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EXPECTED ABSOLUTE DEPARTURE OF CHI-SQUARE FROM ITS MEDIAN

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Abstract

We develop a formula for the expected absolute departure of x^2 from its median.

Key words: chi-square, median, expected absolute departure

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Let D_f denote the median of χ^2_f (chi-square with f degrees of freedom); we here determine $E(|\chi^2_f - D_f|)$. Let D_f denote this quantity. Note that $E(|\chi^2_f - c|)$ is minimal for $c = D_f$.

We first need D_f. One easily obtains $D_1 = Z^2$ with $\Phi(Z) = .75$ and $D_2 = 2 \log 2$; for $f \geqslant 3$ one may base an approximation to D_f on the approximation to χ^2_f of Peizer and Pratt (1968): $D_f = f - 2/3 + .08/f$. To obtain D_f exactly, in essence, as well as to obtain E_f, we make use of the following, familiar results (obtainable from Kennedy and Gentle 1980). For $k \geqslant 1$:

$$P(\chi^2_{2k+1} > c) = P(\chi^2_{2k-1} > c) + a_k$$
$$P(\chi^2_{2k+2} > c) = P(\chi^2_{2k} > c) + b_k \tag{1}$$

with

$$P(\chi^2_1 > c) = 2(1 - \Phi(\sqrt{c})), \quad a_k = a_{k-1}c/(2k-1) \quad (k>1)$$
$$P(\chi^2_2 > c) = \exp(-c/2), \quad b_k = b_{k-1}c/2k \tag{2}$$
$$a_1 = \sqrt{2c/\pi}\ \exp(-c/2), \quad b_0 = \exp(-c/2).$$

Using (1) and (2), we do a binary search of the interval <0,f> (successively cut this interval in half) to determine c such that $P(\chi^2_f > c) = .5$; this gives us D_f.

Let

$$g_f(x) = \frac{1}{2^{f/2}\Gamma(f/2)} \quad x^{f/2 - 1} \exp(-x/2), \tag{3}$$

the density for χ^2_f. The value of E_f is

$$\int_0^{D_f} (D_f - x) g_f(x)dx + \int_{D_f}^{\infty} (x - D_f) g_f(x)dx \tag{4}$$

$$= D_f\left[\int_0^{D_f} g_f(x)dx - \int_{D_f}^{\infty} g_f(x)dx\right] + \int_{D_f}^{\infty} x g_f(x)dx - \int_0^{D_f} x g_f(x)dx. \tag{5}$$

We have

$$xg_f(x) = \frac{2^{f/2+1}\Gamma(f/2+1)}{2^{f/2}\Gamma(f/2)}\, g_{f+2}(x) = fg_{f+2}(x). \qquad (6)$$

Thus, using the definition of D_f, we obtain E_f =

$$D_f(.5 = .5) + f[P(\chi^2_{f+2} > D_f) - P(\chi^2_{f+2} < D_f)] \qquad (7)$$

$$= f[2P(\chi^2_{f+2} > D_f) - 1]. \qquad (8)$$

If f = 2k + 1 (k≥0), we have from (1)

$$E_f = f\{2[P(\chi^2_f > D_f) + a_{k+1}] - 1\}; \qquad (9)$$

a_{k+1} is obtained from (2) with D_f substituted for c. By definition of D_f, again, we are left with $E_f = 2fa_{k+1}$. Likewise, if f = 2k + 2 (k≥0) we have $E_f = 2fb_{k+1}$.

Thus, we have: $D_1 = 0.4549$ and $E_1 = 0.8573$, $D_2 = E_2 = 1.386$, $D_3 = 2.366$ and $E_3 = 1.779$, $D_4 = 3.357$ and $E_4 = 2.103$.

Note correspondences between the formulas for E_f and the formula (Blyth 1980) for Poisson expected absolute departure from the mean: for x with Poisson mean μ, $E(|x - \mu|)$ becomes $2kP(x=k)$ with $k = [\mu] + 1$.

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