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SOME THOUGHTS ON STABILITY
IN NONLINEAR PERIODIC FOCUSING SYSTEMS

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1.) Introduction

In UCRL 17795, it was shown that curves in the x,y plane having reflection symmetry about the positive diagonal are invariant under the transformation:

$$\begin{aligned} x' &= y \\ y' &= f(y) - x \end{aligned}$$

where f(y) is the sum of the two values of x corresponding to the given y. It is required that there be just two values, but the two branches on which they occur are not required to have a common analytic form. An example given was the pair of rectangular hyperbolas $y = 1 - a/(x+1)$ and $y = -1 + a/(1-x)$, with $f(y) = 2ay/(1-y^2)$, mentioned in paragraph 3 and illustrated in Fig. 1. The question whether there are other invariant curves belonging to the same f(y) was left open.

This question was answered by John M. Greene in a letter to L. Jackson Laslett (March 8, 1968). He pointed out that all curves of the form $(1-x^2)(1-y^2)+2axy=const.$ are such invariants. If the constant has the value $2a-a^2$, the equation factors into two equations representing the rectangular hyperbolas, which are now seen to be simply the separatrices of a family of invariant curves. In the course of checking the invariance of "Greene's function" by the methods of UCRL 17195, I found that it is a special case of a broader class, which can be called "double quadratic" curves.

2.) "Double quadratic" curves

Any equation which is quadratic in x can be solved explicitly for x. If x and y occur in it symmetrically, it represents a curve with the required symmetry about the positive diagonal. The most general equation with these properties is:

$$Ax^2y^2 + B(x^2y+xy^2) + C(x^2+y^2) + Dxy + E(x+y) + F = 0, \tag{2}$$

whose solution is:

$$x = \frac{1}{2(Ay^2 + By + C)} \left[-Dy + E \pm \sqrt{D^2 + 4E(Ay^2 + By + C) - 4(Ay^2 + By + C)(Cy^2 + Ey + F)} \right] \tag{3}$$

The sum of the two values of x gives f(y):

$$f(y) = \frac{By^2 + Dy + E}{Ay^2 + By + C} \tag{4}$$

Since f(y) does not depend on F, all members of the family generated by giving different values to F are invariant under the transformation (1), with f(y) given by (4).

We thus have the remarkable result that an f(y) which is the ratio of any two quadratic functions of y leads to a family of invariant curves, with the single restriction that the coefficients of y² in the numerator and of y in the denominator must be of equal magnitude and opposite in sign.

The first order fixed points, if they exist, are at f(y) = 2y, and are therefore the solutions of:

$$2 Ay^3 + 3 By^2 + (2C+D)y + E = 0 \tag{5}$$

The number of parameters in (4) is easily reduced; E can be eliminated by a coordinate displacement along the positive diagonal, either A or B can be made equal to D or E by a change of scale, and any one of the remaining parameters can be set equal to unity. Thus we have a two-parameter system. Some interesting cases are:

(1) A = 1, B = 0, C = -1, D = 2a, E = 0, F = c.

$$x^2y^2 - x^2 - y^2 + 2a xy + c = 0 \quad \text{("Greene's function")}$$

$$f(y) = \frac{2 a y}{1 - y^2}$$

The first order fixed points are at y = 0, $\pm \sqrt{1-a}$.

The separatrices are displaced rectangular hyperbolas, as pointed out above.

(2) A = 1, B = 0, C = 1, D = -2a, E = 0, F = c.

$$x^2y^2 - x^2 - y^2 + 2a xy + c = 0$$

$$f(y) = \frac{2 a y}{1 + y^2}$$

The first order fixed points are at y = 0, + ~~ra~~ 1 .

The separatrix is the curve given by setting c=0.

In cases (1) and (2), if a is negative, the curve is rotated by 90^0 , and the first order fixed points (except the one at $x = 0$) become second order fixed points. (See paragraph 6 and Fig. 3b of UCRL 17795)

$$(3) \quad A = 0, B = 1, C = -1, D = 0, E = 0, F = c.$$

$$x^2y + xy^2 - x^2 - y^2 + c = 0.$$

$$f(Y) = \frac{Y^2}{1 - Y}$$

The first order fixed points are at $y = 0, \frac{2}{3}$.

The separatrices are the curve given by setting $c = \frac{8}{27}$, the line $x+y+2 = 0$, and the curve $xy - x - y + 2 = 0$.

(I thank Dr. Laslett for finding the last two of these.)

$$(4) \quad A = 1, B = -2, C = 1, D = 0, E = 0, F = c.$$

$$x^2y^2 - 2(x^2y + xy^2) + x^2 + y^2 - c = 0.$$

$$f(Y) = \frac{2Y^2 - 2}{(1-Y)}$$

The first order fixed points are at $y = 0, \frac{3 \pm \sqrt{5}}{2}$.

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