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by

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ABSTRACT

The Vasicek single factor model of portfolio credit loss is generalized to include credits with stochastic exposures (*EAD*s) and loss rates (*LGD*s). The model can accommodate any distribution and correlation assumptions for the *LGD*s and *EAD*s and will produce a closed-form expression for an asymptotic portfolio's conditional loss rate. Revolving exposures draw against committed lines of credit. Dependence among defaults, *EAD*s, and *LGD*s are modeled using a single common Gaussian factor. A closed-form expression for an asymptotic portfolio's inverse cumulative conditional loss rate is analyzed for alternative *EAD* and *LGD* assumptions. Positive correlation in individual credits' *EAD* and *LGD* realizations increases portfolio systematic risk, producing wider ranges and increased skewness in portfolio loss distributions.

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A Generalized Single Factor Model of Portfolio Credit Risk

I. INTRODUCTION

The Gaussian asymptotic single factor model of portfolio credit losses (ASFM), developed by Vasicek (1987), Finger (1999), Schönbucher (2001), Gordy (2003), and others, provides an approximation for the loss rate distribution for a credit portfolio in which the dependence among individual defaults is driven by a single common latent factor. The ASFM assumes that the unconditional probability of default on an individual credit (*PD*) is fixed and known. In addition, all obligors' exposures at default (*EAD*) and all loss rates in default (*LGD*) are assumed to be non-stochastic quantities. In a large portfolio of credits, idiosyncratic risk is fully diversified and the only source of risk is the uncertainty in the portfolio default rate that is driven by the common latent Gaussian factor.

The ASFM has been widely applied in the financial industry. It has been used to estimate economic capital allocations [e.g., Finger (1999), Schönbucher (2001), Gordy (2003), and others] and is the model that underlies the Basel II Advanced Internal Ratings-based approach for setting banks' minimum regulatory capital requirements. In addition to capital allocation, the ASFM has been adapted to estimate market price and potential loss distributions for tranches of portfolio credit products [e.g., Li (2000); Andersen, Sidenius, and Basu (2003); Gibson (2004); Gordy and Jones (2002); and others]. But despite the ASFM's widespread use, empirical findings suggest that the model omits important systematic factors that partly determine the characteristics of a portfolio's true underlying credit loss distribution.

Many studies find significant time variability among the realized *LGD*s for a given credit facility or ratings class. The stylized facts hold that default losses increase in periods when default rates are elevated. Studies by Frye (2000b), Hu and Perraudin (2002), Schuermann (2004), Araten, Jacobs, and Varshney (2004), Altman, Brady, Resti, and Sironi (2004), Hamilton, Varma, Ou, and Cantor (2004), Carey and Gordy (2004), Emery, Cantor, and Arnet (2004), and others show pronounced decreases in the recovery rates during periods with elevated default rates. These results suggest the existence of a systematic relationship between default frequencies and default recovery rates that is not captured in the Vasicek ASRM framework.

In addition to issues related to stochastic *LGD*, the ASFM is often used to estimate capital needs for portfolios of revolving credits even though the model is based on the assumption that individual credit *EAD*s are fixed. The available evidence, including studies by Allen and Saunders (2003), Asarnow and Marker (1995), Araten and Jacobs (2001), and Jiménez, Lopez, and Saurina (2006), suggests that obligors draw on committed lines of credit as their credit quality deteriorates. Analysis of creditors' draw rate behavior shows that *EAD*s on revolving exposures are positively correlated with default rates—a correlation suggesting that there is at least one common factor that simultaneously determines portfolio *EAD* and default rate realizations.

The ASFM assumption of fixed *LGD* and *EAD* excludes two important sources of systematic credit risk that are present in historical loss rate data. A number of existing models, including those by Frye (2000a), Pykhtin (2003), Tasche (2004), and Andersen and Sidenius (2005), have extended the Vasicek ASFM framework to include stochastic *LGD* rates. These extensions require complex numerical techniques or restrictive assumptions for the *LGD* distribution to produce tractable expressions for an asymptotic portfolio's loss rate distribution. No existing study extends the ASFM framework to include stochastic *EAD* and *LGD* and produce a closed-form expression for a portfolio loss rate distribution.

In the remainder of this paper, the Gaussian ASFM is extended to incorporate obligors with *EAD*s and *LGD*s that are correlated random variables. In this extension, default is a random event driven by a compound latent factor as in the standard ASFM. Two additional compound latent factors are introduced to drive correlations among individual credits' *EAD*s and *LGD*s. These Gaussian factors are incorporated into *EAD* and *LGD* distributions using the inverse integral transformation. A closed-form expression for the inverse of the portfolio's conditional credit loss distribution is derived for cases when *LGD* and *EAD* probability density functions are continuous. When these distributions are discrete, the integral transformation is inadmissible, and an alternative approach must be used to derive the asymptotic portfolio's loss distribution.

When *LGD* and *EAD* have discrete distributions, a closed-form expression for the inverse of the portfolio's conditional loss rate distribution is constructed using a step function to approximate the underlying *LGD* and *EAD* distributions. The characteristics of the *LGD* and *EAD* distributions that can be modeled using this approach are unrestricted. The approximation error can be made arbitrarily small at the cost of increasing the number of terms in the approximation formula. The step-function approach can also be used to approximate the inverse of a portfolio's conditional loss rate distribution for cases when *LGD* and *EAD* distributions are continuous; and, again, the approximation error can be made arbitrarily small by increasing the number of steps in the approximation algorithm. Using the step-function representation, the calculations needed to approximate the portfolio loss rate distribution are straightforward and can easily be programmed in a financial spreadsheet.

Selected examples of portfolio *EAD*, *LGD*, and overall loss rate distributions are calculated using alternative *EAD* and *LGD* distribution and correlation assumptions that are consistent with stylized representations of corporate and retail portfolios.

2. The Gaussian ASFM Model

The Vasicek single common factor model of portfolio credit risk assumes that uncertainty on credit i is driven by a latent unobserved factor, $\tilde{V}_i$, with the following properties:

$$\begin{aligned} &\tilde{V}_i = \sqrt{\rho_V}\,\tilde{e}_M + \sqrt{1-\rho_V}\,\tilde{e}_{id} \\ &\tilde{e}_M \sim \phi(e_M) \\ &e_{id} \sim \phi(e_{id}), \\ &E(\tilde{e}_{id}\,\tilde{e}_{jd}) = E(\tilde{e}_M\,\tilde{e}_{jd}) = 0, \;\; \forall i, j. \end{aligned} \tag{1}$$

$\phi(\cdot)$ represents the standard normal density function. As expression (1) indicates, $\tilde{V}_i$ is distributed standard normal, $E(\tilde{V}_i) = 0,$ and $\sigma^2(\tilde{V}_i) = E(\tilde{V}_i^2) - E(\tilde{V}_i)^2 = 1.$ $\tilde{e}_M$ is a common factor, and the correlation between individual credits' latent factors is $\rho_V = \dfrac{Cov(\tilde{V}_i, \tilde{V}_j)}{\sigma(\tilde{V}_i)\sigma(\tilde{V}_j)}.$

$\tilde{V}_i$ is often interpreted as a proxy for the market value of the firm that issued credit i.

Credit i is assumed to default when its latent factor takes on a value less than a credit-specific threshold, $\tilde{V}_i < D_i$. The unconditional probability that credit i defaults is $PD = \Phi(D_i),$ where $\Phi(\cdot)$ represents the cumulative standard normal density function. Time is not an independent factor in the ASFM but is implicitly recognized through the calibration of input values for *PD*.

3. A SINGLE FACTOR MODEL OF LOSSES ON A PORTFOLIO OF REVOLVING CREDITS WITH CORRELATED EXPOSURES AND LOSS RATES

3.1 A Model of Stochastic EAD

Assume that a generic revolving credit account, i, has a maximum line of credit, M_i, upon which it may draw. The account utilization rate $\tilde{X}_i \in [0,1]$ is a random variable that determines the end-of-period account exposure, $\tilde{X}_i M_i$. Basel II conventions require that *EAD* be at least as large as initial exposure, and so account-level *EAD* is modeled as an initial outstanding exposure and a random draw rate $\tilde{\delta}_i$ on its remaining line of credit.

Assume that an individual account begins the period with a drawn exposure $d_{i0} M_i$, where d_{i0} is the initial share of the account line of credit that is used. The line of credit that can be drawn by the creditor over the subsequent period is $(1 - d_{i0}) M_i$. Let $\tilde{\delta}_i \in [0,1]$ represent the share of the remaining line of credit that is borrowed over the period, and let $\Omega(\tilde{\delta}_i)$ represent the cumulative density function for $\tilde{\delta}_i$. This representation accommodates the Basel II convention that requires that the exposure at the end of the one-year horizon be at least as large as the initial level of extended credit, $d_{i0} M_i$. The model can be generalized to recognize creditors' ability to reduce or eliminate their outstanding balances by setting $d_{i0} = 0$ and directly modeling an account's end-of-period utilization rate $\tilde{X}_i \in [0,1]$ instead of modeling an account's draw rate $\tilde{\delta}_i$. Under the draw rate specification, the account's end-of-period exposure is

$$M_i \tilde{X}_i = M_i \left(d_{i0} + (1 - d_{i0}) \tilde{\delta}_i \right), \quad \tilde{d}_i \sim \Omega(\delta_i), \quad \delta_i \in [0,1]. \tag{2}$$

Systematic dependence among individual accounts' draw rates is incorporated by assuming that account draw rates are driven by a latent Gaussian factor, $\tilde{Z}_i$, with the following properties:

$$\begin{aligned} &\tilde{Z}_i = \sqrt{\rho_Z}\,\tilde{e}_M + \sqrt{1-\rho_Z}\,\tilde{e}_{iZ} \\ &\tilde{e}_M \sim \phi(e_M) \\ &e_{iZ} \sim \phi(e_{iZ}), \\ &E(\tilde{e}_{iZ}\,\tilde{e}_{jZ}) = E(\tilde{e}_M\,\tilde{e}_{jZ}) = E(\tilde{e}_{iZ}\,\tilde{e}_{jd}) = 0 \ \forall\, i, j. \end{aligned} \tag{3}$$

The correlation between the latent variables that determines each account's draw rate is $\rho_Z = \frac{Cov(\tilde{Z}_i, \tilde{Z}_j)}{\sigma(\tilde{Z}_i)\sigma(\tilde{Z}_j)}$, and the correlation between the latent factors that drive account exposures and defaults is $\sqrt{\rho_Z \rho_V} = \frac{Cov(\tilde{V}_i, \tilde{Z}_j)}{\sigma(\tilde{V}_i)\sigma(\tilde{Z}_j)}$. To induce a positive correlation between a portfolio's default rate and its draw rate, we adopt the normalization convention that higher account draw rates are associated with smaller realizations of the latent variable, $\tilde{Z}_i$.

For any random variable $\tilde{s}$ with continuous density function, $f(\tilde{s})$, the probability integral transformation requires that the random variable $\tilde{S}$ be distributed uniformly over the interval $[0,1]$, when the random variable $\tilde{S}$ is defined by the integral transformation, $S_i = \int_{-\infty}^{s_i} f(s)ds$. Using this transformation, we introduce correlation structure into the realizations of the draw rate process by equating the probability integral transformations for the physical draw rate $\tilde{\delta}_i$ and the latent variable, $\tilde{Z}_i$, $\Omega(\delta_i) = 1 - \Phi(Z_i)$. This transformation implies a one-to-one mapping between $\tilde{Z}_i$ and $\tilde{\delta}_i$,

$$\tilde{\delta}_i = \Omega^{-1}\left(1 - \Phi\left(\tilde{Z}_i\right)\right). \tag{4}$$

3.2 A Model of Stochastic LGD

Let $\tilde{\lambda}_i \in [0,1]$ represent the loss rate that that will be experienced on credit *i*'s outstanding balance should the borrower default. Let $\Theta\left(\tilde{\lambda}_i\right)$ represent the cumulative density function for $\tilde{\lambda}_i$. Systematic dependence among individual credits' loss rates is introduced by assuming that $\tilde{\lambda}_i$ is driven by a latent Gaussian factor, $\tilde{Y}_i$, with the following properties:

$$\begin{aligned}
&\tilde{Y}_i = \sqrt{\rho_Y}\,\tilde{e}_M + \sqrt{1-\rho_Y}\,\tilde{e}_{iY} \\
&\tilde{e}_M \sim \phi(e_M) \\
&e_{iY} \sim \phi(e_{iY}), \\
&E(\tilde{e}_{iY}\,\tilde{e}_{jY}) = E(\tilde{e}_M\,\tilde{e}_{jY}) = E(\tilde{e}_{iY}\,\tilde{e}_{jZ}) = E(\tilde{e}_{iY}\,\tilde{e}_{id}) = 0 \quad \forall\, i, j.
\end{aligned} \tag{5}$$

To induce positive correlation between a portfolio's default rate and its loss rate given default, we adopt the normalization convention that higher account draw rates are associated with smaller realizations of the latent variable, $\tilde{Z}_i$. The correlation between the latent factors that determine default and loss given default is $\sqrt{\rho_V\,\rho_Y} > 0,$ and the correlation between the Gaussian drivers of default and exposure at default is $\sqrt{\rho_V\,\rho_Z} > 0.$ Using the inverse integral transformation to introduce a correlation structure, the mapping between $\tilde{\lambda}_i$ and $\tilde{Y}_i$ is given by

$$\tilde{\lambda}_i = \Theta^{-1}\left(1 - \Phi\left(\tilde{Y}_i\right)\right). \tag{6}$$

3.3 The Loss Rate for an Individual Account

Define an indicator function over the latent variable $\widetilde{V}_i$ that indicates default status

$$1_{D_i}\left(\widetilde{V}_i\right) = \begin{cases} 1 & if \quad \widetilde{V}_i < D_i \\ 0 & otherwise \end{cases}. \tag{7}$$

The indicator function, $1_{D_i}\left(\widetilde{V}_i\right)$, defines a random variable that is distributed binomially with an expected value of $\Phi\left(D_i\right)$.

Let $\Lambda_i\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i\right)$ represent the loss rate for account *i* measured relative to the account's maximum credit limit, M_i. $\Lambda_i\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i\right)$ is defined as

$$\begin{aligned} \Lambda_i\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i\right) &= 1_{D_i}\left(\widetilde{V}_i\right)\left(d_{i0} + \left(1 - d_{i0}\right)\widetilde{\delta}_i\right)\widetilde{\lambda}_i \\ &= 1_{D_i}\left(\widetilde{V}_i\right)\left(d_{i0} + \left(1 - d_{i0}\right)\, \Omega^{-1}\left(1 - \Phi\left(\widetilde{Z}_i\right)\right)\right) \Theta^{-1}\left(1 - \Phi\left(\widetilde{Y}_i\right)\right). \end{aligned} \tag{8}$$

3.4 The Conditional Loss Rate for an Individual Account

Let $1_{D_i}\left(\widetilde{V}_i \mid e_M\right)$ represent the value of the default indicator function conditional on a realized value for e_M, the common latent factor. The conditional expected value of the indicator function is

$$E\left(1_{D_i}\left(\widetilde{V}_i \mid e_M\right)\right) = \Phi\left(\frac{D_i - \sqrt{\rho_V}\; e_M}{\sqrt{1 - \rho_V}}\right). \tag{9}$$

Similarly, let $\widetilde{Z}_i \mid e_M$ and $\widetilde{Y}_i \mid e_M$ represent, respectively, the values of the random draw rate and *LGD* latent variables conditional on a realized value for e_M. These conditional random variables are normally distributed with means, $E\left(\widetilde{Z}_i \mid e_M\right) = \sqrt{\rho_Z} e_M$ and $E\left(\widetilde{Y}_i \mid e_M\right) = \sqrt{\rho_Y} e_M$, and variances, $\sigma^2\left(\widetilde{Z}_i \mid e_M\right) = 1 - \rho_Z$, and $\sigma^2\left(\widetilde{Y}_i \mid e_M\right) = 1 - \rho_Y$.

The cumulative probability $\tilde{Z}_i \le Z_i$ conditional on $\tilde{e}_M = e_M$ is identical to the cumulative probability $\tilde{e}_i \le \frac{Z_i - \sqrt{\rho_Z} e_M}{\sqrt{1-\rho_Z}}$, or, $\Phi\left(\frac{Z_i - \sqrt{\rho_Z} e_M}{\sqrt{1-\rho_Z}}\right)$. A similar relationship defines the cumulative conditional probability distribution for $\tilde{Y}_i$. The cumulative conditional distribution functions are

$$\tilde{Z}_i \mid e_M \sim \Phi\left(\tilde{Z}_i = Z_i \mid \tilde{e}_M = e_M\right) = \Phi\left(\frac{Z_i - \sqrt{\rho_Z} e_M}{\sqrt{1-\rho_Z}}\right) \quad \forall \quad Z_i \in (-\infty, \infty) \text{ and} \tag{10}$$

$$\tilde{Y}_i \mid e_M \sim \Phi\left(\tilde{Y}_i = Y_i \mid \tilde{e}_M = e_M\right) = \Phi\left(\frac{Y_i - \sqrt{\rho_Z} e_M}{\sqrt{1-\rho_Z}}\right) \quad \forall \quad Y_i \in (-\infty, \infty)\ . \tag{11}$$

An individual account's loss rate, conditional on a value for e_M, is

$$\Lambda_i\left(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i \mid e_M\right) = 1_{D_i}\left(\tilde{V}_i \mid e_M\right)\left(d_{i0} + (1-d_{i0})\Omega^{-1}\left(1-\Phi\left(\tilde{Z}_i \mid e_M\right)\right)\right)\Theta^{-1}\left(1-\Phi\left(\tilde{Y}_i \mid e_{M\,i}\right)\right). \tag{12}$$

3.5 The Loss Rate on an Asymptotic Portfolio of Revolving Credits

Consider a portfolio with *N* accounts that have identical credit limits, $M_i = M$, identical initial drawn balances, $d_{i0} M_i = d_0 M$, identical latent factor correlations, $\{\rho_V, \rho_X, \rho_Y\}$, and identical default thresholds, $D_i = D$. Assume that all credits' end-of-period draw rates, $\tilde{\delta}_i$, and loss rates given default, $\tilde{\lambda}_i$, are, respectively, taken from unconditional distributions that are identical across credits (the distributions for $\tilde{\delta}_i$ and $\tilde{\lambda}_i$ generally differ). Under these assumptions, $1_{D_i}\left(\tilde{V}_i\right) = 1_D\left(\tilde{V}_i\right)$ for all i, and the loss rate for an individual credit depends on the identity of the credit only through the idiosyncratic risk factors in the latent variables $\widetilde{V}_i, \widetilde{Z}_i$, and $\widetilde{Y}_i$. As a consequence, the subscript on the loss rate can be eliminated, $\Lambda_i\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i\right) = \Lambda\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i\right)$.

Let $\vec{\widetilde{V}} = \left(\widetilde{V}_1, \widetilde{V}_2, \ldots, \widetilde{V}_N\right)$ represent the vector of N latent variables that determine account defaults. Define $\vec{\widetilde{Y}}$ and $\vec{\widetilde{Z}}$ analogously. Let $\Lambda_p\left(\vec{\widetilde{V}}, \vec{\widetilde{Z}}, \vec{\widetilde{Y}} \mid e_M\right)$ represent the loss rate on the portfolio of N accounts conditional on a realization of e_M,

$$\Lambda_p\left(\vec{\widetilde{V}}, \vec{\widetilde{Z}}, \vec{\widetilde{Y}} \mid e_M\right) = \left(\frac{\sum_{i=1}^{N}\left(M \cdot \Lambda\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i \mid e_M\right)\right)}{N \cdot M}\right) = \left(\frac{\sum_{i=1}^{N}\left(\Lambda\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i \mid e_M\right)\right)}{N}\right). \tag{13}$$

Recall that $\Lambda\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i \mid e_M\right)$ is independent of $\Lambda\left(\widetilde{V}_j, \widetilde{Z}_j, \widetilde{Y}_j \mid e_M\right)$ for all $i \neq j$ and the conditional loss rates for individual credits are identically distributed. Thus, the Strong Law of Large Numbers requires, for any admissible value of e_M,

$$\lim_{N\to\infty} \Lambda_p\left(\vec{\widetilde{V}}, \vec{\widetilde{Z}}, \vec{\widetilde{Y}} \mid e_M\right) = \lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N}\Lambda\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i \mid e_M\right)}{N}\right) \xrightarrow[a.s.]{} E\left(\Lambda\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i \mid e_M\right)\right), \tag{14}$$

where $a.s.$ indicates convergence with probability one. Independence among the conditional indicator functions implies

$$\begin{aligned}
&\lim_{N\to\infty} \Lambda_p\left(\vec{\widetilde{V}}, \vec{\widetilde{Z}}, \vec{\widetilde{Y}} \mid e_M\right) \xrightarrow[a.s]{} \\
&E\left(1_D\left(\widetilde{V}_i \mid e_M\right)\right)\left[d_0 + (1-d_0)E\left(\Omega^{-1}\left(1-\Phi\left(\widetilde{Z}_i \mid e_M\right)\right)\right) \quad E\left(\Theta^{-1}\left(1-\Phi\left(\widetilde{Y}_i \mid e_M\right)\right)\right)\right] \\
&= \Phi\left(\frac{D-\sqrt{\rho_V}\,e_M}{\sqrt{1-\rho_V}}\right)\left[d_0 + (1-d_0)E\left(\Omega^{-1}\left(1-\Phi\left(\widetilde{Z}_i \mid e_M\right)\right)\right)\right] \quad E\left(\Theta^{-1}\left(1-\Phi\left(\widetilde{Y}_i \mid e_M\right)\right)\right)
\end{aligned} \quad . \tag{15}$$

Expression (15) is the inverse of the conditional distribution function for an asymptotic portfolio's loss rate evaluated at $e_M \in (-\infty,+\infty)$. The only random factor driving the unconditional portfolio loss rate distribution is the common latent factor, $\tilde{e}_M$. As a consequence, an asymptotic portfolio's loss rate, $\tilde{\Lambda}_p$, has a density function defined by the implicit function,

$$\tilde{\Lambda}_p \sim \left\{ \Phi\left(\frac{D - \sqrt{\rho_V}\; e_M}{\sqrt{1-\rho_V}} \right) \cdot \left[d_0 + (1-d_0)\; E\left(\Omega^{-1}\left(1-\Phi\left(\tilde{Z}_i \mid e_M\right)\right)\right)\right] E\left(\Theta^{-1}\left(1-\Phi\left(\tilde{Y}_i \mid e_M\right)\right)\right), \phi(e_M) \right\},$$

$$\text{for } e_M \in (-\infty,\infty). \tag{16}$$

3.6 Calculation of the Critical Values of a Portfolio's Loss Rate Distribution

Many risk management functions require estimates for portfolio loss rates that are associated with a particular cumulative probability threshold. Consider, for example, the portfolio loss rate that exceeds a proportion, α, of all potential portfolio credit losses (or alternatively, a loss rate exceeded by at most $1-\alpha$ of all potential portfolio losses). Because the portfolio loss rate function is decreasing in e_M, expression (15) evaluated at $e_M = \Phi^{-1}(1-\alpha)$ is the loss rate consistent with a cumulative probability of α. Using the identities $\Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$ and $D = \Phi^{-1}(PD)$, the portfolio loss rate consistent with a cumulative probability of α is

$$\begin{aligned} \Lambda_p\left(\vec{\tilde{V}}, \vec{\tilde{Z}}, \vec{\tilde{Y}} \mid e_M = \Phi^{-1}(1-\alpha)\right) \xrightarrow[a.s.]{} \\ \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V}\; \Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}} \right) \times \\ \left[d_0 + (1-d_0)\, E\left(\Omega^{-1}\left(1-\Phi\left(\tilde{Z}_i \mid e_M = -\Phi^{-1}(\alpha)\right)\right)\right)\right] \times \\ E\left(\Theta^{-1}\left(1-\Phi\left(\tilde{Y}_i \mid e_M = -\Phi^{-1}(\alpha)\right)\right)\right), \qquad \text{for } \alpha \in [0,1]. \end{aligned} \tag{17}$$

3.7 Discussion

The first term in expression (17), $\Phi\left(\frac{\Phi^{-1}(PD)+\sqrt{\rho_V}\ \Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}}\right)$, is the inverse of an asymptotic portfolio's cumulative default rate distribution evaluated at a probability of α. When *EAD* and *LGD* are both constant as they are in the Vasicek ASFM framework, the formula used to estimate a capital allocation with a coverage rate of α is,

$$\Phi\left(\frac{\Phi^{-1}(PD)+\sqrt{\rho_V}\ \Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}}\right)\cdot LGD\cdot EAD,\ .$$ [1]

The remaining terms in expression (17) are the α-level critical values for the asymptotic portfolio's *EAD* distribution, $d_0+(1-d_0)\,E\left(\Omega^{-1}\left(1-\Phi\left(\tilde{Z}_i \mid e_M=-\Phi^{-1}(\alpha)\right)\right)\right)$, and the asymptotic portfolio's *LGD* distribution, $E\left(\Theta^{-1}\left(1-\Phi\left(\tilde{Y}_i \mid e_M=-\Phi^{-1}(\alpha)\right)\right)\right)$.

In general, the critical values of the asymptotic portfolio exposure and LGD distributions must be calculated using numerical techniques. Although numerical quadrature methods of estimation are preferred on efficiency grounds (i.e., smaller estimation error for a given number of calculations), a simple Monte Carlo estimator of $E\left(\Theta^{-1}\left(1-\Phi\left(\tilde{Y}_i \mid e_M=-\Phi^{-1}(\alpha)\right)\right)\right)$ provides an example that aids in understanding some of the portfolio's distribution properties.

[1] For example, this loss rate formula (with $\alpha=.999$) is used to calculate minimum regulatory capital requirements in the Basel II AIRB approach (see the Basle Committee on Banking Supervision (2006)). The interpretation is that when capital is set at this level, 99.9 percent of all potential portfolio credit losses will be less than the capital allocation.

To construct a simple Monte Carlo estimator, for a given value for α , generate a random sample of size M of standard normal deviates, $\tilde{e}_j \sim \phi(0,1)$, $j = 1,2,3,\ldots,M$. For each observation, e_j, calculate $\hat{\lambda}_j = \Theta^{-1}\left(1-\Phi\left(\sqrt{1-\rho_Y}\, e_j - \sqrt{\rho_Y}\,\Phi^{-1}(\alpha)\right)\right)$. The Monte Carlo estimate of $E\left(\Theta^{-1}\left(1-\Phi\left(\tilde{Y}_i \mid e_M = -\Phi^{-1}(\alpha)\right)\right)\right)$ is $\frac{1}{M}\sum_{j=1}^{M}\hat{\lambda}_j$. The precision of this estimator improves as the Monte Carlo sample size, M, increases,[2]

$$\lim_{M\to\infty} \frac{1}{M}\sum_{j=1}^{M}\hat{\lambda}_j \xrightarrow[a.s]{} E\left(\Theta^{-1}\left(1-\Phi\left(\tilde{Y}_i \mid e_M = -\Phi^{-1}(\alpha)\right)\right)\right). \tag{18}$$

A similar approach can be used to estimate the critical value of the asymptotic portfolio's draw rate distribution, $E\left(\Omega^{-1}\left(1-\Phi\left(\tilde{Z}_i \mid e_M = -\Phi^{-1}(\alpha)\right)\right)\right)$.

The Monte Carlo estimators make two characteristics of the asymptotic portfolio's draw and loss given default rate distributions transparent. First, as the correlations in their latent factors converge to 0, the asymptotic portfolio draw rate and *LGD* distributions converge to a point distribution located at their unconditional expected values:

$$\lim_{\rho_Y\to 0} E\left(\Theta^{-1}\left(1-\Phi\left(\tilde{Y}_i \mid e_M = -\Phi^{-1}(\alpha)\right)\right)\right) = E\left(\tilde{\lambda}\right) \quad \forall \alpha \in [0,1]$$

$$\lim_{\rho_Y\to 0} E\left(\Omega^{-1}\left(1-\Phi\left(\tilde{Z}_i \mid e_M = -\Phi^{-1}(\alpha)\right)\right)\right) = E\left(\tilde{\delta}\right) \quad \forall \alpha \in [0,1]\,. \tag{19}$$

Secondly, as the correlations in these distributions' latent factors approach 1, the distributions of the portfolio draw and *LGD* rate distributions converge to distributions that

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[2] The convergence rate of the Monte Carlo estimator is $O\left(M^{-\frac{1}{2}}\right)$.

characterize the loss or exposure behavior of a single credit (i.e., there is no diversification in the portfolio-level distributions):

$$\lim_{\rho_Y \to 1} E\left(\Theta^{-1}\left(1-\Phi\left(\widetilde{Y}_i \mid e_M = -\Phi^{-1}(\alpha)\right)\right)\right) = \Theta^{-1}(\alpha) \quad \forall \alpha \in [0,1]$$

$$\lim_{\rho_Z \to 1} E\left(\Omega^{-1}\left(1-\Phi\left(\widetilde{Z}_i \mid e_M = -\Phi^{-1}(\alpha)\right)\right)\right) = \Omega^{-1}(\alpha) \quad \forall \alpha \in [0,1]. \tag{20}$$

While it is possible to calculate the full asymptotic portfolio *LGD* or *EAD* distribution function using quadrature, Monte Carlo, or perhaps another numerical method, such methods are computationally intensive. Aside from issues of ease of computation, the derivation of expression (17) is not fully general as it requires use of the integral transform, which is applicable only when the distributions for the individual account *LGD* and draw rates are continuous.

The next section derives an approximation for asymptotic portfolio loss distribution that is applicable in cases when the individual credit *LGD* and draw rate distributions are discrete. A random variable's range of support is divided into equal increments, and these increments are used in conjunction with compound latent Gaussian factors to construct a step-function approximation for individual credit *EAD* and *LGD* distributions. This approach circumvents the need to use numerical techniques to calculate conditional *EAD* and *LGD* expectations, and it can also be used to approximate the portfolio loss distribution when the *LGD* and draw rate distributions are continuous. Because it avoids the need to use numerical methods to calculate expectations, depending on the intended use and accuracy requirements, the step-function formulation of the model may be preferred in some applications.

4. A Step-Function Approximation for an Asymptotic Portfolio's Cumulative Loss Distribution

4.1 Discrete Approximation for a Cumulative Distribution Function

Let $\Xi(\tilde{a})$ represent the cumulative density function for $\tilde{a} \in [0,1]$. Because $\Xi(\tilde{a})$ is a cumulative density function, it is monotonic and non-decreasing in $\tilde{a}$. Over the range of support for $\tilde{a}$, define n equal increments of size $\frac{1}{n}$, and use these increments to define a set of overlapping events that span the random variable range of support:

$\{\mathrm{E}(\tilde{a}, j, n)\}$, $j = 0,1,2,...,n$, where $\mathrm{E}(\tilde{a}, j, n)$ is the event $\tilde{a} \in \left[0, \frac{j}{n}\right]$, for $j = 0,1,2,3,\dots,n.$ By construction, $\Xi\left(\frac{j}{n}\right)$ is the probability that event $\mathrm{E}(\tilde{a}, j, n)$ occurs.

Let $1_{\mathrm{E}(\tilde{a},j,n)}$ be the indicator function for the event $\mathrm{E}(\tilde{a}, j, n)$,

$$1_{\mathrm{E}(\tilde{a},j,n)} = \begin{cases} 1 \text{ if } a_i \in \mathrm{E}(\tilde{a}, j, n) \\ 0 \qquad \text{otherwise} \end{cases} \tag{21}$$

The expected value of the indicator function is the probability of occurrence of the indicated event,

$$E\left(1_{\mathrm{E}(\tilde{a},j,n)}\right) = \Xi\left(\frac{j}{n}\right), \quad j = 0,1,2,3,\cdots,n\,. \tag{22}$$

The correspondence between events and indictor functions in expression (22) is exact for integer values of j, but, for a fixed n, the random variable support may include irrational numbers that cannot be assigned a cumulative probability using expression (22). To construct an approximation for $\Xi(\tilde{a})$ that spans the entire support of $\tilde{a}$, let $a_i = \frac{x_i}{n}$ for any $a_i \in [0,1]$.

For fixed n, x_i need not be integer-valued. Using $x_i = a_i n$, approximate the cumulative distribution function for $\tilde{a}$ as

$$\Xi(a_i) \approx \Xi\left(\frac{]x_i[}{n}\right), \tag{23}$$

where $]g[$ is the so-called ceiling function that returns a value g if g is an integer, and the next-largest integer value if g in not an integer. Using this convention, $\Xi\left(\frac{j}{n}\right)$ is the cumulative probability assigned to all realizations of $\tilde{a}$ in the range, $\frac{j-1}{n} < a_i < \frac{j}{n}$. For non-integer x_i this approximation overstates the true cumulative probability, $\Xi\left(\frac{]x_i[}{n}\right) \geq \Xi\left(\frac{x_i}{n}\right)$, but the magnitude of the approximation error is decreasing in n and can be reduced to any desired degree of precision by choosing n sufficiently large.[3]

Consider the compound event, $\mathrm{E}(\tilde{a}, j-1, n) \cap \mathrm{E}(\tilde{a}, j, n)$, as $n - \propto$. In the limit as $n \to \infty$, $j \to \infty$, and $\frac{1}{n} \to 0$, the ratio $\frac{j}{n}$ remains unchanged, and the compound event $\mathrm{E}(\tilde{a}, j-1, n) \cap \mathrm{E}(\tilde{a}, j, n)$ converges to the point $\frac{j}{n} \in [0,1]$. Using expression (22),

[3] If $a_i \in [0,1]$ is rational, then $a_i = \frac{j}{n}$ for some integers, j and n. If $a_i \in [0,1]$ is irrational, then, Lagrange has shown, $\left|a_i - \frac{j}{n}\right| < \frac{1}{\sqrt{5}\,n^2}$. Thus $a_i \in [0,1]$ can be approximated to any desired degree of accuracy by $a_i \approx \frac{j}{n}$ for some integers, j and n, where the precision of the approximation is increasing in n. See Conway and Guy (1996), pp. 187–89.

$\lim_{n\to\infty} E\left(1_{\mathrm{E}(\tilde{a},j,n)}\right) - E\left(1_{\mathrm{E}(\tilde{a},j-1,n)}\right) = \Xi'\left(\frac{j}{n}\right)$, or in the limit as $n \to \infty$, the difference in the expected values of indicator functions immediately adjacent to the point $\frac{j}{n}$ converges to the value of the probability density of $\tilde{a}$ evaluated at the point $\frac{j}{n}$.

In instances when $\Xi(\tilde{a})$ is discrete, if n is sufficiently large, each point in the support of $\tilde{a}$ can be associated with a unique compound event, $\mathrm{E}(\tilde{a},i,n) \cap \mathrm{E}(\tilde{a},j,n)$, for some integers i and j. Consequently, a discrete distribution $\Xi(\tilde{a})$ can be represented exactly using this representation if sufficient precision (sufficiently large n) is specified.

4.2 Approximating Individual Account EAD

Divide the $[0,1]$ range of support for the draw rate into $n+1$ overlapping regions and define $n+1$ corresponding events: $\{\mathrm{E}(\tilde{\delta},j,n)\}$ is the set of events $\tilde{\delta}_i \in \left[0,\frac{j}{n}\right]$ for $j = 0,1,2,\ldots,n.$

The probability distribution for an account's draw rate is approximated by a uniform-size step function defined on Z_i using $\{\mathrm{E}(\tilde{\delta},j,n)\}$,

$$\tilde{\delta}_i = \begin{cases} 0 & \text{for } \tilde{Z}_i \geq A_{i1} \\ \left(\frac{1}{n}\right) & \text{for } A_{i2} \leq \tilde{Z}_i < A_{i1} \\ \left(\frac{2}{n}\right) & \text{for } A_{i3} \leq \tilde{Z}_i < A_{i2} \\ \vdots & \vdots \\ \left(\frac{j}{n}\right) & \text{for } A_{ij+1} \leq \tilde{Z}_i < A_{ij} \\ \vdots & \vdots \\ \left(\frac{n-1}{n}\right) & \text{or } A_{i\,n} \leq \tilde{Z}_i < A_{i\,n-1} \\ 1 & \text{for } \tilde{Z}_i < A_{i\,n} \end{cases} \tag{24}$$

where $A_{in} < A_{i\,n-1} < \cdots A_{i2} < A_{i1}$. Expression (24) models the draw rate as a monotonically decreasing function of $\tilde{Z}_i$ with $n+1$ distinct draw rates with uniform increments of size $\frac{1}{n}$ beginning at $\delta_i = 1$.

The latent variable thresholds $\{A_{i1}, A_{i2}, \cdots, A_{in}\}$ are defined by equating the Gaussian probabilities for the latent variable thresholds to the probability that the corresponding events occur under $\Omega(\tilde{\delta}_i)$. For example, the equality $1 - \Phi(A_{i1}) = \Omega(0)$ defines $A_{i1} = \Phi^{-1}(1 - \Omega(0))$. Similarly, $1 - \Phi(A_{i2}) = \Omega\left(\frac{1}{n}\right)$ defines $A_{i2} = \Phi^{-1}\left(1 - \Omega\left(\frac{1}{n}\right)\right)$, and so on. The threshold values for the unconditional draw rate distribution approximation are given in Table 1.

Table 1: Step-Function Approximation for an Individual Credit's Draw Rate Distribution

Draw Rate	Event	Cumulative Probability of Draw Rate	Threshold Value for Latent Variable $\tilde{Z}_i$
0	$E(\tilde{\delta},0,n)$	$\Omega(0)$	$A_{i1} = \Phi^{-1}(1-\Omega(0))$
$\frac{1}{n}$	$E(\tilde{\delta},1,n)$	$\Omega\left(\frac{1}{n}\right)$	$A_{i2} = \Phi^{-1}\left(1-\Omega\left(\frac{1}{n}\right)\right)$
$\frac{2}{n}$	$E(\tilde{\delta},2,n)$	$\Omega\left(\frac{2}{n}\right)$	$A_{i3} = \Phi^{-1}\left(1-\Omega\left(\frac{2}{n}\right)\right)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{n-1}{n}$	$E(\tilde{\delta},n-1,n)$	$\Omega\left(\frac{n-1}{n}\right)$	$A_{in} = \Phi^{-1}\left(1-\Omega\left(\frac{n-1}{n}\right)\right)$
1	$E(\tilde{\delta},n,n)$	1	

4.3 Approximating Individual Account LGD

The methodology used to approximate the *LGD* distribution is analogous to the approach used to approximate the draw rate distribution. Divide the interval $[0,1]$ into $n+1$ overlapping regions and define a corresponding set of events, $\left\{E(\tilde{\lambda}, j,n) \ni \tilde{\lambda}_i \in \left[0,\frac{j}{n}\right]\right\}$, for $j = 0,1,2, \ldots, n$.

The model is normalized so that higher realized loss rates are associated with smaller realized values of $\tilde{Y}_i$. Approximate $\Theta(\tilde{\lambda}_i)$ as

$$\tilde{\lambda}_i = \begin{cases} 0 & for\ \tilde{Y}_i \geq B_{i1} \\ \left(\frac{1}{n}\right) & for\ B_{i2} \leq \tilde{Y}_i < B_{i1} \\ \left(\frac{2}{n}\right) & for\ B_{i3} \leq \tilde{Y}_i < B_{i2} \\ \vdots & \vdots \\ \vdots & \vdots \\ \left(\frac{n-1}{n}\right) & for\ B_{i\,n} \leq \tilde{Y}_i < B_{i\,n-1} \\ 1 & for\ \tilde{Y}_i < B_{in} \end{cases} \tag{25}$$

for $B_{in} < B_{in-1} < \cdots < B_{i2} < B_{i1}$. The latent variable thresholds are defined in Table 2.

Table 2: Step-Function Approximation for an Individual Credit's Loss Rate Distribution

Loss Rate	Event	Cumulative Probability of Loss Rate	Threshold Value for Latent Variable $\tilde{Y}_i$
0	$\mathrm{E}(\tilde{\lambda},0,n)$	$\epsilon(0)$	$B_{i1} = \Phi^{-1}(1-\Theta(0))$
$\frac{1}{n}$	$\mathrm{E}(\tilde{\lambda},1,n)$	$\Theta\left(\frac{1}{n}\right)$	$B_{i2} = \Phi^{-1}\left(1-\Theta\left(\frac{1}{n}\right)\right)$
$\frac{2}{n}$	$\mathrm{E}(\tilde{\lambda},2,n)$	$\Theta\left(\frac{2}{n}\right)$	$B_{i3} = \Phi^{-1}\left(1-\Theta\left(\frac{2}{n}\right)\right)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{n-1}{n}$	$\mathrm{E}(\tilde{\lambda},n-1,n)$	$\Theta\left(\frac{n-1}{n}\right)$	$B_{i\,n-1} = \Phi^{-1}\left(1-\Theta\left(\frac{n-2}{n}\right)\right)$
1	$\mathrm{E}(\tilde{\lambda},n,n)$	1	$B_{i\,n} = \Phi^{-1}\left(1-\Theta\left(\frac{n-1}{n}\right)\right)$

4.4 The Loss Rate for an Individual Account

The loss rate distribution for an individual account can be modeled using $2n+1$ indicator functions defined over the latent variables $\tilde{V}_i$, $\tilde{Z}_i$, and $\tilde{Y}_i$. One indicator function indicates default status; n indicator functions are used to approximate the cumulative *EAD* distribution, $\Omega(\tilde{\delta})$; and n indicator functions are used to approximate the cumulative *LGD* distribution, $\Theta(\tilde{\lambda})$,

$$
\begin{aligned}
&1_{D_i}(\tilde{V}_i) = \begin{cases} 1 & if \ \ \tilde{V}_i < D_i \\ 0 & otherwise \end{cases}, \\
&1_{A_{ij}}(\tilde{Z}_i) = \begin{cases} 1 & if \ \ \tilde{Z}_i < A_{ij} \\ 0 & otherwise \end{cases}, \quad for \ \ j = 1,2,3,\ldots,n \\
&1_{B_{ik}}(\tilde{Y}_i) = \begin{cases} 1 & if \ \ \tilde{Y}_i < B_{ik} \\ 0 & otherwise \end{cases}, \quad for \ k = 1,2,3,\ldots,n.
\end{aligned}
\tag{26}
$$

Each indicator function defines a binomial random variable.

Let $\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n)$ represent the approximate loss rate for account *i* measured relative to the account's maximum credit limit, M_i. $\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n)$ is defined as

$$
\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n) = 1_{D_i}(\tilde{V}_i)\left(\left(d_{i0} + (1-d_{i0})\left(\frac{1}{n}\right)\sum_{k=1}^{n} 1_{A_{ik}}(\tilde{Z}_i)\right)\left(\left(\frac{1}{n}\right)\sum_{j=1}^{n} 1_{B_{ij}}(\tilde{Y}_i)\right)\right), \tag{27}
$$

where the notation indicates that the approximation depends on n, the number of increments used to model the account's *LGD* and *EAD* cumulative distribution functions.

5.5 The Conditional Loss Rate for an Individual Account

Let $1_{D_i}(\widetilde{V}_i \mid e_M)$ represent the value of the default indicator function conditional on a realized value for e_M, the common latent factor. Similarly, let $1_{A_{ij}}(\widetilde{Z}_i \mid e_M)$ and $1_{B_{ij}}(\widetilde{Y}_i \mid e_M)$ represent the values of the remaining indicator functions ($j = 1, 2, 3, \ldots, n$) conditional on $\widetilde{e}_M = e_M$. These additional conditional indicator functions define independent binomial random variables with expected values

$$E\left(1_{A_{ij}}\left(\widetilde{Z}_i \mid e_M\right)\right) = \Phi\left(\frac{A_{ij} - \sqrt{\rho_Z}\, e_M}{\sqrt{1-\rho_Z}}\right), \qquad j = 1,2,3,\ldots,n \tag{28}$$

$$E\left(1_{B_{ij}}\left(\widetilde{Y}_i \mid e_M\right)\right) = \Phi\left(\frac{B_{ij} - \sqrt{\rho_Y}\, e_M}{\sqrt{1-\rho_Y}}\right), \qquad j = 1,2,3,\ldots,n\,. \tag{29}$$

An individual account's conditional loss rate is approximated as

$$\Lambda_i^A\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i, n \mid e_M\right) = 1_{D_i}\left(\widetilde{V}_i \mid e_M\right)\left(\left(d_{i0} + (1-d_{i0})\left(\frac{1}{n}\right)\sum_{k=1}^{n} 1_{A_{ik}}\left(\widetilde{Z}_i \mid e_M\right)\right)\left(\left(\frac{1}{n}\right)\sum_{j=1}^{n} 1_{B_{ij}}\left(\widetilde{Y}_i \mid e_M\right)\right)\right). \tag{30}$$

4.6 The Loss Rate on an Asymptotic Portfolio of Revolving Credits

As in Section 3.5, we consider a portfolio composed of *N* accounts that are identical except for their idiosyncratic risk factors. The credits have identical credit limits, initial drawn balances, latent factor correlations, identical default thresholds, and identical *LGD* and draw rate distributions. Under these assumptions, the $2n+1$ threshold values in expression (26) are identical across individual credits, and indicator function subscript *i* no longer is necessary: $1_{D_i}(\widetilde{V}_i) = 1_D(\widetilde{V}_i)$, $1_{A_{ij}}(\widetilde{Z}_i) = 1_{A_j}(\widetilde{Z}_i)$, and $1_{B_{ij}}(\widetilde{Y}_i) = 1_{B_j}(\widetilde{Y}_i)$ for $j = 1,2,3,\cdots,n.$ The loss rate for an individual credit will depend on the identity of the credit only through the idiosyncratic risk factors in the latent variables $\widetilde{V}_i, \widetilde{Z}_i$, and $\widetilde{Y}_i$, and so the account's approximate loss rate can be written without an identifying subscript as $\Lambda^A\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i, n\right)$.

Let $\Lambda_p^A\left(\vec{\widetilde{V}}, \vec{\widetilde{Z}}, \vec{\widetilde{Y}}, n \mid e_M\right)$ represent the approximate loss rate on the portfolio of N accounts conditional on a realization of e_M, and n increments in the step-function approximation

$$\Lambda_p^A\left(\vec{\widetilde{V}}, \vec{\widetilde{Z}}, \vec{\widetilde{Y}}, n \mid e_M\right) = \left(\frac{\sum_{i=1}^{N}\left(M \cdot \Lambda^A\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i, n \mid e_M\right)\right)}{N \cdot M}\right) = \left(\frac{\sum_{i=1}^{N}\left(\Lambda^A\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i, n \mid e_M\right)\right)}{N}\right). \tag{31}$$

Because the individual account conditional loss rates are independent and identically distributed, the Strong Law of Large Numbers requires

$$\begin{aligned} \lim_{N\to\infty} \Lambda_p^A\left(\vec{\widetilde{V}}, \vec{\widetilde{Z}}, \vec{\widetilde{Y}}, n \mid e_M\right) &= \lim_{N\to\infty} \left(\frac{\sum_{i=1}^{N}\Lambda^A\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i, n \mid e_M\right)}{N}\right) \xrightarrow[a.s.]{} E\left(\Lambda^A\left(\widetilde{V}_i, \widetilde{Z}_i, \widetilde{Y}_i, n \mid e_M\right)\right) \\ &\approx E\left(1_D\left(\widetilde{V}_i \mid e_M\right)\right)\left(d_0 + (1-d_0)\left(\frac{1}{n}\right)\sum_{j=1}^{n} E\left(1_{A_j}\left(\widetilde{Z}_i \mid e_M\right)\right)\right)\left(\left(\frac{1}{n}\right)\sum_{j=1}^{n} E\left(1_{B_j}\left(\widetilde{Y}_i \mid e_M\right)\right)\right) \end{aligned} \tag{32}$$

Substitution of the expressions for the conditional expectations yields

$$\begin{aligned} &\lim_{N\to\infty}\left(\Lambda_p^A\left(\vec{\widetilde{V}}, \vec{\widetilde{Z}}, \vec{\widetilde{Y}}, n \mid e_M\right)\right) \approx \\ &\Phi\left(\frac{D - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) \cdot \left(d_0 + (1-d_0)\left(\frac{1}{n}\right)\sum_{j=1}^{n} \Phi\left(\frac{A_j - \sqrt{\rho_Z}\, e_M}{\sqrt{1-\rho_Z}}\right)\right) \cdot \left(\left(\frac{1}{n}\right)\sum_{j=1}^{n} \Phi\left(\frac{B_j - \sqrt{\rho_Y}\, e_M}{\sqrt{1-\rho_Y}}\right)\right). \end{aligned} \tag{33}$$

Expression (33) is an approximation for the inverse of the conditional distribution function for an asymptotic portfolio's loss rate evaluated at $e_M \in (-\infty, +\infty)$. Propositions 2 and 3 in the appendix can be used to show that, in the limit, as $n \to \infty$, the approximation converges to the true underlying asymptotic portfolio conditional loss rate consistent with the model assumptions. Thus, an asymptotic portfolio's loss rate density function can be approximated by the implicit function

$$\tilde{\Lambda}_p^A \sim \left\{ \Phi\left(\frac{D-\sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) \cdot \left(d_0 + (1-d_0)\left(\frac{1}{n}\right)\sum_{j=1}^{n}\Phi\left(\frac{A_j-\sqrt{\rho_Z}\, e_M}{\sqrt{1-\rho_Z}}\right)\right) \cdot \left(\left(\frac{1}{n}\right)\sum_{j=1}^{n}\Phi\left(\frac{B_j-\sqrt{\rho_Y}\, e_M}{\sqrt{1-\rho_Y}}\right)\right), \phi(e_M)\right\},$$

$$\text{for } e_M \in (-\infty, \infty). \tag{34}$$

The portfolio loss rate is decreasing in e_M, so the loss rate consistent with a cumulative probability of α is given by expression (33) evaluated at $e_M = \Phi^{-1}(1-\alpha)$. Using the definitions of the latent variable thresholds in Tables 1 and 2, it follows that an approximation for the portfolio loss rate consistent with a cumulative probability of α is

$$\Lambda_p^A\left(\tilde{\vec{V}}, \tilde{\vec{Z}}, \tilde{\vec{Y}}, n \mid e_M = \Phi^{-1}(1-\alpha)\right) \xrightarrow[a.s.]{}$$
$$\Phi\left(\frac{\Phi^{-1}(PD)+\sqrt{\rho_V}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}}\right)\left(d_0+(1-d_0)\left(\frac{1}{n}\right)A(\alpha)\right)\left(\left(\frac{1}{n}\right)B(\alpha)\right) \quad \text{for } \alpha \in [0,1], \tag{35}$$

where

$$A(\alpha) = \sum_{i=1}^{n}\Phi\left(\frac{\Phi^{-1}\left(1-\Omega\left(\frac{i-1}{n}\right)\right)+\sqrt{\rho_Z}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho_Z}}\right) \tag{36}$$

$$B(\alpha) = \sum_{j=1}^{n}\Phi\left(\frac{\Phi^{-1}\left(1-\Theta\left(\frac{j-1}{n}\right)\right)+\sqrt{\rho_Y}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho_Y}}\right). \tag{37}$$

Proposition 2 in the appendix establishes that $\left(\frac{1}{n}\right)B(\alpha)$ is an approximation for the portfolio's conditional *LGD* rate,

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)B(\alpha)=E\left(\tilde{\lambda}\mid e_M=\Phi^{-1}(1-\alpha)\right), \tag{38}$$

so the probability density of the asymptotic portfolio's *LGD* rate can be approximated by the implicit function

$$\left\{\lim_{n\to\infty}\left(\frac{1}{n}\right)B(\alpha),\ \phi\left(\Phi^{-1}(\alpha)\right)\right\},\quad \forall\ \alpha\in[0,1]. \tag{39}$$

The expression $\left(d_0+(1-d_0)\left(\frac{1}{n}\right)A(\alpha)\right)$ is an approximation for the asymptotic portfolio's conditional utilization rate relative to the portfolio's total committed line of credit; $\left(\frac{1}{n}\right)A(\alpha)$ is an approximation for the portfolio's conditional draw rate. Proposition 2 in the appendix can be used to establish

$$\lim_{n\to\infty}\frac{1}{n}A(\alpha)\to E\left(\tilde{\delta}\mid e_M=\Phi^{-1}(1-\alpha)\right), \tag{40}$$

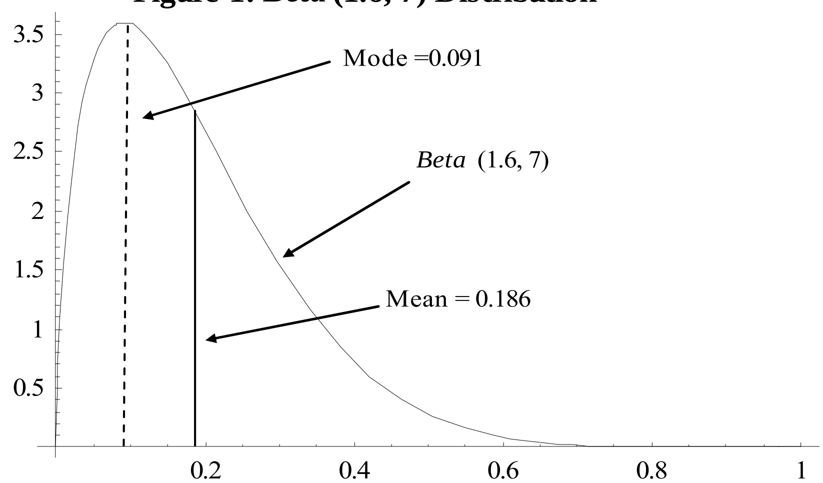
and so the probability density of the portfolio's overall draw rate can be approximated by the implicit function

$$\left\{\lim_{n\to\infty}\left(\frac{1}{n}\right)A(\alpha),\ \phi\left(\Phi^{-1}(1-\alpha)\right)\right\},\quad \forall\ \alpha\in[0,1]. \tag{41}$$

5. Examples of Unconditional Draw Rate and Loss Given Default distributions

In this section, the algorithm developed in Section 4 is applied to approximate portfolio level distributions that are generated by three alternative account-level unconditional distributions. These account-level distributions could represent individual account draw rates or *LGD* rates depending on the specific application. The three distributions considered are all members of the Beta family. The distribution parameters are selected so that one unconditional distribution is skewed right, one is symmetric, and one is skewed left. The analysis demonstrates that skewness and correlation among individual *LGD* and *EAD* distributions are important determinants of the shape of the asymptotic portfolio *LGD* and draw rate distributions. Although the examples could represent either individual account draw or *LGD* rate distributions, to simplify the discussion they are described as if they represent *LGD* distributions.

Figure 1: *Beta* (1.6, 7) Distribution



5.1 Positively Skewed Distribution

The density function for the Beta distribution with the first parameter (alpha) equal to 1.6 and the second parameter (beta) equal to 7, plotted in Figure 1, is

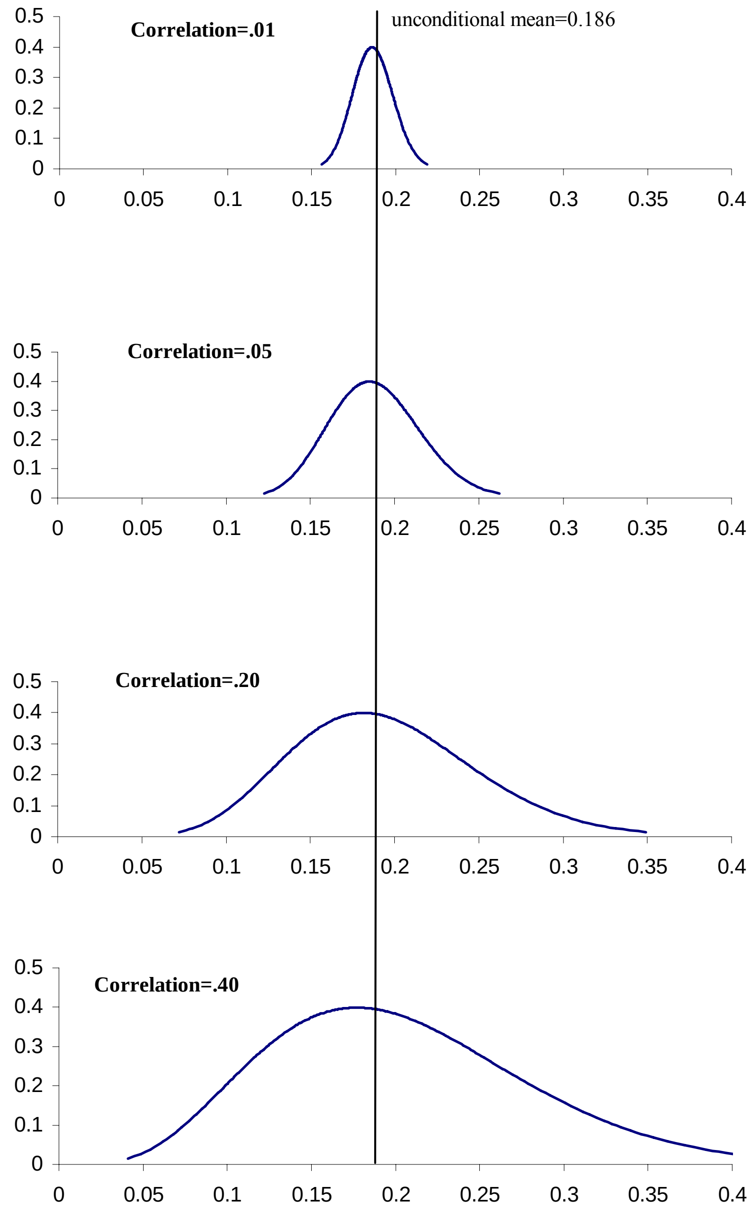
$$\tilde{\lambda} \sim Beta(1.6,7,\tilde{\lambda})$$
$$Beta(1.6,7,\lambda) = \frac{\Gamma(8.6)}{\Gamma(1.6)\Gamma(7)}\lambda^{0.6}(1-\lambda)^{6}, \quad \text{for} \quad 0 < \lambda < 1 \tag{41}$$

where $\Gamma(b)=\int_0^\infty y^{b-1}e^{-y}dy, \quad b>0$, is the mathematical gamma function. This unconditional distribution is skewed right and could, for example, represent the random draw rates on revolving corporate credits or the loss given default rates on wholesale bank loans or alt-A mortgages.

Figure 2 plots the asymptotic portfolio's *LGD* distribution for alternative correlation assumptions assuming that individual *LGD*s are distributed $Beta(1.6,7)$ and that *LGD* correlation is driven by the single common factor structure described above. The asymptotic portfolio's *LGD* distribution is approximated using $\left\{\left(\frac{1}{n}\right)B(\alpha), \phi\left(\Phi^{-1}(1-\alpha)\right)\right\}, \ \alpha=\frac{j}{n}$, $j=1,2,3,\ldots,2500, \ n=2500.$

Figure 2 shows that, when individual credit loss rates are uncorrelated, the portfolio's unconditional *LGD* distribution converges to $E(Beta(1.6,7))=0.1862$. As correlation among individual *LGD* realizations increases, the range of the portfolio *LGD* distribution increases and the distribution becomes increasingly positively skewed. As correlation approaches 1, the ability to diversify *LGD* risk within the portfolio diminishes. When correlation reaches one, the asymptotic portfolio *LGD* distribution converges to the $Beta(1.6,7)$ distribution (not shown).

Figure 2: Asymptotic Portfolio Unconditional LGD or Draw Rate Distribution for Alternative Correlations When Individual Credits Are Distributed *Beta* (1.6, 7)



5.2 Negatively Skewed Distribution

The second unconditional distribution considered is the Beta distribution with parameters alpha = 4 and beta = 1.1:

$$\begin{aligned} &\tilde{\lambda} \sim Beta(4,1.1,\tilde{\lambda}) \\ &Beta(1.6,7,\pi) = \frac{\Gamma(5.1)}{\Gamma(4)\Gamma(1.1)} \lambda^3 (1-\lambda)^{0.1}, \quad \text{for} \quad 0 < \lambda < 1 \end{aligned} \tag{42}$$

This negatively skewed distribution, plotted in Figure 3, could represent individual draw rates or *LGD* rates on sub-prime credit card accounts or other revolving retail credits.

Figure 3: Beta (4,1.1) Distribution

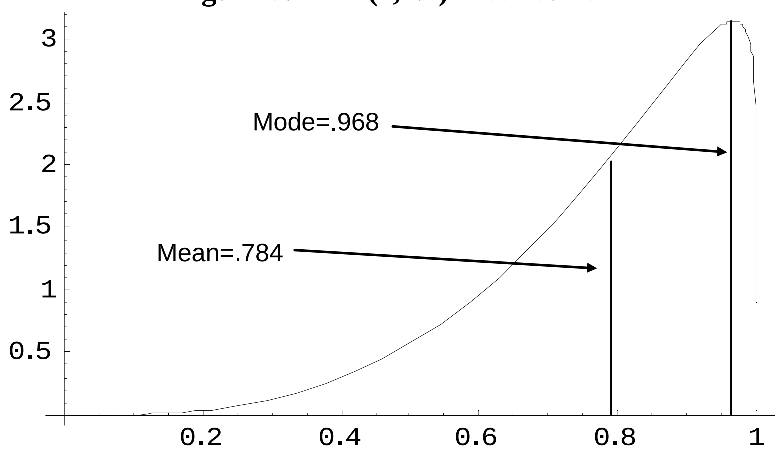


Figure 4 plots the asymptotic portfolio *LGD* distribution generated under different correlation assumptions when individual *LGDs* are distributed $\tilde{\lambda} \sim Beta(4,1.1)$. The unconditional portfolio *LGD* distribution is approximated using the step-function approach with $n = 2500$. When individual *LGD* realizations are uncorrelated, *LGD* risk is completely diversified, and the asymptotic portfolio's *LGD* distribution converges to $E(Beta(4,1.1)) = 0.7845$. As the correlation increases, the portfolio's *LGD* distribution becomes increasingly negatively skewed; it converges to $Beta(4,1.1)$ for $\rho_Y = 1$.

Figure 4: Unconditional Asymptotic Portfolio LGD or Draw Rate Distribution for Alternative Correlations When Individual Credits Are Distributed *Beta* (4, 1.1)

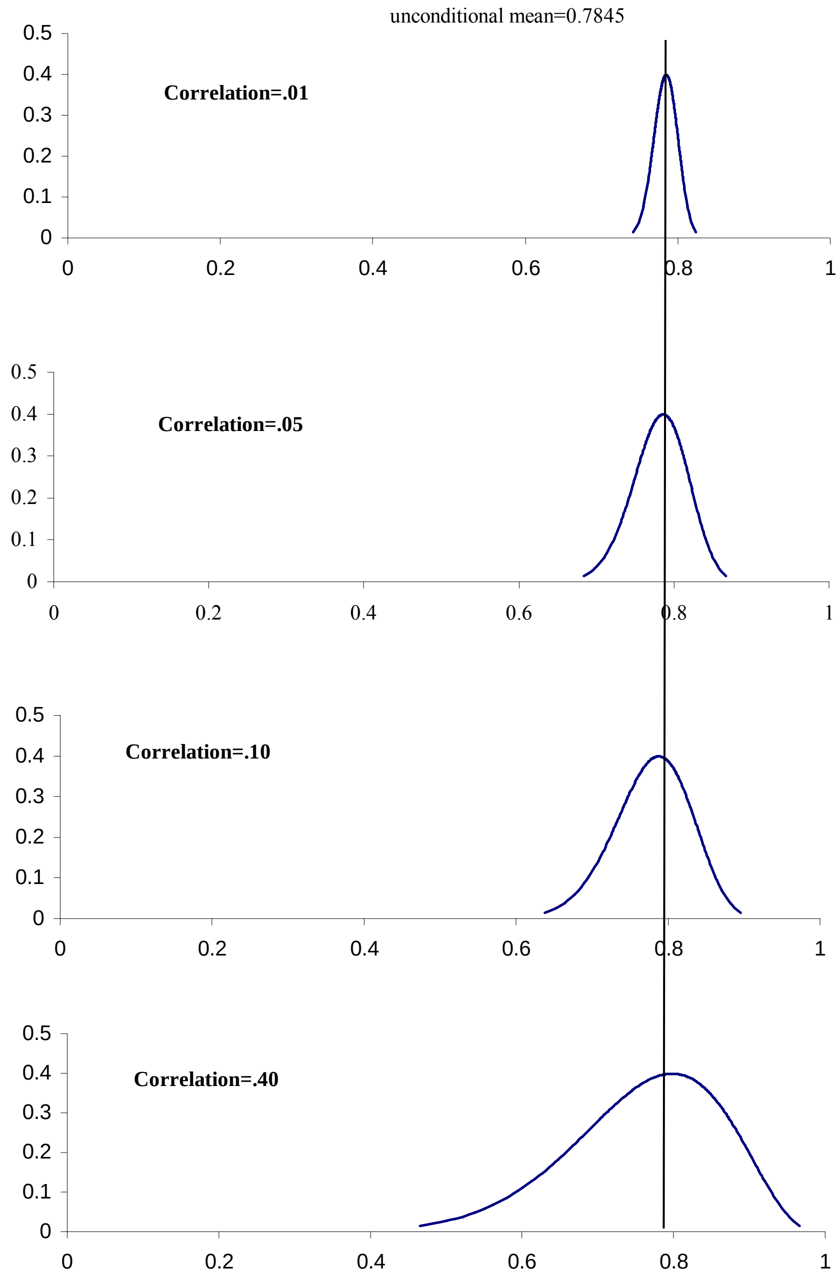
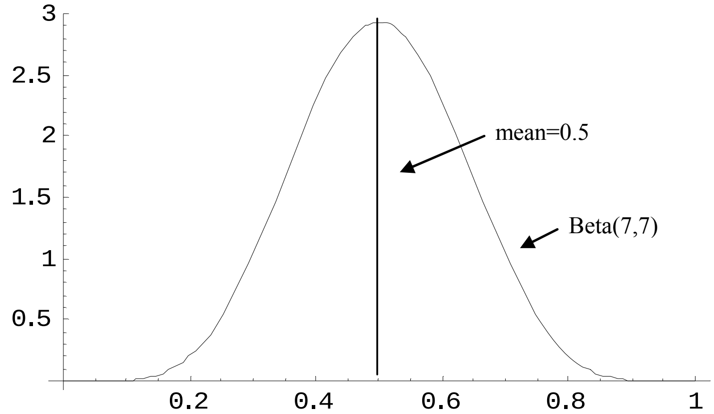


Figure 5: Beta(7,7) Distribution



5.3 Symmetric Distribution

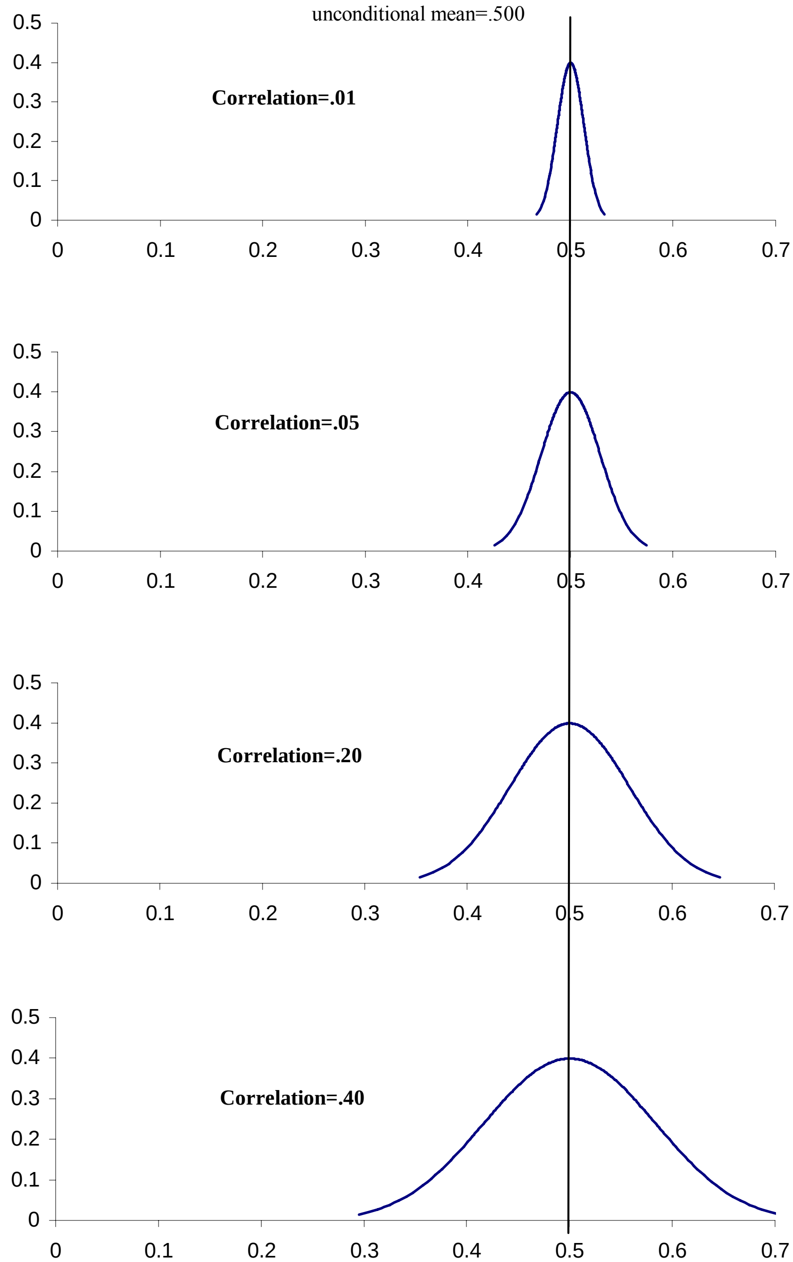
The final example is the Beta distribution with parameters alpha = 7 and beta = 7,

$$\tilde{\lambda} \sim Beta(7,7,\tilde{\lambda})$$
$$Beta(7,7,\lambda) = \frac{\Gamma(14)}{\Gamma(7)\Gamma(7)} \lambda^6 (1-\lambda)^6, \quad \text{for} \quad 0 < \lambda < 1 \tag{43}$$

This distribution, plotted in Figure 5, is symmetric and could represent *LGD* rates on investment-grade corporate debt.

Figure 6 plots, for different correlation assumptions, the unconditional asymptotic portfolio *LGD* distribution approximation ($n = 2500$) that is generated when *LGD*s are distributed $\tilde{\lambda} \sim Beta(7,7)$. The distribution converges to $E(Beta(7,7)) = 0.5$ when individual *LGD* realizations are uncorrelated. As the correlation increases, the range of the asymptotic portfolio *LGD* distribution increases, and the portfolio *LGD* distribution converges to $Beta(7,7)$ as $\rho_Y - 1$.

Figure 6: Unconditional Asymptotic Portfolio LGD or Draw Rate Distribution for Alternative Correlations When Individual Credits Are Distributed *Beta* (7, 7)



5.4 Sample Asymptotic Portfolio Unconditional Loss Rate Distributions

This section illustrates the calculation of unconditional loss rate distributions for alternative asymptotic credit portfolios. The approximations use expression (33) with $n = 2500$. The examples represent hypothetical portfolios that are broadly consistent with the stylized facts associated with selected fixed-term loans and revolving credit facilities for both wholesale and retail credits.

Published evidence on the shape and correlations of individual credits' *LGD* and *EAD* distributions is limited. Few studies characterize the shape of individual account *EAD* distributions, and no study has attempted to estimate the strength of *EAD* correlation in a structural model.[4] A larger number of studies focus on the distribution of *LGD* rates, but the evidence is still sparse and much of it is specialized to default rates for agency-rated credits.

Most studies investigating *LGD* correlation behavior investigate time series correlation estimates between observed default frequencies and default recovery rates. Only one study estimates a structural model *LGD* correlation parameter. Frye (2000b) estimates ρ_Y to be about 20 percent for agency-rated bonds, but his estimate is based on a structural model that assumes that *LGD* distributions are symmetric. It seems likely that alternative specifications for *LGD* that include significant skew in the unconditional *LGD* distribution would produce more modest estimates of correlation, but such issues have yet to be studied. Also, as noted by Carey and Gordy (2004), most *LGD* correlation estimates have been

[4] Araten and Jacobs (2001) provide simple descriptive statistics on a sample of Chase revolving facilities. Jiménez, Lopez, and Saurina (2006) provide an aggregated histogram of all corporate *EAD*s derived from the Spanish credit registry for all credit institution loans in excess of 6000 Euros over a period spanning 1984–2005.

derived from rating-agency bond data, and the correlations for different liability classes are likely to differ according to firm capital structure characteristics and the identity of important stakeholders, including the presence (or absence) of significant banking interests.

A review of the publicly available literature suggests that the shape of individual unconditional *LGD* and *EAD* distributions as well as the magnitudes of their correlations is an open issue. This study will not contribute on the issue of model calibration but, instead, will illustrate the asymptotic portfolio loss rate distributions that are generated under alternative structural model parameterizations.

5.4.1 Hypothetical Portfolio of Term Loans

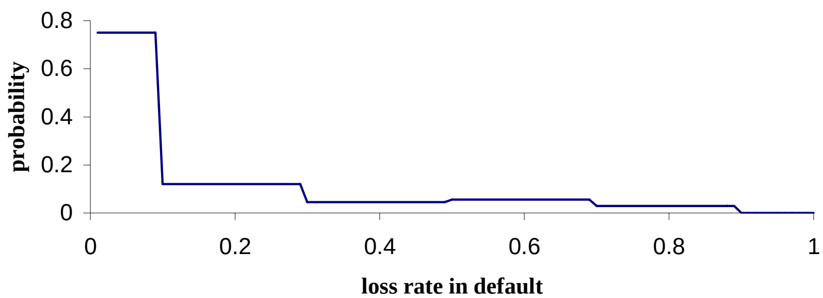
The first example is chosen to represent the portfolio loss rate distribution that may arise on a portfolio of term loans of non-investment-grade senior secured credits. Figure 7 plots the distribution of projected *LGD* rates on loans that receive a recovery rating by *FitchRatings*.[5] A large share of the *FitchRatings* sample of credits are secured first-lien loans, a fact that partly explains the favorable recovery rate distribution. This forward-looking *LGD* rate distribution is not conditioned on any realized state of the economy, so it proxies for an unconditional *LGD* distribution.

The distribution in Figure 7 is not very granular. While this distribution could be used directly in expression (34), it is broadly similar to the continuous $Beta(1.6, 7)$ distribution, and so we will use the latter distribution to represent the *LGD* distribution for individual secured first-lien loans. To construct the asymptotic portfolio loss rate distribution for this class of exposures, we assume that all loans are fully drawn (*EAD* = 1) and that individual credits

[5] Figure 7 was constructed by the author from information provided in *FitchRatings* (2006).

have an unconditional probability of default of 0.5 percent. The default correlation parameter is set at 20 percent $(\rho_V = 0.20)$ to reflect both the wholesale nature of these credits and the corporate correlation used in the Basel AIRB capital framework.

Figure 7: Projected LGD Distribution Loans Rated by *FitchRatings*



The asymptotic portfolio loss distribution is plotted for different *LGD* correlation assumptions (ρ_Y) in Figure 8. The alternative panels in Figure 8 highlight the importance of systematic risk in recovery rates. As the correlation between individual account *LGD* rates increases, the skewness of the asymptotic portfolio's loss rate distribution increases. As correlation increases from 0 to 10 percent, the 99.5 percent critical value of the portfolio loss rate distribution increases by almost 60 percent. When individual *LGD* correlations are 20 percent, the portfolio 99.5 percent loss-coverage rate is about 87.5 percent larger than the estimate produced by the simple Vasicek ASRF formula (the top panel of Figure 8) that assumes uncorrelated *LGD*s.

5.4.2 Hypothetical Portfolio of Revolving Senior Unsecured Credits

A second example is chosen to represent the loss rate distribution of an asymptotic portfolio of revolving senior unsecured bank loans made to investment-grade obligors. The example assumes that portfolio obligors begin with a 30 percent facility utilization rate and draw on their remaining credit line over the subsequent period. Because these are wholesale credits, we use the Basel II default correlation assumption, $\rho_V = 0.20$. We examine the shape of an asymptotic portfolio loss rate distribution under alternative correlation assumptions for *LGD* and *EAD*.

Figure 8: Asymptotic Portfolio Loss Rate Distributions for Fixed EAD Under Alternative Correlation Assumptions. Individual Credits Have *PD* =0.5% and Unconditional *LGD* s~Beta(1.6,7).

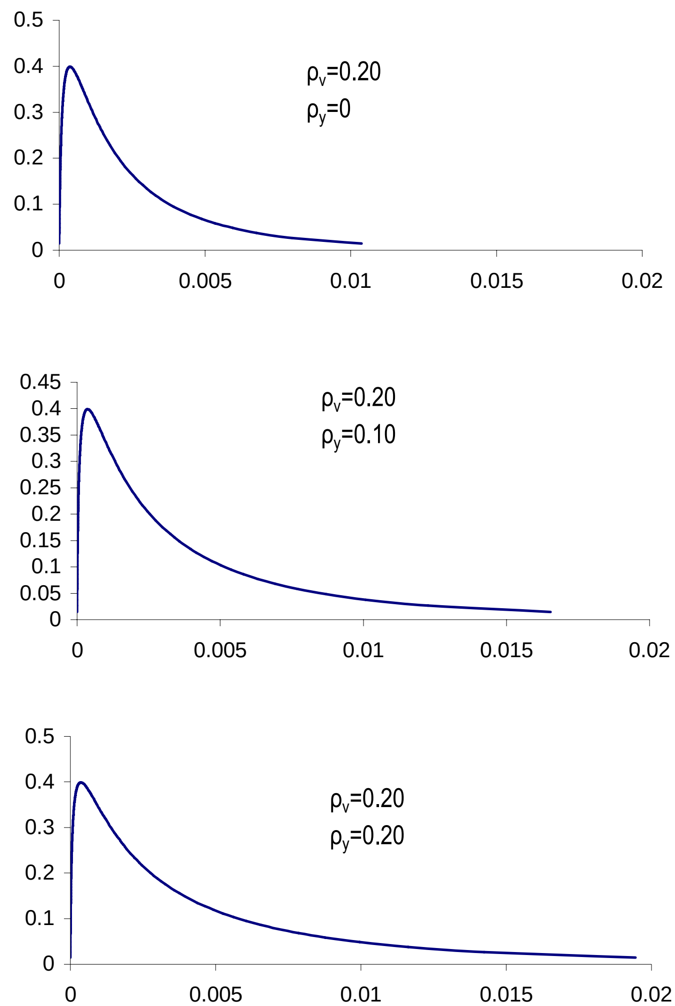
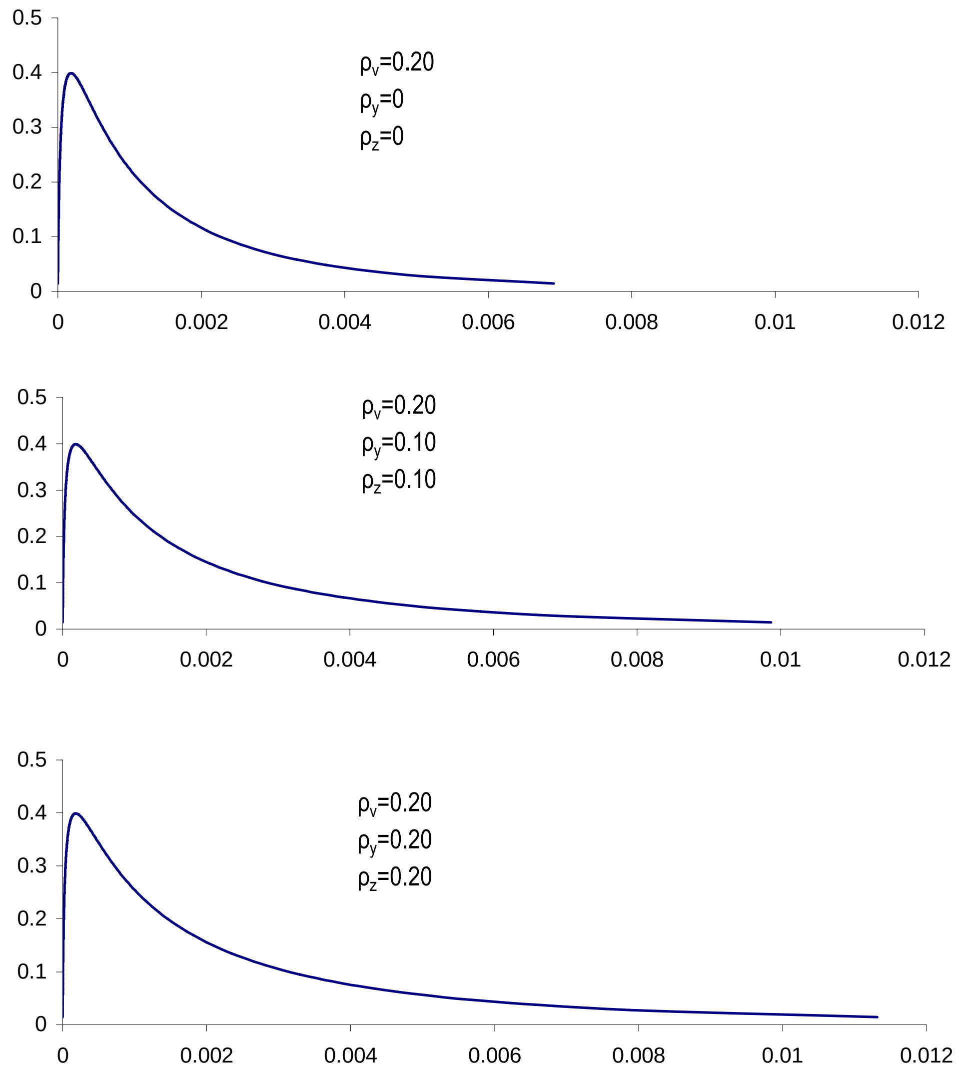


Figure 9: Asymptotic Portfolio Loss Rate Distributions for Alternative Correlations. Individual Credits Have PD=0.25%, 30% Initial Utilization, and 70% Revolving Balances, With Unconditional Draw Rates~*Beta* (1.6,7) and Unconditional LGDs~*Beta* (7,7).



Altman (2006, table 2) reports data that suggest that historical loss rates on senior unsecured bank loans are nearly symmetric, with an average loss rate of about 50 percent and a standard deviation of about 25 percent. The $Beta(7,7)$ distribution provides a reasonable approximation to this *LGD* distribution. Araten and Jacobs (2001, table1) estimate, for the

Chase data they examine, that a credit with a rating of BBB+/BBB has, on average, about a 55 percent loan equivalent value a year before default. We are not aware of any published study that further characterizes the exposure distribution on these types of facilities, but the assumption of an initial utilization rate of 30 percent and a $Beta(1.6,7)$ draw rate distribution matches both the mean of the Araten and Jacobs *EAD* data and conventional wisdom that suggests that bankers are at least partially successful at limiting takedowns by distressed obligors. We assume an unconditional default rate of 0.25 percent.

Figure 9 plots estimates of the asymptotic portfolio loss rate distribution under alternative assumptions for *LGD* (ρ_Y) and *EAD* (ρ_Z) correlations. The panels in Figure 9 show that correlation in individual credit *LGD* and *EAD* distributions has a large effect on the tails of the portfolio's credit loss distribution. As correlation in *EAD*s and *LGD*s increases from 0 to 10 percent (0 to 20 percent), the loss value associated with a 99.5 percent cumulative probability increases by 43 percent (64 percent).

5.4.3 Hypothetical Portfolio of Sub-Prime Retail Credits

The final example is intended to mimic a sub-prime credit card portfolio. Unlike with earlier examples, we are unable to reference a published study to anchor our choice of distributional assumptions. Individual accounts are assumed to have an unconditional probability of default of 4 percent, and default correlations are assumed to be 4 percent, consistent with the Basel AIRB treatment of qualified retail exposures. Customers are assumed to begin the period with 20 percent utilization of their credit limits and are assumed to draw on the remaining 80 percent of their credit limit with a draw rate modeled using the

$Beta(4,1.1)$ distribution. Because these are unsecured credits, recovery rates are low and account LGDs are assumed to follow the $Beta(4,1.1)$ distribution.

Figure 10: Asymptotic Portfolio Loss Rate Distributions for Alternative Correlations. Individual Credits Have PD=4%, 20% Initial Utilization, and 80% Revolving Balance, With Unconditional Draw Rates~*Beta* (4,1.1) and Unconditional LGDs~*Beta* (4,1.1).

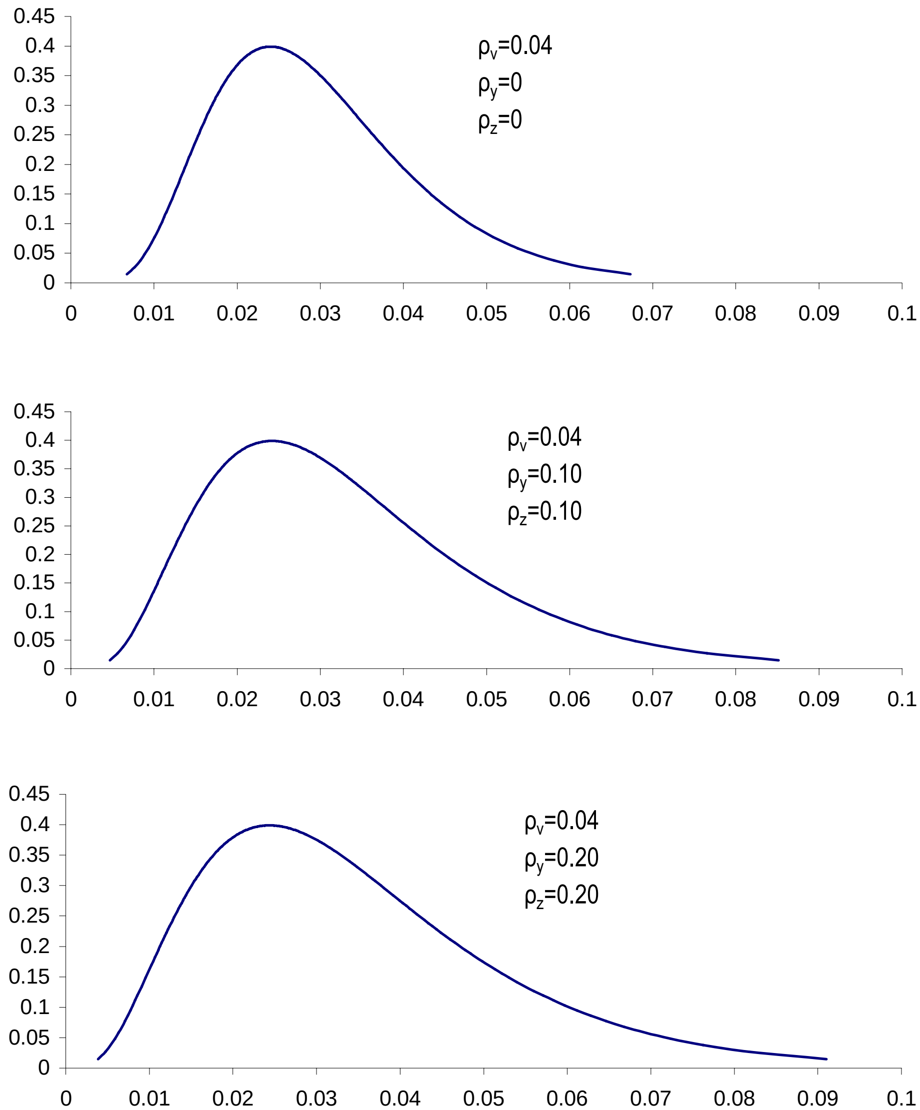


Figure 10 plots estimates of the asymptotic portfolio's loss rate distribution under alternative assumptions for *LGD* and *EAD* correlations. Unlike with the earlier two examples, the panels in Figure 10 show that correlation in individual credit *LGD* and *EAD* distributions has a relatively minor effect on the tails of the portfolio's credit loss distribution. As correlation in *EAD*s and *LGD*s increases from 0 to 10 percent (0 to 20 percent), the loss value associated with a 99.5 percent cumulative probability increases by only 26 percent (35 percent). This result is driven by extreme negative skew in the *LGD* and *EAD* distributions, as most of the probability mass is associated with high account draw and loss rates.

7. Conclusions

This paper has developed a tractable generalization of the single common factor portfolio credit loss model that includes correlated stochastic exposures and loss rates. The model yields an exact closed-form representation when individual account *LGD* and *EAD* distributions are continuous, and an alternative closed-form representation when *LGD* and *EAD* distributions are discrete. The model does not restrict *EAD* or *LGD* distributions or their correlations. The closed-form representation of an asymptotic portfolio's inverse cumulative conditional loss rate can be used to calculate both the unconditional portfolio loss rate distribution and economic capital allocations. Portfolio loss rate distributions are illustrated for representative wholesale and retail portfolios. The results show that the additional systematic risk created by positive correlation among individual account *EAD* and *LGD* realizations increases the skewness of an asymptotic portfolio's loss rate distribution. In turn, this increase in skewness increases the measured risk in lower tranches of collateralized debt obligations (CDOs) and securitizations, and mandates the need for larger economic capital allocations than those calculated with the use of the Vasicek model.

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APPENDIX

Proposition 1: $E(\widetilde{a}) = l = 1 - \lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{j=0}^{n-1} E\left(1_{\mathrm{E}(\widetilde{a},j,n)}\right)$

Using $\lim_{n\to\infty} E\left(1_{\mathrm{E}(\widetilde{a},j,n)}\right) - E\left(1_{\mathrm{E}(\widetilde{a},j-1,n)}\right) = \Xi'\left(\frac{j}{n}\right)$, the mathematical expectation of $\widetilde{a}$ can be written

$$E(\widetilde{a}) = \lim_{n\to\infty} \sum_{j=1}^{n}\left(\frac{j}{n}\right)\left(E\left(1_{\mathrm{E}(\widetilde{a},j,n)}\right) - E\left(1_{\mathrm{E}(\widetilde{a},j-1,n)}\right)\right) = 1 - \lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{j=0}^{n-1} E\left(1_{\mathrm{E}(\widetilde{a},j,n)}\right).$$

Proposition 2: $\lim_{n\to\infty}\left(\frac{1}{n}\right)B(\alpha) = E\left(\widetilde{\lambda} \mid e_M = \Phi^{-1}(1-\alpha)\right)$.

Proposition 1 and expression (25) imply,

$$E\left(\widetilde{\lambda} \mid e_M\right) = \lim_{n\to\infty}\left[1 - \left(\frac{1}{n}\right)\sum_{j=1}^{n} E\left(1_{B_j}\left(\widetilde{Y}_i \mid e_M\right)\right)\right] = \lim_{n\to\infty}\left[1 - \left(\frac{1}{n}\right)\sum_{j=1}^{n}\Phi\left(\frac{B_j - \sqrt{\rho_Y}\, e_M}{\sqrt{1-\rho_Y}}\right)\right].$$

Substitution of the definitions of the B_j values from Table 2 and $e_M = \Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$ establishes the result.

Proposition 3: $\lim_{n\to\infty\, \rho_Y\to 0}\left(\lim_{\rho_Y\to 0}\left(\frac{1}{n}\right)B(\alpha)\right) = E\left(\widetilde{\lambda}\right)$.

Substitution for expression (37) will show

$$\lim_{n\to\infty}\left[\lim_{\rho_Y\to 0}\left(\frac{1}{n}\right)\sum_{j=1}^{n}\Phi\left(\frac{\Phi^{-1}\left(1-\Theta\left(\frac{j-1}{n}\right)\right) + \sqrt{\rho_Y}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho_Y}}\right)\right],$$

$$= 1 - \left(\frac{1}{n}\right)\sum_{j=0}^{n-1}\Theta\left(\frac{j}{n}\right) = 1 - \left(\frac{1}{n}\right)\sum_{j=0}^{n-1} E\left(1_{\mathrm{E}(\widetilde{\lambda},j,n)}\right) \xrightarrow[a.s.]{} E\left(\widetilde{\lambda}\right)$$

where the final step is justified by Proposition 1.

Proposition 4: $\lim_{n\to\infty}\left(\lim_{\rho_Y\to 1}\left(\frac{1}{n}\right)B(\alpha)\right)=\Theta^{-1}(\alpha)$.

To see this, recall $\left(\frac{1}{n}\right)B(\alpha)=\frac{1}{n}\sum_{j=1}^{n}1_{B_j}\left(\tilde{Y}_i \mid e_M=\Phi^{-1}(1-\alpha)\right)$.

When $\rho_Y=1$, $\tilde{Y}_i=\tilde{e}_M$, and consequently $\left(\tilde{Y}_i \mid e_M=\Phi^{-1}(1-\alpha)\right)=\Phi^{-1}(1-\alpha)$. From Table 2, the indicator function thresholds are defined by $B_j=\Phi^{-1}\left(1-\Theta\left(\frac{j}{n}\right)\right)$. The largest increment J for which its indicator function equals one conditional on $\tilde{Y}_i=\Phi^{-1}(1-\alpha)$ is the largest J for which $\Phi^{-1}\left(1-\Theta\left(\frac{J}{n}\right)\right)\leq\Phi^{-1}(1-\alpha)$. For n sufficiently large, the step-function mesh will become fine enough so that the equality will determine the value of J. From $\Phi^{-1}(1-\alpha)=\Phi^{-1}\left(1-\Theta\left(\frac{J}{n}\right)\right)$, it is apparent that, $J=n\cdot\Theta^{-1}(\alpha)$. Consequently,

$$\lim_{n\to\infty}\left(\lim_{\rho_Y\to 1}\left(\frac{1}{n}\right)B(\alpha)\right)=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=1}^{n}1_{B_j}\left(\tilde{Y}_i \mid e_M=\Phi^{-1}(1-\alpha)\right)\right)=\frac{1}{n}J=\Theta^{-1}(\alpha).$$