# Contests with Rank-Order Spillovers

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# Overview: Theory

- Motivation
- Model, Notation, and General Linear Class of Complete Information Contests with Rank-Order Spillovers
- Results
  - Proposition 1: Characterization of symmetric pure-strategy equilibria
  - Proposition 2: Characterization of symmetric mixed-strategy equilibria
  - Proposition 3: Characterization summary

# Overview: Applications

- Results apply/extend a variety of models/results in IO, behavioral economics, and game theory
  - Cover all standard contests and auctions, war of attrition, & all-pay auction
  - More exotic auctions and contests
  - Innovation contests with spillovers
  - Pricing games
  - Price matching policies
  - Behavioral economics (inequality aversion, loss aversion, regret, reference pricing)
  - Evolutionary equilibria (ESS)

# Model and Notation

- Players: $i \in \{1,2\}$
- Actions (bids, prices, effort, etc.): $x_i \in A = [0, \infty)$
- Payoffs (coin-flip tie-breaking rule suppressed):

$$u_i(x_i, x_j) = \begin{cases} v - \beta x_i - \delta x_j & \text{if } x_i > x_j \\ -\gamma - \alpha x_i - \theta x_j & \text{if } x_i < x_j \end{cases}$$

- $v \geq 0$
- $V \equiv v + \gamma > 0$
- $\Gamma$: An arbitrary game with this structure.
- $\eta \equiv \alpha + \theta - \beta - \delta$
- $x^*$: Symmetric pure-strategy (Nash) equilibrium
- $F^*(x)$: Symmetric (non-degenerate) mixed-strategy equilibrium

# Spillover Effects

- Recall payoffs:

$$u_i(x_i, x_j) = \begin{cases} v - \beta x_i - \delta x_j & \text{if } x_i > x_j \\ -\gamma - \alpha x_i - \theta x_j & \text{if } x_i < x_j \end{cases}$$

- First-order $\left\{ \begin{array}{c} \text{positive} \\ \text{negative} \end{array} \right\}$ spillovers as $\delta \left\{ \begin{array}{c} < \\ > \end{array} \right\} 0$
- Second-order $\left\{ \begin{array}{c} \text{positive} \\ \text{negative} \end{array} \right\}$ spillovers as $\theta \left\{ \begin{array}{c} < \\ > \end{array} \right\} 0$
- Simplest example: *Second price auction* ($\gamma = \beta = \alpha = \theta = 0; \delta = 1$), where

$$u_i(x_i, x_j) = \begin{cases} v - x_j & \text{if } x_i > x_j \\ 0 & \text{if } x_i < x_j \end{cases}$$

  - Second-price auction has first-order negative spillovers ($\delta = 1 > 0$)

# Proposition 1: Characterization of Symmetric Pure-Strategy Equilibria

$\Gamma$ *has a symmetric pure-strategy Nash equilibrium if and only if the following three conditions jointly hold:*

(i) $\beta \geq 0$

(ii) $\alpha \leq 0$, *and*

(ii) $\eta < 0$.

*Furthermore, there is but one such equilibrium and it is given by*

$$x^* = -\frac{V}{\eta} \equiv \frac{v + \gamma}{\beta + \delta - \alpha - \theta}.$$

# Proposition 2: Characterization of Symmetric Mixed-Strategy Equilibria

$\Gamma$ *has a nondegenerate symmetric mixed-strategy equilibrium if and only if one of the following three sets of conditions holds:*

(i) $\beta > 0$ *and* $\alpha > 0$; *or*

(ii) $\beta = 0$, $\alpha > 0$ *and either* $\eta\theta = 0$ *or* $\eta < \alpha$; *or*

(iii) $\beta = 0$, $\alpha < 0$ and either $\alpha < \eta < 0$ or $\eta < \theta = 0$.

- *In cases* $(i)$ *and* $(ii)$ *the equilibrium is unique within the class of symmetric equilibria (pure or mixed).*
- *In case* $(iii)$ *there exists a continuum of nondegenerate symmetric mixed-strategy equilibria, as well as a unique symmetric pure-strategy equilibrium (given in Proposition 1).*

# Proposition 2: Characterization of Symmetric Mixed-Strategy Equilibria (Continued)

*The nondegenerate symmetric mixed strategy equilibria are atomless and described by the distribution function $F^*(w)$ on $[m^*, u^*)$, where*

$$F^*(w) = \begin{cases} \frac{\alpha}{\alpha-\beta}\left(1-\left(\frac{V+\eta m^*}{V+\eta w}\right)^{\frac{\alpha-\beta}{\eta}}\right) & \text{if} \quad \eta \neq 0; \alpha-\beta \neq 0 \\ \frac{\alpha}{\theta-\delta}\ln\left(\frac{V+(\theta-\delta)w}{V}\right) & \text{if} \quad \eta \neq 0; \alpha-\beta = 0 \\ \frac{\alpha}{\alpha-\beta}\left(1-\exp\left(-\frac{\alpha-\beta}{V}w\right)\right) & \text{if} \quad \eta = 0; \alpha-\beta \neq 0 \\ \frac{\alpha}{V}w & \text{if} \quad \eta = 0; \alpha-\beta = 0 \end{cases},$$

# Proposition 2: Characterization of Symmetric Mixed-Strategy Equilibria (Continued)

$[m^*, u^*)$ *is the interval defined by the lower bound*

$$m^* = \begin{cases} 0 & \text{if} \quad \alpha > 0 \\ m' \in \left(-\frac{V}{\eta}, \infty\right) & \text{if} \quad \alpha < 0 \end{cases},$$

*and upper bound*

$$u^* = \begin{cases} -\frac{V}{\eta} & \text{if} \quad \alpha > 0; \beta = 0; \eta < 0 \\ \frac{V}{\eta}\left((\alpha/\beta)^{\frac{\eta}{\alpha-\beta}} - 1\right) & \text{if} \quad \alpha > 0; \beta > 0; \alpha \neq \beta; \eta \neq 0 \\ \frac{V}{\eta}\left(\exp\left(\eta/\alpha\right) - 1\right) & \text{if} \quad \alpha = \beta > 0; \eta \neq 0 \\ \frac{V}{\alpha-\beta}\ln\frac{\alpha}{\beta} & \text{if} \quad \alpha > 0; \beta > 0; \alpha \neq \beta; \eta = 0 \\ V/\alpha & \text{if} \quad \alpha > 0; \beta > 0; \alpha = \beta; \eta = 0 \\ \infty & \text{if} \quad \text{otherwise} \end{cases}$$

# Proposition 2: Characterization of Symmetric Mixed-Strategy Equilibria (Finale)

*Finally, the corresponding equilibrium payoffs are given by*

$$EU^* = \begin{cases} \frac{\theta v+\delta\gamma}{\theta-\delta}+\frac{\theta\beta V}{\eta(\theta-\delta)}\left(1-\left(\frac{\alpha}{\beta}\right)^{\frac{\eta}{\alpha-\beta}}\right) & \text{if} \quad \eta \neq 0; \alpha \neq \beta; \theta \neq \delta; \beta \neq 0 \\ -\gamma+\frac{\theta V}{\eta}-\frac{\theta V}{\eta}\frac{\alpha}{\eta}\ln\frac{\alpha}{\beta} & \text{if} \quad \eta \neq 0; \alpha \neq \beta; \theta = \delta; \beta \neq 0 \\ \frac{\theta v+\delta\gamma}{\theta-\delta}+\frac{\alpha\delta}{\theta-\delta}m^* & \text{if} \quad \eta \neq 0; \alpha \neq \beta; \theta \neq 0; \beta = 0 \\ -\gamma-\alpha m^* & \text{if} \quad \eta \neq 0; \alpha \neq \beta; \theta = 0; \beta = 0 \\ \frac{\theta v+\delta\gamma}{\theta-\delta}+\frac{\theta\beta V}{(\theta-\delta)^2}\left(1-\exp\left(\frac{\theta-\delta}{\beta}\right)\right) & \text{if} \quad \eta \neq 0; \alpha-\beta = 0 \\ \frac{\theta v+\delta\gamma}{\theta-\delta}+\frac{\theta\beta V}{(\theta-\delta)^2}\ln\left(\frac{\alpha}{\beta}\right) & \text{if} \quad \eta = 0; \alpha-\beta \neq 0; \beta \neq 0 \\ \frac{\theta v+\delta\gamma}{\theta-\delta} & \text{if} \quad \eta = 0; \alpha-\beta \neq 0; \beta = 0 \\ -\gamma-\frac{\theta}{2\alpha}V & \text{if} \quad \eta = 0; \alpha-\beta = 0 \end{cases}$$

# Proposition 3: Summary Characterization

*The symmetric equilibria to* $\Gamma$ *are characterized as follows:*

(a) *The unique symmetric equilibrium is in pure strategies if and only if one of the following three conditions holds* $(i)$ $\beta > 0$, $\alpha \leq 0$, *and* $\eta < 0$; $(ii)$ $\beta = 0$, $\alpha = 0$, *and* $\eta < 0$; *or* $(iii)$ $\beta = 0$, $\eta \leq \alpha < 0$, *and* $\theta \neq 0$;

(b) *The unique symmetric equilibrium is in nondegenerate mixed strategies if and only one of the following two conditions holds:* $(i)$ $\beta > 0$ *and* $\alpha > 0$; *or* $(ii)$ $\beta = 0$, $\alpha > 0$ *and either* $\eta\theta = 0$ *or* $\eta < \alpha$;

(c) *There is a unique symmetric pure-strategy equilibrium and a continuum of nondegenerate symmetric mixed-strategy equilibria if and only if* $\beta = 0$, $\alpha < 0$ *and either* $\alpha < \eta < 0$ *or* $\eta < \theta = 0$;

(d) *If none of the conditions in* $(a)$ *through* $(c)$ *hold,* $\Gamma$ *does not have a symmetric equilibrium (either pure or mixed).*

# Example: Partnership Dissolution (The Self-Auction)

- Two partners wish to dissolve a partnership each values at $v > 0$.
- Submit bids simultaneously; high bidder pays other partner her bid to gain ownership:

$$u_i(x_i, x_j) = \begin{cases} v - x_i & \text{if} \quad x_i > x_j \\ x_j & \text{if} \quad x_\iota < x_j \end{cases}$$

- $\Gamma$ is covered by Proposition 1, since $\beta = -\theta = 1$, $\gamma = \alpha = \delta = 0$, and $\eta = -2$.
- Proposition 1 implies that the unique symmetric pure-strategy equilibrium is

$$x^* = -\frac{v + \gamma}{\beta + \delta - \alpha - \theta} = \frac{v}{2}.$$

- Proposition 2 implies absence of any non-degenerate symmetric mixed-strategy equilibria.

# Example: An Innovation Contest with Spillovers

- Extend Dasgupta's (1986) all-pay auction innovation contest model
  - Each firm's expenditure on R&D has beneficial spillover on rival

$$u_i(x_i, x_j) = \begin{cases} v - x_i - \delta x_j & \text{if} \quad x_i > x_j \\ -x_i - \theta x_j & \text{if} \quad x_\iota < x_j \end{cases}.$$

  - Greater benefit to winner than loser ( $\delta < \theta < 0$)
- This is a $\Gamma$ with $\delta < \theta < 0$, $V = v > 0 = \gamma$, and $\alpha = \beta = 1$.
- Since $\alpha - \beta = 0$ and $\eta > 0$, Propositions 2 and 3 imply that the unique symmetric equilibrium is

$$F^*(x) = \frac{1}{\theta - \delta} \ln\left(1 + \frac{\theta - \delta}{v} x\right) \text{ on } \left[0, \frac{v}{\theta - \delta}\left(\exp\left(\frac{\theta - \delta}{\alpha}\right) - 1\right)\right].$$

- Essentially an all-pay auction with asymmetric first-and second-order *positive* spillovers
- When $\delta = \theta$, can use Proposition 2 to show strategies identical to those in a standard all-pay auction: $F^*(x) = x/v$.

# Example: Varian/Rosenthal Sales Models

- $L > 0$ : "loyal" consumers, unit demand, choke price $r > 0$
- $S > 0$ : "shoppers" purchase from firm charging lowest price
- Price setting firms, zero cost, and payoffs

$$\pi_i(p_i, p_j) = \begin{cases} (S+L)\,p_i & \text{if} \quad p_i < p_j \\ Lp_i & \text{if} \quad p_i > p_j \end{cases}$$

- Letting $x_i \equiv r - p_i \geq 0$, we can rewrite payoffs as

$$u_i(x_i, x_j) = \begin{cases} (S+L)\,r - (S+L)\,x_i & \text{if} \quad x_i > x_j \\ rL - Lx_i & \text{if} \quad x_i < x_j \end{cases}$$

- Here, $\Gamma$ has $v = (S+L)\,r$, $\gamma = -rL$, etc., so Proposition 2 implies

$$F^*(x) = \frac{L}{S}\left(\frac{r}{r-x} - 1\right) \text{ on } \left[0, \frac{rS}{S+L}\right].$$

- Use fact that $G^*(p) = \Pr(P \leq p) = 1 - F^*(r-p)$:

$$G^*(p) = 1 - \frac{L}{S}\left(\frac{r-p}{p}\right) \text{ on } \left[r\frac{L}{S+L}, r\right].$$

# Example: Inequality Aversion in a Job Tournament

- Two workers compete in a winner-take-all fashion for a promotion worth $\mu > 0$, but get disutility from effort inequality:

$$u_i(x_i, x_j) = \begin{cases} \mu - x_i - b(x_i - x_j) & \text{if} \quad x_i > x_j \\ -x_i - a(x_j - x_i) & \text{if} \quad x_i < x_j \end{cases}$$

- $0 < b$ (winner gets disutility from "slaughtering" the loser) and $0 < a < 1$ (loser gets disutility from being "slaughtered."
- Rewrite as

$$u_i(x_1, x_2) = \begin{cases} \mu - (1 + b) x_i + b x_j & \text{if} \quad x_i > x_j \\ -(1 - a) x_i - a x_j & \text{if} \quad x_i < x_j \end{cases} .$$

- This is a $\Gamma$ with $V = v = \mu > 0$, $\gamma = 0$, $\alpha = 1 - a > 0$, $\delta = -b < 0$, etc.

# Example: Inequality Aversion in a Job Tournament (Continued)

- Propositions 1, 2 and 3 imply that the unique symmetric equilibrium is in mixed strategies and given by

$$F^*(x) = \frac{1-a}{a+b}\left(\exp\left(\frac{a+b}{\mu}x\right) - 1\right)$$

on $\left[0, \frac{\mu}{a+b} \ln \frac{1+b}{1-a}\right]$.

- When the winner enjoys "slaughtering" the loser, such that $a \in (0,1)$ and $b \in (-1,0)$:
  - If $b \neq -a$, solution identical to that above.
  - If $b = -a$, the equilibrium distribution of effort takes on the all-pay auction form

    $$F^*(x) = \frac{1-a}{\mu} \text{ on } \left[0, \frac{\mu}{1-a}\right].$$

    Expected payoffs are *not* zero (as in the standard all-pay auction), but

    $$EU^* = -\frac{a}{2(1-a)}\mu$$

# Example: Loss Aversion in a Job Tournament

- Two workers compete in a winner-take-all fashion for a bonus valued at $\mu > 0$. Worker $i$'s income is thus

$$y_i = \begin{cases} \mu - x_i & \text{if} \quad x_i > x_j \\ -x_i & \text{if} \quad x_\imath < x_j \end{cases}$$

- Worker utility (over income) is $u_i = y_i$ if player $i$ wins, and $\lambda y_i$ if player $i$ loses, where $\lambda > 1$. Hence, utility (as a function of effort) is

$$u_i\left(x_i, x_j\right) = \begin{cases} \mu - x_i & \text{if} \quad x_i > x_j \\ -\lambda x_i & \text{if} \quad x_\imath < x_j \end{cases}$$

- This is a $\Gamma$ with $v = \mu > 0$, $\gamma = 0$, $\alpha = \lambda > 0$, $\beta = 1 > 0$, $\theta = \delta = 0$, and $\eta = \alpha - \beta = \lambda - 1 > 0$.
- Propositions 2 and 3 reveal that the unique nondegenerate symmetric mixed-strategy equilibrium is

$$F^*\left(x\right) = \frac{\lambda x}{\mu + (\lambda - 1)x} \text{ on } \left[0, \mu\right].$$

# Example: Winner Regret in Auctions

- First price auction with regret (Engelbrecht-Wiggans (1989), Engelbrecht-Wiggans and Katok (2007), and Filiz-Ozbay and Ozbay (2007)):

$$u_i(x_1, x_2) = \begin{cases} v - x_i - \mu(x_i - x_j) & \text{if} \quad x_i > x_j \\ 0 & \text{if} \quad x_i < x_j \end{cases}.$$

- $x_i$ is player $i$'s bid, $v > 0$ the value of the item, and $\mu > 0$ a "regret" parameter
- Winner regret refers to the fact that the high bidder derives disutility from leaving money on the table (the difference between the winning and losing bid). The payoffs may be rewritten as

$$u_i(x_1, x_2) = \begin{cases} v - (\mu + 1)x_i + \mu x_j & \text{if} \quad x_i > x_j \\ 0 & \text{if} \quad x_i < x_j \end{cases}$$

- $V = v$, $\alpha = \theta = 0$, $\beta = (1 + \mu) > 0$, $\delta = -\mu$, and $\eta = -1$.
- Propositions 1 and 3 imply the unique symmetric equilibrium is $x^* = v$.

# Example: Auctions with Winner and Loser Regret

- Can show both *first and second order spillover* effects arise in this case

$$u_i(x_1, x_2) = \begin{cases} v - x_i(\mu + 1) + \mu x_j & \text{if} \quad x_i > x_j \\ -\rho(v - x_j) & \text{if} \quad x_i < x_j \end{cases}.$$

- This is a $\Gamma$ with $V = (1+\rho)v$, $\alpha = (1+\rho) > 0$, $\beta = (1+\mu) > 0$, $\theta = -\rho$, $\delta = -\mu$, and $\eta = 0$.
- When $\rho \neq \mu$, Propositions 2 and 3 imply the unique symmetric equilibrium is

$$F^*(x) = \left(\frac{1+\rho}{\rho-\mu}\right)\left(1 - \exp\left(-\frac{\rho-\mu}{1+\rho}\frac{x}{v}\right)\right)$$

on $\left[0, \frac{1-\delta}{\delta-\mu} \ln\left(\frac{1+\rho}{1+\mu}\right)\right]$

- When $\rho = \mu$, Proposition 2 yields the standard all-pay auction form: $F^*(x) = x/v$, but $EU^* = -\rho v/2$.

# Example: Evolutionary Stationary Strategies (ESS) in the All-Pay Auction

- One can also use Proposition 2 to find the unique symmetric ESS equilibrium in the standard two-player all-pay auction
- Finite agent ESS equilibrium of Schaffer (1988) requires each player maximize difference in payoffs:

$$u_i(x_1, x_2) = \begin{cases} v - x_i - (-x_j) & \text{if} \quad x_i > x_j \\ -x_i - (v - x_j) & \text{if} \quad x_i < x_j \end{cases}$$

- This is a $\Gamma$ with payoffs

$$u_i(x_1, x_2) = \begin{cases} v - x_i + x_j & \text{if} \quad x_i > x_j \\ -v - x_i + x_j & \text{if} \quad x_i < x_j \end{cases}.$$

  and $V = 2v > 0$, $\beta = \alpha = -\theta = -\delta = 1$, $\alpha - \beta = 0$ and $\eta = 0$.
- Proposition 2 implies the unique symmetric ESS equilibrium to the original game is $F^*(x) = \frac{x}{2v}$ on $[0, 2v]$.
- Entails overdissipation of rents, as Hehenkamp, Leininger, and Possajennikov's (2004) showed for a Tullock contest.

# Concluding Remarks

- Characterized symmetric equilibria for wide class of complete information contests with rank-order spillovers
- Simple closed form expressions for equilibrium strategies and payoffs
- May be used to establish uniqueness of symmetric equilibria in existing models, as well as closed-form expressions for equilibrium behavior
- Useful for examining implications of behavioral economics on contests and auctions
- Evolution in contests
- Caveats
    - Complete information; but Baye, Kovenock, and de Vries (2005) consider the (simpler) incomplete information case, as do Lizzeri and Perisco (2000)
    - Two players (genuine)
    - Symmetric equilibria (genuine)