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Modeling Stock Trading Day Effects Under Flow Day-of-Week Constraints

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Abstract

From an invertible linear relation between stock and flow trading day regression coefficients that is derived, it is shown how flow day-of-week effect constraints can be imposed upon the day-of-week effect component of the stock trading day model of Bell (1984) used in X-12-ARIMA. We illustrate the use of the general formulas obtained by deriving the one-coefficient stock regression model determined by the constraints that give rise to the one-coefficient weekday-weekend-contrast flow trading day model of TRAMO and X-12-ARIMA.

Key Words: Time series; Trading day adjustment; One-coefficient stock trading day model

1 Formulas Relating Flow and Stock Trading Day Coefficients

With $i = 1, \ldots, 7$ indexing the weekdays from Monday through Sunday, and with $t = 1, 2, \ldots$ indexing the months of the span of interest for a monthly time series, let $X_{i,t}$ denote the number of times the i-th weekday occurs in month t. Then $\sum_{i=1}^{7} \beta_i X_{i,t}$ is the basic formula for flow series trading day effects from which regression models are derived for estimation of such effects with regARIMA models, see Bell and Hillmer (1983) and Findley, Monsell, Bell, Otto and Chen (1998), for example. From the decomposition of $\sum_{i=1}^{7}\beta_i X_{i,t}$ into day-of-week and length-of-month effects,

$$\sum_{i=1}^{7}\beta_i X_{i,t}=\sum_{i=1}^{7}\tilde{\beta}_i X_{i,t}+\bar{\beta}m_t, \tag{1}$$

where $\tilde{\beta}_i=\beta_i-\bar{\beta}$ with $\bar{\beta}=\frac{1}{7}\sum_{i=1}^{7}\beta_i$ and $m_t=\sum_{i=1}^{7}X_{i,t}$ (the length of month t), Bell (1984,1995) derived a regression model for the cumulative end-of-month stock trading day effects

$$\sum_{j=1}^{t}\sum_{i=1}^{7}\beta_i X_{i,j}. \tag{2}$$

The day-of-week component of Bell's model was derived from the day-of-week component of (1),

$$\sum_{i=1}^{7}\tilde{\beta}_i X_{i,t}=\sum_{i=1}^{6}\tilde{\beta}_i X_{i,t}^{*}, \tag{3}$$

where $X_{i,t}^{*}=X_{i,t}-X_{7,t}$, $i=1,\ldots,6$. The right hand side of (3), which arises from

$$\sum_{i=1}^{7}\tilde{\beta}_i=0, \tag{4}$$

defines the regression model for the day-of-week component of (1). In this section, we derive complementary formulas connecting the coefficient vector

$$\tilde{\beta}=\left[\begin{array}{cccc}\tilde{\beta}_1 & \tilde{\beta}_2 & \ldots & \tilde{\beta}_6\end{array}\right]' \tag{5}$$

of this model and the coefficient vector of Bell's day-of-week effect model.

To present Bell's formula for the stock day-of-week effects, for $k=1,\ldots,7$, we define $I_t(k)=1$ if the stock is measured on the k-th weekday in month t and $I_t(k)=0$ otherwise. Let k_0 be the index of the type of day on which the stock is measured in the month preceding month 1. From (3), the derivation on pp. 5–7 of Bell (1984), but with the more general definition of k_0 just given to permit any pattern of days for stock measurements, shows that the (detrended) day-of-week effect of (2) is given by

$$\sum_{k=1}^{7}\tilde{\gamma}_k I_t(k), \tag{6}$$

with $\tilde{\gamma}_k = \gamma_k - \bar{\gamma}$ and $\bar{\gamma} = \frac{1}{7}\sum_{k=1}^{7}\gamma_k$, where

$$\gamma_7 = -\sum_{i=1}^{k_0}\tilde{\beta}_i$$

and

$$\gamma_k = \sum_{i=1}^{k}\tilde{\beta}_i + \gamma_7,\ k = 1, \ldots, 6. \tag{7}$$

Since

$$\sum_{k=1}^{7}\tilde{\gamma}_k = 0, \tag{8}$$

setting $I_t^*(k) = I_t(k) - I_t(7)$, $k = 1, \ldots, 6$, we have

$$\sum_{k=1}^{7}\tilde{\gamma}_k I_t(k) = \sum_{k=1}^{6}\tilde{\gamma}_k I_t^*(k)\,. \tag{9}$$

In the case of end-of-month stocks, or of $\bar{w}$-th day of the month stocks, where, for fixed $1 \leq w \leq 31$, $\bar{w} = w$ in months with at least w days, and $\bar{w}$ is the final day of the month for shorter months, the r.h.s. of (9) defines the regression model for stock day-of-week effect regression models used by X-12-ARIMA (Findley et al., 1998). A. Maravall has informed us that the same regression models will be implemented in a future version of TRAMO (Gómez and Maravall, 1996). For these cases, the day-of-week effect defined by (6) has no seasonal component, see the Remark below.

To obtain the invertible linear relation between the coefficient vector

$$\tilde{\gamma} = \begin{bmatrix} \tilde{\gamma}_1 & \tilde{\gamma}_2 & \ldots & \tilde{\gamma}_6 \end{bmatrix}'$$

and $\tilde{\beta}$, note first that, with

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

(7) is equivalent to

$$L\tilde{\beta} = \begin{bmatrix} \gamma_1 - \gamma_7 & \gamma_2 - \gamma_7 & \dots & \gamma_6 - \gamma_7 \end{bmatrix}'.$$

Next, using (8), observe that

$$\gamma_k - \gamma_7 = \tilde{\gamma}_k - \tilde{\gamma}_7 = \tilde{\gamma}_k + \sum_{j=1}^{6} \tilde{\gamma}_j = 2\tilde{\gamma}_k + \sum_{j \neq k} \tilde{\gamma}_j.$$

Thus, defining

$$M = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix},$$

we have

$$L\tilde{\beta} = M\tilde{\gamma}.$$

Since

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

we are led to

$$\tilde{\beta} = N\tilde{\gamma}, \tag{10}$$

with

$$N = L^{-1}M = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{11}$$

Also, since

$$M^{-1} = \frac{1}{7} \begin{bmatrix} 6 & -1 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 & -1 \\ -1 & -1 & 6 & -1 & -1 & -1 \\ -1 & -1 & -1 & 6 & -1 & -1 \\ -1 & -1 & -1 & -1 & 6 & -1 \\ -1 & -1 & -1 & -1 & -1 & 6 \end{bmatrix},$$

a special case of Ex. 5.18 of Noble (1969, p. 148), we have

$$\tilde{\gamma} = N^{-1}\tilde{\beta}, \tag{12}$$

with

$$N^{-1} = M^{-1}L = \frac{1}{7} \begin{bmatrix} 1 & -5 & -4 & -3 & -2 & -1 \\ 1 & 2 & -4 & -3 & -2 & -1 \\ 1 & 2 & 3 & -3 & -2 & -1 \\ 1 & 2 & 3 & 4 & -2 & -1 \\ 1 & 2 & 3 & 4 & 5 & -1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}.$$

Remark. When precise interpretations of the estimated seasonal factors of a time series are desired, it could be important that the day-of-week factors be free of seasonal effects. The argument on p. 7 of Bell (1984) reveals that the stock day-of-week effects (6) always have this property if and only if the $I_t(k)$ have identical long-term calendar month means,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} I_{j+12n}(k) = \frac{1}{7},\ 1 \le j \le 12, \tag{13}$$

for $k = 1, \ldots, 7$. That is, in each of the twelve calendar months, over time the seven days of the week must be stock days with the equal frequency. This happens with end-of-month and $\bar{w}$-th day of month stocks because the monthly calendar repeats every twenty-eight years (ignoring an exception every four hundred years). (13) does not hold, for example when $I_t(7)$ is defined to be zero for all t in the situation in which Sunday stocks are never measured. When (13) fails for some k, then to obtain regressors that yield day-of-week factors with no seasonal component the long-term calendar-month means of the $I_t^*(k)$ must be removed; see Bell (1984, pp. 1-3).

2 The Effect of Flow-Coefficient Constraints

With stock series, it can happen that there is information about the associated flow series which constrains the coefficients β_i in (1). When the constraint is linear with coefficients summing to zero, i.e., is a contrast, it is equivalent to a constraint on $\tilde{\beta}$ of the form

$$H\tilde{\beta} = 0, \tag{14}$$

for some matrix H. From (10) and (12), the constraint (14) on $\tilde{\beta}$ is equivalent to the constraint

$$HN\tilde{\gamma} = 0 \tag{15}$$

on $\tilde{\gamma}$.

2.1 An example with one constraint

We first consider the simple contrast

$$\beta_6 - \beta_7 = 0, \tag{16}$$

used for series in which the level of economic activity can be assumed to be the same on Saturday and Sunday. It is equivalent to $\tilde{\beta}_6 - \tilde{\beta}_7 = 0$, and, from (4), also to $\sum_{i=1}^{5} \tilde{\beta}_j + 2\tilde{\beta}_6 = 0$. This is the same as (14) with

$$H = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 2 \end{array}\right]. \tag{17}$$

From (11),

$$HN = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 3 \end{array}\right],$$

so, from (15), the constraint (16) is equivalent to

$$\tilde{\gamma}_6 = -\frac{1}{3}\sum_{k=1}^{4} \tilde{\gamma}_k.$$

Consequently, the regression model for the constrained stock day-of-week effect is given by the r.h.s. of

$$\sum_{k=1}^{6} \tilde{\gamma}_k I_t^* (k) = \sum_{k=1}^{5} \tilde{\gamma}_k D_t (k),$$

with

$$D_t(k) = I_t^*(k) - \frac{1}{3} I_t^*(6), \; k = 1, \ldots, 4,$$

and

$$D_t(5) = I_t^*(5).$$

2.2 Models from multiple constraints

In order to illustrate a general approach to obtaining constrained regression models outlined in Silvey (1975, p. 60), we now consider the one-coefficient weekday-weekend-contrast flow day-of-week effect model of TRAMO and X-12-ARIMA. This model imposes constraints on the weekday coefficients $\beta_1, \ldots, \beta_5$ as well as on β_6 and β_7,

$$\beta_1 = \beta_2 = \cdots = \beta_5, \; \beta_6 = \beta_7, \tag{18}$$

resulting in the constraint matrices

$$H = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

and

$$HN = \begin{bmatrix} 3 & 0 & 1 & 1 & 1 & 1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 3 \end{bmatrix}$$

for $\tilde{\beta}$ and $\tilde{\gamma}$, respectively.

To obtain the regression model resulting from imposing (18) on $\sum_{k=1}^{6} \tilde{\gamma}_k I_t^*(k)$, we create an auxiliary matrix by adding a row to HN in such way that an invertible matrix results: with

$$J = \begin{bmatrix} 3 & 0 & 1 & 1 & 1 & 1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

we have

$$J^{-1}=\frac{1}{35}\begin{bmatrix} 12 & -3 & -10 & -9 & -4 & -21 \\ 9 & 24 & 10 & 2 & -3 & -7 \\ 6 & 16 & 30 & 13 & -2 & 7 \\ 3 & 8 & 15 & 24 & -1 & 21 \\ 0 & 0 & 0 & 0 & 0 & 35 \\ -10 & -15 & -15 & -10 & 15 & 0 \end{bmatrix}.$$

Defining the row vector

$$I_t^* = \left[\begin{array}{cccccc} I_t^*(1) & I_t^*(2) & I_t^*(3) & I_t^*(4) & I_t^*(5) & I_t^*(6) \end{array}\right],$$

observe that

$$\sum_{k=1}^{6} \tilde{\gamma}_k I_t^*(k) = I_t^* \tilde{\gamma} = \left(I_t^* J^{-1}\right)(J\tilde{\gamma}). \tag{19}$$

Due to (15),

$$J\tilde{\gamma} = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \tilde{\gamma}_5 \end{array}\right]',$$

so the sixth column of J^{-1} defines the one-coefficient regressor D_t for the constrained regression model defined by the r.h.s. of (19):

$$\sum_{k=1}^{6} \tilde{\gamma}_k I_t^*(k) = \tilde{\gamma}_5 D_t,$$

with

$$D_t = -\frac{3}{5} I_t^*(1) - \frac{1}{5} I_t^*(2) + \frac{1}{5} I_t^*(3) + \frac{3}{5} I_t^*(4) + I_t^*(5). \tag{20}$$

From this we obtain

$$\tilde{\gamma} = \left[\begin{array}{cccccc} -\frac{3}{5} & -\frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 1 & 0 \end{array}\right]' \tilde{\gamma}_5, \tag{21}$$

showing that $\tilde{\gamma}_6 = 0$ (so Saturday is an average day, $\beta_6 = \bar{\beta}$) and, from (8), that $\tilde{\gamma}_7 = -\tilde{\gamma}_5$.

Alternatively, given the constrained form of $\tilde{\beta}$, i.e. $\tilde{\beta} = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & -\frac{5}{2} \end{array}\right]' \tilde{\beta}_5$, which follows from (18) and (4), it is simpler to obtain (21), and thus also (20), from (12). Thus, from (12),

$$\tilde{\gamma} = \frac{1}{7}\begin{bmatrix} 1 & -5 & -4 & -3 & -2 & -1 \\ 1 & 2 & -4 & -3 & -2 & -1 \\ 1 & 2 & 3 & -3 & -2 & -1 \\ 1 & 2 & 3 & 4 & -2 & -1 \\ 1 & 2 & 3 & 4 & 5 & -1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -\frac{5}{2} \end{bmatrix}\tilde{\beta}_5 = \frac{1}{2}\begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \\ 5 \\ 0 \end{bmatrix}\tilde{\beta}_5,$$

which yields $\tilde{\gamma}_5 = \frac{5}{2}\tilde{\beta}_5$ and (21).

In general, if the constraint matrix H in (14) has r rows and full rank, then $6-r$ rows must be added to HN to obtain the auxiliary matrix J for (19). The last $6-r$ rows of $I_t^* J^{-1}$ define a constrained model regressor set. Adding rows to form J that are zero except in a single location, as above, has the important advantage that the nonzero entries of $J\tilde{\gamma}$ are entries of $\tilde{\gamma}$ and are therefore immediately interpretable.

For implementation of constrained estimation in practice, we note that, from a preliminary unconstrained estimation of $\tilde{\gamma}$, X-12-ARIMA can output a regression matrix file containing the values of $I_t^*(k)$, $k = 1, \ldots, 6$ for all t. (Use the command `save=rmx` in the `regression` spec.) From the latter file, a file with the constrained model's regression matrix can be constructed. This file can be input into X-12-ARIMA (or TRAMO) to obtain estimated coefficients of the constrained model and its day-of-week adjustment factors.

It should be pointed out that the general approach illustrated above can as well be applied to find the flow series regressors associated with the constraint (14). In this case, a nonsingular J is obtained by adding $6-r$ rows to H. Then, with

$$X_t^* = \left[\begin{array}{cccccc} X_{1,t}^* & X_{2,t}^* & X_{3,t}^* & X_{4,t}^* & X_{5,t}^* & X_{6,t}^* \end{array} \right],$$

we have

$$\sum_{i=1}^{6} \tilde{\beta}_i X_{i,t}^* = X_t^* \tilde{\beta} = \left(X_t^* J^{-1}\right)\left(J\tilde{\beta}\right),$$

and the last $6-r$ rows of $X_t^* J^{-1}$ define a regressor set for the constrained flow trading day model, whose coefficients are the last $6-r$ entries of $J\tilde{\beta}$.

3 Final Remarks

The approach to finding regressors for the constrained model by means of the inverse of an augmented constraint matrix J is appealing because of its generality. But, in practice, it is usually not difficult to obtain these regressors without such a matrix inversion, as we illustrated.

Bell (1984,1995) also provides a model for the detrended and deseasonalized component of the end-of-month stocks, $\bar{\beta}\sum_{j=1}^{t} m_j$ accumulated from the length-of-month effects $\bar{\beta}m_t$ in (1). This model is not affected by constraints (15) and can be modified to apply to $\bar{w}$-th day-of-month stocks, by redefining the length of month t to be the number of days between the stock measurements of months $t-1$ and t in the derivation of the mode. Bell (1995) has discussions for and against estimation of this component. A future version of X-12-ARIMA may provide a regressor for its estimation.

For day-of-week effect models for quarterly series, the only changes required to the formulas of this report are (i) quantities that were defined in terms of months, e.g. m_t, must be redefined in terms of quarters; and (ii) $I_{j+12n}(k)$, $1 \leq j \leq 12$ in (13) must be replaced by $I_{j+4n}(k)$, $1 \leq j \leq 4$.

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References

[1] Bell, W.R. (1984), Seasonal Decomposition of Deterministic Effects. Statistical Research Division. U.S. Bureau of the Census Statistical Research Division Report Number: Census/SRD/RR-84/01. Available from http://www.census.gov/srd/papers/pdf/rr84-1.pdf

[2] Bell, W.R. (1995), Correction to "Seasonal Decomposition of Deterministic Effects." Statistical Research Division. U.S. Bureau of the Census Statistical Research Division Report Number: Census/SRD/RR-95/01. Available from http://www.census.gov/srd/papers/pdf/rr95-1.pdf

[3] Bell, W.R. and Hillmer, S.C. (1983), Modeling Time Series with Calendar Variation. Journal of the American Statistical Association, 78, 526-534.

[4] Findley, D.F., Monsell, B.C., Bell, W.R.,Otto, M.C. and Chen, B.C. (1998), New Capabilities and Methods of the X-12-ARIMA Seasonal Adjustment Program. Journal of Business and Economic Statistics 16, 127-177 (with discussion).

[5] Gómez, V. and Maravall, A. (1996), Programs TRAMO and SEATS: Instructions for the User. Working Paper 9628. Sevicio de Estudios, Banco de España.

[6] Noble, B. (1969), Applied Linear Algebra. Englewood Cliffs: Prentice Hall.

[7] Silvey, S.D. (1975), Statistical Inference. London: Halstead Press