}{2}i}I_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{-\frac{\pi}{2}i}}{a|\beta|}\right) - e^{-\frac{\nu\pi}{2}i}I_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{\frac{\pi}{2}i}}{a|\beta|}\right)\right]d\lambda\Bigg\}.$$

Using the property

$$I_\nu\left(e^{\pm\frac{\pi}{2}i}a\right) = e^{\pm i\frac{\nu\pi}{2}}J_\nu(a),$$

we can verify that the first and third integrals vanish, and the remaining integral takes the form:

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{st}(\mathcal{R}_s 1)(x)ds = x^{\frac{1}{2}-c}\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}}\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(\frac{c}{|\beta|}+1\right)$$

$$\times \int_0^\infty e^{-(\lambda+b)t}\lambda^{-\left(1+\frac{2c-1}{4|\beta|}\right)} J_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)\left[\frac{e^{-\left(\frac{1}{2|\beta|}-1\right)\pi i}-e^{\left(\frac{1}{2|\beta|}-1\right)\pi i}}{2\pi i}\right] d\lambda.$$

Finally, using the identity $\pi/\sin(\pi\nu) = -\Gamma(-\nu)\Gamma(1+\nu)$, we obtain:

$$\mathbb{Q}(\zeta > t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{st}(\mathcal{R}_s 1)(x)ds$$

$$= x^{\frac{1}{2}-c}\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}} \frac{\Gamma\left(\frac{c}{|\beta|}+1\right)}{\Gamma\left(\frac{1}{2|\beta|}\right)}\int_0^\infty e^{-(\lambda+b)t}\lambda^{-\left(1+\frac{2c-1}{4|\beta|}\right)} J_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right) d\lambda,$$

which completes the proof. □

A.9 Proof of Theorem 8.4

Since the payoff $f(x) = (k-x)^+$ is in the Hilbert space $L^2\left((0,\infty),\mathfrak{m}\right)$, we can apply the spectral expansion approach as described in section 6. (i) When $\mu + b > 0$, by Theorem 8.2(i) the spectrum is purely discrete with eigenvalues and eigenfunctions (8.11). We need to compute the eigenfunction expansion coefficients:

$$c_n = \int_0^k (k-y)\,\varphi_n(y)\mathfrak{m}(y)dy$$

$$= \frac{2A^{\frac{\nu}{2}}}{a^2}\sqrt{\frac{(n-1)!(\mu+b)}{\Gamma(\nu+n)}}\left(k\int_0^k y^{2c-2\beta-1}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)dy - \int_0^k y^{2c-2\beta}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)dy\right).$$

Using the integral (B.17), we obtain:

$$\int_0^k y^{2c-2\beta-1}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)dy = \frac{k^{2c-2\beta}\,(1+\nu)_{n-1}}{2\,(c+|\beta|)\,(n-1)!}\,{}_2F_2\left(\begin{array}{cc}1-n, & 1+\frac{c}{|\beta|}\\ 1+\nu, & 2+\frac{c}{|\beta|}\end{array};Ak^{-2\beta}\right),$$

and

$$\int_0^k y^{2c-2\beta}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)dy = \frac{k^{2c+1-2\beta}\,(1+\nu)_{n-1}}{(2c+1+2|\beta|)\,(n-1)!}\,{}_1F_1\left(1-n;2+\nu;Ak^{-2\beta}\right).$$

Finally, using the identities ${}_1F_1(a,b;z) = \frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)}L_{-a}^{b-1}(z)$ and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we obtain the expression (8.15).

(ii) For $\mu + b = 0$ we need to compute the integral:

$$c(\lambda) = \frac{1}{a^2|\beta|}\left(k\int_0^k y^{c-\frac{3}{2}-2\beta}J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)dy - \int_0^k y^{c-\frac{1}{2}-2\beta}J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)dy\right).$$

Using the integral (B.14), we obtain:

$$\int_0^k y^{c-\frac{3}{2}-2\beta}J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)dy = \frac{k^{2c-2\beta}}{2\,(c+|\beta|)\,\Gamma(1+\nu)}\left(\frac{\sqrt{\lambda}}{\sqrt{2}a|\beta|}\right)^\nu {}_1F_2\left(\begin{array}{cc}1+\frac{c}{|\beta|} & \\ 2+\frac{c}{|\beta|}, & 1+\nu\end{array};-\frac{k^{-2\beta}\lambda}{2\,(a|\beta|)^2}\right),$$

and

$$\int_0^k y^{c-\frac{1}{2}-2\beta} J_\nu \left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right) dy = \frac{k^{2c+1-2\beta}}{(2c+1+2|\beta|)\,\Gamma(1+\nu)} \left(\frac{\sqrt{\lambda}}{\sqrt{2}a|\beta|}\right)^\nu {}_0F_1\left(;2+\nu;-\frac{k^{-2\beta}\lambda}{2\,(a|\beta|)^2}\right).$$

Substituting these into the integral for $c(\lambda)$, we obtain:

$$c\,(\lambda) = \frac{k^{2c+1-2\beta}\lambda^{\nu/2}}{\left(\sqrt{2}a|\beta|\right)^{\nu+2}\Gamma(1+\nu)}$$

$$\times\left(\frac{|\beta|}{(c+|\beta|)}\,{}_1F_2\left(\begin{matrix}1+\frac{c}{|\beta|} & \\ 2+\frac{c}{|\beta|}, & 1+\nu\end{matrix};-\frac{k^{-2\beta}\lambda}{2\,(a|\beta|)^2}\right) - \frac{1}{(\nu+1)}\,{}_0F_1\left(;2+\nu;-\frac{k^{-2\beta}\lambda}{2\,(a|\beta|)^2}\right)\right).$$

Using the identity

$${}_0F_1\,(;b;z) = (-z)^{\frac{1-b}{2}}\Gamma(b)J_{b-1}\left(2\sqrt{-z}\right),$$

we finally obtain the explicit expression (8.17). □

B Special Functions

This Appendix collects some facts about special functions appearing in the solution of the time changed JDCEV model in this paper. The reader is referred to Abramowitz and Stegun (1972), Buchholz (1969), Gradshteyn and Ryzhik (1994), Prudnikov et al. (1990) and Slater (1960) for further details. All the special functions in this Appendix are available as built-in functions in *Mathematica* and *Maple* software packages. To compute these functions efficiently, these packages use a variety of integral and asymptotic representations given in the above references in addition to the defining hypergeometric series presented here.

B.1 Hypergeometric Functions

The generalized hypergeometric function is defined by the generalized hypergeometric series:

$${}_pF_q(a_1,...,a_p;b_1,...,b_q;z) \equiv {}_pF_q\left(\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix};x\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{z^n}{n!}, \qquad \text{(B.1)}$$

where $(a)_n = a(a+1)...(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol (and $\Gamma(z)$ is the Gamma function). The regularized function ${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)/(\Gamma(b_1)...\Gamma(b_q))$ is analytic for all values of $a_1,...,a_p,b_1,...,b_q$, and z real or complex. The most well-known special cases are the Gauss hypergeometric function ${}_2F_1(a_1,a_2;b;z)$ and the Kummer confluent hypergeometric function ${}_1F_1(a;b;z)$.

The second confluent hypergeometric function (Tricomi function) is defined in terms of the Kummer function:

$$U(a,b,z) = \frac{\pi}{\sin(\pi b)}\left\{\frac{{}_1F_1(a;b;z)}{\Gamma(1+a-b)\Gamma(b)} - \frac{z^{1-b}\,{}_1F_1(1+a-b;2-b;z)}{\Gamma(a)\Gamma(2-b)}\right\}. \qquad \text{(B.2)}$$

It is analytic for all values of a, b, and z real or complex even when b is zero or a negative integer, for in these cases it can be defined in the limit $b \to \pm n$ or 0. The confluent hypergeometric functions are solutions of the confluent hypergeometric equation:

$$z\frac{d^2u}{dz^2} + (b-z)\frac{du}{dz} - au = 0. \qquad \text{(B.3)}$$

B.2 Whittaker Functions

The Whittaker functions arise as solutions to the Whittaker equation:

$$\frac{d^2w}{dz^2}(z) + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) w(z) = 0. \tag{B.4}$$

They can be expressed in terms of the confluent hypergeometric functions:

$$M_{k,m}(z) = e^{-z/2} z^{m+1/2} \, {}_1F_1(1/2 + m - k; 2m + 1; z) \tag{B.5}$$

$$W_{k,m}(z) = e^{-z/2} z^{m+1/2} U(1/2 + m - k, 2m + 1, z) \tag{B.6}$$

In turn the confluent hypergeometric functions can be expressed in terms of the Whittaker functions:

$${}_1F_1(a; b; \pm z) = z^{-b/2} e^{\pm z/2} M_{b/2-a,(b-1)/2}(z) \tag{B.7}$$

$$U(a, b, z) = z^{-b/2} e^{z/2} W_{b/2-a,(b-1)/2}(z) \tag{B.8}$$

Due to (B.2), the Whittaker function W can be expressed in terms of the function M:

$$W_{k,m}(x) = \frac{\pi}{\sin(2m\pi)} \left[\frac{M_{k,-m}(x)}{\Gamma(1/2 + m - k)\Gamma(1 - 2m)} - \frac{M_{k,m}(x)}{\Gamma(1/2 - m - k)\Gamma(1 + 2m)}\right]. \tag{B.9}$$

For details on the Whittaker functions and their properties see Slater (1960) and Buchholz (1969).

The Whittaker function $M_{k,m}(x)$ satisfies the following multiplication identity (Slater (1960), Eq.(2.6.18), p.30):

$$M_{k,m}(xy) = e^{\frac{1}{2}x(y-1)} y^{-k} \sum_{n=0}^{\infty} \frac{(y-1)^n}{n! y^n} (1/2 + m + k)_n \, M_{k+n,m}(x) \tag{B.10}$$

Using the multiplication theorem and the connection between the Whittaker and Kummer functions (B.5), one can show that the eigenfunction expansion for the survival probability (8.13) collapses to the closed-form expression (5.14) obtained in Carr and Linetsky (2006).

B.3 Integrals with Special Functions

In this subsection we collect a number of integrals necessary for the proofs of Theorems 8.2–8.4.

B.3.1 Integrals with Whittaker functions.

Prudnikov et al. (1990), Eq.(1.13.1.1), p.39:

$$\int_0^x x^{\alpha-1} e^{\pm\frac{a}{2}x} M_{\rho,\sigma}(ax)\, dx = \frac{2a^{\sigma+\frac{1}{2}} x^{\alpha+\sigma+\frac{1}{2}}}{2\alpha + 2\sigma + 1} \, {}_2F_2\left(\begin{matrix} \alpha + \sigma + \frac{1}{2}, & \sigma + \frac{1}{2} \mp \rho \\ \alpha + \sigma + \frac{3}{2}, & 2\sigma + 1 \end{matrix}; \pm ax\right)$$

valid for $\Re(\alpha + \sigma + \frac{1}{2}) > 0$ and $x > 0$. Changing the integration variable $x = y^\delta$ and setting $\gamma := \alpha\delta - 1$, we obtain the following integral:

$$\int_0^y y^\gamma e^{\pm\frac{a}{2}y^\delta} M_{\rho,\sigma}\left(ay^\delta\right) dy \tag{B.11}$$

$$= \frac{2a^{\sigma+\frac{1}{2}} y^{(\gamma+1)+\delta\left(\sigma+\frac{1}{2}\right)}}{2(\gamma+1)+\delta(2\sigma+1)} \, {}_2F_2 \left(\begin{matrix} \frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}, & \sigma+\frac{1}{2}\mp\rho \\ \frac{\gamma+1}{\delta}+\sigma+\frac{3}{2}, & 2\sigma+1 \end{matrix} ; \pm a y^{\delta} \right)$$

valid for $\Re\left(\frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}\right)>0$, $\delta>0$ and $y>0$.

Prudnikov et al. (1990), Eq.(1.13.2.1), p.40:

$$\int_x^{\infty} x^{\alpha-1} e^{\pm\frac{a}{2}x} W_{\rho,\sigma}(ax)\,dx = -\frac{2a^{\sigma+\frac{1}{2}} x^{\alpha+\sigma+\frac{1}{2}}}{2\alpha+2\sigma+1} \frac{\Gamma(-2\sigma)}{\Gamma\left(\frac{1}{2}-\rho-\sigma\right)} \, {}_2F_2 \left(\begin{matrix} \alpha+\sigma+\frac{1}{2}, & \sigma+\frac{1}{2}\mp\rho \\ 2\sigma+1, & \alpha+\sigma+\frac{3}{2} \end{matrix} ; \pm ax \right)$$

$$-\frac{2a^{\frac{1}{2}-\sigma} x^{\alpha-\sigma+\frac{1}{2}}}{2\alpha-2\sigma+1} \frac{\Gamma(2\sigma)}{\Gamma\left(\frac{1}{2}-\rho+\sigma\right)} \, {}_2F_2 \left(\begin{matrix} \alpha-\sigma+\frac{1}{2}, & \frac{1}{2}-\sigma\mp\rho \\ 1-2\sigma, & \alpha-\sigma+\frac{3}{2} \end{matrix} ; \pm ax \right) + a^{-\alpha} A_{\pm},$$

where:

$$A_+ = \frac{\Gamma\left(\alpha+\sigma+\frac{1}{2}\right)\Gamma\left(\alpha-\sigma+\frac{1}{2}\right)\Gamma(-\alpha-\rho)}{\Gamma\left(\sigma+\frac{1}{2}-\rho\right)\Gamma\left(\frac{1}{2}-\rho-\sigma\right)}, \quad A_- = \frac{\Gamma\left(\alpha+\sigma+\frac{1}{2}\right)\Gamma\left(\alpha-\sigma+\frac{1}{2}\right)}{\Gamma(\alpha-\rho+1)}.$$

The integral is valid for $\Re(a)>0$, $\Re(\alpha+\rho)<0$, $x>0$, $|\arg(a)|<\frac{3\pi}{2}$. Changing the integration variable $x=y^{\delta}$ and setting $\gamma=\alpha\delta-1$, we obtain the following integral:

$$\int_y^{\infty} y^{\gamma} e^{\pm\frac{a}{2}y^{\delta}} W_{\rho,\sigma}\left(a y^{\delta}\right) dy \tag{B.12}$$

$$= -\frac{2a^{\sigma+\frac{1}{2}} y^{\gamma+1+\delta\left(\sigma+\frac{1}{2}\right)}}{2(\gamma+1)+\delta(2\sigma+1)} \times \frac{\Gamma(-2\sigma)}{\Gamma\left(\frac{1}{2}-\rho-\sigma\right)} \, {}_2F_2 \left(\begin{matrix} \frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}, & \sigma+\frac{1}{2}\mp\rho \\ 2\sigma+1, & \frac{\gamma+1}{\delta}+\sigma+\frac{3}{2} \end{matrix} ; \pm a y^{\delta} \right)$$

$$-\frac{2a^{\frac{1}{2}-\sigma} y^{\gamma+1-\delta\left(\sigma-\frac{1}{2}\right)}}{2(\gamma+1)-\delta(2\sigma-1)} \frac{\Gamma(2\sigma)}{\Gamma\left(\frac{1}{2}-\rho+\sigma\right)} \, {}_2F_2 \left(\begin{matrix} \frac{\gamma+1}{\delta}-\sigma+\frac{1}{2}, & \frac{1}{2}-\sigma\mp\rho \\ 1-2\sigma, & \frac{\gamma+1}{\delta}-\sigma+\frac{3}{2} \end{matrix} ; \pm a y^{\delta} \right) + \frac{a^{-\frac{\gamma+1}{\delta}}}{\delta} A_{\pm},$$

where:

$$A_+ = \frac{\Gamma\left(\frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}\right)\Gamma\left(\frac{\gamma+1}{\delta}-\sigma+\frac{1}{2}\right)\Gamma\left(-\frac{\gamma+1}{\delta}-\rho\right)}{\Gamma\left(\sigma+\frac{1}{2}-\rho\right)\Gamma\left(\frac{1}{2}-\rho-\sigma\right)}$$

$$A_- = \frac{\Gamma\left(\frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}\right)\Gamma\left(\frac{\gamma+1}{\delta}-\sigma+\frac{1}{2}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}-\rho+1\right)}.$$

The integral (B.12) is valid for $\Re(a)>0$, $\delta>0$, $\Re\left(\frac{\gamma+1}{\delta}+\rho\right)<0$, $y>0$, $|\arg(a)|<\frac{3\pi}{2}$.

B.3.2 Integrals with Bessel functions.

Gradshteyn and Ryzhik (2000), Eq.(6.643.1), p.701:

$$\int_0^{\infty} e^{-\alpha x} x^{\mu-1/2} J_{2\nu}\left(2\beta\sqrt{x}\right) dx = \frac{\Gamma(\mu+\nu+1/2)}{\beta\Gamma(2\nu+1)} e^{-\frac{\beta^2}{2\alpha}} \alpha^{-\mu} M_{\mu,\nu}\left(\frac{\beta^2}{\alpha}\right). \tag{B.13}$$

Prudnikov et al. (1988), Eq.(2.12.3.1), p.175 (set $\beta=1$):

$$\int_0^a x^{\alpha-1} J_{\nu}(cx)dx = \left(\frac{c}{2}\right)^{\nu} \frac{a^{\alpha+\nu}}{\Gamma(\nu+1)(\alpha+\nu)} \, {}_1F_2 \left(\begin{matrix} \frac{\alpha+\nu}{2}, & \\ \nu+1, & \frac{\alpha+\nu}{2}+1 \end{matrix} ; -\frac{a^2c^2}{4} \right)$$

valid for $a > 0,\ \Re(\alpha+\nu) > 0$. Introducing a new integration variable $x = y^\delta$ with $\delta > 0$ and setting $a = b^\delta$, we obtain the following integral:

$$\int_0^b y^\gamma J_\nu(cy^\delta)dy = \left(\frac{c}{2}\right)^\nu \frac{b^{\gamma+1+\delta\nu}}{\Gamma(\nu+1)(\gamma+1+\delta\nu)}\, {}_1F_2\left(\begin{matrix} \frac{\gamma+1+\delta\nu}{2\delta}, \\ \nu+1, \quad \frac{\gamma+1+\delta\nu}{2\delta}+1 \end{matrix}; -\frac{b^{2\delta}c^2}{4}\right) \quad \text{(B.14)}$$

valid for $b^\delta > 0,\ \Re(\gamma+1+\delta\nu) > 0$.

Prudnikov et al. (1988), Eq.(2.15.2.5), p.302 (set $\beta = 1$):

$$\int_0^a x^{\alpha-1} I_\nu(cx)dx = 2^{-\nu-1}a^{\alpha+\nu}c^\nu \frac{\Gamma\left(\frac{\alpha+\nu}{2}\right)}{\Gamma(\nu+1)\Gamma\left(1+\frac{\alpha+\nu}{2}\right)}\, {}_1F_2\left(\begin{matrix} \frac{\nu+\alpha}{2}, \\ 1+\nu, \quad 1+\frac{\nu+\alpha}{2} \end{matrix}; \frac{(ac)^2}{4}\right)$$

valid for $a > 0,\ \Re(\alpha+\nu) > 0$. Introducing a new integration variable $x = y^\delta$ with $\delta > 0$ and setting $a = b^\delta$, we obtain the following integral:

$$\int_0^b y^\gamma I_\nu(cy^\delta)dy = \frac{b^{\gamma+1+\delta\nu}c^\nu}{2^{\nu+1}\delta} \frac{\Gamma\left(\frac{\gamma+1+\delta\nu}{2\delta}\right)}{\Gamma(\nu+1)\Gamma\left(1+\frac{\gamma+1+\delta\nu}{2\delta}\right)}\, {}_1F_2\left(\begin{matrix} \frac{\gamma+1+\delta\nu}{2\delta}, \\ 1+\nu, \quad 1+\frac{\gamma+1+\delta\nu}{2\delta} \end{matrix}; \frac{c^2b^{2\delta}}{4}\right) \quad \text{(B.15)}$$

valid for $b^\delta > 0,\ \Re(\gamma+1+\delta\nu) > 0$.

Prudnikov et al. (1988), Eq.(2.16.3.7), p.345 (set $\beta = 1$):

$$\int_a^\infty x^{\alpha-1} K_\nu(cx)dx = 2^{\nu-2}a^{\alpha-\nu}c^{-\nu}\Gamma(\nu)\frac{\Gamma\left(\frac{\nu-\alpha}{2}\right)}{\Gamma\left(1+\frac{\nu-\alpha}{2}\right)}\, {}_1F_2\left(\begin{matrix} \frac{\alpha-\nu}{2}, \\ 1-\nu, \quad 1+\frac{\alpha-\nu}{2} \end{matrix}; \frac{(ac)^2}{4}\right)$$

$$+2^{-\nu-2}a^{\alpha+\nu}c^\nu\Gamma(-\nu)\frac{\Gamma\left(-\frac{\nu+\alpha}{2}\right)}{\Gamma\left(1-\frac{\nu+\alpha}{2}\right)}\, {}_1F_2\left(\begin{matrix} \frac{\nu+\alpha}{2}, \\ 1+\nu, \quad 1+\frac{\nu+\alpha}{2} \end{matrix}; \frac{(ac)^2}{4}\right)+2^{\alpha-2}c^{-\alpha}\Gamma\left(\frac{\nu+\alpha}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right)$$

valid for $a > 0,\ \Re(c) > 0$. Introducing a new integration variable $x = y^\delta$ with $\delta > 0$ and setting $a = b^\delta$, we obtain the following integral:

$$\int_b^\infty y^\gamma K_\nu(cy^\delta)dy \quad \text{(B.16)}$$

$$= \frac{1}{\delta}2^{\nu-2}b^{\gamma+1-\delta\nu}c^{-\nu}\Gamma(\nu)\frac{\Gamma\left(\frac{\delta\nu-(\gamma+1)}{2\delta}\right)}{\Gamma\left(1+\frac{\delta\nu-(\gamma+1)}{2\delta}\right)}\, {}_1F_2\left(\begin{matrix} \frac{(\gamma+1)-\delta\nu}{2\delta}, \\ 1-\nu, \quad 1+\frac{(\gamma+1)-\delta\nu}{2\delta} \end{matrix}; \frac{c^2b^{2\delta}}{4}\right)$$

$$+\frac{1}{\delta}2^{-\nu-2}b^{\gamma+1+\delta\nu}c^\nu\Gamma(-\nu)\frac{\Gamma\left(-\frac{\nu\delta+\gamma+1}{2\delta}\right)}{\Gamma\left(1-\frac{\nu\delta+\gamma+1}{2\delta}\right)}\, {}_1F_2\left(\begin{matrix} \frac{\nu\delta+\gamma+1}{2\delta}, \\ 1+\nu, \quad 1+\frac{\nu\delta+\gamma+1}{2\delta} \end{matrix}; \frac{c^2b^{2\delta}}{4}\right)$$

$$+\frac{1}{\delta}2^{\frac{\gamma+1}{\delta}-2}c^{-\frac{\gamma+1}{\delta}}\Gamma\left(\frac{\nu\delta+\gamma+1}{2\delta}\right)\Gamma\left(\frac{(\gamma+1)-\delta\nu}{2\delta}\right)$$

valid for $b^\delta > 0,\ \Re(c) > 0$.

B.3.3 Integrals with Generalized Laguerre Polynomials.

Prudnikov et al. (1988), Eq.(1.14.3.3), p.51:

$$\int_{x_1}^{x_2} x^{\lambda} L_n^{(\alpha)}(ax)dx = \pm \frac{(\alpha+1)_n}{n!(\lambda+1)} x^{\lambda+1} {}_2F_2\left(\begin{array}{cc} -n, & \lambda+1 \\ \alpha+1, & \lambda+2 \end{array} ; ax \right), \tag{B.17}$$

where $x_1 = 0$, $x_2 = x$ and $\Re(\lambda) > -1$ (plus sign) or $x_1 = x$, $x_2 = \infty$ and $\Re(\lambda) < -n-1$ (minus sign). Introducing a new integration variable $x = y^{\delta}$ with $\delta > 0$ and setting $x_1 = b_1^{\delta}$, $x_2 = b_2^{\delta}$, and $\gamma = (\lambda+1)\delta - 1$, we obtain:

$$\int_{b_1}^{b_2} y^{\gamma} L_n^{(\alpha)}(ay^{\delta})dy = \pm \frac{(\alpha+1)_n}{n!\,(\gamma+1)} b^{\gamma+1} {}_2F_2\left(\begin{array}{cc} -n, & \frac{\gamma+1}{\delta} \\ \alpha+1, & \frac{\gamma+1}{\delta}+1 \end{array} ; ab^{\delta} \right) \tag{B.18}$$

where $b_1 = 0$, $b_2 = b$ and $\Re\left(\frac{\gamma+1}{\delta}\right) > 0$ (plus sign in (B.17)) or $b_1 = b$, $b_2 = \infty$ and $\Re\left(\frac{\gamma+1}{\delta}\right) < -n$ (minus sign in (B.17)).

References

[1] Abramowitz, M., and I.A. Stegun, 1972, *Handbook of Mathematical Functions*, Dover, New York.

[2] Albanese, C., and A. Kuznetsov, 2004, "Unifying the Three Volatility Models," *RISK*, 17 (3), 94-98.

[3] Amrein, W.O., A.M. Hinz, and D.B. Pearson, Eds., 2005, *Sturm Liouville Theory*, Birkhauser, Basel.

[4] Applebaum, D., 2004, *Lévy Processes and Stochastic Calculus*, Cambridge University Press, Cambridge.

[5] Barndorff-Nielsen, O.E., 1998, "Process of Normal Inverse Gaussian Type," *Finance and Stochastics*, 2, 41-68.

[6] Barndorff-Nielsen, O.E., E. Nicolato, and N. Shepherd, 2002, "Some Recent Developments in Stochastic Volatility Modeling," *Quantitative Finance*, 2, 11-23.

[7] Bertoin, J., 1996, *Lévy Processes*, Cambridge University Press, Cambridge.

[8] Bertoin, J., 1999, *Subordinators: Examples and Applications*, Lecture Notes in Mathematics, 1727, Springer, Berlin.

[9] Bielecki, T., and M. Rutkowski, 2002, *Credit Risk: Modeling, Valuation and Hedging*, Springer, Berlin.

[10] Bochner, S., 1948, "Diffusion Equation and Stochastic Processes," *Proc. Nat. Acad. Sci. USA*, 35, 368-370.

[11] Bochner, S., 1955, *Harmonic Analysis and the Theory of Probability*, University of California Press.

[12] Borodin, A.N., and P. Salminen, 2002, *Handbook of Brownian Motion*, 2nd Ed., Birkhauser, Boston.

[13] Boyarchenko, S.I., and S.Z. Levendorskiyǐ, 2002, *Non-Gaussian Merton-Black-Scholes Theory*, World Scientific, Singapore.

[14] Boyarchenko, N., and S.Z. Levendorskiyǐ, 2007, "The eigenfunction expansion method in multi-factor quadratic term structure models," *Mathematical Finance*, 17 (4), 503-539.

[15] Buchholz, H., 1969, *The Confluent Hypergeometric Function*, Springer, Berlin.

[16] Carr, P., H. Geman, D.B. Madan, and M. Yor, 2002, "The Fine Structure of Asset Returns: An Empirical Investigation," *Journal of Business*, 75(2), 305-332.

[17] Carr, P., H. Geman, D.B. Madan, and M. Yor, 2003, "Stochastic Volatility for Levy Processes," *Mathematical Finance*, 13 (3), 345-382.

[18] Carr, P., and V. Linetsky, 2006, "A Jump-to-Default Extended Constant Elasticity of Variance Model: An Application of Bessel Processes," *Finance and Stochastics*, 10(3), 303-330.

[19] Carr, P., and L. Wu, 2004, "Time-Changed Levy Processes and Option Pricing," *Journal of Financial Economics*, 71 (1), 113-141.

[20] Carr, P., and L. Wu, 2006, "Stock Options and Credit Default Swaps: A Joint Framework for Valuation and Estimation," Working Paper.

[21] Chen, Z.Q., R. Song, 2005, "Two-sided Eigenvalue Estimates for Subordinate Processes in Domains," *Journal of Functional Analysis*, 226, 90-113.

[22] Chen, Z.Q., and R. Song, 2007, "Spectral Properties of Subordinate Processes in Domains," In *Stochastic Analysis and Partial Differential Equations*, 77-84. (Eds. G-Q Chen, E. Hsu and M. Pinsky.) AMS Contemp. Math. 429.

[23] Clark, P.K., 1973, "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices," *Econometrica*, Vol. 41, No. 1, 135-155.

[24] Cont, R., and P. Tankov, 2004, *Financial Modeling with Jump Processes*, Chapman & Hall/CRC, Florida.

[25] Cox, J.C., 1975, "Notes on Option Pricing I: Constant Elasticity of Variance Diffusions", Working Paper, Stanford University (reprinted in *Journal of Portfolio Management*, 1996, 22, 15-17).

[26] Cox, J.C., J.E. Ingersoll and S.A. Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385-407.

[27] Davydov, D., and V. Linetsky, 2001, "The Valuation and Hedging of Barrier and Lookback Options under the CEV Process," *Management Science*, 47, 949-965.

[28] Davydov, D., and V. Linetsky, 2003, "Pricing Options on Scalar Diffusions: An Eigenfunction Expansion Approach," *Operations Research*, 51, 185-209.

[29] Ding, X., K. Giesecke, and P. Tomecek, 2006, "Time-Changed Birth Processes and Multi-name Credit," working paper, Stanford University.

[30] Doetsch, G., 1974, *Introduction to the Theory and Application of the Laplace Transformation*, Springer-Verlag, Berlin.

[31] Duffie, D., 2001, *Dynamics Asset Pricing*, 3rd Ed., Princeton U. Press, Princeton.

[32] Duffie, D., D. Filipovic and W. Schachermayer, 2003, "Affine Processes and Applications in Finance," *Annals of Applied Probability*, Volume 13, (2003), pp. 984-1053.

[33] Duffie, D., J. Pan, and K. Singleton, 2000, "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 68, 1343-1376.

[34] Duffie, D., and K. Singleton, 2003, *Credit Risk*, Princeton University Press, Princeton, NJ.

[35] Eberlein, E., U. Keller, and K. Prause, 1998, "New Insights into Smile, Mispricing, and Value at Risk: The Hyperbolic Model," *The Journal of Business*, 71(3), 371-405.

[36] Ethier, S.N., and T.G. Kurtz, 1986, *Markov Processes: Characterization and Convergence*, New York, Wiley.

[37] Fukushima, M., Y. Oshima, and M. Takeda, 1994, *Dirichlet Forms and Symmetric Markov Processes*, W. de Gruyter, Berlin.

[38] Gatheral, J., 2006, *The Volatility Surface: A Practitioner's Guide*, Wiley Finance.

[39] Gradshteyn, I.S., and I.M. Ryzhik, 1994, *Tables of Integrals, Series and Products*, Academic Press, New York.

[40] Geman, H., D.B. Madan, and M. Yor, 2001, "Time Changes for Lévy Processes," *Mathematical Finance*, 11(1), 79-96.

[41] Hagan, P. S., Kumar, D., Lesniewski, A. S., and D.E. Woodward, 2002, "Managing Smile Risk," *WILMOTT Magazine*, September, 84-108.

[42] Heston, S.L., 1993, "A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6, 327-343.

[43] Ito, K., and H. McKean, 1974, *Diffusion Processes and their Sample Paths*, 2nd Printing, Springer, Berlin.

[44] Karlin, S., and H.M. Taylor, 1981, *A Second Course in Stochastic Processes*, Academic Press, San Diego.

[45] Kou, S.G., 2002, "A Jump-Diffusion Model for Option Pricing," *Management Science*, 48(8), 1086-1101.

[46] Kou, S.G., and H. Wang, 2004, "Option Pricing under a Double Exponential Jump Diffusion Model," *Management Science*, 50(9), 1178-1192.

[47] Lando, D., 2004, *Credit Risk Modeling*, Princeton University Press, Princeton, NJ.

[48] Leippold, M., and L. Wu, 2002, "Asset Pricing Under The Quadratic Class," *Journal of Financial and Quantitative Analysis*, 37(2), 271–295.

[49] Lewis, A., 1998, "Applications of Eigenfunction Expansions in Continuous-Time Finance," *Mathematical Finance*, 8, 349-383.

[50] Lewis, A., 2000, *Option Valuation under Stochastic Volatility*, Finance Press, CA.

[51] Linetsky, V., 2004a, "The Spectral Decomposition of the Option Value," *International Journal of Theoretical and Applied Finance*, 7 (3), 337-384.

[52] Linetsky, V., 2004b, "Lookback Options and Diffusion Hitting Times: A Spectral Expansion Approach," *Finance and Stochastics*, 8 (3), 373-398.

[53] Linetsky, V., 2004c, "Computing Hitting Time Densities for CIR and OU Diffusions: Applications to Mean-Reverting Models," *Journal of Computational Finance*, 7 (4), 1-22.

[54] Linetsky, V., 2006, "Pricing Equity Derivatives subject to Bankruptcy," *Mathematical Finance*, 16 (2), 255-282.

[55] Linetsky, V., 2007, "Spectral Methods in Derivatives Pricing, In *Handbook of Financial Engineering*, J.R. Birge and V. Linetsky, Editors, Volume 15 in the series *Handbooks in Operations Research and Management Science*, Elsevier, Amsterdam.

[56] Lipton, A., 2002, "The Volatility Smile Problem," *RISK*, February, 61-65.

[57] Lipton, A., and W. McGhee, 2002, "Universal Barriers," *RISK*, May, 81-85.

[58] Madan, D.B., P. Carr, and E. Chang, 1998, "The Variance Gamma Process and Option Pricing," *European Finance Review*, 2, 79-105.

[59] Madan, D.B., and F. Milne, 1991, "Option Pricing with V.G. Martingale Components," *Mathematical Finance*, 1(4), 39-55.

[60] Madan, D.B., and E. Seneta, 1990, "The Variance Gamma (V.G.) Model for Share Market Returns," *Journal of Business*, 63(4), 511-524

[61] Madan, D.B., and M. Yor, 2006, "Representing CGMY and Meixner Processes as Time Changed Brownian Motions," Working Paper.

[62] McKean, H., 1956, "Elementary Solutions for Certain Parabolic Partial Differential Equations," *Transactions of the American Mathematical Society*, 82, 519-548.

[63] Merton, R., 1976, "Option Pricing When Underlying Stock Returns Are Discontinuous," *Journal of Financial Economics*, 3, 125-144.

[64] Okura, H., 2002, "Recurrence and Transience Criteria for Subordinated Symmetric Markov Processes," *Forum Mathematicum*, 14, 121-146.

[65] Pazy, A., 1983, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, Vol. 44, Springer.

[66] Phillips, R.S., 1952, "On the Generation of Semigroups of Linear Operators," *Pacific J. of Math.*, 2, 343-369.

[67] Prudnikov, A.P., Yu.A. Brychkov, and O.I. Marichev, 1986, *Integrals and Series*, Vol. 2, Gordon and Breach, New York.

[68] Prudnikov, A.P., Yu.A. Brychkov, and O.I. Marichev, 1990, *Integrals and Series*, Vol. 3, Gordon and Breach, New York.

[69] Sato, K., 1999, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge.

[70] Schoutens, W., 2003, *Lévy Processes in Finance: Pricing Financial Derivatives*, John Wiley & Sons, West Sussex.

[71] Slater, L.J., 1960, *Confluent Hypergeometric Functions*, Cambridge University Press.

[72] Wong, E., 1964, "The Construction of A Class of Stationary Markoff Processes," In: Bellman, R. (Ed.), Sixteenth Symposium in Applied Mathematics – Stochastic Processes in Mathematical Physics and Engineering, American Mathematical Society, Providence, RI, pp. 264-276.