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An Accuracy Algorithm for an Atomic Time Scale

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Abstract

The accuracy of the rate or frequency of an atomic time scale is the degree to which its unit agrees with the SI second. Primary frequency standards are constructed in such a manner that they provide the most accurate possible physical realization of the SI second. These standards are then used to calibrate or construct an atomic time scale which may also be used as a stable reference standard.

Mathematical models characterizing the performance of both the primary frequency standards and reference standards are developed, and based on these models a current best estimate of the SI second is derived utilizing current and previous calibrations.

The modeling techniques and theory are applied to the NBS primary frequency standards and atomic time scale system and a significant improvement is realized in the accuracy of the frequency estimate so derived. We estimate that the second used by the Bureau International de l'Heure in generating TAI and UTC was too short by about 9 ± 2 parts in 10^{13} during the fall of 1974.

Introduction

In generating an accurate time scale there are two basic operational necessities: first, as the "seconds" are accumulated in constructing the scale, each one should conform as closely as possible with the definition of the SI second [1]; this conformity not only guarantees accuracy for any time interval but also uniformity for the scale. Second, the scale needs an explicit origin; this is usually provided by a straight-forward definition; i.e., the "seconds" are accumulated starting from some well defined event.

The most accurate physical realizations of the second are by evaluable, primary laboratory cesium beam frequency standards. Because of the complexity of these standards, they are often not conveniently usable as clocks; i.e., it is difficult to run them continuously and reliably. Rather, these primary standards are typically used to calibrate the rate of secondary standards which run continuously and reliably and hence can be used as clocks. In other words the primary standard is utilized to obtain a physical realization of the definition of the second, which is "... the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom", and then to determine if the second used in the atomic time scale is too long or too short. Typically these calibrations are not continuous, but are aperiodic and may be made even with different devices. One would like to optimally utilize all of these calibrations.

The logical combination of standards into a single standard has been considered by Crow [2], and the efficiency of the various methods of doing so has been given by him for a variety of distributions. His results may be used in combining the calibrations involving primary frequency standards if two additional problems are considered: first, calibrations may be separated in space, and second, they may be separated in time. For example, in practice one calibration is compared to another, when spatially separated, by using some time signal propagation system such as Loran-C; and when temporally separated, one calibration may be referred to the next via some secondary frequency standard, e.g. an ensemble of clocks in a time scale system.

As to the first problem Guetrot and others [3 to 5] have considered ways of optimum or near optimum filtering of noise introduced by some of the more precise methods of communicating time and frequency, via Loran-C, VLF, and TV. Particular signal and noise models have been deduced respectively for the clocks and propagation media, and are assumed to apply generally in the above considerations. Thus the spatial separation problem has been treated and solved to the degree that the models developed actually describe the signal and noise respectively of the clocks and propagation medium.

As to the second problem it is fundamentally one of the frequency stability of the secondary reference frequency standard used in comparing the frequencies of two or more calibrations, and several papers have been written on the capabilities and stability of such references [6 to 17].

Yoshimura [6] considers in detail calibrations of a clock ensemble by a primary frequency standard involving the assumption of a particular statistical model of noise processes. One of these assumptions was that the stability of a reference clock ensemble may be characterized by

$$S_y(f) = h_0 + h_{-1} f^{-1}, \quad f < 1 \text{ Hz} \tag{1}$$

where $S_y(f)$ is the spectral density of the fractional frequency fluctuation, y; f is the Fourier frequency, and h_0 and h_{-1} are the intensities of the white noise frequency modulation (FM) and flicker noise FM respectively. A further assumption was that the frequency calibrations with a primary standard had errors which were random uncorrelated from one calibration to any other calibration. With these assumptions he showed that such calibrations could improve the stability of an atomic time scale over periods of years but that there was some degradation of stability for shorter times (months).

This paper will add some additional elements to the Yoshimura model: consideration of frequency steps, frequency drift and probable correlations among calibrations made by the same primary frequency standard or other similar ones, and probable contributors to the instability of a reference clock ensemble. In addition, a method will be presented whereby the stability of stable reference clocks may be efficiently utilized while simultaneously utilizing the accuracy given by all the calibrations by the primary standards — the goal being to have both short and long term stability as well as accuracy for the atomic time scale.

Model for Clocks and Standards

A. A stability model for a clock ensemble

As some years of data are now available on commercial cesium beam frequency standards, it has become evident that over periods of years frequency drift becomes a predominant term in the time dispersion of a clock or of an ensemble [7, 8, 9, 10, 14, 15]. The apparent magnitude of this drift is of the order of parts in 10^{13} per year. The direction of the drift quite typically tends to be negative, but positive drifts have been noted [15].

Occasional frequency steps seem also to occur in these standards and methods of handling these to some degree have been published [9, 10, 14, 15]. Sufficient data are not available on these frequency steps to ascertain the underlying processes involved. One can easily model the process of a step, however.

If the above considerations are added to the Yoshimura model for the instabilities of a reference cesium clock ensemble, one may write the following equation as a measure of the time-domain stability:

$$\sigma_y^2(\tau) = \frac{h_0}{2\tau} + h_{-1}\, 2 \ln 2 + h_{-2}\, \frac{(2\pi)^2 \tau}{6} + \frac{D^2 \tau^2}{2} \qquad (2)$$

where $\sigma_y^2(\tau)$, h_0, h_{-1}, h_{-2} are as defined in reference 13, and D is the linear frequency drift, i.e. fractional frequency change per unit of τ. An example of Eq. (2) is shown in Fig. 1 for an individual commercial cesium clock. Another individual atomic clock or an ensemble of such clocks may differ from this characterization and Fig. 1 is given only as an example to illustrate the theory.

B. An accuracy Model for a Frequency Standard

Also shown in Fig. 1 is a point at $\sigma_y(\tau = 30 \text{ days}) = 2 \times 10^{-13}$ representing a monthly frequency calibration error due to the primary frequency standard whose errors in the calibration are random uncorrelated from one to any other. If these calibration errors could be averaged perfectly in time, they would average as $\tau^{-1/2}$ and the dashed line in Fig. 1 illustrates the principle that at some sample time τ the average error of several such calibrations will fluctuate less than an ensemble of commercial standards.

Such an accuracy model is subject to two imperfections; first, the calibrations cannot be averaged perfectly, i.e. there exists no perfect reference; and second, for a given primary standard and even for a set of primary standards the errors of one calibration may well be correlated (in space or time) with some other calibration. A more general model for the errors involved in any given calibration may be written;

$$\sigma_s^2(l) = \sigma_{\mathrm{ruc}}^2(l) + \sigma_{\overline{\mathrm{ruc}}}^2(l) \quad \text{for the } l^{\text{th}} \text{ calibration,} \qquad (3)$$

where $\sigma_s^2(l)$ is the overall accuracy for the calibration,

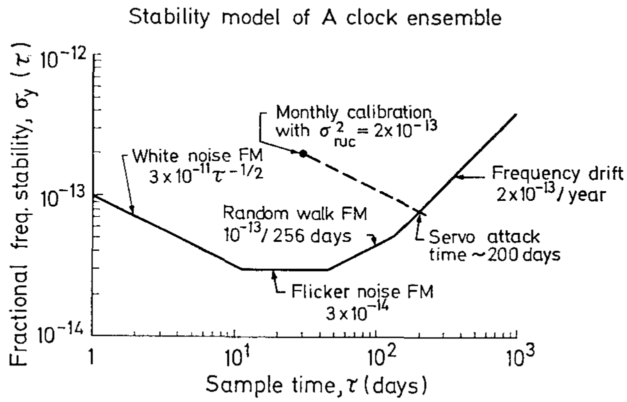

Fig. 1. Fractional frequency stability model for a reference clock ensemble and for calibrations with a primary standard whose inaccuracies are uncorrelated from one calibration to any other; $\sigma_y(\tau)$ is the square root of a 2 sample variance according to the recommendation of the IEEE sub-committee on frequency stability [13] and τ is the frequency sample time or interval. The stability level for the random walk FM was generated by introducing a simulated random frequency step occurring on the average every 256 days

$\sigma_{\mathrm{ruc}}(l)$ is an estimate of the random uncorrelated errors (often these are the primary contribution to an error budget generated during the complete evaluation of a primary standard), $\sigma_{\overline{\mathrm{ruc}}}(l)$ is an estimate of errors that are correlated with some of the past calibrations or with some other primary standard due to similarity of design or evaluation procedure. The bar over ruc "$\overline{\mathrm{ruc}}$" denotes the logical "not".

Correlated errors are sometimes assessed as part of the error budget [18]; for example, when the Ramsey cavity phase shift is measured, the value corresponds to some frequency error which is part of the bias corrections used in the actual realization of the SI second. The "true" value is obtained by correcting the measured value, and if this cavity phase shift is not measured in an uncorrelated way from one calibration to another this may give rise to a significant contribution to $\sigma_{\overline{\mathrm{ruc}}}(l)$. It will be shown later that other experience with a particular standard may give insight into its correlated contribution to the errors in a sequence of calibrations.

Assessing all the contributions to $\sigma_{\overline{\mathrm{ruc}}}(l)$ is obviously very difficult; but as an example comparisons among several different standards may reveal some difficulties — previously unknown — in a particular standard.

Theoretical Development

A. Best Estimate of frequency (or of the SI second)*

We define a *fractional frequency offset* for a frequency $\nu(t)$ to be $(\nu(t) - \nu_{\mathrm{Cs}})/\nu_{\mathrm{Cs}}$, where ν_{Cs} is derived from the true cesium frequency. We wish to develop an algorithm that will approach a best estimate of ν_{Cs} within some practical limits. Let us define a calibration as a measurement of the frequency of a standard with some evaluable primary frequency standard. Consider the l^{th} calibration, and let $y_s(l)$ be the measured value of the fractional frequency offset of a reference

* Best estimate is used in this paper as the minimum squared error.

standard (such as TAI or a clock ensemble) *minus* the fractional frequency offset of an evaluable primary standard. The words "reference standard" are here used to mean that it is continuously available. Let $\hat{y}_e(l)$ be an estimate of the fractional frequency offset of the reference standard based on all *previous* calibrations up to but not including the l^{th} calibration, defined recursively by

$$\hat{y}_e(l) = \beta(l-1)\cdot\hat{y}_e(l-1) + \big(1-\beta(l-1)\big)\cdot y_s(l-1) + \Delta_e(l-1, l), \tag{4}$$

where $\Delta_e(l-1, l)$ is the fractional frequency fluctuation introduced by the reference standard between the $(l-1)^{\text{th}}$ and l^{th} calibrations (assumed to be random and uncorrelated* with respect to the fluctuations in any $y_s(l)$ and to have zero mean), and the $\beta(l-1)$ are previously determined weighting factors. It follows that some linear combination of the values $\hat{y}_e(l)$, $y_s(l)$:

$$\hat{y}(l) \equiv \beta(l)\cdot\hat{y}_e(l) + \big(1-\beta(l)\big)\cdot y_s(l) \tag{5}$$

will have a minimum variance for some optimum value of $\beta(l)$. The $\hat{y}(l)$ of Eq. (5) is defined as a best estimate of the fractional frequency offset at l. This value of $\beta(l)$ may then be used in Eq. (4) to estimate $\hat{y}_e(l+1)$.

To determine the value for $\beta(l)$ take the variance of $y(l)$ and minimize with respect to $\beta(l)$; i.e.

$$\frac{\partial}{\partial\beta(l)}\bigg|\ \mathrm{Var}\ \hat{y}(l) = \beta^2(l)\cdot\sigma_e^2(l) + \big(1-\beta(l)\big)^2\cdot\sigma_s^2(l) + 2\,\beta(l)\cdot\big(1-\beta(l)\big)\ \mathrm{Var}\big[\hat{y}_e(l)\cdot y_s(l)\big] = 0 \tag{6}$$

Now $\sigma_e^2(l)$ — the estimated accuracy from past calibrations combined and remembered by the reference standard predictor $\hat{y}_e(l)$ — is

$$\sigma_e^2(l) = \sigma^2(l-1) + \langle\sigma_{y_e}^2(N=2, T, \tau, f_h)\rangle \tag{7}$$

where $\sigma(l-1)$ is the accuracy of the last best estimate and the 2nd term on the right of Eq. (7) is the time average of the squared error contributed by the *reference standard* from the last calibration $(l-1)$ to the current one (l). The time interval T may reasonably** be taken as the time from the middle of the $(l-1)^{\text{th}}$ calibration to the middle of the l^{th} calibration; τ is the $(l-1)^{\text{th}}$ calibration interval and f_h is the measurement system bandwidth for the calibration (typically negligible for the kinds of noise shown in Fig. 1). If $\sigma_{y_e}(\tau)$ is known for the reference standard as in Fig. 1, then the 2nd term in Eq. (7) may be estimated using reference [19]. The value for $\sigma_s^2(l)$ is given by Eq. (3). Now $y_s(l)$ may be separated into that part which is random and uncorrelated from one calibration to the next and into that part which is not; therefore,

$$y_s(l) = y_{\mathrm{ruc}}(l) + y_{\overline{\mathrm{ruc}}}(l) \quad \text{for any } l \tag{8}$$

Using Eqs. (8) and (4) one may calculate the third term in Eq. (6) (call it the correlation term), $2\cdot\beta(l)\cdot(1-\beta(l))\,C(l)$ obtaining the following:

$$C(l) = \sum_{i=1}^{l-1}\left(1-\beta(l-i)\right)\cdot\gamma_{\overline{\mathrm{ruc}}}(l, l-i)\cdot\prod_{j=1}^{i-1}\beta(l-j), \tag{9}$$

where $\gamma_{\overline{\mathrm{ruc}}}(l, l-i)$ is the cross-covariance between the l^{th} and $(l-i)^{\text{th}}$ frequency calibrations.

One cannot measure this cross-covariance, but one may write the following relationship:

$$|\gamma_{\overline{\mathrm{ruc}}}(l, l-i)| \le \sigma_{\overline{\mathrm{ruc}}}(l)\cdot\sigma_{\overline{\mathrm{ruc}}}(l-i). \tag{10}$$

In general $\gamma_{\overline{\mathrm{ruc}}}(l, l-i)$ may be positive or negative, but in practice where a series of calibrations may involve the same or similar primary standards it will tend to be positive. We tested the effects of setting it near zero, of using the equality, and of using some values in between and observed negligible differences. We chose $\gamma_{\overline{\mathrm{ruc}}}(l, l-i)$ equal to $^1/_2\,\sigma_{\overline{\mathrm{ruc}}}(l)\cdot\sigma_{\overline{\mathrm{ruc}}}(l-i)$ as a reasonable estimate in the NBS application.

The next step is to solve for $\beta(l)$, which gives:

$$\beta(l) = \frac{\sigma_s^2(l) - C(l)}{\sigma_e^2(l) + \sigma_s^2(l) - 2\,C(l)} \tag{11}$$

Thus, the best estimate of frequency is given by substituting Eq. (11) into Eq. (4):

$$\hat{y}(l) = \frac{(\sigma_s^2(l) - C(l))\cdot\hat{y}_e(l) + (\sigma_e^2(l) - C(l))\cdot y_s(l)}{\sigma_e^2(l) + \sigma_s^2(l) - 2\,C(l)}. \tag{12}$$

The accuracy for the estimate is given by the variance of $\hat{y}(l)$, i.e.:

$$\sigma^2(l) = \frac{\sigma_e^2(l)\cdot\sigma_s^2(l) - C^2(l)}{\sigma_e^2(l) + \sigma_s^2(l) - 2\,C(l)} \tag{13}$$

Of course, if the calibration fluctuations are totally random and uncorrelated then $C(l) = 0$, and the accuracy takes on a more simple form:

$$\sigma^2(l) = \frac{\sigma_e^2(l)\cdot\sigma_s^2(l)}{\sigma_e^2(l) + \sigma_s^2(l)}. \tag{14}$$

From Eq. (14) it is easy to see that, as expected, the optimum accuracy, $\sigma(l)$, is better than that from the current calibration, $\sigma_s(l)$, as well as the memory ability of the reference standard predictor $\sigma_e(l)$.

Aspects of the theory developed by Crow [2] may be applied in an approximate and operational sense by letting $\sigma_{\overline{\mathrm{ruc}}}(l)$ in Eq. (10) above take on the value of the deviation of the l^{th} calibration as given by the standard from the last best estimate via the memory of the reference standard; i.e., $\sigma_{\overline{\mathrm{ruc}}}(l) = |y_s(l) - y_e(l)|$. Such a procedure may be appropriate as shown by Crow, where standards separated in space are employed. An example of this would be where the Bureau International de l'Heure obtains data from the primary frequency standards of various national laboratories and wishes to combine them to get a best estimate of the SI second. Since we are limited to very few primary standards, two of which are at NBS, we have choosen to estimate $\sigma_{\overline{\mathrm{ruc}}}(l)$ as well as possible from all pertinent data characterizing the standard. We believe this latter approach gives us a more sensitive method of ascertaining systematic difficulties than the former approach.

Application using NBS-4, NBS-5, and the NBS clock ensemble

An estimation of the model parameters in Eq. (2) for the NBS clock ensemble gives: $\sqrt{h_0/2} \simeq 6\times10^{-12}$ [$\mathrm{s}^{1/2}$], $\sqrt{h_{-1}\,2\ln 2} \simeq 3\times10^{-14}$, $2\pi\sqrt{h_{-2}/6} \lesssim 2\times10^{-17}$ [$\mathrm{s}^{-1/2}$], and $|D| \approx 1\times10^{-13}$/year. In Table 1 are listed the pertinent calibration data in which the NBS primary frequency standard was used to measure the

* This assumption will be good to the degree that $\Delta_e(l-1, l) \ll y_s(l)$ on the average.

** The value of T is commonly taken from the beginning of one measurement to the beginning of the next, but with variable calibration intervals and calibration deviations the above usage is more appropriate.

frequency (clock rate) of the NBS clock ensemble. The calibration periods were typically 3 days or longer giving a measurement precision of about 1×10^{-14}. The NBS clock ensemble consists of eight commercial cesium beam frequency standards operating as clocks where data are statistically analyzed and optimally combined in an algorithm which weights each clock appropriately in order to generate a near optimally stable Atomic Time Scale, AT_0(NBS), from the available data [20]. The rate (frequency) of this scale has *not* been changed* as a result of any of the listed calibration data, but has been maintained — as nearly as possible — as an independent ongoing stable frequency reference. The listings in Table 1 for $y_S(l)$ are $\nu_{AT_0(NBS)} - \nu_{Cs}/\nu_{Cs}$, at the time of the l^{th} calibration, where ν_{Cs} is the frequency of either NBS-III, NBS-4 or NBS-5 as indicated in the third column. The above expression may equivalently be written: $y_S(l) = y_{AT_0\text{-}(NBS)}(l) - y_{Cs}(l)$. The column for $\sigma_{ruc}(l)$ is estimated primarily from the primary standard's error budget as in reference [18]. The estimation of $\sigma_{\overline{ruc}}(l)$ is much more difficult as it relates to unknown biases as well as known biases in a given standard that may unavoidably persist from one calibration to the next. The size of this inaccuracy contribution may decrease as one gains experience with a given standard as it did at NBS. One of the main factors leading to the reduction of this term in the evaluations of the NBS primary standards NBS-4 and NBS-5 was the choice of three independent evaluation techniques [18, 21, 22, 23]. Calibrations 2, 3, 4, 6, 8, and 9 employed the "Power Shift" method; calibration 5 used a microwave "pulse" method; and calibration 1 and 7 cesium "beam reversal" method for evaluation of the primary standard. Calibrations 10 through 19 depended on the complete evaluation performed for calibration 9, but these calibrations were independent of each other in the measurement of the perturbing influence due to the magnetic field and

* Also, these calibrations have never been used to alter the rate of UTC(NBS) which also was independently maintained. For coordination purposes during 1974 and 1975 to date the frequency of UTC(NBS) was constrained to be 8 parts in 10^{13} higher than that of AT_0(NBS).

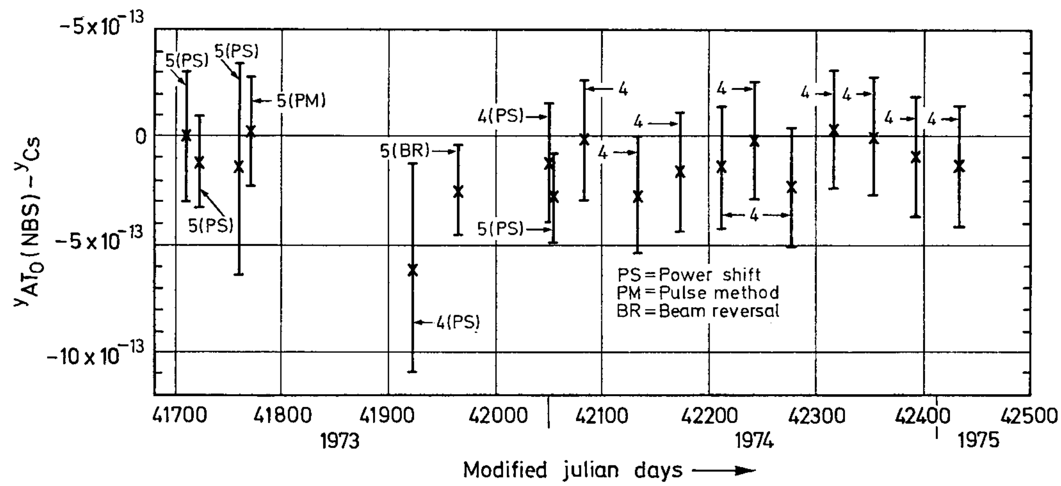

Fig. 2. A plot of the fractional frequency deviations of the rate of the time scale AT_0(NBS) as calibrated with NBS-4 or NBS-5 ($y_{AT_0(NBS)}$-y_{Cs}) versus time (Modified Julian Day). The uncertainties are $\pm\,\sigma_{ruc}(l)$ for calibrations $l = 2$ through $l = 19$

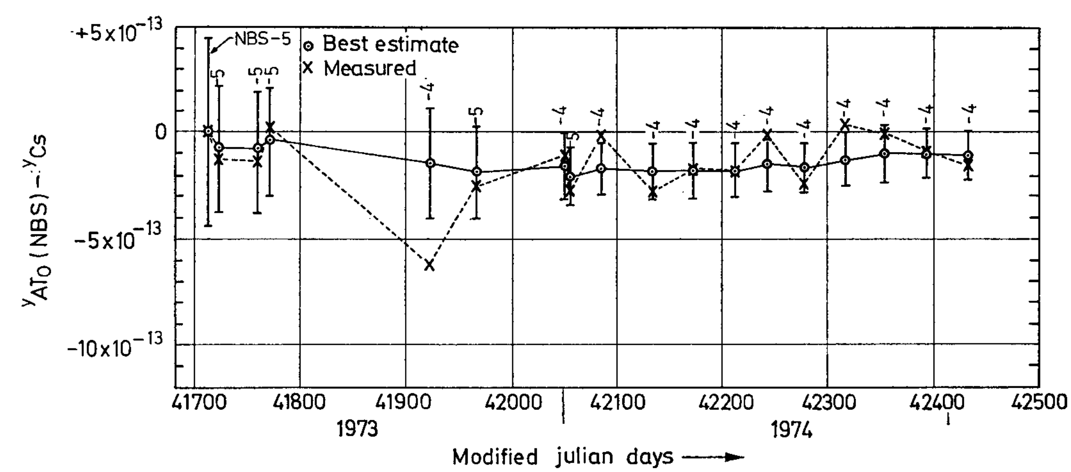

Fig. 3. A plot of the best estimate and measured fractional frequency deviation of the rate of the AT_0(NBS) scale with respect to the NBS primary frequency standards NBS-4 and NBS-5 ($y_{AT_0(NBS)}$-y_{Cs}) versus running time. The uncertainty bars are calculated from Eq. (13) and represent $\pm\,\sigma(l)$, the accuracy best estimate. The number above the uncertainty bars indicate whether the last calibration at the time of the best estimate was with NBS-4 or NBS-5

in the "optimum" microwave power setting needed for each calibration. An experimental estimate for $\sigma_{\overline{\mathrm{ruc}}}(l)$ was not possible — the present model could not retroactively be applied to the calibration with NBS-III. The long spacing to that calibration would decrease the correlation; in addition, calibration No. 1 is the only one involving NBS-III. An arbitrary value of $\sigma_{\overline{\mathrm{ruc}}}(l) = 2 \times 10^{-13}$ was chosen as a compromise between zero and an unknown small value.

One may note the large frequency dispersion uncertainty at calibration 2 for the ensemble, AT_0(NBS); this was due to the long period since the last calibration with NBS-III, $T = 1350$ days.

Fig. 2 is a current plot of the raw calibration values (excluding No. 1); the confidence intervals shown are $\pm\,\sigma_{\mathrm{ruc}}(l)$. These data along with the other data in Table 1 were processed through Eqs. (9) through (13) inclusive, and Fig. 3 is a plot of the best estimate of the obtained frequency and uncertainty of AT_0(NBS), so obtained. The "X"s are plotted in Fig. 3 as the raw calibrations for comparison.

The circles in Fig. 4 are a plot of the weight given each calibration in determining the best estimate of frequency after the 19th calibration. The solid line is a theoretical example for the 19th calibration where the inaccuracy of the primary standard is twice that of the inaccuracy of the best estimate as remembered through the reference standard, and where zero correlation is assumed between any of the calibrations. The peak at the 1st calibration is similar to the results of Yoshimura [6], and arises because the first calibration carries the weight for the infinite past — there being no previous calibration. The peak does not occur in the results obtained from processing the real data because of the dispersion of the frequency of the reference standard over the approximately three and one-half year interval between calibrations 1 and 2.

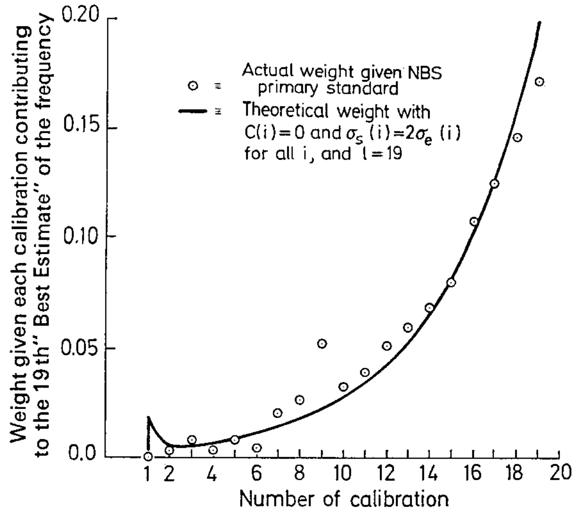

Fig. 4. The circles are a plot of the weights assigned the last (calibration No. 19) calibration and each preceding calibration to get the frequency best estimate, $\hat{y}(l = 19)$. The solid line is theoretically determined assuming $C(i) = 0$ and $\sigma_s(i) = 2\,\sigma_e(i)$ for $i = 2, \ldots, 19$

Once the best estimate of frequency is known, there are a variety of ways to servo a time scale in order to obtain the stability shown in Fig. 1. First, one needs to decide how the servoing will be accomplished, e.g. in an analog or digital fashion. As specific examples one may choose frequency drifts or steps. The equation for a frequency drift which gives rise to a quadratic in the time deviation, adds an additional term as compared to one with dated frequency steps; hence, one may choose the latter approach in order to simplify the description. As long as the frequency steps are small compared to the FM flicker noise of the reference

Table of NBS Frequency Calibration Data

Listed successively in the columns for each calibration are: the calibration number, l; the Modified Julian Day at the midpoint of each calibration interval; the NBS primary standard involved; the fractional frequency (rate) of the AT_0(NBS) scale minus that of the primary standard involved; an estimate of the one-sigma random uncorrelated errors associated with each calibration; an estimate of the one-sigma errors that were not random uncorrelated associated with each calibration; an estimate of the one-sigma fractional frequency dispersion of the reference standard (the AT_0(NBS) scale) during the interval between the $(l-1)^{\mathrm{th}}$ and l^{th} calibration; the fractional frequency (rate) best estimate of the AT_0(NBS) scale as defined by the accuracy algorithm in the text; and the accuracy corresponding to this best estimate. All fractional frequencies are in parts in 10^{13}. MJD 42431 corresponds with 19 January 1975.

			$\times 10^{13}$					
No. (l)	MJD	NBS-Std	$y_s(l)$	$\sigma_{\mathrm{ruc}}(l)$	$\sigma_{\overline{\mathrm{ruc}}}(l)$	$\langle\sigma_{y_e}{}^2\ (2, T, \tau)\rangle^{1/2}$	$\hat{y}(l)$	$\sigma(l)$
1	40360	NBS-3	0	5	?	—	0	5*
2	41711	NBS-5	+ 0.1	3	3.5	8.7	0.1	4.5
3	41726	NBS-5	− 1.2	2.1	2.5	1	− 0.8	3.0
4	41761	NBS-5	− 1.4	5	2.5	1	− 0.9	2.9
5	41777	NBS-5	0.2	2.5	2.5	0.4	− 0.5	2.6
6	41926	NBS-4	− 6.2	5	2.5	1	− 1.4	2.6
7	41964	NBS-5	− 2.6	2	2.0	0.4	− 1.9	2.2
8	42049	NBS-4	− 1.2	2.8	0.5	0.6	− 1.7	1.8
9	42050	NBS-5	− 2.7	2	0.5	0.1	− 2.1	1.4
10	42086	NBS-4	− 0.1	2.8	0.5	0.5	− 1.7	1.3
11	42130	NBS-4	− 2.7	2.8	0.5	0.6	− 1.9	1.3
12	42172	NBS-4	− 1.7	2.8	0.5	0.7	− 1.9	1.3
13	42211	NBS-4	− 1.8	2.8	0.5	0.5	− 1.8	1.3
14	42241	NBS-4	− 0.2	2.8	0.5	0.5	− 1.5	1.2
15	42276	NBS-4	− 2.3	2.8	0.5	0.5	− 1.7	1.2
16	42319	NBS-4	+ 0.4	2.8	0.5	0.7	− 1.3	1.3
17	42354	NBS-4	− 0.0	2.8	0.5	0.5	− 1.1	1.2
18	42396	NBS-4	− 1.0	2.8	0.5	0.5	− 1.0	1.2
19	42431	NBS-4	− 1.4	2.8	0.5	0.5	− 1.1	1.2

standard, near-optimum long and short term stability as well as accuracy of the time scale may be realized. Of course, if a systematic frequency drift is observed over a sustained period in the reference standard, a compensating frequency drift should be subtracted. The coefficient for this drift could be determined by a linear least squares fit to the frequency. The apparent decrease in the frequency drift of AT_0(NBS) which occurred about August 1973 was coincident with a change of procedure. At this date we began once a month to adjust to zero the voltage applied to the varicap of the quartz oscillator in the commercial cesium standards. This procedure was initiated for each commercial cesium standard which was a member of the NBS clock ensemble.

If the step-in-frequency approach were used then the recommended step size of appropriate sign would be given by:

$$|\Delta y| = \frac{T}{\tau_A} \sigma_{y_e}(\tau = \tau_A), \tag{15}$$

where T is the time since the last calibration and τ_A is an estimate of the servo attack time depicted in Fig. 1. If a frequency ramp were already built in the description of the time scale, one may choose to vary the ramp by an amount $\frac{\Delta y}{T}$ to accomodate the step size given by Eq. (15). Now if the frequency drift, D, is adequately removed then τ_A becomes longer and $\sigma_{y_e}(\tau = \tau_A)$ becomes smaller according to the model depicted in Fig. 7 — resulting in an even more uniform time scale.

Conclusions

We have shown that it is possible to have both an accurate and a stable atomic time scale based on an optimum realization of the SI second. This realization is obtained by applying optimum filtering techniques to the averaging of calibrations of a suitable reference standard. The calibrations are obtained from primary frequency standards. A two- to threefold accuracy improvement was obtained over that for an individual calibration when the accuracy algorithm developed in the text was applied to the NBS Atomic Time Scale Primary Frequency Standards system.

The time scale UTC(NBS) used to control the time and frequency signals broadcast by the NBS radio stations, WWV, WWVH, and WWVB, is kept nominally synchronous with the UTC scale generated by BIH. As of January 1975 we found that the second used in generating UTC(NBS) was too short by 6.9 ± 1.2 parts in 10^{13}. Using UTC(NBS) as a transfer scale and Loran-C [5] as the mode for comparing time intervals, we estimate that the second used by the BIH in generating TAI and UTC to be also too short, but by about 9 ± 2 parts in 10^{13} during the fall of 1974. During the same period of time the Physikalisch Technische Bundesanstalt (PTB) in the Federal Republic of Germany measured the TAI and UTC second to be 10 parts in 10^{13} too short with respect to their primary standard. The NBS value accounts for a 1.8×10^{-13} fractional frequency correction needed to account for the "gravitational red shift" due to the elevation of the NBS, Boulder, laboratories above the BIH. During the fall of 1973 the primary standards of NBS, the National Research Council (NRC) in Canada, [24] and PTB were used to calibrate the TAI and UTC second and measured it to be too short by 10, 10, and 11 parts in 10^{13} respectively [18].

Also, if NBS-4, NBS-5 and the PTB and NRC primary standards are compared via TAI as the reference standard, the standard deviation of their comparison values is about 1×10^{-13}, which is in good agreement with the accuracy derived using the accuracy algorithm developed in the text.

The second used in generating the time scale AT(NBS) was brought into agreement with the NBS "best estimate" of the SI second as of 1 January 1975 (MJD 42413). Starting with this date and using the accuracy algorithm in the text an effort will be made to cause AT(NBS) to be as accurate and stable (over any sample time) as reasonably possible. In contrast, the time scale AT_0(NBS), used in the text as the reference standard for comparing calibrations, uses a second determined by a time scale algorithm [20] which combines the times of an ensemble of commercial clocks — totally independent of calibrations with the NBS primary frequency standards.

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