Handbook of Mathematical Functions

The *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* [1] was the culmination of a quarter century of NBS work on core mathematical tools. Evaluating commonly occurring mathematical functions has been a fundamental need as long as mathematics has been applied to the solution of practical problems. In 1938, NBS initiated its Mathematical Tables Project to satisfy the increasing demand for extensive and accurate tables of functions [2]. Located in New York and administered by the Works Projects Administration, the project employed not only mathematicians, but also a large number of additional staff who carried out hand computations necessary to produce tables. From 1938 until 1946, 37 volumes of the NBS Math Tables Series were issued, containing tables of trigonometric functions, the exponential function, natural logarithms, probability functions, and related interpolation formulae. In 1947, the Math Tables Project was moved to Washington to form the Computation Laboratory of the new National Applied Mathematics Laboratories of NBS. Many more tables subsequently were published in the NBS Applied Mathematics Series; the first of these, containing tables of Bessel functions [3], appeared in 1948.

On May 15, 1952, the NBS Applied Mathematics Division convened a Conference on Tables. Milton Abramowitz of NBS, who had been a member of the technical planning staff for the Math Tables Project, described preliminary plans for a compendium of mathematical tables and related material. Abramowitz indicated that the Bureau was in need of both technical advice and financial support to carry out the project. With the support of the National Science Foundation (NSF), a two-day Conference on Tables was held at the Massachusetts Institute of Technology on September 15-16, 1954, to discuss the prospects for such an undertaking. Twenty-eight persons attended, including both table producers and users from the science and engineering community. The report of the conference concluded that

> "an outstanding need is for a Handbook of Tables for the Occasional Computer, with tables of usually encountered functions and a set of formulas and tables for interpolation and other techniques useful to the occasional computer."

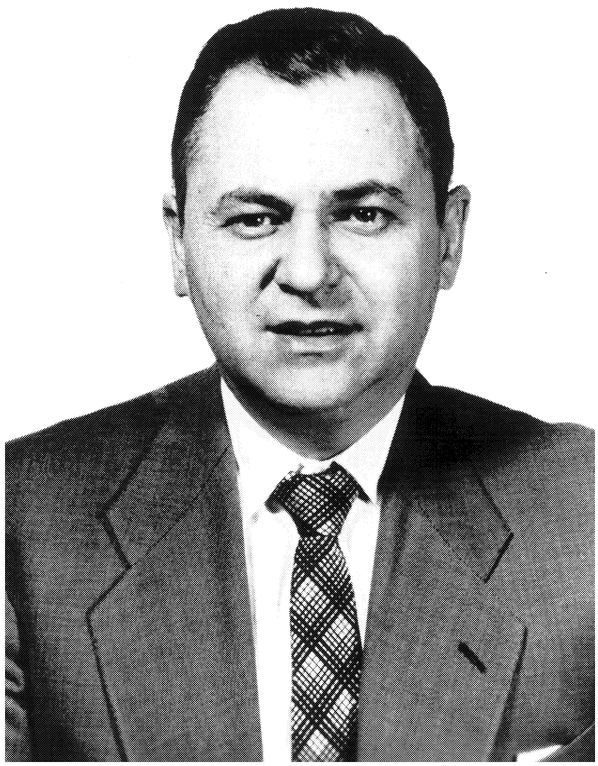

Fig. 1. Portrait of Milton Abramowitz.

(Note that here the term computer refers to a person performing a calculation by hand.) The report recommended that NBS manage the production of the *Hand book* and that NSF provide financial assistance. The conference elected the following committee to carry out its recommendations: P.M. Morse (Chair), M. Abramowitz, J.H. Curtiss, R.W. Hamming, D. H. Lehmer, C.B. Tompkins, and J.W. Tukey. The committee was successful in persuading both NBS and NSF to support the project, and it began officially in December of 1956.

The Mathematics Division of the National Research Council also had an interest in mathematical tables. Since 1943, they had been publishing a quarterly journal entitled *Mathematical Tables and Other Aids to Computation* (today known as *Mathematics of Computation).* To provide technical assistance to NBS, as well as independent oversight for NSF, the NRC established a Committee on Revision of Mathematical Tables. Its members were P. M. Morse (Chair), A. Erdélyi, M. C.

Gray, N. C. Metropolis, J. B. Rosser, H. C. Thacher, Jr., John Todd, C. B. Tompkins, and J. W. Tukey. This group of luminaries in the fields of applied mathematics and physics provided guidance to NBS throughout the project to produce the *Handbook*.

Milton Abramowitz, who was then Chief of the Computation Laboratory of the NBS Applied Mathematics Division, led the project. Abramowitz was born in Brooklyn, NY, in 1915. He received a B. A. from Brooklyn College in 1937 and an M. A. in 1940. He joined the NBS Math Tables Project in 1938 and in 1948 received a Ph.D. in Mathematics from New York University. Abramowitz' dedication, enthusiasm, and boundless energy led to substantial progress in the project during its first year. The proposed outline for the *Handbook* called for a series of some 20 chapters, each with a separate author. Authors were drawn from NBS staff and guest researchers, as well as external researchers working under contract. Most chapters would focus on a particular class of functions, providing formulas, graphs, and tables. Listed formulas would include differential equations, definite and indefinite integrals, inequalities, recurrence relations, power series, asymptotic expansions, and polynomial and rational approximations. Material would be carefully selected in order to provide information most important in applications, especially in physics. Consequently, the higher mathematical functions, such as Bessel functions, hypergeometric functions, and elliptic functions, would form the core of the work. Additional chapters would provide background on interpolation in tables and related numerical methods for differentiation and quadrature.

Philip J. Davis of NBS first prepared Chapter 6, on the gamma and related functions, to serve as a model for other authors. This chapter portrayed the telegraphic style that is a hallmark of the Handbook, i.e., the material is displayed with a minimum of textual description. In the course of developing his chapter, Davis became interested in the history of the topic. This led to a historical profile published in 1959 [4], which won the prestigious Chauvenet Prize for distinguished mathematical exposition from the Mathematical Association of America.

The *Handbook* project occurred during the period when general-purpose electronic computing machinery was first coming into use in government research laboratories. (Early computer development of SEAC at NBS is described elsewhere in this volume.) Nevertheless, most of the tables in the *Handbook* were generated by hand on desk calculators. However, even at that time it was clear to the developers of the Handbook that the need for tables themselves would eventually be superseded by computer programs which could evaluate functions for specified arguments on demand.

By the summer of 1958, substantial work had been completed on the project. Twelve chapters had been completed, and the remaining ones were well underway. The project experienced a shocking setback one weekend in July 1958 when Abramowitz suffered a heart attack and died. Irene Stegun, who was Assistant Chief of the Computation Laboratory, took over management of the project. Stegun, who was born in Yonkers, NY in 1919, had received an M. A. from Columbia University in 1941, and joined NBS in 1943. The exacting work of assembling the many chapters, checking tables and formulas, and preparing the work for printing took much longer than anticipated. Nevertheless, the *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables* was finally issued as Applied Mathematics Series Number 55 in June 1964 [1]. The volume, which is still in print at the U.S. Government Printing Office and stocked by many bookstores and online booksellers, is 1046 pages in length. The chapters and authors are as follows.

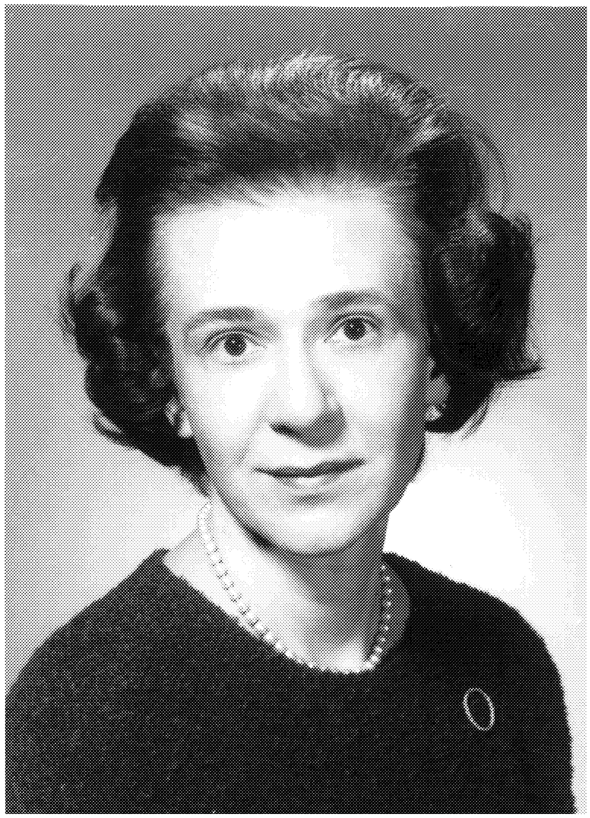

Fig. 2. Portrait of Irene Stegun.

1. *Mathematical Constants,* D. S. Liepman.
2. *Physical Constants and Conversion Factors,* A. G. McNish.
3. *Elementary Analytical Methods,* M. Abramowitz.
4. *Elementary Transcendental Functions,* R. Zucker.
5. *Exponential Integral and Related Functions,* W. Gautschi (American University) and William F. Cahill.
6. *Gamma Function and Related Functions,* P. J. Davis.
7. *Error Function and Fresnel Integrals,* W. Gautschi (American University).
8. *Legendre Functions,* I. A. Stegun.
9. *Bessel Functions of Integer Order,* F. W. J. Olver.
10. *Bessel Functions of Fractional Order,* H. A. Antosiewicz.
11. *Integrals of Bessel Functions,* Y. L. Luke.
12. *Struve Functions and Related Functions,* M. Abramowitz.
13. *Confluent Hypergeometric Functions,* L. J. Slater (Cambridge University).
14. *Coulomb Wave Functions,* M. Abramowitz.
15. *Hypergeometric Functions,* F. Oberhettinger.
16. *Jacobian Elliptic Functions and Theta Functions,* L. M. Milne-Thomson (University of Arizona).
17. *Elliptic Integrals,* L. M. Milne-Thomson (University of Arizona).
18. *Weierstrass Elliptic and Related Functions,* T. H. Southard.
19. *Parabolic Cylinder Functions,* J. C. P. Miller (Cambridge University).
20. *Mathieu Functions,* G. Blanch (Wright-Patterson Air Force Base).
21. *Spheroidal Wave Functions,* A. N. Lowan (Yeshiva University).
22. *Orthogonal Polynomials,* U. W. Hochstrasser (American University).
23. *Bernoulli and Euler Polynomials—Riemann Zeta Function,* E. V. Haynsworth and K. Goldberg.
24. *Combinatorial Analysis,* K. Goldberg, M. Newman, and E. Haynsworth.
25. *Numerical Interpolation, Differentiation, and Integration,* P. J. Davis and I. Polonsky.
26. *Probability Functions,* M. Zelen and N. C. Severo
27. *Miscellaneous Functions,* I. A. Stegun.
28. *Scales of Notation,* S. Peavy and A. Schopf (American University).
29. *Laplace Transforms.*

The public reaction to the publication of the *Handbook* was overwhelmingly positive. In a preface to the ninth printing in November 1970, NBS Director Lewis Branscomb wrote

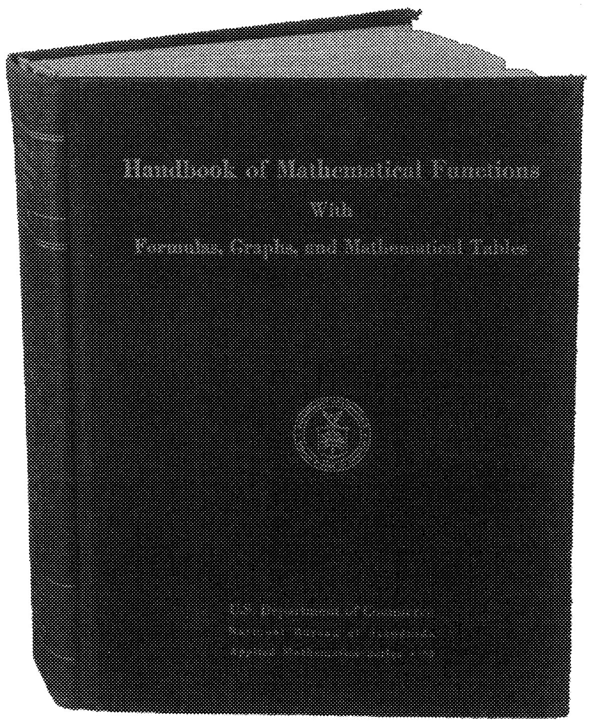

Fig. 3. Photograph of *Handbook.*

> "The enthusiastic reception accorded the *'Handbook of Mathematical Functions'* is little short of unprecedented in the long history of mathematical tables that began when John Napier published his tables of logarithms in 1614. Only four and one-half years after the first copy came from the press in 1964, Myron Tribus, the Assistant Secretary for Commerce for Science and Technology, presented the 100,000th copy of the *Handbook* to Lee A. DuBridge, then Science Advisor to the President."

The *Handbook* has had enormous impact on science and engineering. Likely the most widely distributed NBS/NIST technical publication of all time, the government edition has never gone out of print, and it has appeared as a Dover reprint since 1965. It has been reprinted (in all or part) by other publishers, such as Moscow Nauka, Verlag Harri Deutsch, and Wiley Interscience. Government sales exceed 150,000 copies, with commercial sales estimated at three to six times this number. The *Handbook's* citation record is also remarkable. More than 23,000 citations have been logged by Science Citation Index (SCI) since 1973. Remarkably, the number of citations to the *Handbook* continues to grow, not only in absolute numbers, but also as a fraction of the total number of citations made in the sciences and engineering. During the mid-1990s, for example, about once every 1.5 hours of each working day some author, somewhere, made sufficient use of the *Handbook* to list it as a reference. The success of the *Handbook* was due to several factors. It collected in one place, and in a well-organized way, the most important information needed to make use of mathematical functions in practical applications. It served to standardize notations and normalizations for the special functions of applied mathematics, thus easing the communication of scientific results. In 1965, Irene Stegun was awarded a Gold Medal from the Department of Commerce for her efforts in completing the project.

A number of difficult mathematical problems that emerged in the course of developing the *Handbook* engaged researchers in the NBS Applied Mathematics Division for a number of years after its publication. Two of these are especially noteworthy, the first having to do with stability of computations and the second with precision.

Mathematical functions often satisfy recurrence relations (difference equations) that have great potential for use in computations. However, if used improperly, recurrence relations can quickly lead to ruinous errors. This phenomenon, known as instability, has tripped up many a computation that appeared, superficially, to be straightforward. The errors are the result of subtle interactions in the set of all possible solutions of the difference equation. Frank Olver, who wrote the *Handbook's* chapter on Bessel functions of integer order, studied this problem in great detail. In a paper published in 1967 [5], Olver provided the first (and only) stable algorithm for computing all types of solutions of a difference equation with three different kinds of behavior: strongly growing, strongly decaying, and showing moderate growth or decay. Part of the impact of this work is reflected today in the existence of robust software for higher mathematical functions. Olver worked on such topics in the Mathematical Analysis Division of NBS, and this work provided the foundation for his very influential later book on asymptotic analysis

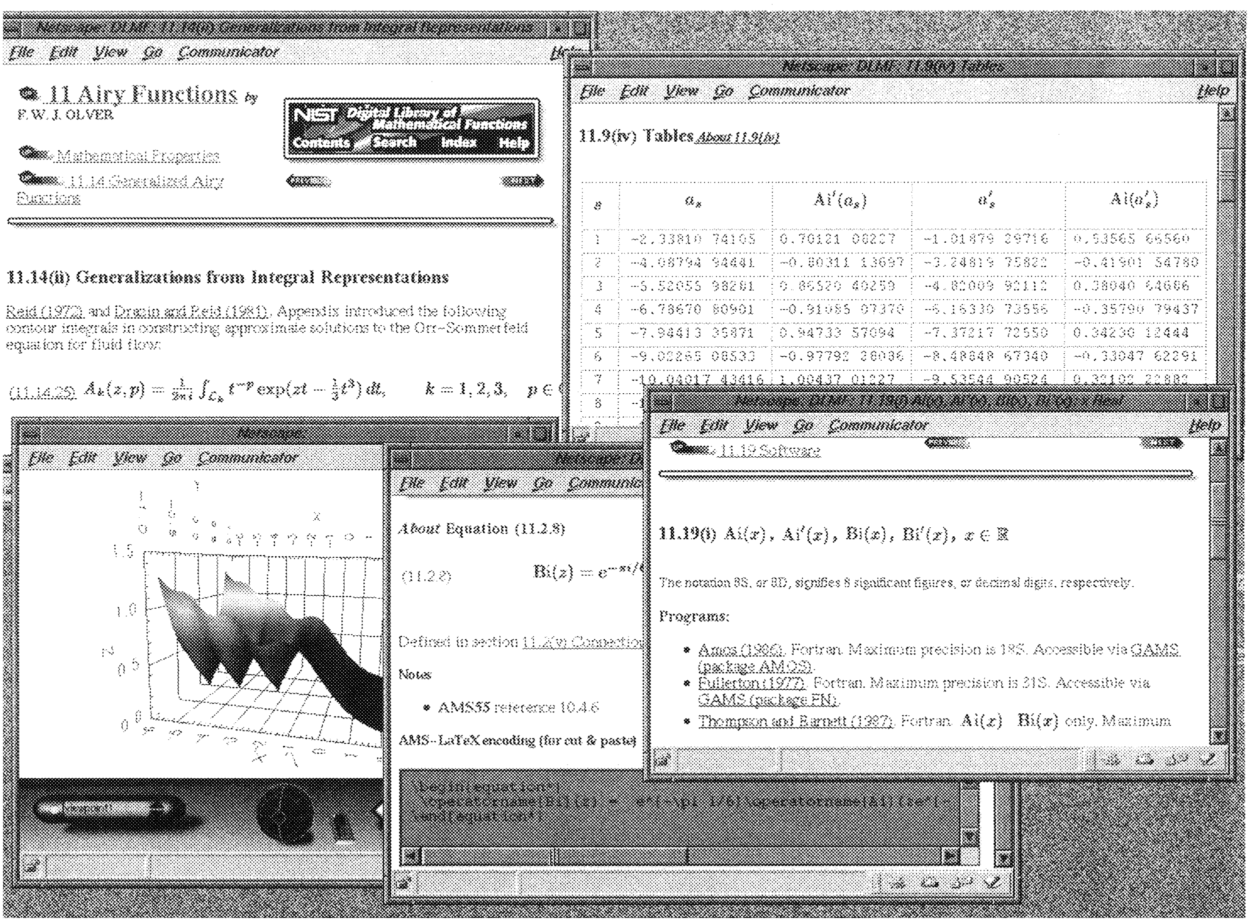

Fig. 4. Screen shot of the NIST Digital Library of Mathematical Functions.

and special functions [6]. This book has been cited more than 800 times, according to SCI.

Another important problem in mathematical computation is the catastrophic loss of significance caused by the fixed length requirement for numbers stored in computer memory. Morris Newman, who co-authored the *Handbook's* chapter on combinatorial analysis, sought to remedy this situation. He proposed storing numbers in a computer as integers and performing operations on them exactly. This contrasts with the standard approach in which rounding errors accumulate with each arithmetic operation. Newman's approach had its roots in classical number theory: First perform the computations modulo a selected set of small prime numbers, where the number of primes required is determined by the problem. These computations furnish a number of local solutions, done using computer numbers represented in the normal way. At the end, only one multilength computation is required to construct the global solution (the exact answer) by means of the Chinese Remainder Theorem. This technique was first described in a paper by Newman in 1967 [7]; it was employed with great success in computing and checking the tables in Chapter 24 of the *Handbook.* Today, this technique remains a standard method by which exact computations are performed. Newman's research on this and other topics, performed at NBS, formed the basis for his 1972 book [8], which quickly became a standard reference in the applications of number theory to computation.

Research into the functions of applied mathematics has continued actively in the 36 years since the *Handbook* appeared. New functions have emerged in importance, and new properties of well-known functions have been discovered. In spite of the fact that sophisticated numerical methods have been embodied in well-designed commercial software for many functions, there continues to be a need for a compendium of information on the properties of mathematical functions. To address this need, NIST is currently developing a successor to the *Handbook* to be known as the *Digital Library of Mathematical Functions (DLMF)* [9]. Based upon a completely new survey of the literature, the DLMF will provide reference data in the style of the *Handbook* in a freely available online format, with sophisticated mathematical search facilities and interactive three-dimensional graphics.

Prepared by Ronald F. Boisvert and Daniel W. Lozier.

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