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Capital Allocation for Portfolio Credit Risk

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by

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ABSTRACT

Capital allocation rules are derived that maximize leverage while maintaining a target solvency rate for credit portfolios where risk is driven by a single common factor and idiosyncratic risk is fully diversified. Equilibrium conditions ensure that capital allocations depend on interest earnings as well as credits' probability of default, endogenous loss given default, and asset correlation. Capitalization rates exceed those estimated using Gaussian credit loss models. Results demonstrate that credit risk is undercapitalized by the Basel II AIRB approach in part because of ambiguities regarding the definition of loss given default. An alternative proposed capital rule removes this bias.

Key words: economic capital, credit risk internal models, Basel II Internal Ratings Approach

JEL Classification: G12, G20, G21, G28

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1. Introduction

The market value of equity in a bank's capital structure functions as a buffer that protects all bank creditors from potential loss.[1] Other things equal, an increase in bank equity raises the probability the bank will fully perform on its contractual obligations. In the risk management literature, bank equity is often called economic capital, and the process of selecting the amount of equity in the bank's capital structure is called capital allocation.

Survey data suggest that many banks use value-at-risk (VaR) techniques to set economic capital allocations.[2] VaR-based methods of setting capital attempt to maximize bank leverage while ensuring that the potential default rate on a bank's outstanding debt is below a maximum target rate selected by management.[3] This paper develops capital allocation rules for portfolios of risky credits that are consistent with the implicit objective function underlying VaR capital methods. In the context of Black and Scholes (1973) and Merton (1974) (BSM) model, capital is set to maximize the amount of debt used to finance a credit portfolio while maintaining a pre-determined target solvency rate on the bank's financing debt.

In contrast to the Gaussian credit loss model (GCLM) approach pioneered by Vasicek (1991) and extended by Finger (1999), Schönbucher (2000), Gordy (2003) and others, this paper derives capital requirements using a full equilibrium structural model of credit risk. The use of equilibrium pricing restrictions ensures that the capital allocation rule is consistent with equilibrium relationships that exist between probability of default (PD), loss given default (LGD), yield to maturity, (YTM), and asset correlations—relationships that

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[1] The capital allocation issues discussed herein apply to non-bank firms as well.

[2] See, for example, the Basel Committee on Banking Supervision (1999).

[3] The constraint sets a minimum bank solvency rate (1 minus the bank's expected default rate).

determine the benefits that can be achieved by diversifying idiosyncratic default risk and trading off interest income against credit losses.

The survey data documenting the widespread use of VaR suggests that it is common to set capital equal to estimates of unexpected credit loss (UL). Such a methodology is, for example, used to set regulatory capital requirements for banks under the Basel II Internal Ratings Based (IRB) approach. Kupiec (2004a) demonstrates that the UL approach does not set capital consistent with the objective of maximizing leverage while maintaining a minimum solvency target. To satisfy this objective, capital must be set equal to the sum of expected loss (EL), UL, and the interest that will accrue on the bank's funding debt over the horizon of interest.

Capital allocation rules that incorporate equilibrium pricing conditions are computationally cumbersome relative to the GCLM. They require inputs that are not direct measures of credit risk. The BSM approach can be translated into an applications-friendly model using a Gaussian representation of the default process, albeit at a cost in terms of model accuracy. This Gaussian copula approximation, termed the Gaussian credit return model (GCRM), uses individual credit's PD, LGD, YTM and asset correlation as inputs. GCRM capital rules are easily calculated, but they are biased. The sign and magnitude of the bias depends on the target solvency margin, but the bias can be attenuated by incorporating a scalar adjustment factor once the solvency margin is selected.

The GCRM capital rule, while biased, outperforms the GCLM capital allocation rule proposed in the literature and implemented in the Advanced Internal Models Approach (AIRB) of Basel II. When capital allocations are set equal to UL estimated in a GCLM, capital shortfalls are large and the magnitude of the shortfalls vary across credit risk profiles. On average, capital recommended by the GCLM is only a fraction of the magnitude needed to generate the true target solvency rate.

The bias in the GCLM methodology can be attributed to a number of sources, the most obvious being the use of UL to estimate capital requirements. Capital allocations must cover EL, UL and interest on the bank's funding debt. The omission of EL from the capital calculation is a large source of bias that causes capital GCLM estimates to be understated.[4]

An equally important source of GCLM bias relates to the measurement of LGD—or more accurately, ambiguity regarding how LGD should be measured. The equilibrium model developed herein sets capital requirements using a credit portfolio's return distribution. Returns are measured relative to a portfolio's initial market value and negative portfolio returns represent portfolio credit losses. In the GCRM framework, LGD is measured as loss relative to a credit's initial market value. This current exposure measure of LGD is commonly used in practice and is fully compliant with Basel II AIRB guidance, but it is not the measure that should be used in the AIRB capital rule.

An alternative way to measure default loss is to measure loss relative to the total promised future value (future exposure). Here LGD is measured as a shortfall relative to a credit's principal plus accrued interest at the end of the capital allocation horizon. While the GCLM literature is not prescriptive as to how LGD should be measured in applications, it is shown that if LGD is measured relative to future exposure, provided one includes capital for EL, the GCLM converges to the GCRM. This convergence result not only provides the missing economic foundation for the GCLM, but it adds an important prescriptive result on the definition of LGD that has been missing in the GCLM literature.

In practice, there are significant issues surrounding the definition of LGD used in bank and vendor capital allocation models. For example, a recent report produced jointly by the International Association of Credit Portfolio Managers (IACPM) and the International Swap Dealers Association (ISDA) (2006) finds that the definition of LGD varies widely across vendor and bank internal capital models. The report concludes that variation in the definition of LGD is an important source of dispersion in capital estimates produced by the alternative vendor and bank models included in their study.

[4] In the Basel AIRB application of the GCLM, a bank must have loan loss reserves equal to EL or adjust its capital base. Reserves are a dynamic feature not incorporated into the static analysis in this paper, but corrections are made to account the absence reserve accounts.

Both simplified Gaussian models (the CGLM and CGRM) are unable to accurately reproduce the negative return tail of a credit portfolio's return (loss) distribution. The inaccuracy arises in part because these models assume fixed LGD. The full equilibrium structural model includes fully endogenous LGDs and recognizes statistical co-dependence among LGDs as well as between LGDs and PDs. These features generate return distribution characteristics that cannot be accurately reproduced in the simplified Gaussian framework. This bias in GCRM capital estimates can be effectively attenuated using a scaling factor that is calibrated according to the selected solvency target. The bias in the GCLM is not simply repaired as this rule excludes EL which varies with PD and LGD.

The results in this paper help to explain concerns that have been raised regarding the capital requirements that are set by the Basel II AIRB approach. Quantitative Impact Studies (QIS) conducted both in the US and in other countries have found substantial declines in most banks' required minimum capital requirements under the AIRB framework.[5] Across reporting banks, the QIS results show wide variation in capital requirements for positions with similar risk. The AIRB formula is based on the GCLM UL framework and includes the biases that are identified in this study. AIRB capital requirements are very sensitive to the definition of LGD and use of the current exposure measure of LGD will result in undercapitalized exposures. The variation in industry practice regarding estimation of LGD identified in the ICAPM-ISDA study is a potential explanation for both the level and dispersion in QIS results.

An outline of this paper follows. Section 2 summarizes the methodology for constructing optimal capital allocations and demonstrates that UL measures of capital are downward biased. Section 3 revisits the capital allocation problem in the context of the Black-Scholes-Merton (BSM) model. Section 4 derives closed-form portfolio-invariant capital allocation rules for a single common factor version of the BSM model in which idiosyncratic risk is fully diversified. Section 5 reviews Gaussian copula methods and derives the GCLM, GCRM, and the respective capital allocation functions. Section 6 reports the

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[5] See, *Summary Findings of the Fourth Quantitative Impact Study* (2006), and BCBS, *Results of the fifth quantitative impact study* (2006a).

results of a calibration exercise in which alternative capital allocation rules are compared. Section 7 discusses the importance of the definition of LGD and reviews evidence on market practices. Section 8 concludes the paper.

2. Optimal Capital Allocation for Credit Risks

It is assumed that capital allocations are set to maximize leverage using a single class of discount funding debt while ensuring that the bank is able to meet all the associated interest and principal payments with a minimum probability of α. α is the bank's target solvency rate; $1-\alpha$. is the bank's *ex ante* target probability of default

Let T represent the capital allocation horizon of interest. The purchased asset A, has an initial market value A_0, a time T random value of $\tilde{A}_T$ with a cumulative density function $\Psi(\tilde{A}_T, A_T)$, and a probability density function $\psi(\tilde{A}_T, A_T)$. Let $\Psi^{-1}(\tilde{A}_T, 1-\alpha)$ represent the inverse of the cumulative density function of $\tilde{A}_T$ evaluated at $(1-\alpha)$, $\alpha \in [0,1]$. Define an α coverage VaR measure, $VaR(\alpha)$, as,

$$VaR(\alpha) = A_0 - \Psi^{-1}(\tilde{A}_T, 1-\alpha) \tag{1}$$

$VaR(\alpha)$ is the loss amount that is exceeded by at most $(1-\alpha)$ of all potential future value realizations of $\tilde{A}_T$. Expression (1) measures value-at-risk relative to the initial market value of the asset.

Consider a capital allocation rule that sets equity capital equal to $VaR(\alpha)$. By construction, the probability of realizing a loss larger than $VaR(\alpha)$ at time T is bounded above by $100(1-\alpha)$ percent. The probability that the bank will experience a loss that exceeds its initial equity value is at most $100(1-\alpha)$ percent. Under this capital allocation rule, the bank must borrow $A_0 - VaR(\alpha)$ to finance the investment. If the bank borrows $A_0 - VaR(\alpha)$, it must promise to pay back *more* than $A_0 - VaR(\alpha)$ if equilibrium interest rates and credit risk compensation are positive. Because the $VaR(\alpha)$ capital allocation rule ignores time and the equilibrium returns that are required by bank creditors, the probability that the bank will default on its funding debt is greater than $1-\alpha$ if the bank's debts can only be satisfied by raising funds through the sale of $\tilde{A}_T$ at time T. Thus a $VaR(\alpha)$ capital allocation rule does not meet management's capital allocation objective of maximizing debt subject to maintaining a minimum solvency rate α .

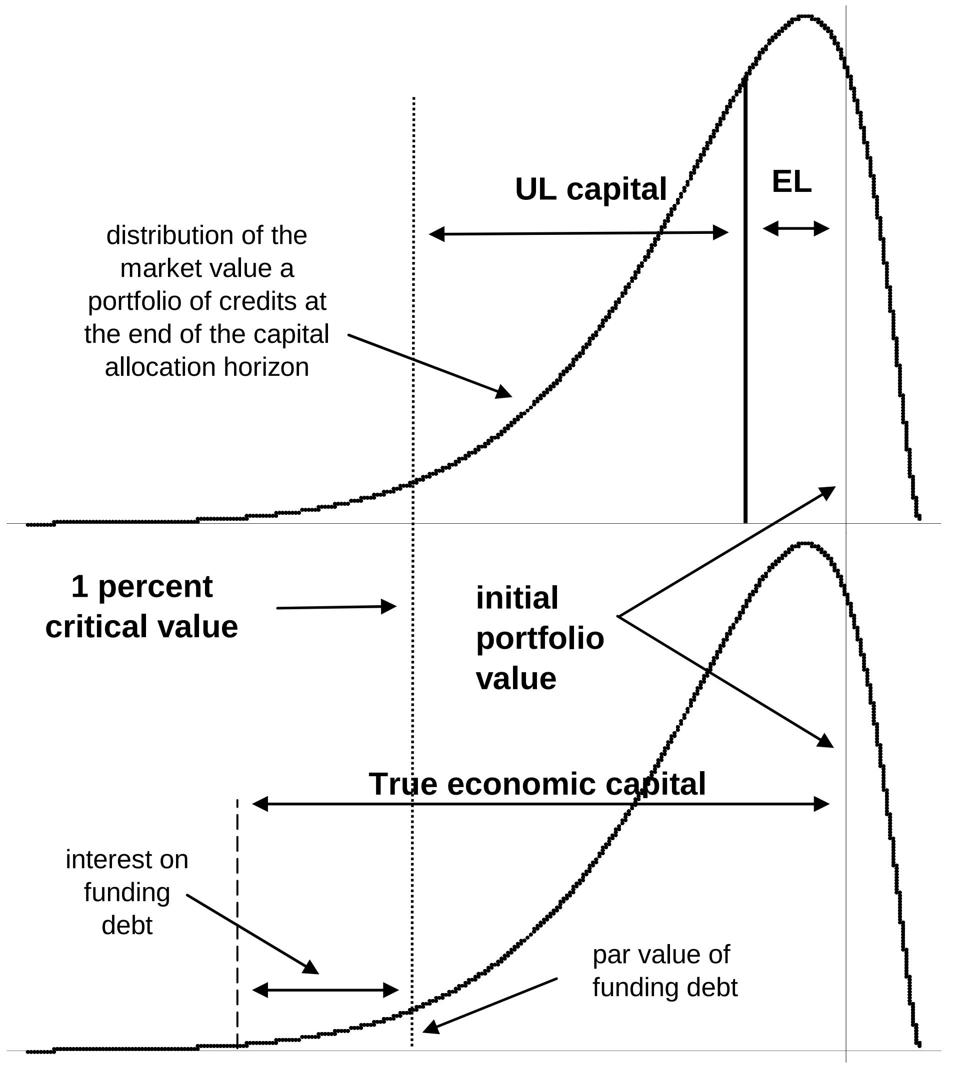
In order to meet the objective of maximizing leverage subject to maintaining a minimum solvency rate, the required capital allocation rule is: set equity capital equal to $VaR(\alpha)$ plus the interest that will accrue on the bank's borrowings. $VaR(\alpha)$ is measured relative to initial asset value and thus capital includes both EL and UL. An equivalent allocation is achieved by setting the par (maturity) value of the funding debt equal to $VaR(\alpha)$ and estimating the funding debt's market value at issuance. The difference between the initial market value of the purchased asset and the proceeds from the funding debt issue is the economic capital needed to fund the investment.

UL Capital Allocation Rules

The literature on capital allocation often recommends setting economic capital equal to unexpected loss which may be defined as, $UL(\alpha) = VaR(\alpha) - EL$, where,

$EL = A_0 - \left(\int_{-\infty}^{A_0} \psi(\tilde{A}_T, A_T)\, dA_T \right)^{-1} \int_{-\infty}^{A_0} A_T\, \psi(\tilde{A}_T, A_T)\, dA_T$. The second term in the expression for EL is the expected end-of-period asset value conditional on the asset posting a loss relative to the asset's initial value. Because $0 \le EL \le A_0$, $VaR(\alpha) - EL \le VaR(\alpha)$. Since the default rate associated with a $VaR(\alpha)$ capital allocation rule exceeds $(1-\alpha)$ when interest rates are positive, it follows that the true default rate associated with a $UL(\alpha)$ capital allocation rule will exceed $(1-\alpha)$ if $EL>0$. The UL methodology is compared to the accurate approach for setting economic capital in Figure 1 for a 99 percent target solvency rate.

Figure 1: Alternative Capital Allocation Methodolgies



3. Optimal Capital Allocation in the Black-Scholes-Merton (BSM) Model

Estimation of the equilibrium interest cost on funding debt requires the use of formal asset pricing models or an empirical approximation to value a bank's funding debt. If the risk-free term structure is flat and a firm issues only pure discount bonds, and asset values follow geometric Brownian motion, under simplifying assumptions,[6] BSM establish that the market value of a firm's debt issue is equal to the discounted value of the bond's par value (at the risk free rate), less the market value of a Black-Scholes put option written on the value of the firm's assets. The put option has a maturity identical to the bond's maturity, and a strike price equal to the par value of the bond.

Consider a bank whose only asset is a risky BSM discount bond that matures at date M and is issued by an unrelated counterparty. Assume that the bank will fund this bond with its own discount debt and equity securities. In this setting, the bank's funding debt issue is a compound option.

Let $\tilde{A}_T$ and Par_P represent, respectively, the time T value of the assets that support the discount debt investment and the par value of the purchased bond,

$\tilde{A}_T \sim A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\tilde{z}}$ where $\tilde{z}$ is a standard normal variable and $\sigma > 0$ is the asset's instantaneous geometric return volatility. Equilibrium conditions restrict the asset's physical drift rate, $\mu = r_f + \lambda\sigma,$ where λ is the market price of risk and r_f is the instantaneous risk free rate.

Let Par_F represent the par value of the discount bond that is issued by the bank to fund the investment. To simplify the discussion, we restrict attention to the case where the maturity of the bank's funding debt matches the maturity of the BSM asset and both equal to M.[7] The end-of-period cash flows that accrue to the bank's debt holders are,

$$Min\left[Min\left(\tilde{A}_M, Par_P\right), Par_F\right]. \qquad (2)$$

[6] There are no taxes, transactions are costless, short sales are possible, trading takes place continuously, if borrowers and savers have access to the debt market on identical risk-adjusted terms, and investors in asset markets act as perfect competitors.

[7] See Kupiec (2004a) for pricing when the funding debt matures before the investment.

The initial equilibrium market value of the bank's discount bond issue is the discounted (at the risk free rate) expected value of the end-of-period funding debt cash flows taken with respect to the equivalent martingale probability distribution for $\tilde{A}_M^Q$,

$$\tilde{A}_M^Q \sim A_0\, e^{\left(r_f - \frac{\sigma^2}{2}\right)M + \sigma\sqrt{M}\,\tilde{z}} \tag{3}$$

The initial market value of the bank's funding debt is,

$$E\left[Min\left[Min\left(\tilde{A}_M^Q, Par_P\right), Par_F\right]\right] e^{-r_f M} \tag{4}$$

At maturity, the payoff of the bank's purchased bond is $Min\left[Par_P, \tilde{A}_M\right]$. Let $\Phi(x)$ represent the cumulative standard normal distribution function evaluated at x, and $\Phi^{-1}(\alpha)$ represent the inverse of this function for $\alpha \in [0,1]$. The upper bound on the funding debt par value consistent with the target solvency constraint is,

$$Par_F(\alpha) = \Psi^{-1}\left(\tilde{A}_T, 1-\alpha\right) = A_0\, e^{\left[\mu - \frac{\sigma^2}{2}\right]M + \sigma\sqrt{M}\Phi^{-1}(1-\alpha)}. \tag{5}$$

The initial market value of this funding debt issue is, $B_{F0}(\alpha)$,

$$B_{F0}(\alpha) = E\left[Min\left[Min\left(\tilde{A}_M^Q, Par_P\right), Par_F(\alpha)\right]\right] e^{-r_f M}. \tag{6}$$

The initial equity allocation consistent with the target solvency rate α, $E(\alpha)$, is,

$$E(\alpha) = B_0 - E\left[Min\left[Min\left(\tilde{A}_M^Q, Par_P\right), Par_F(\alpha)\right]\right] e^{-r_f M}. \tag{7}$$

When the probability of default on the purchased bond exceeds $(1-\alpha)$, capital is required, and $Par_F(\alpha) < Par_P$.[8] In this case, expression (7) simplifies and the bank's debt is valued as simple BSM bond with a par value of $Par_F(\alpha)$. In this case, the dollar value of required equity is,

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[8] In the single asset case, when the probability of default on the purchased bond is less than or equal to $(1-\alpha)$, the bond can be financed 100 percent with bank debt ($Par_F(\alpha) = Par_P$) without violating the solvency constraint.

$$B_0 - Par_F(\alpha)\, e^{-r_f}\, \Phi\left(\frac{\ln(A_0) - \ln(Par_F(\alpha)) + \left(r_f - \frac{\sigma^2}{2} \right)}{\sigma} \right) - A_0\, \Phi\left(\frac{\ln(Par_F(\alpha)) - \ln(A_0) - \left(r_f + \frac{\sigma^2}{2} \right)}{\sigma} \right) \tag{8}$$

Portfolio Capital

The portfolio capital calculation is analogous to the calculation for a single asset. In most cases, credit portfolios do not have "user-friendly" density functions that admit a closed-form expression for either the par value of the funding debt or its initial market value. Monte Carlo simulation is often required to estimate $VaR(\alpha)$ and the par value of the funding debt. Pricing the funding debt may require numerical evaluation of a high order integral. The next section considers portfolio capital allocation under assumptions that reduce significantly the complexity of portfolio capital calculations.

4. Optimal Capital Allocation under Asymptotic Single Factor Assumptions

Optimal capital allocation calculations are simplified if a portfolio is perfectly diversified and asset values are driven by a single common factor in addition to individual idiosyncratic factors. Let dW_M represents a standard Wiener that is common in all asset price dynamics, and dW_i represents a standard Weiner process idiosyncratic to the price dynamics of asset i. Asset price dynamics for firm i are given by,

$$dA_i = \mu\, A_i\, dt + \sigma_M\, A_i\, dW_M + \sigma_i A_i\, dW_i, \tag{9}$$

$$dW_i\, dW_j = \rho_{ij} = 0, \quad \forall\, i, j.$$
$$dW_i\, dW_M = \rho_{im} = 0, \forall\, i.$$

Under these dynamics, asset prices are log normally distributed,

$$\tilde{A}_{iT} = A_{i0}\, e^{\left[r_f + \lambda\, \sigma_M - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right) \right] T + \left(\sigma_M \tilde{z}_M + \sigma_i \tilde{z}_i\right)\sqrt{T}}, \tag{10}$$

where $\tilde{z}_M$ and $\tilde{z}_i$ are independent standard normal random variables. Under the equivalent martingale change of measure, asset values at time T are distributed,

$$\tilde{A}_{iT}^{Q} = A_{i0}\, e^{\left[r_f - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right)\right]T + \left(\sigma_M \tilde{z}_M + \sigma_i \tilde{z}_i\right)\sqrt{T}} . \tag{11}$$

Under these price dynamics, the correlation between geometric asset returns is,

$$Corr\left[\frac{1}{T}\ln\left(\frac{\tilde{A}_{it}}{A_{i0}}\right), \frac{1}{T}\ln\left(\frac{\tilde{A}_{jt}}{A_{j0}}\right)\right] = \frac{\sigma_M^2}{\left(\sigma_M^2 + \sigma_i^{\,2}\right)^{\frac{1}{2}}\left(\sigma_M^2 + \sigma_j^{\,2}\right)^{\frac{1}{2}}},\ \forall i, j. \tag{12}$$

If the model is further specialized so that the volatilities of assets' idiosyncratic factors are assumed identical, $\sigma_i = \sigma_j = \overline{\sigma}, \forall\, i, j,$ the pair-wise asset return correlations are,

$$\rho = Corr\left[\frac{1}{T}\ln\left(\frac{\tilde{A}_{it}}{A_{i0}}\right), \frac{1}{T}\ln\left(\frac{\tilde{A}_{jt}}{A_{j0}}\right)\right] = \frac{\sigma_M^2}{\sigma_M^2 + \overline{\sigma}^2}\quad \forall i, j. \tag{13}$$

Asset Return Distributions

The T-period rate of return on BSM risky bond i that is held to maturity is,

$$\tilde{M}_{iT} = \frac{1}{B_{i0}}\left(Min\left(\tilde{A}_{iT}, Par_i\right)\right) - 1 . \tag{14}$$

$\tilde{M}_{iT}$ is bounded in the interval $[-1, a_i]$, where $a_i = \frac{Par_i}{B_{i0}} - 1$. A bond's physical return distribution (14) has an associated equivalent martingale return distribution,

$$\tilde{M}_{iT}^{Q} = \frac{1}{B_{i0}}\left(Min\left(\tilde{A}_{iT}^{Q}, Par_i\right)\right) - 1 . \tag{15}$$

By construction, expressions (14) and (15) have identical support.

Asymptotic Portfolio Return Distribution

The *T*-period return on a portfolio of *n* risky individual credits, ${}_P\tilde{M}_T$, is

$$ {}_P\tilde{M}_T \equiv \frac{\sum_{i=1}^{n} \tilde{M}_{iT} B_{i0}}{\sum_{i=1}^{n} B_{i0}} \tag{16} $$

Let $\left({}_P\tilde{M}_T \middle| \tilde{z}_M = z_M\right) = {}_P\tilde{M}_T \middle| z_M$ represent the portfolio return conditional on a realization of the common market factor, $\tilde{z}_m = z_M$,

$$ {}_P\tilde{M}_T \Big| z_M = \frac{\sum_{i=1}^{n} \left(\tilde{M}_{iT} \middle| z_M\right) \cdot B_{i0}}{\sum_{i=1}^{n} B_{i0}} \tag{17} $$

Let $\psi\left(\tilde{M}_{iT} \middle| z_M\right)$ represents the conditional return density function. Under the single common factor assumption, $\psi\left(\tilde{M}_{iT} \middle| z_M\right)$ and $\psi\left(\tilde{M}_{jT} \middle| z_M\right)$ are independent for $\forall i \neq j.$[9]

Consider a portfolio composed of equal investments in individual bonds that share identical *ex ante* credit risk profiles; that is, the bonds in the portfolio are identical regarding par value $\{Par_i = Par_j, \forall i, j\}$, maturity $\{T\}$, and volatility characteristics, $\{\sigma_i = \sigma_j = \overline{\sigma}, \forall i, j\}$. The bonds have conditional returns that are independent and

[9] Independence in this non-Gaussian setting requires that an observation of the return to bond j be uninformative regarding the conditional distribution function for bond i, $\Pr\left(\tilde{M}_{it} \mid z_M\right) < a = \Pr\left(\left(\tilde{M}_{it} \mid z_M\right) < a \text{ given that } \tilde{M}_{jt} = M_{jt}\right), \forall a, i \neq j.$ This condition is satisfied under the single common factor model assumption.

identically distributed with finite means. As the number of bonds in portfolio, N, grows without bound, the Strong Law of Large Numbers requires,

$$\underset{n\to\infty}{Lim}\left[{}_P\tilde{M}_T \mid z_M\right] = \underset{n\to\infty}{Lim}\left[\frac{\sum_{i=1}^{n}\tilde{M}_{iT}\Big|z_M}{n}\right] \xrightarrow[a.s]{} E\left[\psi\left(\tilde{M}_{iT}\Big|z_M\right)\right] \;\forall z_M \tag{18}$$

The notation $a.s.$ indicates convergence with probability one. Under the BSM single factor assumptions, expression (18) is,

$$\begin{aligned}\underset{n\to\infty}{Lim}\left[{}_P\tilde{M}_T\Big|z_M\right] &= \frac{Par_i}{B_{i0}}\left[1-\Phi\left(w_{iT}^{*}(z_M)\right)\right] \\ &+ \frac{G(z_M)}{B_{i0}}\left[1-\Phi\left(-w_{iT}^{*}(z_M)+\gamma_{iT}\right)\right]-1\end{aligned} \tag{19}$$

where,

$$\mu_{iT}(z_M) = \ln[A_{i0}] + \left[r_f + \lambda\sigma_M - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right)\right]T + z_M\sigma_M\sqrt{T}$$

$$\gamma_{iT} = \sigma_i\sqrt{T}$$

$$w_{iT}^{*}(z_M) = \frac{\ln[Par_i] - \mu_{iT}(z_M)}{\gamma_{iT}}$$

$$G(z_M) = e^{\mu_{iT}(z_M) + \frac{\gamma_{iT}^2}{2}}$$

The conditional equivalent martingale portfolio return distribution is given by,

$$\begin{aligned}\underset{n\to\infty}{Lim}\left[{}_P\tilde{M}_T^{Q}\Big|z_M\right] &= \frac{Par_i}{B_{i0}}\left[1-\Phi\left(w_{iT}^{Q*}(z_M)\right)\right] \\ &+ \frac{G^{Q}(z_M)}{B_{i0}}\left[1-\Phi\left(-w_{iT}^{Q*}(z_M)+\gamma_{iT}\right)\right]-1\end{aligned} \tag{20}$$

where,

$$\mu_{iT}^{Q}\left(z_M\right)=\ln\left[A_{i0}\right]+\left[r_f-\frac{1}{2}\left(\sigma_M^2+\sigma_i^2\right)\right]T+z_M\sigma_M\sqrt{T}$$

$$w_{iT}^{Q*}\left(z_M\right)=\frac{\ln[Par_i]-\mu^{Q}{}_{iT}\left(z_M\right)}{\gamma_{iT}}$$

$$G^{Q}(z_M)=e^{\mu_{iT}^{Q}\left(z_M\right)+\frac{\gamma_{iT}^2}{2}}$$

Optimal Portfolio Capital Allocation

Expressed as a proportion of the portfolio's initial market value, the optimal par value of funding debt can be determined by setting $z_M=\Phi^{-1}(1-\alpha)$ and using expression (19) to solve for the end-of-horizon portfolio critical value,

$$par_F^P(\alpha)=\left(\begin{array}{l}\frac{Par_i}{B_{i0}}\left[1-\Phi\left(w_{iT}^{*}\left(\Phi^{-1}(1-\alpha)\right)\right)\right]\\ \quad+\frac{Q(z_M)}{B_{i0}}\left[1-\Phi\left(-w_{iT}^{*}\left(\Phi^{-1}(1-\alpha)\right)+\gamma_{iT}\right)\right]\end{array}\right) \qquad (21)$$

To determine the market value of the funding debt, it is necessary to solve for the limits of integration, $\hat{z}_M$, under the equivalent martingale measure,

$$\hat{z}_M=\Phi^{-1}(1-\alpha)+\frac{\lambda}{\sqrt{T}} \qquad (22)$$

Expressed as a proportion of the investment portfolio's market value, the initial market value of the funding issue, $b_{F0}^{P}(\alpha)$, is ,

$$b_{F0}^{P}(\alpha) = e^{-r_f T} \begin{pmatrix} \int_{-\infty}^{\hat{z}_M} \left[\frac{Par_i}{B_{i0}} \left[1 - \Phi\left(w_{iT}^{Q*}(z_M)\right)\right] \right] \phi(z_M)\, dz_M \\ + \int_{-\infty}^{\hat{z}_M} \left[\frac{G^Q(z_M)}{B_{i0}} \left[1 - \Phi\left(-w_{iT}^{Q*}(z_M) + \gamma_{iT}\right)\right] \right] \phi(z_M)\, dz_M \\ + par_F^P(\alpha)\left[1 - \Phi(\hat{z}_M)\right] \end{pmatrix} \tag{23}$$

The economic capital allocation for the portfolio, expressed as a proportion of the portfolio's initial market value, $K_{BSM}^{P}(\alpha)$ is,

$$K_{BSM}^{P}(\alpha) = 1 - e^{-r_f T} \begin{pmatrix} \int_{-\infty}^{\hat{z}_M} \left[\frac{Par_i}{B_{i0}} \left[1 - \Phi\left(z_{iT}^{Q*}(z_M)\right)\right] \right] \phi(z_M)\, dz_M \\ + \int_{-\infty}^{\hat{z}_M} \left[\frac{G^Q(z_M)}{B_{i0}} \left[1 - \Phi\left(-z_{iT}^{Q*}(z_M) + \gamma_{iT}\right)\right] \right] \phi(z_M)\, dz_M \\ + par_F^P(\alpha)\left[1 - \Phi(\hat{z}_M)\right] \end{pmatrix} \tag{24}$$

The dollar value capital requirement is $\sum_{i=1}^{n} B_{i0}\, K_{BSM}^{P}(\alpha)$.

Because idiosyncratic risk is fully diversified, when an additional credit is added to the portfolio, the marginal capital required to maintain the target solvency margin is equal to the portfolio's average capitalization rate multiplied by the market value of the marginal credit added to the portfolio. Expression (24) represents the required capitalization rate for both the average and the marginal credit in an asymptotic portfolio when credit risks are priced to satisfy BSM equilibrium conditions. While the evaluation of expression (24) is straightforward as a numerical exercise, the expression for optimal capital does not include inputs that are recognizable and familiar to risk managers (e.g., PD, LGD, asset correlation). This can be addressed by approximating the optimal capitalization rule using a Gaussian asymptotic single factor approximation. The resulting model, developed in the following section, is new and will be referred to as the Gaussian credit return model, or GCRM.

5. The Gaussian Single Risk Factor Framework

A Gaussian factor model approach can be used to construct a simplified version of the BSM structural model of credit risk. The standard Gaussian specification for modeling portfolio credit risk is uses a standardized normal random variable with the properties,

$$
\begin{aligned}
&\tilde{V}_i = \sqrt{\rho}\,\tilde{e}_M + \sqrt{1-\rho}\,\tilde{e}_i \\
&\tilde{e}_M \sim \phi(e_M) \\
&e_i \sim \phi(e_i), \\
&E(\tilde{e}_i\,\tilde{e}_j) = E(\tilde{e}_M\,\tilde{e}_j) = 0\ \forall\, i,\, j.
\end{aligned}
\tag{25}
$$

$\tilde{V}_i$ is normally distributed with $E\left(\tilde{V}_i\right)=0,$ and $E\left(\tilde{V}_i^{\,2}\right)=1.$ $\tilde{e}_M$ is the market factor common to all firm asset values. The correlation between asset values is $\rho.$

Firm *i* defaults when $\tilde{V}_i < D_i$. The unconditional probability that firm *i* will default is, $PD = \Phi\left(D_i\right).$ The loss incurred should the firm default, *LGD*, is specified exogenously and does not include restrictions as to how *LGD* is measured. In most applications, *LGD* is set equal to a constant value calibrated from historical loss data. Time does not play an independent role in these models but is implicitly recognized through the calibration of input values; generally *PD* differs according to the capital allocation horizon.

Consider a portfolio composed of *N* credits with identical initial market values, promised future values, correlations, ρ, and default thresholds, $D_i = D.$ The loss distribution for a portfolio is defined using an indicator function,

$$\tilde{I}_i = \begin{cases} 1 & if \quad \tilde{V}_i < D \\ 0 & otherwise \end{cases} \tag{26}$$

$\tilde{I}_i$ has a binomial distribution with an expected value of $\Phi(D)$. Define $\tilde{I}_i \mid e_M$ to be the value of the indicator function conditional on a realized value for e_M. Conditional default indicators are independent and identically distributed binomial random variables,

$$\begin{aligned} &E\left(\tilde{I}_i \mid e_M\right) = \Phi\left(\frac{D - \sqrt{\rho}\, e_M}{\sqrt{1-\rho}}\right), \quad \forall\, i \\ &E\left(\tilde{I}_i \mid e_M - E\left(\tilde{I}_i \mid e_M\right)\right)^2 = \Phi\left(\frac{D - \sqrt{\rho}\, e_M}{\sqrt{1-\rho}}\right)\left(1 - \Phi\left(\frac{D - \sqrt{\rho}\, e_M}{\sqrt{1-\rho}}\right)\right), \quad \forall\, i \\ &E\left(\left(\tilde{I}_i \mid e_M\right)\left(\tilde{I}_j \mid e_M\right)\right) = 0, \quad \forall\, i \neq j. \end{aligned} \tag{27}$$

Define $\tilde{X} \mid e_M$ as the proportion of credits in the portfolio that default conditional on a realization of e_M, $\tilde{X} \mid e_M = \frac{\sum_{i=1}^{n}\left(\tilde{I}_i \mid e_M\right)}{n}$. Because $\tilde{I}_i \mid e_M$ are independent and identically distributed, the Strong Law of Large Numbers requires, for all e_M,

$$\lim_{n\to\infty}\left(\tilde{X} \mid e_M\right) = \lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n}\left(\tilde{I}_i \mid e_M\right)}{n}\right) \xrightarrow[a.s.]{} E\left(\tilde{I}_i \mid e_M\right) = \Phi\left(\frac{D - \sqrt{\rho}\, e_M}{\sqrt{1-\rho}}\right) \tag{28}$$

The unconditional distribution function of $\tilde{X}$ can be derived using expression (28) and information on the density of $\tilde{e}_M$. Because realized values of *X* are monotonically decreasing in e_M,

$$
\begin{aligned}
\Pr\left[\tilde{X} \le x\right] &= \Pr\left[\tilde{e}_M \ \ge \frac{D - \Phi^{-1}(x)\sqrt{1-\rho}}{\sqrt{\rho}}\right] \\
&= \Pr\left[-\tilde{e}_M \ \le \frac{\sqrt{1-\rho}\ \Phi^{-1}(x) - D}{\sqrt{\rho}}\right] = \Phi\left(\frac{\sqrt{1-\rho}\ \Phi^{-1}(x) - D}{\sqrt{\rho}}\right)
\end{aligned}
\tag{29}
$$

Substituting for the default barrier, $D = \Phi^{-1}(PD)$, the unconditional cumulative distribution function for $\tilde{X}$, the proportion of portfolio defaults, is given by,

$$
\Pr\left[\tilde{X} \le x\right] = \Phi\left(\frac{\sqrt{1-\rho}\ \Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right), \quad x \in [0,1] \tag{30}
$$

Capital Allocation under the Gaussian Credit Loss Model (GCLM)

If LGD represents the percentage of a credit's value that is lost if a credit defaults, the portfolio credit loss rate is defined by the distribution of $\tilde{X}$,

$$
\Pr\left[LGD \cdot \tilde{X} \le LGD \cdot x\right] = \Phi\left(\frac{\Phi^{-1}(x)\sqrt{1-\rho} - \Phi^{-1}(PD)}{\sqrt{\rho}}\right), \quad x \in [0,1] \tag{31}
$$

The CGLM literature is not specific as to how LGD should be defined. LGD can be defined as a loss rate measured relative to credit's promised future value (principal + accrued interest) or a loss rate relative to a credit's initial value, or perhaps in other ways.

The GCLM model (expression (30)), introduced by Vasicek (1991), is widely used for modeling portfolio credit losses and setting credit risk capital allocations. Credit value-at-risk (VaR) techniques often recommend setting economic capital equal to unexpected credit loss (*UL)*. For a solvency margin target of α , $UL(\alpha)$ is defined as the difference between the $(1-\alpha)$ critical value of the GCLM loss distribution, less the portfolio's expected loss, $LGD \cdot PD.$ The $UL(\alpha)$ capital allocation measured as a percentage of the investment portfolio's initial value is,

$$K_{UL}(\alpha) = LGD \cdot \Phi\left(\frac{\sqrt{\rho}\ \Phi^{-1}(\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}} \right) - LGD \cdot PD \tag{32}$$

The Basel II AIRB approach is based on expression (32). The AIRB capital rule is for corporate, bank and sovereign credits is,

$$K = EAD \cdot \left[LGD \times \Phi\left[\frac{1}{\sqrt{1-R}} \Phi^{-1}(PD) + \sqrt{\frac{R}{1-R}} \Phi^{-1}(.999) \right] - PD \times LGD \right] \left(\frac{1+(M-2.5)b}{1-1.5b} \right) \tag{33}$$

$$R = 0.12\left(\frac{1-e^{-50\,PD}}{1-e^{-50}} \right) + 0.24\left(1 - \frac{1-e^{-50\,PD}}{1-e^{-50}} \right),\ b = (0.11852 - .05478\,Ln(PD))^2,$$

where *EAD* is exposure at default, *PD* is a credit's probability of default expressed as a percentage, *LGD* is a credit's expected loss given default expressed as a percentage, *M* is the credit's maturity in measured in years, and *K* represents the dollar capital requirement. The *R* function and maturity adjustment factor and are *ad hoc* functions that were introduced by the BCBS as a means for "tuning" the capital calibration. The *R* function is a regulatory rule that links a portfolio's asset correlation to the PD of its individual credits–low PD credits are specified to have higher asset correlation values. The finial term in parenthesis in equation (33) is the maturity adjustment factor. When $M = 1$, there is no capital adjustment for maturity, $\frac{1+(M-2.5)\,b}{1-1.5b} = 1.$

Capital Allocation under the Gaussian Credit Return Model (GCRM)

In order to estimate an optimal capital allocation in a manner consistent with the BSM model, it is necessary to derive the end-of-horizon *return* distribution for a portfolio of credits.[10] Let *YTM* represent the yield to maturity calculated using initial market value of an individual credit. Let *LGD* represent the loss from initial loan value should a loan default. Conditional on a realized value of $\tilde{X} = X$, the end-of-horizon return on the portfolio is given by,

$$\left(R_p \mid X\right) = YTM - (YTM + LGD)\, X \tag{34}$$

The portfolio's conditional end-of-period return is monotonically decreasing in the portfolio default rate X. The unconditional cumulative return distribution for the portfolio, $\tilde{R}_P$ is,

$$\begin{aligned} \Pr\left[\tilde{R}_p \le R_P\right] &= 1 - \Pr\left[\tilde{X} < \left(\frac{YTM - R_p}{YTM + LGD}\right)\right] \\ &= \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{1-\rho}\;\Phi^{-1}\left(\frac{YTM - R_p}{YTM + LGD}\right)}{\sqrt{\rho}}\right), \quad R_p \in \left[-(1 - LGD), YTM\right] \end{aligned} \tag{34}$$

Under a target solvency margin of α, the par value of the funding debt is determined by the $(1-\alpha)$ critical value of expression (35). Measured as a proportion of the investment portfolio's initial value, the par value of the funding debt is,

$$par_F^{GCRM}(\alpha) = \left(1 + YTM - (YTM + LGD)\Phi\left(\frac{\sqrt{\rho}\;\Phi^{-1}(\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)\right) \tag{36}$$

Because the GCRM does not include dynamic restrictions that ensure absence of arbitrage equilibrium conditions hold, the model does not contain the information necessary to construct the equivalent martingale measure to price the funding debt. If the portfolio is

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[10] In the remainder of the discussion, consistent with Basle II capital rules, the horizon is assumed to be 1 year.

100 percent debt financed, in the absence of taxes or government safety net subsidies to the bank, the Modigliani-Miller (1958) theorem ensures that equilibrium interest rate required on the bank's funding debt at the time of issuance is equal to the *YTM* on the credits in the banks investment portfolio.[11] When the share of equity funding is increased above zero, the equilibrium issuance *YTM* on the bank's funding debt will decline. Using *YTM* as a conservative estimate of the required market rate of return on the bank's funding debt at issuance, the initial market value of the funding debt measured as a proportion of the investment portfolio's initial market value is,

$$b_{F0}^{GCRM} \approx \frac{1}{1+YTM}\left(1+YTM-\left(YTM+LGD\right)\Phi\left(\frac{\sqrt{\rho}\ \Phi^{-1}(\alpha)+\Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)\right) \tag{37}$$

The required capitalization rate for the investment portfolio and its constituent credits is,

$$\hat{K}_{GCRM}(\alpha) \approx \frac{YTM+LGD}{1+YTM}\Phi\left(\frac{\sqrt{\rho}\ \Phi^{-1}(\alpha)+\Phi^{-1}(PD)}{\sqrt{1-\rho}}\right) \tag{38}$$

6. Capital Allocation Performance

In the analysis that follows, portfolio capital requirements are calculated using the full structural equilibrium model (expression (24)), the GCLM (expression (32)), and the GCRM (expression (38)) for asymptotic portfolios of BSM credits where the credits have a wide range of risk characteristics. Consistent with BCBS Basel II objectives, capital allocations are

[11] Merton (1974) provides a more modern proof of the Modigliani-Miller theorem. Kupiec (2004b) discusses the implications of non-priced implicit or explicit safety net guarantees on a bank's capital allocation process.

estimated for the 99.9 percent solvency margin and BSM capital allocations are taken as the benchmark or "true" capital required.

The asset price dynamics that are maintained throughout the analysis appear in Table 1. All individual credits have identical firm specific risk factor volatilities of 20 percent and a common factor volatility of 10 percent; these imply an asset return correlation of 20 percent. The market price of risk is 10 percent and the risk free rate is 5 percent.

All credits in an asymptotic portfolio have the same initial value, identical promised par values and all share identical *ex ante* credit risk profiles. The par values of individual credits are altered to change the credit risk characteristics of an asymptotic portfolio. The calibration analysis focuses on a one-year capital allocation horizon for one-year credits.

Table 1: Calibration Assumptions

risk free rate	$r_f = .05$
market price of risk	$\lambda = .10$
market factor volatility	$\sigma_M = .10$
Firm specific volatility	$\overline{\sigma}_i = .20$
Initial market value of assets	$A_0 = 100$
correlation between asset returns	$\rho = .20$

Under the single common factor BSM model assumptions, the physical probability that a bond defaults is,

$$PD = \Phi\left(z_i^{df}\right)$$

$$z_i^{df} = \frac{Log\left(Par_i\right) - Log\left(A_{i0}\right) - \left(r_f + \lambda\sigma_M - \frac{\sigma_M^2 + \bar{\sigma}_i^2}{2}\right)T}{\sqrt{T}\sqrt{\sigma_M^2 + \bar{\sigma}_i^2}} \qquad (39)$$

The expected value of the bond's payoff given a default is,

$$E\left[Min\left(\tilde{A}_{iT}, Par_i\right) | A_{iT} < Par_i\right] = \frac{1}{\Phi\left(z_i^{df}\right)} \int_{-\infty}^{z_i^{df}} A_{i0} e^{\left(r_f + \lambda\sigma_M - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right)\right)T + \sqrt{T\left(\sigma_M^2 + \sigma_i^2\right)}\,z} \phi(z)\,dz\,. \qquad (40)$$

A bond's LGD measured from initial market value is,

$$LGD = 1 - \frac{1}{B_{i0}\Phi\left(z_i^{df}\right)} \int_{-\infty}^{z_i^{df}} A_{i0} e^{\left(r_f + \lambda\sigma_M - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right)\right)T + \sqrt{T\left(\sigma_M^2 + \sigma_i^2\right)}\,z} \phi(z)\,dz \qquad (41)$$

Each row in Table 2 describes the characteristic of the credits in a different asymptotic portfolio. Individual credit PDs range from 23 basis points—for a bond with par values of 55, to 3.99 percent for a bond with a par value of 70. LGD characteristics (measured from initial market value) range from 1.40 percent to 3.28 percent. Measured as loss relative to par value, loss rates range from 6.22 to 8.34 percent. While the LGDs of the bonds examined in this analysis are modest relative to the observed default loss history on corporate bonds, the GCLM and GCRM capital allocation rules explicitly account for loss given default, so *a priori*, there is no reason any specific set of LGD values should compromise the performance of these capital rules.[12]

[12] Some industry credit risk models include a stochastic default barrier such as in the Black and Cox (1976) model to increase the LGD relative to a basic BSM model and thereby improve correspondence with observed market data.

Table 2: Credit Risk Characteristics of 1-Year Credits

par value	initial market value	probability of default in percent	expected value given default	in percent: loss given default from initial value	in percent: loss given default from par value	in percent: yield to maturity
55	52.31	0.233	51.58	1.40	6.22	5.142
56	53.26	0.298	52.45	1.53	6.35	5.145
57	54.20	0.379	53.31	1.64	6.47	5.166
58	55.15	0.476	54.17	1.78	6.60	5.168
59	56.10	0.593	55.03	1.91	6.73	5.169
60	57.04	0.732	55.88	2.03	6.87	5.189
61	57.98	0.896	56.73	2.16	7.00	5.209
62	58.92	1.088	57.57	2.29	7.14	5.227
63	59.86	1.311	58.41	2.42	7.28	5.246
64	60.80	1.568	59.25	2.55	7.43	5.263
65	61.73	1.862	60.08	2.68	7.57	5.297
66	62.66	2.196	60.90	2.80	7.72	5.330
67	63.59	2.574	61.73	2.93	7.87	5.362
68	64.51	2.997	62.54	3.05	8.03	5.410
69	65.43	3.469	63.35	3.17	8.18	5.456
70	66.34	3.992	64.16	3.28	8.34	5.517

The alternative recommendations for capital consistent with a 99.9 percent solvency margin are reported in Table 3. Capital requirements generated under the GCLM unexpected loss capital rule (expression (32)) are substantially smaller than the capital needed to achieve the regulatory target default rate of 0.1 percent. The shortfall depends on the characteristics of the credit portfolio. If one constructs a multiplier to correct for the GCLM bias, the multiplier is very unstable and ranges from 3.8 to 5.7 for the credits analyzed in this calibration exercise.

GCRM capital estimates (expression (38)), while downward biased, are closer to achieving the target solvency objective. Because the bias is stable across portfolios, a multiplier can be used correct the bias, but the multiplier will vary according to the target solvency margin selected. Calibration results for a target 99.9 percent solvency margin reported in Table 3, suggest that the multiplier varies between 1.16 and 1.31. A capital allocation rule based on expression (38) with a multiplier $M \approx 1.26$ would produce capital allocations that are very close to the targeted solvency margin of 99.9 percent.

Table 3: Capital Allocation for a 99.9 Solvency Margin Under Alternative Models

par value	probability of default in percent	BSM structural model capital	GCRM capital estimate	GCLM capital estimate	implied GCRM multiplier	implied GCLM multiplier
55	0.233	0.396	0.325	0.070	1.217	5.660
56	0.298	0.487	0.402	0.092	1.210	5.272
57	0.379	0.593	0.486	0.117	1.221	5.076
58	0.476	0.715	0.584	0.149	1.224	4.803
59	0.593	0.854	0.734	0.184	1.164	4.652
60	0.732	1.011	0.809	0.225	1.249	4.501
61	0.896	1.187	0.951	0.274	1.249	4.336
62	1.088	1.384	1.100	0.328	1.258	4.224
63	1.311	1.601	1.264	0.388	1.267	4.125
64	1.568	1.839	1.445	0.456	1.273	4.030
65	1.862	2.098	1.639	0.530	1.280	3.958
66	2.196	2.379	1.852	0.610	1.285	3.900
67	2.574	2.681	2.073	0.696	1.293	3.850
68	2.997	3.005	2.316	0.789	1.298	3.811
69	3.469	3.348	2.567	0.885	1.304	3.784
70	3.992	3.712	2.831	0.983	1.311	3.776
				average multiplier	1.256	4.360
				minimum multiplier	1.164	3.776
				maximum multiplier	1.311	5.660

Table 4: Capital Allocation for a 98 Percent Solvency Margin Under Alternative Models

par value	probability of default in percent	BSM structural model capital	GCRM capital estimate	GCLM capital estimate	implied GCRM multiplier	implied GCLM multiplier
55	0.233	0.095	0.100	0.019	0.950	5.000
56	0.298	0.121	0.129	0.027	0.938	4.481
57	0.379	0.152	0.163	0.035	0.933	4.343
58	0.476	0.19	0.204	0.046	0.931	4.130
59	0.593	0.235	0.248	0.059	0.948	3.983
60	0.732	0.287	0.304	0.075	0.944	3.827
61	0.896	0.348	0.370	0.095	0.941	3.663
62	1.088	0.418	0.443	0.117	0.944	3.573
63	1.311	0.498	0.527	0.143	0.945	3.483
64	1.568	0.588	0.623	0.174	0.944	3.379
65	1.862	0.69	0.730	0.208	0.945	3.317
66	2.196	0.804	0.851	0.247	0.945	3.255
67	2.574	0.93	0.982	0.290	0.947	3.207
68	2.997	1.069	1.132	0.338	0.944	3.163
69	3.469	1.221	1.291	0.390	0.946	3.131
70	3.992	1.387	1.465	0.446	0.947	3.110
				average multiplier	0.943	3.690
				minimum multiplier	0.931	3.110
				maximum multiplier	0.950	5.000

Table 4 reports capital estimates for a 98 percent target solvency margin. At the 98 percent solvency level, the GCLM still understates capital by a wide and variable margin. In contrast, the GCRM capital allocation estimator (expression (38)) overstates the true capital required and the overstatement is again fairly uniform. At a 98 percent solvency rate, the multiplier for expression (38) is slightly less than 1.

7. THE DEFINITION OF LOSS GIVEN DEFAULT

The BSM model (expression (24)) sets capital requirements by referencing a credit portfolio's return distribution. Returns are measured relative to a portfolio's initial market value (current exposure) and negative portfolio returns represent portfolio credit losses. The GCRM framework is analogous by construction, and so the only measure of LGD consistent with the GCRM is loss relative to a credit's initial market value. Define this measure of loss given default as current exposure *LGD*, or LGD^{CE} . An alternative way to measure LGD is to measure shortfall from promised future value (principal plus accrued interest) at the end of the capital allocation horizon. Define this measure of loss given default as LGD^{FE} where the *FE* superscript denotes future exposure basis. These alternative LGD measures satisfy,

$$LGD^{FE} = \frac{LGD^{CE} + YTM}{1 + YTM} \tag{42}$$

A comparison of expression (32) with expression (38) will show that, provided the EL adjustment is excluded from the GCLM capital calculation, the GCLM and GCRM model capital allocations converge provided LGD^{FE} is used in the GCLM.

The GCLM literature is not prescriptive as to how LGD should be measured. The Basel Committee on Banking Supervision (2006b, Paragraph 297) specifies that, ”*LGD* be measured as… a percentage of the *EAD*,” and, "…banks must estimate *EAD* at no less than the current drawn amount, subject to recognizing the effects of on-balance sheet netting...(paragraph 474) ". The U.S. Basel II NPR (p.123) defines "*EAD* for the on-balance sheet component of a wholesale or retail exposure means (i) the bank's carrying value for the exposure (including accrued but unpaid interest and fees)…". Basel II documents do not include guidance that suggests that *EAD* must be higher than the current carrying value of a simple fully-drawn loan. Since the AIRB framework does not restrict when default is assumed to occur during the year, the recommended *LGD* measure does not clearly recommend the future exposure measure and yet it clearly does allow use of the current exposure measure–a measure which always produces a smaller *LGD* estimate.

A recent report produced jointly by the International Association of Credit Portfolio Managers (IACPM) and the International Swap Dealers Association (ISDA) (2006) studies the performance of capital allocation vendor models and the internal capital models used by 28 participant banks for a selected portfolio of credit positions. The study finds wide variation in the definition of *LGD* across vendor and bank internal capital models. Many banks use vendor models in their capital calculations either directly, in modified form, or as inputs into their internal processes. Bank models that are similar to Moody's KMV Portfolio Manager use an *LGD* measure that is similar to a future exposure measure and include coupon EAD payments in loss estimates. Models similar to RiskMetrics Group's Credit Manager and Credit Suisse First Boston's CreditRisk+ appear to use *LGD* measures that are close to a current exposure measure where interest earnings are excluded from loss estimates. Model differences in *LGD* are an important source of variation in bank capital estimates. For an identical portfolio, differences in the *LGD* definitions resulted in a 20 percent variation in assigned capital across the study's participating banks.

Figure 2 compares the capital allocations that are recommended by the alternative approaches discussed in this study with capital allocations that are set using the Basel II AIRB approach. To control for the absence of reserves in this static model, Basel AIRB capital requirements are modified to include UL and EL. All estimates assume a maturity of one year, and the AIRB is estimated for two different measures of LGD, LGD^{CE} and LGD^{FE}. The AIRB capital rules plotted are,

$$K_i^{AIRB\ LGD^{CE}\ (UL+EL)} = \left[LGD_i^{CE} \times \Phi\left[\frac{1}{\sqrt{1-R}} \Phi^{-1}(PD) + \sqrt{\frac{R}{1-R}} \Phi^{-1}(.999) \right] \right] \qquad (41)$$

$$K_i^{AIRB\ LGD^{FE}\ (UL+EL)} = \frac{Par_i}{B_{i0}} \left[LGD_i^{FE} \times \Phi\left[\frac{1}{\sqrt{1-R}} \Phi^{-1}(PD) + \sqrt{\frac{R}{1-R}} \Phi^{-1}(.999) \right] \right] \qquad (42)$$

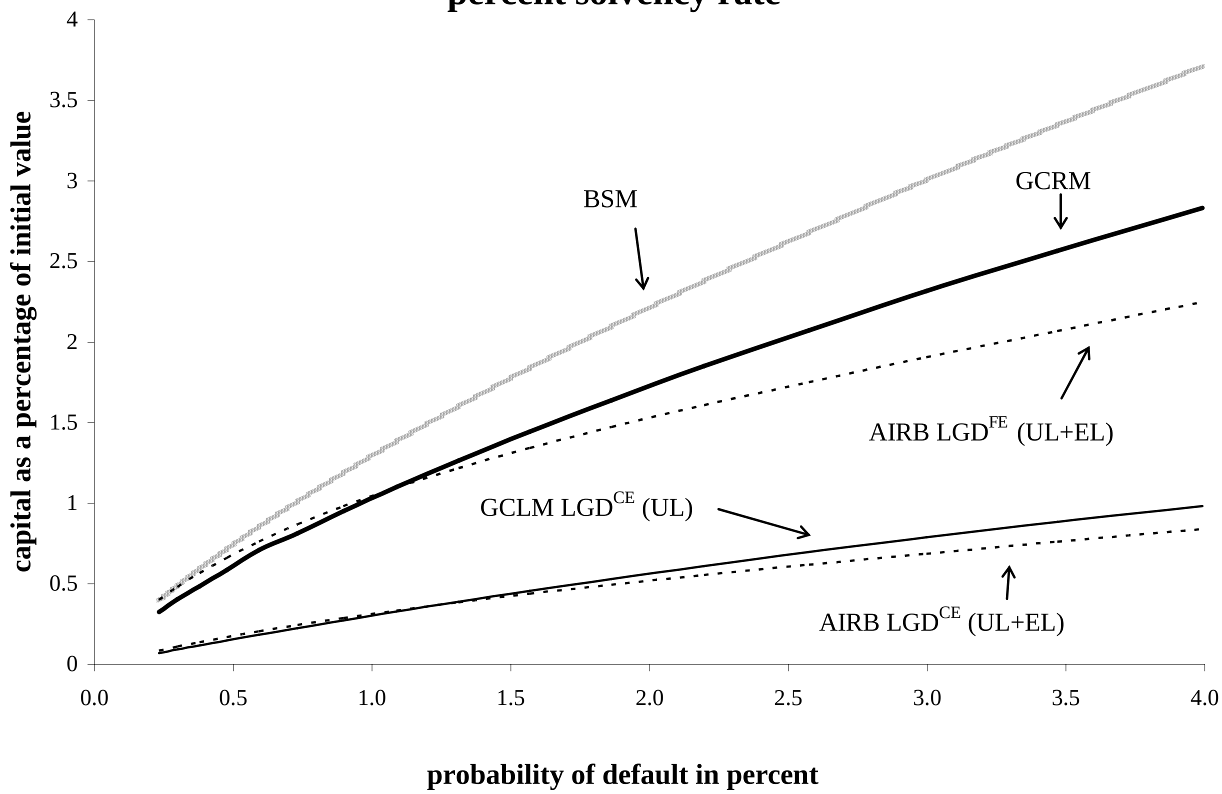
where, $R = 0.12\left(\frac{1-e^{-50PD}}{1-e^{-50}} \right) + 0.24\left(1 - \frac{1-e^{-50PD}}{1-e^{-50}} \right)$.

Capital requirement estimates are also plotted for the BSM model (expression (24)), forthe uncorrected GCRM (expression (38)), and for the GCLM, (expression (32)), where GCLM estimatea use the current exposure measure of *LGD*.

The plots in Figure 2 show that the Basel AIRB approach undercapitalizes credit risk relative to the BSM model benchmark. If the AIRB approach is applied using a current exposure measure of LGD, a measure that is both permitted under existing Basel II guidance and is commonly used by many banks, the AIRB substantially understates portfolio capital needs. Relative to Vasicek GCLM capital estimates, this implementation of the AIRB rule understates capital needs for credits with high PDs because the regulatory correlation function understates correlations.

If alternatively, banks were required to use a future exposure measure of LGD, AIRB requirements would be roughly equivalent to capital calculated using the GCRM for credits with PD less than 0.81 percent. For these credits, if the AIRB were augmented with a multiplier equal to about 1.26, the AIRB would approximate the capital required by the BSM benchmark. For credits with higher PDs, the AIRB approach requires less capital than the GCRM because the regulatory correlation function reduces implied correlations below 20 percent. For these higher risk credits, the 1.26 multiplier correction would be insufficient.

Figure 2: Alternative Capital Requirement Estimates for 99.9 percent solvency rate



A particularly interesting feature of Figure 2 is the range of variability of capital requirements under the AIRB. By changing the definition of *LGD* used in the AIRB capital rule and moving from LGD^{CE} to LGD^{FE}, capital requirements may increase by more than 350 percent. The sensitivity of AIRB capital to the definition of *LGD* may explain in part the variability of the capital estimates reported in QIS 4 and QIS 5.

8. Conclusions

Although it has become common practice to use Gaussian copula methods to model portfolio credit risk and estimate capital allocations, these methods do not produce accurate estimates of portfolio credit risk and capital requirements. Comparisons with capital allocations estimated using a full BSM equilibrium model show that the Vasicek GCLM approach produces downward biased capital estimates. The bias is substantial in magnitude as estimates may be only one-fifth as large as the true capital needed to achieve targeted solvency rates.

An alternative model, the Gaussian credit return model or GCRM, can reproduce capital requirements that are accurate relative to capital requirements calculated using a portfolio model consistent with the BSM for pricing credit risk. The improvement in accuracy is achieved without an increase in computational complexity over the GCLM approach. This new alternative capital rule uses the YTM of individual credits as well as PD, current exposure LGD, and asset correlations as inputs into the capital allocation assignment function. A multiplier is employed to improve the accuracy of the capital estimator relative to the allocations set using the full equilibrium model of credit risk.

Analysis of the GCRM identifies important sources of bias in the Basel II AIRB approach. The AIRB approach is derived from the Vasicek GCLM and thereby includes the biases identified in this study. The GCLM is focused on the distribution of the default rate for an asymptotic portfolio and is silent on the definition of LGD that should be used in CGLM capital calculation. Depending on the definition of LGD used in CGLM capital calculations, capital estimates may vary by more than 350 percent. A comparison with the GCRM shows that adequate capitalization under a GCLM capital rule can be achieved ii the GCLM capital rule is employed using the future exposure measure of LGD and a regulatory multiplier of about 1.26. The modified capital rule calls for a substantial increase in minimum capital requirements over the existing Basel II A-IRB regulatory capital function.

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