Formulation and Numerical Implementation of the 2D/3D ADCIRC Finite Element Model Version 44.XX

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TABLE OF CONTENTS

1.0 CONTINUITY EQUATION

Both the vertically-integrated (ADCIRC-2DDI) and the fully three-dimensional (ADCIRC-3D) versions of ADCIRC solve a vertically-integrated continuity equation for water surface elevation. To avoid the spurious oscillations that are associated with a primitive Galerkin finite element formulation of this equation, ADCIRC utilizes the Generalized Wave Continuity Equation (GWCE) formulation. Development of the weak weighted residual form of the GWCE used in ADCIRC is described below.

The vertically-integrated continuity equation is

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(UH) + \frac{\partial}{\partial y}(VH) = 0 \tag{1.1}$$

where

$U,V \equiv \frac{1}{H}\int_{-h}^{\zeta} u,v\ dz =$ *depth-averaged velocities in the x,y directions*

$u,v =$ *vertically-varying velocities in the x,y directions*

$H \equiv \zeta + h =$ *total water column thickness*

$h =$ *bathymetric depth (distance from the geiod to the bottom)*

$\zeta =$ *free surface departure from the geoid*

Take $\partial/\partial t$ of Eq. (1.1), add to this Eq. (1.1) multiplied by the parameter τ_o (which may be variable in space), assume a bathymetric depth that does not change in time, (i.e., $\partial H/\partial t = \partial\zeta/\partial t$) and rearrange using the chain rule

$$\frac{\partial^2 \zeta}{\partial t^2} + \tau_o \frac{\partial \zeta}{\partial t} + \frac{\partial \tilde{J}_x}{\partial x} + \frac{\partial \tilde{J}_y}{\partial y} - UH\frac{\partial \tau_o}{\partial x} - VH\frac{\partial \tau_o}{\partial y} = 0 \tag{1.2}$$

where

$$\tilde{J}_x \equiv \frac{\partial}{\partial t}(UH) + \tau_o UH \tag{1.3}$$

$$= \frac{\partial Q_x}{\partial t} + \tau_o Q_x \tag{1.4}$$

$$= H\frac{\partial U}{\partial t} + U\frac{\partial \zeta}{\partial t} + \tau_o UH \tag{1.5}$$

$$\tilde{J}_y \equiv \frac{\partial}{\partial t}\left(VH\right) + \tau_o VH \tag{1.6}$$

$$= \frac{\partial Q_y}{\partial t} + \tau_o Q_y \tag{1.7}$$

$$= H\frac{\partial V}{\partial t} + V\frac{\partial \zeta}{\partial t} + \tau_o VH \tag{1.8}$$

$Q_x, Q_y \equiv UH, VH = x, y$ *- directed fluxes per unit width*

Note that Eqs. (1.3) - (1.5) are equivalent as are Eqs. (1.6) - (1.8).

The weighted residual method is applied to Eq. (1.2) by multiplying each term by a weighting function ϕ_j and integrating over the horizontal computational domain Ω.

$$\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_j \right\rangle + \left\langle \tau_o \frac{\partial \zeta}{\partial t}, \phi_j \right\rangle + \left\langle \frac{\partial \tilde{J}_x}{\partial x}, \phi_j \right\rangle + \left\langle \frac{\partial \tilde{J}_y}{\partial y}, \phi_j \right\rangle - \left\langle UH\frac{\partial \tau_o}{\partial x}, \phi_j \right\rangle - \left\langle VH\frac{\partial \tau_o}{\partial y}, \phi_j \right\rangle = 0 \tag{1.9}$$

where, the inner product notation $\langle \ \rangle$ is defined as

$$\left\langle \Upsilon, \phi_j \right\rangle \equiv \int_\Omega \Upsilon \phi_j \, d\Omega \tag{1.10}$$

Integrating the terms involving $\tilde{J}_x$ and $\tilde{J}_y$ by parts, yields a weak form of this equation

$$\begin{aligned} &\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_j \right\rangle + \left\langle \tau_o \frac{\partial \zeta}{\partial t}, \phi_j \right\rangle - \left\langle \tilde{J}_x, \frac{\partial \phi_j}{\partial x} \right\rangle - \left\langle \tilde{J}_y, \frac{\partial \phi_j}{\partial y} \right\rangle \\ &\qquad - \left\langle UH\frac{\partial \tau_o}{\partial x}, \phi_j \right\rangle - \left\langle VH\frac{\partial \tau_o}{\partial y}, \phi_j \right\rangle + \int_\Gamma \left[\frac{\partial Q_N}{\partial t} + \tau_o Q_N \right] \phi_j d\Gamma = 0 \end{aligned} \tag{1.11}$$

The integration by parts introduces an integral along the boundary of the computational domain, Γ, involving the components of $\tilde{J}_x$ and $\tilde{J}_y$ normal to the boundary. Using Eqs. (1.4) and (1.7), this can be converted to the integral of the outward flux per unit width normal to the boundary, Q_N, contained in Eq. (1.11).

The GWCE derivation is completed by substituting the vertically-integrated momentum equations in conservative form, (Eqs. **(2.2)**) into Eqs. (1.4) and (1.7) or in non-conservative form (Eqs. **(2.1)**) into Eqs. (1.5) and (1.8). Kolar et al, (ref) has shown that the form of the momentum equations used in the GWCE should match that used for the momentum (velocity) solution (see Section 2). The original version of ADCIRC-2DDI uses the non-conservative momentum equations, although a conservative formulation has been added to ADCIRC (version 44.15). Making this substitution and isolating the linear free surface gravity wave terms gives:

$$\begin{aligned}\tilde{J}_x &= J_x - gh\frac{\partial \zeta}{\partial x}\\ \tilde{J}_y &= J_y - gh\frac{\partial \zeta}{\partial y}\end{aligned} \tag{1.12}$$

where for the non-conservative formulation:

$$\begin{aligned}J_x &= -Q_x\frac{\partial U}{\partial x} - Q_y\frac{\partial U}{\partial y} + fQ_y - \frac{g}{2}\frac{\partial \zeta^2}{\partial x} - gH\frac{\partial\left[P_s/g\rho_o - \alpha\eta\right]}{\partial x} + \frac{\tau_{sx}}{\rho_o} - \frac{\tau_{bx}}{\rho_o}\\ &\quad + M_x - D_x - B_x + U\frac{\partial \zeta}{\partial t} + \tau_o Q_x\\ J_y &= -Q_x\frac{\partial V}{\partial x} - Q_y\frac{\partial V}{\partial y} - fQ_x - \frac{g}{2}\frac{\partial \zeta^2}{\partial y} - gH\frac{\partial\left[P_s/g\rho_o - \alpha\eta\right]}{\partial y} + \frac{\tau_{sy}}{\rho_o} - \frac{\tau_{by}}{\rho_o}\\ &\quad + M_y - D_y - B_y + V\frac{\partial \zeta}{\partial t} + \tau_o Q_y\end{aligned} \tag{1.13}$$

and for the conservative formulation:

$$\begin{aligned}J_x &= -\frac{\partial UQ_x}{\partial x} - \frac{\partial UQ_y}{\partial y} + fQ_y - \frac{g}{2}\frac{\partial \zeta^2}{\partial x} - gH\frac{\partial\left[P_s/g\rho_o - \alpha\eta\right]}{\partial x} + \frac{\tau_{sx}}{\rho_o} - \frac{\tau_{bx}}{\rho_o}\\ &\quad + M_x - D_x - B_x + \tau_o Q_x\\ J_y &= -\frac{\partial VQ_x}{\partial x} - \frac{\partial VQ_y}{\partial y} - fQ_x - \frac{g}{2}\frac{\partial \zeta^2}{\partial y} - gH\frac{\partial\left[P_s/g\rho_o - \alpha\eta\right]}{\partial y} + \frac{\tau_{sy}}{\rho_o} - \frac{\tau_{by}}{\rho_o}\\ &\quad + M_y - D_y - B_y + \tau_o Q_y\end{aligned} \tag{1.14}$$

Substituting Eqs. (1.12) into Eq. (1.11) and rearranging yields the weighted residual form of the GWCE that is solved by ADCIRC:

$$\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_j \right\rangle + \left\langle \tau_o \frac{\partial \zeta}{\partial t}, \phi_j \right\rangle + \left\langle gh \frac{\partial \zeta}{\partial x}, \frac{\partial \phi_j}{\partial x} \right\rangle + \left\langle gh \frac{\partial \zeta}{\partial y}, \frac{\partial \phi_j}{\partial y} \right\rangle$$

$$= \left\langle J_x, \frac{\partial \phi_j}{\partial x} \right\rangle + \left\langle J_y, \frac{\partial \phi_j}{\partial y} \right\rangle + \left\langle Q_x \frac{\partial \tau_o}{\partial x}, \phi_j \right\rangle + \left\langle Q_y \frac{\partial \tau_o}{\partial y}, \phi_j \right\rangle - \int_\Gamma \left[\frac{\partial Q_N}{\partial t} + \tau_o Q_N \right] \phi_j \, d\Gamma = 0 \qquad (1.15)$$

Term by term integration of Eq. (1.15) yields:

$$\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_j \right\rangle_\Omega \equiv \sum_{n=1}^{NE_j} \int_{\Omega_n} \frac{\partial^2 \zeta}{\partial t^2} \phi_j \, d\Omega = \sum_{n=1}^{NE_j} \sum_{i=1}^{3} \left(\frac{\partial^2 \zeta_i}{\partial t^2} \right)_n \int_{\Omega_n} \phi_i \phi_j \, d\Omega = \sum_{n=1}^{NE_j} \frac{\mathrm{A}_n}{12} \left(\sum_{i=1}^{3} \varphi_{i,j} \frac{\partial^2 \zeta_i}{\partial t^2} \right)_n$$

$$\left\langle \tau_o \frac{\partial \zeta}{\partial t}, \phi_j \right\rangle_\Omega \equiv \sum_{n=1}^{NE_j} \int_{\Omega_n} \overline{\tau}_{on} \frac{\partial \zeta}{\partial t} \phi_j \, d\Omega = \sum_{n=1}^{NE_j} \sum_{i=1}^{3} \left(\overline{\tau}_{on} \frac{\partial \zeta_i}{\partial t} \right)_n \int_{\Omega_n} \phi_i \phi_j \, d\Omega = \sum_{n=1}^{NE_j} \frac{\mathrm{A}_n \overline{\tau}_{on}}{12} \left(\sum_{i=1}^{3} \varphi_{i,j} \frac{\partial \zeta_i}{\partial t} \right)_n$$

$$\left\langle gh \frac{\partial \zeta}{\partial x}, \frac{\partial \phi_j}{\partial x} \right\rangle \equiv \sum_{n=1}^{NE_j} \int_{\Omega_n} gh \frac{\partial \zeta}{\partial x} \frac{\partial \phi_j}{\partial x} \, d\Omega = \sum_{n=1}^{NE_j} \left(\frac{\partial \zeta}{\partial x} \right)_n \frac{\partial \phi_j}{\partial x} \sum_{i=1}^{3} g h_i \int_{\Omega_n} \phi_i \, d\Omega$$

$$= \sum_{n=1}^{NE_j} \mathrm{A}_n g \overline{h}_n \left(\frac{\partial \zeta}{\partial x} \right)_n \frac{\partial \phi_j}{\partial x} = \sum_{n=1}^{NE_j} \frac{g \overline{h}_n}{4 \mathrm{A}_n} b_j \left(\sum_{i=1}^{3} \zeta_i b_i \right)_n$$

$$\left\langle gh \frac{\partial \zeta}{\partial y}, \frac{\partial \phi_j}{\partial y} \right\rangle \equiv \sum_{n=1}^{NE_j} \int_{\Omega_n} gh \frac{\partial \zeta}{\partial y} \frac{\partial \phi_j}{\partial y} \, d\Omega = \sum_{n=1}^{NE_j} \left(\frac{\partial \zeta}{\partial y} \right)_n \frac{\partial \phi_j}{\partial y} \sum_{i=1}^{3} g h_i \int_{\Omega_n} \phi_i \, d\Omega$$

$$= \sum_{n=1}^{NE_j} \mathrm{A}_n g \overline{h}_n \left(\frac{\partial \zeta}{\partial y} \right)_n \frac{\partial \phi_j}{\partial y} = \sum_{n=1}^{NE_j} \frac{g \overline{h}_n}{4 \mathrm{A}_n} a_j \left(\sum_{i=1}^{3} \zeta_i a_i \right)_n$$

$$\left\langle J_x, \frac{\partial \phi_j}{\partial x} \right\rangle = \sum_{n=1}^{NE_j} \int_{\Omega_n} J_x \frac{\partial \phi_j}{\partial x} \, d\Omega = \sum_{n=1}^{NE_j} \sum_{i=1}^{3} \left(J_{xi} \right)_n \int_{\Omega_n} \phi_i \frac{\partial \phi_j}{\partial x} \, d\Omega = \sum_{n=1}^{NE_j} \mathrm{A}_n \overline{J}_{xn} \frac{\partial \phi_j}{\partial x} = \sum_{n=1}^{NE_j} \frac{\overline{J}_{xn}}{2} b_j$$

$$\left\langle J_y, \frac{\partial \phi_j}{\partial y} \right\rangle = \sum_{n=1}^{NE_j} \int_{\Omega_n} J_y \frac{\partial \phi_j}{\partial y} \, d\Omega = \sum_{n=1}^{NE_j} \sum_{i=1}^{3} \left(J_{y_i} \right)_n \int_{\Omega_n} \phi_i \frac{\partial \phi_j}{\partial y} \, d\Omega = \sum_{n=1}^{NE_j} \mathrm{A}_n \overline{J}_{y_n} \frac{\partial \phi_j}{\partial y} = \sum_{n=1}^{NE_j} \frac{\overline{J}_{y_n}}{2} a_j$$

$$\left\langle Q_x \frac{\partial \tau_o}{\partial x}, \phi_j \right\rangle = \sum_{n=1}^{NE_j} \int_{\Omega_n} Q_x \frac{\partial \tau_o}{\partial x} \phi_j \, d\Omega = \sum_{n=1}^{NE_j} \overline{Q}_{xn} \left(\sum_{i=1}^{3} \tau_{oi} \int_{\Omega_n} \frac{\partial \phi_i}{\partial x} \phi_j \, d\Omega \right)_n$$

$$= \sum_{n=1}^{NE_j} \overline{Q}_{xn} \frac{\mathrm{A}_n}{3} \sum_{i=1}^{3} \left(\tau_{oi} \right)_n \frac{\partial \phi_i}{\partial x} = \sum_{n=1}^{NE_j} \overline{Q}_{xn} \left(\sum_{i=1}^{3} \tau_{oi} \frac{b_i}{6} \right)_n$$

$$\left\langle Q_y \frac{\partial \tau_o}{\partial y}, \phi_j \right\rangle = \sum_{n=1}^{NE_j} \int_{\Omega_n} Q_y \frac{\partial \tau_o}{\partial y} \phi_j \, d\Omega \;= \sum_{n=1}^{NE_j} \bar{Q}_{y_n} \left(\sum_{i=1}^{3} \tau_{oi} \int_{\Omega_n} \frac{\partial \phi_i}{\partial y} \phi_j \, d\Omega \right)_n$$

$$= \sum_{n=1}^{NE_j} \bar{Q}_{y_n} \frac{A_n}{3} \sum_{i=1}^{3} \left(\tau_{oi} \right)_n \frac{\partial \phi_i}{\partial y} = \sum_{n=1}^{NE_j} \bar{Q}_{y_n} \left(\sum_{i=1}^{3} \tau_{oi} \frac{a_i}{6} \right)_n$$

$$\int_{\Gamma} \left[\frac{\partial Q_N}{\partial t} + \tau_o Q_N \right] \phi_j d\Gamma = \sum_{n=1}^{2} \sum_{i=1}^{2} \left[\frac{\partial Q_{Ni}}{\partial t} + \bar{\tau}_{on} Q_{Ni} \right]_n \int_{\Gamma_n} \phi_i \phi_j d\Gamma \;= \sum_{n=1}^{2} \frac{L_n}{6} \left(\sum_{i=1}^{2} \varphi_{i,j} \left[\frac{\partial Q_{Ni}}{\partial t} + \bar{\tau}_{on} Q_{Ni} \right] \right)_n$$

where

A_n = *area of element n*

$A_{NE_j} \equiv \sum_{n=1}^{NE_j} A_n$ = *area of all elements containing node j*

NE_j = *number of elements containing node j*

$L_n =$ *length of element leg n*

$\bar{h}_n \equiv \frac{1}{3}\sum_{i=1}^{3} h_i =$ *average bathymetric water depth over element n*

$\bar{\tau}_{on} \equiv \frac{1}{3}\sum_{i=1}^{3} \tau_{oi} =$ *average* τ_o *over element n*

$\bar{J}_{xn}, \bar{J}_{yn} \equiv \frac{1}{3}\sum_{i=1}^{3} J_{xi}, J_{yi} =$ *average* J_x, J_y *over element n*

$\bar{Q}_{xn}, \bar{Q}_{yn} \equiv \frac{1}{3}\sum_{i=1}^{3} Q_{xi}, Q_{yi} =$ *average* Q_x, Q_y *over element n*

$$\varphi_{i,j} \equiv \begin{cases} 1 & if \quad i \neq j \\ 2 & if \quad i = j \end{cases}$$

$\phi_j =$ *horizontal weighting function, =1 at node j, =0 at all other nodes, varies linearly between adjacent nodes*

$$\frac{\partial \phi_j}{\partial x}, \frac{\partial \phi_j}{\partial y} = \frac{b_j}{2A_n}, \frac{a_j}{2A_n}$$

$$\left(\frac{\partial \zeta}{\partial x}\right)_n \equiv \frac{1}{2A_n}\left(\sum_{i=1}^{3} \zeta_i b_i\right)_n; \qquad \left(\frac{\partial \zeta}{\partial y}\right)_n \equiv \frac{1}{2A_n}\left(\sum_{i=1}^{3} \zeta_i a_i\right)_n$$

$a_1 \equiv x_3 - x_2;\ a_2 \equiv x_1 - x_3;\ a_3 \equiv x_2 - x_1$

$b_1 \equiv y_2 - y_3;\ b_2 \equiv y_3 - y_1;\ b_3 \equiv y_1 - y_2$

$x_i, y_i =$ *horizontal coordinates of node i*

The definition of the weighting function ϕ_j reduces integration over the horizontal domain Ω to integration over only the NE_j elements containing node *j*. Also, we assume a Galerkin finite element formulation in which the basis and weighting functions vary linearly within an element. Therefore, spatial derivatives are constant within an element and can be pulled out of elemental integrations.

After integration, Eq. (1.15) becomes

$$\begin{aligned} \sum_{n=1}^{NE_j}\left\{\frac{A_n}{12}\left[\sum_{i=1}^{3}\varphi_{i,j}\frac{\partial^2 \zeta_i}{\partial t^2} + \bar{\tau}_{on}\sum_{i=1}^{3}\varphi_{i,j}\frac{\partial \zeta_i}{\partial t}\right] + \frac{g\bar{h}_n}{4A_n}\left[b_j\sum_{i=1}^{3}\zeta_i b_i + a_j\sum_{i=1}^{3}\zeta_i a_i\right]\right\}_n &= \\ \sum_{n=1}^{NE_j}\frac{1}{2}\left\{\bar{J}_{xn} b_j + \bar{J}_{yn} a_j + \bar{Q}_{xn}\sum_{i=1}^{3}\tau_{oi}\frac{b_i}{3} + \bar{Q}_{yn}\sum_{i=1}^{3}\tau_{oi}\frac{a_i}{3}\right\}_n & \\ -\sum_{n=1}^{2}\frac{L_n}{6}\left\{\sum_{i=1}^{2}\varphi_{i,j}\left[\frac{\partial Q_{Ni}}{\partial t} + \bar{\tau}_{on} Q_{Ni}\right]\right\}_n & \end{aligned} \quad (1.16)$$

Equation (1.16) presents the spatially discretized solution for elevation at horizontal node *j* used by ADCIRC. This equation is discretized in time using a three time level scheme at the past (*s*-1), present (*s*) and future (*s*+1) times as described below:

$$\frac{\partial^2 \zeta_i}{\partial t^2} = \frac{\zeta_i^{s+1} - 2\zeta_i^{s} + \zeta_i^{s-1}}{\Delta t^2}$$

$$\frac{\partial \zeta_i}{\partial t} = \frac{\zeta_i^{s+1} - \zeta_i^{s-1}}{2\Delta t}$$

$$\zeta_i = \alpha_1 \zeta_i^{s+1} + \alpha_2 \zeta_i^{s} + \alpha_3 \zeta_i^{s-1}$$

$$\frac{\partial Q_{N_i}}{\partial t} = \frac{Q_{N_i}^{s+1} - Q_{N_i}^{s-1}}{2\Delta t}$$

$$Q_{N_i} = \alpha_1 Q_{N_i}^{s+1} + \alpha_2 Q_{N_i}^{s} + \alpha_3 Q_{N_i}^{s-1}$$

$$\bar{J}_{xn}, \bar{J}_{y_n} = \bar{J}_{xn}^{s}, \bar{J}_{y_n}^{s}$$

$$\bar{Q}_{xn}, \bar{Q}_{y_n} = \bar{Q}_{xn}^{s}, \bar{Q}_{y_n}^{s}$$

Substituting these time discretizations into Eq. (1.16) and re-arranging yields:

$$\begin{aligned}
&\sum_{n=1}^{NE_j} \left\{ \begin{aligned} &\frac{\mathrm{A}_n}{12\Delta t}\left(\frac{1}{\Delta t} + \frac{\bar{\tau}_{on}}{2}\right) \sum_{i=1}^{3} \varphi_{i,j} \zeta_i^{*s+1} \\ &\quad + \frac{g\bar{h}_n \alpha_1}{4\mathrm{A}_n}\left[b_j \sum_{i=1}^{3} \zeta_i^{*s+1} b_i + a_j \sum_{i=1}^{3} \zeta_i^{*s+1} a_i \right] \end{aligned} \right\} = \\
&\sum_{n=1}^{NE_j} \left\{ \begin{aligned} &\frac{\mathrm{A}_n}{12\Delta t}\left(\frac{1}{\Delta t} - \frac{\bar{\tau}_{on}}{2}\right) \sum_{i=1}^{3} \varphi_{i,j} \zeta_i^{*s} \\ &\quad - \frac{g\bar{h}_n}{4\mathrm{A}_n}\left[(\alpha_1 + \alpha_2)\left(b_j \sum_{i=1}^{3} \zeta_i^{s} b_i + a_j \sum_{i=1}^{3} \zeta_i^{s} a_i \right) + \alpha_3 \left(b_j \sum_{i=1}^{3} \zeta_i^{s-1} b_i + a_j \sum_{i=1}^{3} \zeta_i^{s-1} a_i \right) \right] \\ &\quad + \frac{1}{2}\left[\bar{J}_{xn}^{s} b_j + \bar{J}_{y_n}^{s} a_j \right] + \frac{1}{6}\left[\bar{Q}_{xn}^{s} \sum_{i=1}^{3} \tau_{oi} b_i + \bar{Q}_{y_n}^{s} \sum_{i=1}^{3} \tau_{oi} a_i \right] \end{aligned} \right\} \\
&- \sum_{n=1}^{2} \left\{ \frac{\mathrm{L}_n}{6} \sum_{i=1}^{2} \varphi_{i,j} \left[\frac{Q_{N_i}^{s+1} - Q_{N_i}^{s-1}}{2\Delta t} + \bar{\tau}_{on} Q_{N_i}^{s} \right] \right\}
\end{aligned} \tag{1.17}$$

where

$$\zeta^{*s+1}_i = \zeta^{s+1}_i - \zeta^s_i$$
$$\zeta^{*s}_i = \zeta^s_i - \zeta^{s-1}_i \qquad (1.18)$$

The left side of Eq. (1.17) is a sparse symmetric matrix (number of nodes x number of nodes) and the right side is a vector. The normal flux terms are only included in equations corresponding to boundary nodes.

Eq. (1.18) requires evaluation of $\bar{J}^s_{xn}, \bar{J}^s_{y_n}$ as defined in Eqs. (1.13) and (1.14).

For the non-conservative formulation:

$$\begin{aligned}
\bar{J}^s_{xn} &= -\frac{\bar{Q}^s_{xn}}{2A_n}\sum_{i=1}^{3} U^s_i b_i - \frac{\bar{Q}^s_{y_n}}{2A_n}\sum_{i=1}^{3} U^s_i a_i + f\bar{Q}^s_{y_n} - \frac{g}{4A_n}\sum_{i=1}^{3}\zeta^{2s}_i b_i - \frac{g\bar{H}^s_i}{2A_n}\sum_{i=1}^{3}\left[P_s/g\rho_o - \alpha\eta\right]^s_i b_i \\
&\quad + \overline{\left(\frac{\tau_{sx}}{\rho_o}\right)}^s_n - \overline{\left(\frac{\tau_{bx}}{\rho_o}\right)}^s_n + \bar{M}^s_{xn} - \bar{D}^s_{xn} - \bar{B}^s_{xn} + \bar{U}^s_n \frac{\bar{\zeta}^{*s}_n}{\Delta t} + \overline{\left(\tau_o Q_x\right)}^s_n \\
\bar{J}^s_{y_n} &= -\frac{\bar{Q}^s_{xn}}{2A_n}\sum_{i=1}^{3} V^s_i b_i - \frac{\bar{Q}^s_{y_n}}{2A_n}\sum_{i=1}^{3} V^s_i a_i - f\bar{Q}^s_{xn} - \frac{g}{4A_n}\sum_{i=1}^{3}\zeta^{2s}_i a_i - \frac{g\bar{H}^s_i}{2A_n}\sum_{i=1}^{3}\left[P_s/g\rho_o - \alpha\eta\right]^s_i a_i \\
&\quad + \overline{\left(\frac{\tau_{sy}}{\rho_o}\right)}^s_n - \overline{\left(\frac{\tau_{by}}{\rho_o}\right)}^s_n + \bar{M}^s_{y_n} - \bar{D}^s_{y_n} - \bar{B}^s_{y_n} + \bar{V}^s_n \frac{\bar{\zeta}^{*s}_n}{\Delta t} + \overline{\left(\tau_o Q_y\right)}^s_n
\end{aligned} \qquad (1.19)$$

For the conservative formulation (**version 1**):

$$\begin{aligned}
\bar{J}^s_{xn} &= -\frac{1}{2A_n}\sum_{i=1}^{3} Q^s_{xi} U^s_i b_i - \frac{1}{2A_n}\sum_{i=1}^{3} Q^s_{y_i} U^s_i a_i + f\bar{Q}^s_{y_n} - \frac{g}{4A_n}\sum_{i=1}^{3}\zeta^{2s}_i b_i \\
&\quad - \frac{g\bar{H}^s_i}{2A_n}\sum_{i=1}^{3}\left[P_s/g\rho_o - \alpha\eta\right]^s_i b_i + \overline{\left(\frac{\tau_{sx}}{\rho_o}\right)}^s_n - \overline{\left(\frac{\tau_{bx}}{\rho_o}\right)}^s_n + \bar{M}^s_{xn} - \bar{D}^s_{xn} - \bar{B}^s_{xn} + \overline{\left(\tau_o Q_x\right)}^s_n \\
\bar{J}^s_{y_n} &= -\frac{1}{2A_n}\sum_{i=1}^{3} Q^s_{xi} V^s_i b_i - \frac{1}{2A_n}\sum_{i=1}^{3} Q^s_{y_i} V^s_i a_i - f\bar{Q}^s_{xn} - \frac{g}{4A_n}\sum_{i=1}^{3}\zeta^{2s}_i a_i \\
&\quad - \frac{g\bar{H}^s_i}{2A_n}\sum_{i=1}^{3}\left[P_s/g\rho_o - \alpha\eta\right]^s_i a_i + \overline{\left(\frac{\tau_{sy}}{\rho_o}\right)}^s_n - \overline{\left(\frac{\tau_{by}}{\rho_o}\right)}^s_n + \bar{M}^s_{y_n} - \bar{D}^s_{y_n} - \bar{B}^s_{y_n} + \overline{\left(\tau_o Q_y\right)}^s_n
\end{aligned} \qquad (1.20)$$

A second conservative formulation (**version 2)** is obtained by expanding the advective terms using the product rule:

$$\bar{J}^s_{xn} = -\frac{\bar{Q}^s_{xn}}{2A_n}\sum_{i=1}^{3} U^s_i b_i - \frac{\bar{Q}^s_{y_n}}{2A_n}\sum_{i=1}^{3} U^s_i a_i - \frac{\bar{U}^s_n}{2A_n}\sum_{i=1}^{3} Q^s_{xi} b_i - \frac{\bar{U}^s_n}{2A_n}\sum_{i=1}^{3} Q^s_{y_i} a_i + f\bar{Q}^s_{y_n} - \frac{g}{4A_n}\sum_{i=1}^{3} \zeta^{2^s}_i b_i$$
$$-\frac{g\bar{H}^s_i}{2A_n}\sum_{i=1}^{3}\left[P_s/g\rho_o - \alpha\eta\right]^s_i b_i + \overline{\left(\frac{\tau_{sx}}{\rho_o}\right)}^s_n - \overline{\left(\frac{\tau_{bx}}{\rho_o}\right)}^s_n + \bar{M}^s_{xn} - \bar{D}^s_{xn} - \bar{B}^s_{xn} + \overline{\left(\tau_o Q_x\right)}^s_n \tag{1.21}$$
$$\bar{J}^s_{y_n} = -\frac{\bar{Q}^s_{xn}}{2A_n}\sum_{i=1}^{3} V^s_i b_i - \frac{\bar{Q}^s_{y_n}}{2A_n}\sum_{i=1}^{3} V^s_i a_i - \frac{\bar{V}^s_n}{2A_n}\sum_{i=1}^{3} Q^s_{xi} b_i - \frac{\bar{V}^s_n}{2A_n}\sum_{i=1}^{3} Q^s_{y_i} a_i - f\bar{Q}^s_{xn} - \frac{g}{4A_n}\sum_{i=1}^{3} \zeta^{2^s}_i a_i$$
$$-\frac{g\bar{H}^s_i}{2A_n}\sum_{i=1}^{3}\left[P_s/g\rho_o - \alpha\eta\right]^s_i a_i + \overline{\left(\frac{\tau_{sy}}{\rho_o}\right)}^s_n - \overline{\left(\frac{\tau_{by}}{\rho_o}\right)}^s_n + \bar{M}^s_{y_n} - \bar{D}^s_{y_n} - \bar{B}^s_{y_n} + \overline{\left(\tau_o Q_y\right)}^s_n$$

Using definitions and expressions for the various terms in the momentum equations presented in Section 2.0, the evaluation of J_x, J_y using Eqs. (1.19) - (1.21) is straightforward with the exception of the vertically-integrated lateral stress gradient terms, M_x, M_y, that are defined as:

$$M_x \equiv \frac{\partial H\tau_{xx}}{\partial x} + \frac{\partial H\tau_{yx}}{\partial y}$$
$$M_y \equiv \frac{\partial H\tau_{xy}}{\partial x} + \frac{\partial H\tau_{yy}}{\partial y} \tag{1.22}$$

The vertically-integrated, lateral stresses, $H\tau_{xx}$, $H\tau_{yx} = H\tau_{xy}$, $H\tau_{yy}$, derive from time averaging the advection terms in the momentum equations. They are due to high frequency fluctuations in the flow field that are not explicitly included in the model solution and they have no absolute relationship to the time averaged variables that are solved for. Rather, they must be approximated using a closure assumption. It is usually assumed that their significance is small compared to the other terms in the momentum equations, yet in practice most models depend on these terms to stabilize the numerical solution. While the use of a diffusive-type expression for these terms is standard, the exact form is equivocal.

The original version of ADCIRC represents these terms in the GWCE as:

$$H\tau_{xx} = E_h \frac{\partial Q_x}{\partial x} \qquad H\tau_{yx} = E_h \frac{\partial Q_x}{\partial y}$$
$$H\tau_{xy} = E_h \frac{\partial Q_y}{\partial x} \qquad H\tau_{yy} = E_h \frac{\partial Q_y}{\partial y} \tag{1.23}$$

As described in Section 2, several alternative lateral stress closures have been added to more recent version of ADCIRC.

Substituting Eqs. (1.23) (or one of the alternates) into Eq. (1.22) generates terms containing second derivatives of Q_x, Q_y or U, V. This requires additional consideration because second derivative terms can not be represented directly using linear basis functions (i.e., the second derivative of a linear function is zero).

Kolar and Gray (1990) proposed a solution to this difficulty provided the lateral stresses are computed using Eq. (1.23) and the lateral stress coefficient, E_h, is constant in space. Isolating the lateral stress gradient terms from J_x, J_y in Eq. (1.15) yields:

$$\left\langle M_x, \frac{\partial \phi_j}{\partial x} \right\rangle + \left\langle M_y, \frac{\partial \phi_j}{\partial y} \right\rangle \tag{1.24}$$

Integrating by parts:

$$\left\langle M_x, \frac{\partial \phi_j}{\partial x} \right\rangle + \left\langle M_y, \frac{\partial \phi_j}{\partial y} \right\rangle = -\left\langle \frac{\partial M_x}{\partial x}, \phi_j \right\rangle - \left\langle \frac{\partial M_y}{\partial y}, \phi_j \right\rangle + \int_\Gamma M_N \phi_j d\Gamma \tag{1.25}$$

where M_N is the component of the lateral stress gradient normal to the boundary.

Inserting the definition of the lateral stress gradients, Eq. (1.22), and the closure in Eq. (1.23) into Eq. (1.25) and rearranging terms gives:

$$-\left\langle \left\{ \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(E_h \frac{\partial Q_x}{\partial x} \right) + \frac{\partial}{\partial y}\left(E_h \frac{\partial Q_y}{\partial x} \right) \right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(E_h \frac{\partial Q_x}{\partial y} \right) + \frac{\partial}{\partial y}\left(E_h \frac{\partial Q_y}{\partial y} \right) \right] \right\}, \phi_j \right\rangle + \int_\Gamma M_N \phi_j d\Gamma$$

Using the product rule and substituting in the depth-averaged continuity equation, yields:

$$-\left\langle \left\{ \frac{\partial}{\partial x}\left[\frac{\partial E_h}{\partial x}\frac{\partial Q_x}{\partial x} + \frac{\partial E_h}{\partial y}\frac{\partial Q_y}{\partial x} - E_h \frac{\partial}{\partial x}\left(\frac{\partial \zeta}{\partial t} \right) \right] + \frac{\partial}{\partial y}\left[\frac{\partial E_h}{\partial x}\frac{\partial Q_x}{\partial y} + \frac{\partial E_h}{\partial y}\frac{\partial Q_y}{\partial y} - E_h \frac{\partial}{\partial y}\left(\frac{\partial \zeta}{\partial t} \right) \right] \right\}, \phi_j \right\rangle$$
$$+ \int_\Gamma M_N \phi_j d\Gamma$$

This can be condensed to

$$-\left\langle \frac{\partial M'_x}{\partial x}, \phi_j \right\rangle - \left\langle \frac{\partial M'_y}{\partial y}, \phi_j \right\rangle + \int_\Gamma M_N \phi_j d\Gamma \tag{1.26}$$

by defining modified lateral stress gradient terms:

$$
\begin{aligned}
M'_x &\equiv \frac{\partial E_h}{\partial x}\frac{\partial Q_x}{\partial x} + \frac{\partial E_h}{\partial y}\frac{\partial Q_y}{\partial x} - E_h \frac{\partial}{\partial x}\left(\frac{\partial \zeta}{\partial t}\right) \\
M'_y &\equiv \frac{\partial E_h}{\partial x}\frac{\partial Q_x}{\partial y} + \frac{\partial E_h}{\partial y}\frac{\partial Q_y}{\partial y} - E_h \frac{\partial}{\partial y}\left(\frac{\partial \zeta}{\partial t}\right)
\end{aligned}
\tag{1.27}
$$

Integrating Eq. (1.26) by parts yields:

$$
\left\langle M'_x, \frac{\partial \phi_j}{\partial x} \right\rangle + \left\langle M'_y, \frac{\partial \phi_j}{\partial y} \right\rangle + \int_{\Gamma} M_N \phi_j d\Gamma - \int_{\Gamma} M'_N \phi_j d\Gamma \tag{1.28}
$$

where M'_N is the component of the modified lateral stress gradient normal to the boundary.

Neglecting the two boundary integral terms in Eq. (1.28), reduces Eq. (1.28) to Eq. (1.24) and suggests that $M_x \approx M'_x, M_y \approx M'_y$. Boundary integrals of lateral stress gradient terms are also neglected in the development of the momentum equations in Section 2. Discretizing in time and averaging in space on an element yields final expressions for the lateral stress gradient terms:

$$
\begin{aligned}
\bar{M}^s_x &\approx \frac{\partial E^s_h}{\partial x}\frac{\partial Q^s_x}{\partial x} + \frac{\partial E^s_h}{\partial y}\frac{\partial Q^s_y}{\partial x} - \bar{E}^s_h \frac{\partial \zeta^{*s}}{\partial x} \\
\bar{M}^s_y &\approx \frac{\partial E^s_h}{\partial x}\frac{\partial Q^s_x}{\partial y} + \frac{\partial E^s_h}{\partial y}\frac{\partial Q^s_y}{\partial y} - \bar{E}^s_h \frac{\partial \zeta^{*s}}{\partial y}
\end{aligned}
\tag{1.29}
$$

If E_h is constant in space, Eq. (1.29) is equivalent to the lateral stress gradient terms derived by Kolar and Gray (1990) and implemented in the original version of ADCIRC.

An alternative, two part approach for evaluating the lateral stress gradient terms is first to compute the lateral stresses, $H\tau_{xx}$, $H\tau_{yx}$, $H\tau_{xy}$, and $H\tau_{yy}$, at the nodes and second to expand these values using linear basis functions, thereby allowing spatial gradients to be computed. This approach has a considerable advantage over the previous approach because it is not restricted to a specific lateral stress closure.

For purposes of illustration the first step is applied to $H\tau_{xx}$ in Eq. (1.23). Multiplying by a weighting function and integrating across the domain gives:

$$
\left\langle H\tau_{xx}, \phi_j \right\rangle - \left\langle E_h \frac{\partial Q_x}{\partial x}, \phi_j \right\rangle = 0
$$

The first term is integrated using mass lumping (i.e., Rule 1 described in APPENDIX – BASIC CALCULATIONS ON LINEAR TRIANGLES). The second term is integrated consistently (i.e., Rule 2). The resulting vertically-integrated lateral stress at node *j* is:

$$\left(H\tau_{xx}\right)_j = \frac{\bar{E}_h \sum_{n=1}^{NE_j} \sum_{i=1}^{3} Q_{xi} b_i}{2 \sum_{n=1}^{NE_j} \mathrm{A}_n}$$

2.0 2D MOMENTUM EQUATIONS

Both the vertically-integrated (ADCIRC-2DDI) and the fully three-dimensional (ADCIRC-3D) versions of ADCIRC substitute the vertically-integrated momentum equations into the continuity equation to form the GWCE as described in the previous section. The GWCE is solved to determine the new free surface elevation. ADCIRC-2DDI solves the vertically-integrated momentum equations to determine the depth-averaged velocity. The vertically-integrated, momentum equations can be written in either non-conservative form:

$$
\begin{aligned}
&\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} - fV = -g\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial x} + \frac{\tau_{sx}}{H\rho_o} - \frac{\tau_{bx}}{H\rho_o} + \frac{M_x}{H} - \frac{D_x}{H} - \frac{B_x}{H} \\
&\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + fU = -g\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial y} + \frac{\tau_{sy}}{H\rho_o} - \frac{\tau_{by}}{H\rho_o} + \frac{M_y}{H} - \frac{D_y}{H} - \frac{B_y}{H}
\end{aligned}
\quad (2.1)
$$

or conservative form,

$$
\begin{aligned}
&\frac{\partial Q_x}{\partial t} + \frac{\partial UQ_x}{\partial x} + \frac{\partial VQ_x}{\partial y} - fQ_y = -gH\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial x} + \frac{\tau_{sx}}{\rho_o} - \frac{\tau_{bx}}{\rho_o} + M_x - D_x - B_x \\
&\frac{\partial Q_y}{\partial t} + \frac{\partial UQ_y}{\partial x} + \frac{\partial VQ_y}{\partial y} + fQ_x = -gH\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial y} + \frac{\tau_{sy}}{\rho_o} - \frac{\tau_{by}}{\rho_o} + M_y - D_y - B_y
\end{aligned}
\quad (2.2)
$$

where,

$Q_x, Q_y \equiv UH, VH = x, y$ *- directed flux per unit width*

$D_x \equiv \frac{\partial D_{uu}}{\partial x} + \frac{\partial D_{uv}}{\partial y} =$ *momentum dispersion*

$D_y \equiv \frac{\partial D_{uv}}{\partial x} + \frac{\partial D_{vv}}{\partial y} =$ *momentum dispersion*

$D_{uu} \equiv \int_{-h}^{\zeta}(u-U)(u-U)\,dz$

$D_{uv} \equiv \int_{-h}^{\zeta}(u-U)(v-V)\,dz$

$D_{vv} \equiv \int_{-h}^{\zeta}(v-V)(v-V)\,dz$

$M_x \equiv \frac{\partial H\tau_{xx}}{\partial x} + \frac{\partial H\tau_{yx}}{\partial y} =$ *vertically-integrated lateral stress gradient*

$M_y \equiv \frac{\partial H\tau_{xy}}{\partial x} + \frac{\partial H\tau_{yy}}{\partial y} =$ *vertically-integrated lateral stress gradient*

$B_x \equiv \int_{-h}^{\zeta} b_x \, dz$ *= vertically-integrated baroclinic pressure gradient*

$B_y \equiv \int_{-h}^{\zeta} b_y \, dz$ *= vertically-integrated baroclinic pressure gradient*

$b_x \equiv g \frac{\partial}{\partial x} \int_z^{\zeta} \frac{(\rho - \rho_o)}{\rho_o} dz$ *= baroclinic pressure gradient*

$b_y \equiv g \frac{\partial}{\partial y} \int_z^{\zeta} \frac{(\rho - \rho_o)}{\rho_o} dz$ *= baroclinic pressure gradient*

$f = 2\Omega \sin\phi$, *Coriolis parameter,* $\Omega = 7.29212x10^{-5}$ *rad* s^{-1}, ϕ *= degrees latitude*

ρ *= time and spatially varying density of water due to salinity and temperature variations*

ρ_o *= reference density of water*

$H\tau_{xx}, H\tau_{yx} = H\tau_{xy}, H\tau_{yy}$ *= vertically integrated lateral stresses*

τ_{sx}, τ_{sy} *= imposed surface stresses*

τ_{bx}, τ_{by} *= bottom stress components, suitably defined, e.g., using a linear or quadratic drag law*

P_s *= atmospheric pressure at the sea surface*

η *= Newtonian equilibrium tide potential*

E_h = vertically integrated lateral stress coefficient (often called the horizontal eddy viscosity)

Evaluation of the momentum dispersion terms requires knowledge of the vertical profile of the horizontal velocity. This is available only from a three-dimensional model solution utilizing the three-dimensional momentum equations described in the next section. Consequently, the momentum dispersion terms are retained only in the GWCE for ADCIRC-3D. In ADCIRC-2DDI, they are assumed negligible and dropped from both the GWCE and the momentum equations.

The vertically-integrated, lateral stresses, $H\tau_{xx}, H\tau_{yx} = H\tau_{xy}, H\tau_{yy}$, derive from time averaging the advection terms in the momentum equations. They are due to high frequency fluctuations in the flow field that are not explicitly included in the model solution and they have no absolute relationship to the time averaged variables that are solved for. Rather, they must be approximated using a closure assumption. It is usually assumed that their significance is small compared to the other terms in the momentum equations, yet in practice most models depend on these terms to stabilize the numerical solution. While the use of a diffusive-type expression for these terms is standard, the exact form is equivocal.

The original version of ADCIRC represents these terms in the momentum equations as:

$$
\begin{aligned}
H\tau_{xx} &= HE_h \frac{\partial U}{\partial x} & H\tau_{yx} &= HE_h \frac{\partial U}{\partial y} \\
H\tau_{xy} &= HE_h \frac{\partial V}{\partial x} & H\tau_{yy} &= HE_h \frac{\partial V}{\partial y}
\end{aligned}
\qquad (2.3)
$$

Several alternative expressions have been added to more recent version of ADCIRC (version 44.15):

$$
\begin{aligned}
H\tau_{xx} &= E_h \frac{\partial Q_x}{\partial x} & H\tau_{yx} &= E_h \frac{\partial Q_x}{\partial y} \\
H\tau_{xy} &= E_h \frac{\partial Q_y}{\partial x} & H\tau_{yy} &= E_h \frac{\partial Q_y}{\partial y}
\end{aligned}
\qquad (2.4)
$$

and (version 44.XX):

$$
\begin{aligned}
H\tau_{xx} &= 2E_h \frac{\partial Q_x}{\partial x} & H\tau_{yx} &= E_h \left[\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right] \\
H\tau_{xy} &= E_h \left[\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right] & H\tau_{yy} &= 2E_h \frac{\partial Q_y}{\partial y}
\end{aligned}
\qquad (2.5)
$$

$$
\begin{aligned}
H\tau_{xx} &= 2HE_h \frac{\partial U}{\partial x} & H\tau_{yx} &= HE_h \left[\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right] \\
H\tau_{xy} &= HE_h \left[\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right] & H\tau_{yy} &= 2HE_h \frac{\partial V}{\partial y}
\end{aligned}
\qquad (2.6)
$$

Eqs. (2.5) and (2.6) are conceptually more attractive than Eqs. (2.3) and (2.4) because they maintain the theoretical condition that $\tau_{yx} = \tau_{xy}$.

The buoyancy terms can be simplified from the form shown above by recognizing that there is no *z*-dependence in a 2DDI model and using Leibnitz's rule. Thus we can integrate these terms in the vertical:

$$B_x = g\int_{-h}^{\zeta}\frac{\partial}{\partial x}\left[\frac{(\rho_{2D}-\rho_o)}{\rho_o}(\zeta - z)\right]dz = g\frac{\partial}{\partial x}\left[\frac{(\rho_{2D}-\rho_o)}{\rho_o}\int_{-h}^{\zeta}(\zeta - z)\,dz\right] - gH\frac{(\rho_{2D}-\rho_o)}{\rho_o}\frac{\partial h}{\partial x}$$

$$= gH\left[\frac{(\rho_{2D}-\rho_o)}{\rho_o}\frac{\partial \zeta}{\partial x} + \frac{H}{2}\frac{\partial}{\partial x}\frac{(\rho_{2D}-\rho_o)}{\rho_o}\right]$$

$$B_y = gH\left[\frac{(\rho_{2D}-\rho_o)}{\rho_o}\frac{\partial \zeta}{\partial y} + \frac{H}{2}\frac{\partial}{\partial y}\frac{(\rho_{2D}-\rho_o)}{\rho_o}\right]$$

where ρ_{2D} represents the vertically constant, depth-averaged density that is represented by a 2DDI model.

ADCIRC-2DDI utilizes a generalized slip formulation for the bottom stress term:

$$\frac{\tau_{bx}}{\rho_o} = K_{slip}U = \frac{K_{slip}Q_x}{H}; \qquad \frac{\tau_{by}}{\rho_o} = K_{slip}V = \frac{K_{slip}Q_y}{H}$$

where,

$K_{slip} = constant,$ = *linear slip boundary condition, (*K_{slip}*= linear drag coefficent)*

$K_{slip} = C_d\sqrt{U^2+V^2},$ = *quadratic slip boundary condition, (*C_d *= quadratic drag coefficent)*

The weighted residual method is applied to Eqs. (2.1) or (2.2) by multiplying each term by a weighting function ϕ_j and integrating over the horizontal computational domain Ω. Thus the momentum equations become in non-conservative form:

$$\left\langle \frac{\partial U}{\partial t},\phi_j \right\rangle + \left\langle U\frac{\partial U}{\partial x},\phi_j \right\rangle + \left\langle V\frac{\partial U}{\partial y},\phi_j \right\rangle - \left\langle fV,\phi_j \right\rangle = -\left\langle g\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial x},\phi_j \right\rangle$$
$$+\left\langle \frac{\tau_{sx}}{H\rho_o},\phi_j \right\rangle - \left\langle \frac{K_{slip}U}{H\rho_o},\phi_j \right\rangle + \left\langle \frac{M_x}{H},\phi_j \right\rangle - \left\langle \frac{B_x}{H},\phi_j \right\rangle$$
$$\left\langle \frac{\partial V}{\partial t},\phi_j \right\rangle + \left\langle U\frac{\partial V}{\partial x},\phi_j \right\rangle + \left\langle V\frac{\partial V}{\partial y},\phi_j \right\rangle + \left\langle fU,\phi_j \right\rangle = -\left\langle g\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial y},\phi_j \right\rangle$$
$$+\left\langle \frac{\tau_{sy}}{H\rho_o},\phi_j \right\rangle - \left\langle \frac{K_{slip}V}{H\rho_o},\phi_j \right\rangle + \left\langle \frac{M_y}{H},\phi_j \right\rangle - \left\langle \frac{B_y}{H},\phi_j \right\rangle \qquad (2.7)$$

and in conservative form:

$$
\begin{aligned}
&\left\langle \frac{\partial Q_x}{\partial t},\phi_j \right\rangle + \left\langle \frac{\partial U Q_x}{\partial x},\phi_j \right\rangle + \left\langle \frac{\partial V Q_x}{\partial y},\phi_j \right\rangle - \left\langle f Q_y,\phi_j \right\rangle = -\left\langle gH \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial x},\phi_j \right\rangle \\
&\qquad + \left\langle \frac{\tau_{sx}}{\rho_o},\phi_j \right\rangle - \left\langle \frac{K_{slip}Q_x}{H},\phi_j \right\rangle + \left\langle M_x,\phi_j \right\rangle - \left\langle B_x,\phi_j \right\rangle \\
&\left\langle \frac{\partial Q_y}{\partial t},\phi_j \right\rangle + \left\langle \frac{\partial U Q_y}{\partial x},\phi_j \right\rangle + \left\langle \frac{\partial V Q_y}{\partial y},\phi_j \right\rangle + \left\langle f Q_x,\phi_j \right\rangle = -\left\langle gH \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial y},\phi_j \right\rangle \\
&\qquad + \left\langle \frac{\tau_{sy}}{\rho_o},\phi_j \right\rangle - \left\langle \frac{K_{slip}Q_y}{H},\phi_j \right\rangle + \left\langle M_y,\phi_j \right\rangle - \left\langle B_y,\phi_j \right\rangle
\end{aligned}
\tag{2.8}
$$

where, the inner product notation $\langle\ \rangle$ is defined by Eq. (1.10).

Integrations in Eqs. (2.7) and (2.8) are carried out using one of two basic integration rules as noted in the text. These rules are described in APPENDIX - BASIC CALCULATIONS ON LINEAR TRIANGLES.

Term by term integrations of Eqs. (2.7) and (2.8) are presented below (only the x-component equations are presented as the y-component equations are fully analogous).

Integration of the transient terms in Eqs. (2.7) and (2.8) utilizes **Rule 1**:

$$
\left\langle \frac{\partial U}{\partial t},\phi_j \right\rangle_\Omega = \frac{A_{NE\,j}}{3}\frac{\partial U_j}{\partial t}
$$

$$
\left\langle \frac{\partial Q_x}{\partial t},\phi_j \right\rangle_\Omega = \frac{A_{NE\,j}}{3}\frac{\partial Q_{x\,j}}{\partial t}
$$

Integration of the advection terms in Eq. (2.7) utilizes **Rule 2** and assumes the un-differentiated terms are elementally averaged (i.e., $\bar{U}_n \equiv \frac{1}{3}\sum_{i=1}^{3} U_i$, $\bar{V}_n \equiv \frac{1}{3}\sum_{i=1}^{3} V_i$,)

$$
\left\langle \left(U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right),\phi_j \right\rangle_\Omega = \sum_{n=1}^{NE_j} \frac{A_n}{3}\left[\bar{U}_n\left(\frac{\partial U}{\partial x}\right)_n + \bar{V}_n\left(\frac{\partial U}{\partial y}\right)_n \right]
$$

Two different integrations have been used for the advection terms in Eq. (2.8). **Version 1** uses **Rule 2** and a linear expansion in space for the conservative flux terms UQ_x, VQ_x:

$$\left\langle \left(\frac{\partial U Q_x}{\partial x} + \frac{\partial V Q_x}{\partial y} \right), \phi_j \right\rangle_\Omega = \sum_{n=1}^{NE_j} \frac{A_n}{3} \left[\left(\frac{\partial U Q_x}{\partial x} \right)_n + \left(\frac{\partial V Q_x}{\partial y} \right)_n \right]$$

Version 2 expands the advection terms with the product rule, utilizes **Rule 2** on the derivative terms and assumes the un-differentiated terms are elementally averaged:

$$\left\langle \left(\frac{\partial U Q_x}{\partial x} + \frac{\partial V Q_x}{\partial y} \right), \phi_j \right\rangle_\Omega = \sum_{n=1}^{NE_j} \frac{A_n}{3} \left[\bar{U}_n \left(\frac{\partial Q_x}{\partial x} \right)_n + \bar{V}_n \left(\frac{\partial Q_x}{\partial y} \right)_n + \bar{Q}_{xn} \left(\frac{\partial U}{\partial x} \right)_n + \bar{Q}_{xn} \left(\frac{\partial V}{\partial y} \right)_n \right]$$

Integration of the Coriolis terms in Eqs. (2.7) and (2.8) utilizes **Rule 1**:

$$\left\langle fV, \phi_j \right\rangle_\Omega = \frac{A_{NE\,j}}{3} fV_j$$

$$\left\langle f Q_x, \phi_j \right\rangle_\Omega = \frac{A_{NE\,j}}{3} f Q_{x\,j}$$

Integration of the combined barotropic pressure (i.e., the free surface elevation, atmospheric pressure and tidal potential) gradient terms in Eqs. (2.7) and (2.8) utilizes **Rule 2** and assumes the undifferentiated total water depth term in the conservative form of the equations is elementally averaged (i.e., $\bar{H}_n \equiv \frac{1}{3} \sum_{i=1}^{3} H_i$):

$$\left\langle g \frac{\partial \left[\zeta + P_s / g \rho_o - \alpha \eta \right]}{\partial x}, \phi_j \right\rangle_\Omega = \sum_{n=1}^{NE_j} \frac{A_n}{3} \left(g \frac{\partial \left[\zeta + P_s / g \rho_o - \alpha \eta \right]}{\partial x} \right)_n$$

$$\left\langle gH \frac{\partial \left[\zeta + P_s / g \rho_o - \alpha \eta \right]}{\partial x}, \phi_j \right\rangle_\Omega = \sum_{n=1}^{NE_j} g \frac{A_n \bar{H}_n}{3} \left(\frac{\partial \left[\zeta + P_s / g \rho_o - \alpha \eta \right]}{\partial x} \right)_n$$

Integration of the surface and bottom stress terms in Eqs. (2.7) and (2.8) utilizes **Rule 1**:

$$\left\langle \frac{\tau_{sx}}{H \rho_o}, \phi_j \right\rangle_\Omega - \left\langle \frac{K_{slip} U}{H \rho_o}, \phi_j \right\rangle_\Omega = \frac{A_{NE\,j}}{3} \left[\left(\frac{\tau_{sx}}{H \rho_o} \right)_j - \left(\frac{K_{slip} U}{H \rho_o} \right)_j \right]$$

$$\left\langle \frac{\tau_{sx}}{\rho_o}, \phi_j \right\rangle_\Omega - \left\langle \frac{K_{slip} Q_x}{H \rho_o}, \phi_j \right\rangle_\Omega = \frac{A_{NE\,j}}{3} \left[\left(\frac{\tau_{sx}}{\rho_o} \right)_j - \left(\frac{K_{slip} Q_x}{H \rho_o} \right)_j \right]$$

The vertically-integrated, baroclinic pressure gradient terms in Eqs. (2.7) and (2.8) are assumed to vary linearly across an element. Integration of these terms utilizes **Rule 2**:

$$\left\langle \frac{B_x}{H}, \phi_j \right\rangle_\Omega = \sum_{n=1}^{NE_j} \frac{A_n}{3} \left(\frac{B_x}{H} \right)_n$$

$$\left\langle B_x, \phi_j \right\rangle_\Omega = \sum_{n=1}^{NE_j} \frac{A_n}{3} \left(B_x \right)_n$$

The lateral stress gradient terms in Eqs. (2.7) and (2.8) are initially integrated by parts to eliminate the second derivatives of flux or velocity that result from the lateral stress closure:

$$\left\langle \frac{M_x}{H}, \phi_j \right\rangle_\Omega = \left\langle \frac{1}{H} \left[\frac{\partial H \tau_{xx}}{\partial x} + \frac{\partial H \tau_{yx}}{\partial y} \right], \phi_j \right\rangle_\Omega$$

$$= -\left\langle H\tau_{xx}, \frac{\partial}{\partial x}\left(\frac{\phi_j}{H}\right) \right\rangle_\Omega - \left\langle H\tau_{yx}, \frac{\partial}{\partial y}\left(\frac{\phi_j}{H}\right) \right\rangle_\Omega + \int_\Gamma \frac{E_h}{H} \frac{\partial \tau_{N_x}}{\partial N} \phi_j d\Gamma$$

$$\left\langle M_x, \phi_j \right\rangle_\Omega = \left\langle \left[\frac{\partial H \tau_{xx}}{\partial x} + \frac{\partial H \tau_{yx}}{\partial y} \right], \phi_j \right\rangle_\Omega$$

$$= -\left\langle H\tau_{xx}, \frac{\partial \phi_j}{\partial x} \right\rangle_\Omega - \left\langle \partial H\tau_{yx}, \frac{\partial \phi_j}{\partial y} \right\rangle_\Omega + \int_\Gamma E_h \frac{\partial \tau_{N_x}}{\partial N} \phi_j d\Gamma$$

where Γ represents the external boundary of the computational domain. In both cases we assume that the lateral stresses are small along all external boundary segments and therefore that the boundary integral term can be neglected. In addition we replace the depth by the central nodal depth and assume that the lateral stress is spatially constant across an element:

$$\left\langle \frac{M_x}{H}, \phi_j \right\rangle_\Omega = -\frac{1}{H_j} \sum_{n=1}^{NE_j} A_n \left(H\tau_{xx} \frac{\partial \phi_j}{\partial x} + H\tau_{yx} \frac{\partial \phi_j}{\partial y} \right)_n$$

$$\left\langle M_x, \phi_j \right\rangle_\Omega = -\sum_{n=1}^{NE_j} A_n \left(H\tau_{xx} \frac{\partial \phi_j}{\partial x} + H\tau_{yx} \frac{\partial \phi_j}{\partial y} \right)_n$$

Following integration and multiplication by $3/A_{NE\,j}$, the non-conservative Eq. (2.7) becomes:

$$\begin{aligned}
&\frac{\partial U_j}{\partial t}+\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left[\bar{U}_n\left(\frac{\partial U}{\partial x}\right)_n+\bar{V}_n\left(\frac{\partial U}{\partial y}\right)_n\right]-fV_j=-\frac{g}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial x}\right)_n\\
&\qquad+\left(\frac{\tau_{sx}}{H\rho_o}\right)_j-\left(\frac{K_{slip}U}{H\rho_o}\right)_j-\frac{3}{H_jA_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xx}\frac{\partial\phi_j}{\partial x}+H\tau_{yx}\frac{\partial\phi_j}{\partial y}\right)_n-\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(\frac{B_x}{H}\right)_n\\
&\frac{\partial V_j}{\partial t}+\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left[\bar{U}_n\left(\frac{\partial V}{\partial x}\right)_n+\bar{V}_n\left(\frac{\partial V}{\partial y}\right)_n\right]+fU_j=-\frac{g}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial y}\right)_n\\
&\qquad+\left(\frac{\tau_{sy}}{H\rho_o}\right)_j-\left(\frac{K_{slip}V}{H\rho_o}\right)_j-\frac{3}{H_jA_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xy}\frac{\partial\phi_j}{\partial x}+H\tau_{yy}\frac{\partial\phi_j}{\partial y}\right)_n-\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(\frac{B_y}{H}\right)_n
\end{aligned}\tag{2.9}$$

the conservative Eq. (2.8) for **version 1** becomes:

$$\begin{aligned}
&\frac{\partial Q_{x\,j}}{\partial t}+\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left[\left(\frac{\partial UQ_x}{\partial x}\right)_n+\left(\frac{\partial VQ_x}{\partial y}\right)_n\right]-fQ_{y\,j}=\\
&\qquad-\frac{g}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\bar{H}_n\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial x}\right)_n+\left(\frac{\tau_{sx}}{\rho_o}\right)_j-\left(\frac{K_{slip}Q_x}{H}\right)_j\\
&\qquad-\frac{3}{H_jA_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xx}\frac{\partial\phi_j}{\partial x}+H\tau_{yx}\frac{\partial\phi_j}{\partial y}\right)_n-\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_nB_{x\,n}\\
&\frac{\partial Q_{y\,j}}{\partial t}+\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left[\left(\frac{\partial UQ_y}{\partial x}\right)_n+\left(\frac{\partial VQ_y}{\partial y}\right)_n\right]+fQ_{x\,j}=\\
&\qquad-\frac{g}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\bar{H}_n\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial y}\right)_n+\left(\frac{\tau_{sy}}{\rho_o}\right)_j-\left(\frac{K_{slip}Q_y}{H}\right)_j\\
&\qquad-\frac{3}{H_jA_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xy}\frac{\partial\phi_j}{\partial x}+H\tau_{yy}\frac{\partial\phi_j}{\partial y}\right)_n-\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_nB_{y\,n}
\end{aligned}\tag{2.10}$$

and the conservative Eq. (2.8) for **version 2** becomes:

$$
\begin{aligned}
&\frac{\partial Q_{xj}}{\partial t}+\frac{1}{A_{NEj}}\sum_{n=1}^{NE_j}A_n\left[\bar{U}_n\left(\frac{\partial Q_x}{\partial x}\right)_n+\bar{V}_n\left(\frac{\partial Q_x}{\partial y}\right)_n+\bar{Q}_{xn}\left(\frac{\partial U}{\partial x}\right)_n+\bar{Q}_{xn}\left(\frac{\partial V}{\partial y}\right)_n\right]-fQ_{yj}=\\
&\qquad -\frac{g}{A_{NEj}}\sum_{n=1}^{NE_j}A_n\bar{H}_n\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial x}\right)_n+\left(\frac{\tau_{sx}}{\rho_o}\right)_j-\left(\frac{K_{slip}Q_x}{H}\right)_j\\
&\qquad -\frac{3}{H_jA_{NEj}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xx}\frac{\partial\phi_j}{\partial x}+H\tau_{yx}\frac{\partial\phi_j}{\partial y}\right)_n-\frac{1}{A_{NEj}}\sum_{n=1}^{NE_j}A_nB_{xn}\\
&\frac{\partial Q_{yj}}{\partial t}+\frac{1}{A_{NEj}}\sum_{n=1}^{NE_j}A_n\left[\bar{U}_n\left(\frac{\partial Q_y}{\partial x}\right)_n+\bar{V}_n\left(\frac{\partial Q_y}{\partial y}\right)_n+\bar{Q}_{yn}\left(\frac{\partial U}{\partial x}\right)_n+\bar{Q}_{yn}\left(\frac{\partial V}{\partial y}\right)_n\right]+fQ_{xj}=\\
&\qquad -\frac{g}{A_{NEj}}\sum_{n=1}^{NE_j}A_n\bar{H}_n\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial y}\right)_n+\left(\frac{\tau_{sy}}{\rho_o}\right)_j-\left(\frac{K_{slip}Q_y}{H}\right)_j\\
&\qquad -\frac{3}{H_jA_{NEj}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xy}\frac{\partial\phi_j}{\partial x}+H\tau_{yy}\frac{\partial\phi_j}{\partial y}\right)_n-\frac{1}{A_{NEj}}\sum_{n=1}^{NE_j}A_nB_{yn}
\end{aligned}
\tag{2.11}
$$

As noted above, early versions of ADCIRC-2DDI used an approximation to the exact integration contained in integration **Rule 2**. If integration **Rule 2a** is used instead of **Rule 2**, the non-conservative Eqs. (2.9) become:

$$
\begin{aligned}
&\frac{\partial U_j}{\partial t}+\frac{1}{NE_j}\sum_{n=1}^{NE_j}\left[\bar{U}_n\left(\frac{\partial U}{\partial x}\right)_n+\bar{V}_n\left(\frac{\partial U}{\partial y}\right)_n\right]-fV_j=-\frac{g}{NE_j}\sum_{n=1}^{NE_j}\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial x}\right)_n\\
&\qquad +\left(\frac{\tau_{sx}}{H\rho_o}\right)_j-\left(\frac{K_{slip}U}{H\rho_o}\right)_j-\frac{3}{H_jNE_j}\sum_{n=1}^{NE_j}\left(H\tau_{xx}\frac{\partial\phi_j}{\partial x}+H\tau_{yx}\frac{\partial\phi_j}{\partial y}\right)_n-\frac{1}{NE_j}\sum_{n=1}^{NE_j}\left(\frac{B_x}{H}\right)_n\\
&\frac{\partial V_j}{\partial t}+\frac{1}{NE_j}\sum_{n=1}^{NE_j}\left[\bar{U}_n\left(\frac{\partial V}{\partial x}\right)_n+\bar{V}_n\left(\frac{\partial V}{\partial y}\right)_n\right]+fU_j=-\frac{g}{NE_j}\sum_{n=1}^{NE_j}\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial y}\right)_n\\
&\qquad +\left(\frac{\tau_{sy}}{H\rho_o}\right)_j-\left(\frac{K_{slip}V}{H\rho_o}\right)_j-\frac{3}{H_jNE_j}\sum_{n=1}^{NE_j}\left(H\tau_{xy}\frac{\partial\phi_j}{\partial x}+H\tau_{yy}\frac{\partial\phi_j}{\partial y}\right)_n-\frac{1}{NE_j}\sum_{n=1}^{NE_j}\left(\frac{B_y}{H}\right)_n
\end{aligned}
\tag{2.12}
$$

The formulated using approximate integration **Rule 2a** for the conservative equations is not presented.

Equations (2.9) - (2.12) present four spatially discretized, vertically-integrated versions of the momentum equations that may be used to solve for velocity at horizontal node *j*. A two level time discretization at the present (*s*) and future (*s*+1) time levels is described below (only the x-component equations are presented as the y-component equations are fully analogous):

Non-conservative transient term: $\dfrac{U_j^{s+1} - U_j^s}{\Delta t}$

Conservative transient term: $\dfrac{Q_{x\,j}^{\,s+1} - Q_{x\,j}^{\,s}}{\Delta t}$

Non-conservative horizontal advection: $\dfrac{1}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left[\bar{U}_n^s \left(\dfrac{\partial U^s}{\partial x} \right)_n + \bar{V}_n^s \left(\dfrac{\partial U^s}{\partial y} \right)_n \right]$

Conservative horizontal advection, **version 1**: $\dfrac{1}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left[\left(\dfrac{\partial U Q_x}{\partial x} \right)_n^s + \left(\dfrac{\partial V Q_x}{\partial y} \right)_n^s \right]$

Conservative horizontal advection, **version 2**:

$$\frac{1}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left[\bar{U}_n^s \left(\frac{\partial Q_x}{\partial x} \right)_n^s + \bar{V}_n^s \left(\frac{\partial Q_x}{\partial y} \right)_n^s + \bar{Q}_{x\,n}^{s} \left(\frac{\partial U}{\partial x} \right)_n^s + \bar{Q}_{x\,n}^{s} \left(\frac{\partial V}{\partial y} \right)_n^s \right]$$

Non-conservative Coriolis: $\dfrac{1}{2}\left(fV_j^{s+1} + fV_j^s \right)$

Conservative Coriolis: $\dfrac{1}{2}\left(f Q_{y\,j}^{\,s+1} + f Q_{y\,j}^{\,s} \right)$

Non-conservative barotropic pressure gradient:

$$\frac{1}{A_{NE\,j}} \sum_{n=1}^{NE_j} g \frac{A_n}{2} \left(\frac{\partial \left[\zeta + P_s / g\rho_o - \alpha\eta \right]^s}{\partial x} + \frac{\partial \left[\zeta + P_s / g\rho_o - \alpha\eta \right]^{s+1}}{\partial x} \right)_n$$

Conservative barotropic pressure gradient:

$$\frac{1}{A_{NE\,j}} \sum_{n=1}^{NE_j} g \frac{A_n}{2} \left(\bar{H}_n^s \frac{\partial \left[\zeta + P_s / g\rho_o - \alpha\eta \right]^s}{\partial x} + \bar{H}_n^{s+1} \frac{\partial \left[\zeta + P_s / g\rho_o - \alpha\eta \right]^{s+1}}{\partial x} \right)_n$$

Non-conservative free surface stress: $\frac{1}{2}\left[\frac{\tau_{sx\,j}^{s+1}}{H_j^{s+1}\rho_o}+\frac{\tau_{sx\,j}^{s}}{H_j^{s}\rho_o}\right]$

Conservative free surface stress: $\frac{1}{2}\left[\frac{\tau_{sx\,j}^{s+1}}{\rho_o}+\frac{\tau_{sx\,j}^{s}}{\rho_o}\right]$

Non-conservative bottom stress: $\frac{K_{slip\,j}^{s}}{2}\left[\frac{U_j^{s+1}}{H_j^{s+1}}+\frac{U_j^{s}}{H_j^{s}}\right]$

Conservative bottom stress: $\frac{K_{slip\,j}^{s}}{2}\left[\frac{Q_{x\,j}^{\;s+1}}{H_j^{s+1}}+\frac{Q_{x\,j}^{\;s}}{H_j^{s}}\right]$

Non-conservative baroclinic pressure gradient: $\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(\frac{B_x}{H}\right)_n^s$

Conservative baroclinic pressure gradient: $\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n B_{x\,n}^{\;s}$

Non-conservative lateral stress: $\frac{3}{H_j^s A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xx}^s\frac{\partial\phi_j}{\partial x}+H\tau_{yx}^s\frac{\partial\phi_j}{\partial y}\right)_n$

Conservative lateral stress: $\frac{3}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xx}^s\frac{\partial\phi_j}{\partial x}+H\tau_{yx}^s\frac{\partial\phi_j}{\partial y}\right)_n$

These time discretizations are substituted into the spatially discretized equations, multiplied by Δt and grouped at time levels $s+1$ and s, to yield the fully discretized equations.

Non-conservative, exact integration, (Eq. (2.9)):

$$
\begin{aligned}
&\left[1+\frac{\Delta t K_{slip\,j}^{s}}{2H_j^{s+1}}\right]U_j^{s+1}-\frac{f\Delta t}{2}V_j^{s+1}= \\
&\quad\left[1-\frac{\Delta t K_{slip\,j}^{s}}{2H_j^{s+1}}\right]U_j^{s}-\frac{\Delta t}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left[\bar{U}_n^s\left(\frac{\partial U^s}{\partial x}\right)_n+\bar{V}_n^s\left(\frac{\partial U^s}{\partial y}\right)_n\right]+\frac{\Delta t\, fV_j^s}{2} \\
&\quad-\frac{g\Delta t}{2A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]^s}{\partial x}+\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]^{s+1}}{\partial x}\right)_n \\
&\quad+\frac{\Delta t}{2}\left[\frac{\tau_{sx\,j}^{s+1}}{H_j^{s+1}\rho_o}+\frac{\tau_{sx\,j}^{s}}{H_j^{s}\rho_o}\right]-\frac{3\Delta t}{H_j^s A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xx}^s\frac{\partial\phi_j}{\partial x}+H\tau_{yx}^s\frac{\partial\phi_j}{\partial y}\right)_n \\
&\quad-\frac{\Delta t}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(\frac{B_x}{H}\right)_n^s
\end{aligned}
$$

$$
\begin{aligned}
&\left[1+\frac{\Delta t K_{slip\,j}^{s}}{2H_j^{s+1}}\right]V_j^{s+1}+\frac{f\Delta t}{2}U_j^{s+1}= \\
&\quad\left[1-\frac{\Delta t K_{slip\,j}^{s}}{2H_j^{s+1}}\right]V_j^{s}-\frac{\Delta t}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left[\bar{U}_n^s\left(\frac{\partial V^s}{\partial x}\right)_n+\bar{V}_n^s\left(\frac{\partial V^s}{\partial y}\right)_n\right]-\frac{\Delta t\, fU_j^s}{2} \\
&\quad-\frac{g\Delta t}{2A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]^s}{\partial y}+\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]^{s+1}}{\partial y}\right)_n \\
&\quad+\frac{\Delta t}{2}\left[\frac{\tau_{sy\,j}^{s+1}}{H_j^{s+1}\rho_o}+\frac{\tau_{sy\,j}^{s}}{H_j^{s}\rho_o}\right]-\frac{3\Delta t}{H_j^s A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(H\tau_{xy}^s\frac{\partial\phi_j}{\partial x}+H\tau_{yy}^s\frac{\partial\phi_j}{\partial y}\right)_n \\
&\quad-\frac{\Delta t}{A_{NE\,j}}\sum_{n=1}^{NE_j}A_n\left(\frac{B_y}{H}\right)_n^s
\end{aligned}
\tag{2.13}
$$

Conservative **version 1**, exact integration (Eq. (2.10)):

$$
\begin{aligned}
&\left[1+\frac{\Delta t K_{slip\,j}^{s}}{2H_j^{s+1}}\right] Q_{x\,j}^{\;s+1} - \frac{f\Delta t}{2} Q_{y\,j}^{\;s+1} = \\
&\qquad \left[1-\frac{\Delta t K_{slip\,j}^{s}}{2H_j^{s+1}}\right] Q_{x\,j}^{\;s} - \frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left[\left(\frac{\partial UQ_x}{\partial x}\right)_n^s + \left(\frac{\partial VQ_x}{\partial y}\right)_n^s\right] + \frac{f\Delta t}{2} Q_{y\,j}^{\;s+1} \\
&\qquad -\frac{g\Delta t}{2A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left(\bar{H}_n^s \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^s}{\partial x} + \bar{H}_n^{s+1} \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s+1}}{\partial x}\right)_n \\
&\qquad +\frac{\Delta t}{2}\left[\frac{\tau_{sx\,j}^{s+1}}{\rho_o} + \frac{\tau_{sx\,j}^{s}}{\rho_o}\right] - \frac{3\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left(H\tau_{xx}^s \frac{\partial\phi_j}{\partial x} + H\tau_{yx}^s \frac{\partial\phi_j}{\partial y}\right)_n \\
&\qquad -\frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n B_{x\,n}^{\;s} \\
&\left[1+\frac{\Delta t K_{slip\,j}^{s}}{2H_j^{s+1}}\right] Q_{y\,j}^{\;s+1} + \frac{f\Delta t}{2} Q_{x\,j}^{\;s+1} = \\
&\qquad \left[1-\frac{\Delta t K_{slip\,j}^{s}}{2H_j^{s+1}}\right] Q_{y\,j}^{\;s} - \frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left[\left(\frac{\partial UQ_y}{\partial x}\right)_n^s + \left(\frac{\partial VQ_y}{\partial y}\right)_n^s\right] - \frac{f\Delta t}{2} Q_{x\,j}^{\;s+1} \\
&\qquad -\frac{g\Delta t}{2A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left(\bar{H}_n^s \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^s}{\partial y} + \bar{H}_n^{s+1} \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s+1}}{\partial y}\right)_n \\
&\qquad +\frac{\Delta t}{2}\left[\frac{\tau_{sy\,j}^{s+1}}{\rho_o} + \frac{\tau_{sy\,j}^{s}}{\rho_o}\right] - \frac{3\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left(H\tau_{xy}^s \frac{\partial\phi_j}{\partial x} + H\tau_{yy}^s \frac{\partial\phi_j}{\partial y}\right)_n \\
&\qquad -\frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n B_{y\,n}^{\;s}
\end{aligned}
\tag{2.14}
$$

Conservative **version 2**, exact integration (Eq. (2.11)):

$$
\begin{aligned}
&\left[1+\frac{\Delta t K^{s}_{slip\,j}}{2H^{s+1}_{j}}\right] Q_{x\,j}^{\,s+1} - \frac{f\Delta t}{2} Q_{y\,j}^{\,s+1} = \\
&\quad \left[1-\frac{\Delta t K^{s}_{slip\,j}}{2H^{s+1}_{j}}\right] Q_{x\,j}^{\,s} - \frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left[\bar{U}^{s}_{n}\left(\frac{\partial Q_x}{\partial x}\right)^{s}_{n} + \bar{V}^{s}_{n}\left(\frac{\partial Q_x}{\partial y}\right)^{s}_{n} + \bar{Q}^{s}_{x\,n}\left(\frac{\partial U}{\partial x}\right)^{s}_{n} + \bar{Q}^{s}_{x\,n}\left(\frac{\partial V}{\partial y}\right)^{s}_{n} \right] \\
&\quad + \frac{f\Delta t}{2} Q_{y\,j}^{\,s+1} \\
&\quad - \frac{g\Delta t}{2A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left(\bar{H}^{s}_{n} \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s}}{\partial x} + \bar{H}^{s+1}_{n} \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s+1}}{\partial x} \right)_{n} \\
&\quad + \frac{\Delta t}{2}\left[\frac{\tau^{s+1}_{sx\,j}}{\rho_o} + \frac{\tau^{s}_{sx\,j}}{\rho_o} \right] - \frac{3\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left(H\tau^{s}_{xx}\frac{\partial \phi_j}{\partial x} + H\tau^{s}_{yx}\frac{\partial \phi_j}{\partial y} \right)_{n} - \frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n B_{x\,n}^{\,s} \\
&\left[1+\frac{\Delta t K^{s}_{slip\,j}}{2H^{s+1}_{j}}\right] Q_{y\,j}^{\,s+1} + \frac{f\Delta t}{2} Q_{x\,j}^{\,s+1} = \\
&\quad \left[1-\frac{\Delta t K^{s}_{slip\,j}}{2H^{s+1}_{j}}\right] Q_{y\,j}^{\,s} - \frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left[\bar{U}^{s}_{n}\left(\frac{\partial Q_y}{\partial x}\right)^{s}_{n} + \bar{V}^{s}_{n}\left(\frac{\partial Q_y}{\partial y}\right)^{s}_{n} + \bar{Q}^{s}_{y\,n}\left(\frac{\partial U}{\partial x}\right)^{s}_{n} + \bar{Q}^{s}_{y\,n}\left(\frac{\partial V}{\partial y}\right)^{s}_{n} \right] \\
&\quad - \frac{f\Delta t}{2} Q_{x\,j}^{\,s+1} \\
&\quad - \frac{g\Delta t}{2A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left(\bar{H}^{s}_{n} \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s}}{\partial y} + \bar{H}^{s+1}_{n} \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s+1}}{\partial y} \right)_{n} \qquad (2.15) \\
&\quad + \frac{\Delta t}{2}\left[\frac{\tau^{s+1}_{sy\,j}}{\rho_o} + \frac{\tau^{s}_{sy\,j}}{\rho_o} \right] - \frac{3\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n \left(H\tau^{s}_{xy}\frac{\partial \phi_j}{\partial x} + H\tau^{s}_{yy}\frac{\partial \phi_j}{\partial y} \right)_{n} - \frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} A_n B_{y\,n}^{\,s}
\end{aligned}
$$

Non-conservative, approximate integration (Eq. (2.12)):

$$
\begin{aligned}
&\left[1+\frac{\Delta t K^{s}_{slip\,j}}{2H^{s+1}_{j}}\right]U^{s+1}_{j}-\frac{f\Delta t}{2}V^{s+1}_{j}= \\
&\qquad \left[1-\frac{\Delta t K^{s}_{slip\,j}}{2H^{s+1}_{j}}\right]U^{s}_{j}-\frac{\Delta t}{NE_j}\sum_{n=1}^{NE_j}\left[\bar{U}^{s}_{n}\left(\frac{\partial U^{s}}{\partial x}\right)_{n}+\bar{V}^{s}_{n}\left(\frac{\partial U^{s}}{\partial y}\right)_{n}\right]+\frac{\Delta t\, fV^{s}_{j}}{2} \\
&\qquad -\frac{g\Delta t}{2NE_j}\sum_{n=1}^{NE_j}\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]^{s}}{\partial x}+\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]^{s+1}}{\partial x}\right)_{n} \\
&\qquad +\frac{\Delta t}{2}\left[\frac{\tau^{s+1}_{sx\,j}}{H^{s+1}_{j}\rho_o}+\frac{\tau^{s}_{sx\,j}}{H^{s}_{j}\rho_o}\right]-\frac{3\Delta t}{H^{s}_{j}NE_j}\sum_{n=1}^{NE_j}\left(H\tau^{s}_{xx}\frac{\partial\phi_j}{\partial x}+H\tau^{s}_{yx}\frac{\partial\phi_j}{\partial y}\right)_{n} \\
&\qquad -\frac{\Delta t}{NE_j}\sum_{n=1}^{NE_j}\left(\frac{B_x}{H}\right)^{s}_{n} \\
&\left[1+\frac{\Delta t K^{s}_{slip\,j}}{2H^{s+1}_{j}}\right]V^{s+1}_{j}+\frac{f\Delta t}{2}U^{s+1}_{j}= \\
&\qquad \left[1-\frac{\Delta t K^{s}_{slip\,j}}{2H^{s+1}_{j}}\right]V^{s}_{j}-\frac{\Delta t}{NE_j}\sum_{n=1}^{NE_j}\left[\bar{U}^{s}_{n}\left(\frac{\partial V^{s}}{\partial x}\right)_{n}+\bar{V}^{s}_{n}\left(\frac{\partial V^{s}}{\partial y}\right)_{n}\right]-\frac{\Delta t\, fU^{s}_{j}}{2} \\
&\qquad -\frac{g\Delta t}{2NE_j}\sum_{n=1}^{NE_j}\left(\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]^{s}}{\partial y}+\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]^{s+1}}{\partial y}\right)_{n} \\
&\qquad +\frac{\Delta t}{2}\left[\frac{\tau^{s+1}_{sy\,j}}{H^{s+1}_{j}\rho_o}+\frac{\tau^{s}_{sy\,j}}{H^{s}_{j}\rho_o}\right]-\frac{3\Delta t}{H^{s}_{j}NE_j}\sum_{n=1}^{NE_j}\left(H\tau^{s}_{xy}\frac{\partial\phi_j}{\partial x}+H\tau^{s}_{yy}\frac{\partial\phi_j}{\partial y}\right)_{n} \\
&\qquad -\frac{\Delta t}{NE_j}\sum_{n=1}^{NE_j}\left(\frac{B_y}{H}\right)^{s}_{n}
\end{aligned}
\tag{2.16}
$$

Each momentum equation discretization requires the solution of a 2x2 matrix at every node *j* in the model domain. This is accomplished in ADCIRC-2DDI using Kramer's rule.

The original version of ADCIRC-2DDI uses the non-conservative, approximate integration presented in Eq. (2.16). The other formulations have been added as of ADCIRC version 44.15.

3.0 3D MOMENTUM EQUATIONS

ADCIRC uses the shallow water form of the momentum equations (applying the Boussinesq and hydrostatic pressure approximations).

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv &= -g\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial x}+\frac{\partial}{\partial z}\left(\frac{\tau_{zx}}{\rho_o}\right)-b_x+m_x \\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu &= -g\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial y}+\frac{\partial}{\partial z}\left(\frac{\tau_{zy}}{\rho_o}\right)-b_y+m_y
\end{aligned} \qquad (3.1)$$

where,

u, v, w = velocity components in the coordinate directions x, y, z

$\frac{\tau_{zx}}{\rho_o}=E_z\frac{\partial u}{\partial z}=\text{vertical stress}$

$\frac{\tau_{zy}}{\rho_o}=E_z\frac{\partial v}{\partial z}=\text{vertical stress}$

$E_z=$ *vertical eddy viscosity*

$m_x\equiv\frac{\partial}{\partial x}\left(E_\ell\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(E_\ell\frac{\partial u}{\partial y}\right)=\text{lateral stress gradient}$

$m_y\equiv\frac{\partial}{\partial x}\left(E_\ell\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(E_\ell\frac{\partial v}{\partial y}\right)=\text{lateral stress gradient}$

$E_\ell\equiv$ *lateral stress coefficient (often called the lateral eddy viscosity)*

$b_x\equiv g\frac{\partial}{\partial x}\int_z^\zeta\frac{(\rho-\rho_o)}{\rho_o}dz=\text{baroclinic pressure gradient}$

$b_y\equiv g\frac{\partial}{\partial y}\int_z^\zeta\frac{(\rho-\rho_o)}{\rho_o}dz=\text{baroclinic pressure gradient}$

All horizontal derivatives in Eq. (3.1) and the accompanying definitions are computed in a level or "z" coordinate system. ADCIRC utilizes a generalized stretched vertical coordinate system

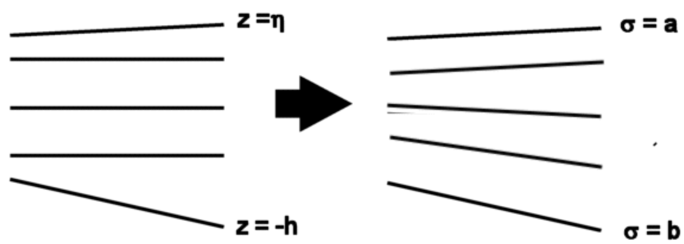


Figure 1. Schematic of level and stretched coordinates

$$\sigma = a + \left(\frac{a-b}{H}\right)(z-\zeta) \tag{3.2}$$

$$z = \left(\frac{\sigma-a}{a-b}\right)H + \zeta \tag{3.3}$$

(Figure 1) in which the vertical dimension is transformed from z, ranging from -*h* to ζ, to σ, ranging from *b* to *a*, where *b* and *a* are arbitrary constants. (Most models assume *b=-1, a=0*. ADCIRC assumes *b=-1, a=1*.) While ADCIRC uses the variable σ to represent the stretched vertical coordinate, a traditional "σ" coordinate system implies that the nodes are spaced uniformly over the vertical at any given horizontal location. ADCIRC does not carry this limitation, but rather nodes can be distributed over the vertical in any manner desired.

Using the chain rule we can relate derivatives along level (*z*) surfaces to derivatives along the stretched (σ) surfaces:

$$\frac{\partial}{\partial x_z} = \frac{\partial}{\partial x_\sigma} - \left[\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial x_z} + \left(\frac{\sigma-a}{a-b}\right)\frac{\partial h}{\partial x_z}\right]\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial y_z} = \frac{\partial}{\partial y_\sigma} - \left[\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial y_z} + \left(\frac{\sigma-a}{a-b}\right)\frac{\partial h}{\partial y_z}\right]\frac{\partial}{\partial z} \tag{3.4}$$

$$\frac{\partial}{\partial z} = \left(\frac{a-b}{H}\right)\frac{\partial}{\partial\sigma}$$

where for clarity, σ subscripts have been used on the horizontal derivatives computed along the stretched surfaces in Eqs. (3.4).

Considerable discussion exists in the literature regarding the generation of spurious circulation due to the use of stretched vertical coordinates. Most of this attention has focused on problems arising from the baroclinic pressure gradient terms and to a lesser extent the lateral stress terms. In ADCIRC we apply the stretched coordinate system to all but the baroclinic pressure gradient terms resulting in the following transformed momentum equations:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x_\sigma}+v\frac{\partial u}{\partial y_\sigma}+w_\sigma\left(\frac{a-b}{H}\right)\frac{\partial u}{\partial \sigma}-fv= \\
&\qquad -g\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial x}+\left(\frac{a-b}{H}\right)\frac{\partial}{\partial\sigma}\left(\frac{\tau_{zx}}{\rho_o}\right)-b_x+m_{x\sigma} \\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x_\sigma}+v\frac{\partial v}{\partial y_\sigma}+w_\sigma\left(\frac{a-b}{H}\right)\frac{\partial v}{\partial \sigma}+fu= \\
&\qquad -g\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial y}+\left(\frac{a-b}{H}\right)\frac{\partial}{\partial\sigma}\left(\frac{\tau_{zy}}{\rho_o}\right)-b_y+m_{y\sigma}
\end{aligned} \tag{3.5}$$

Note that the first term on the right hand side of each equation is not a function of depth and therefore horizontal derivatives in level coordinates are identical to horizontal derivatives in stretched coordinates.

Introduction of the stretched coordinate system in the advection terms produces similar-looking advection terms in the stretched coordinate system, Eqs. (3.5), provided a stretched-coordinate, vertical velocity, w_σ, is introduced that is related to the true vertical velocity by:

$$w_\sigma \equiv w-\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial t}-u\left[\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial x}+\left(\frac{\sigma-a}{a-b}\right)\frac{\partial h}{\partial x}\right]-v\left[\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial y}+\left(\frac{\sigma-a}{a-b}\right)\frac{\partial h}{\partial y}\right] \tag{3.6}$$

ADCIRC does not formally transform the lateral stress terms (m_x, m_y) in Eqs. (3.4) to obtain equivalent terms in Eqs. (3.5). Rather, the original lateral stress terms (along horizontal surfaces) are approximated as lateral stresses "along stretched surfaces", i.e.,

$$\begin{aligned}
m_{x\sigma} &\equiv \frac{\partial}{\partial x_\sigma}\left(E_\ell\frac{\partial u}{\partial x_\sigma}\right)+\frac{\partial}{\partial y_\sigma}\left(E_\ell\frac{\partial u}{\partial y_\sigma}\right)= \textit{lateral stress gradients along stretched surface} \\
m_{y\sigma} &\equiv \frac{\partial}{\partial x_\sigma}\left(E_\ell\frac{\partial v}{\partial x_\sigma}\right)+\frac{\partial}{\partial y_\sigma}\left(E_\ell\frac{\partial v}{\partial y_\sigma}\right)= \textit{lateral stress gradients along stretched surface}
\end{aligned} \tag{3.7}$$

The generation of spurious circulation because of this assumption has also been discussed in the literature. ADCIRC uses the lateral stress gradient terms purely to dampen numerical noise in the solution and therefore assumes a lateral stress coefficient that is as small as possible. This should minimize the generation of spurious circulation by these terms.

The weighted residual method is applied to Eqs. (3.5) by multiplying each term by a horizontal weighting function ϕ_j and integrating over the horizontal computational domain Ω and then multiplying the result by a vertical weighting function ψ_k and integrating over the vertical domain, Z. By constructing the grid so that the vertical nodes line up vertically beneath each horizontal node, the horizontal and vertical integrations can be performed independently.

$$\left\langle\left\langle \frac{\partial u}{\partial t},\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z + \left\langle\left\langle\left(u\frac{\partial u}{\partial x_\sigma}+v\frac{\partial u}{\partial y_\sigma}+w_\sigma\left(\frac{a-b}{H}\right)\frac{\partial u}{\partial\sigma}\right),\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z - \left\langle\left\langle fv,\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z =$$
$$-\left\langle\left\langle g\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial x},\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z + \left\langle\left\langle\left(\frac{a-b}{H}\right)\frac{\partial}{\partial\sigma}\left(\frac{\tau_{zx}}{\rho_o}\right),\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z \tag{3.8}$$
$$-\left\langle\left\langle b_x,\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z + \left\langle\left\langle m_{x\sigma},\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z$$

$$\left\langle\left\langle \frac{\partial v}{\partial t},\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z + \left\langle\left\langle\left(u\frac{\partial v}{\partial x_\sigma}+v\frac{\partial v}{\partial y_\sigma}+w_\sigma\left(\frac{a-b}{H}\right)\frac{\partial v}{\partial\sigma}\right),\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z + \left\langle\left\langle fu,\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z =$$
$$-\left\langle\left\langle g\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial y},\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z + \left\langle\left\langle\left(\frac{a-b}{H}\right)\frac{\partial}{\partial\sigma}\left(\frac{\tau_{zy}}{\rho_o}\right),\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z \tag{3.9}$$
$$-\left\langle\left\langle b_y,\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z + \left\langle\left\langle m_{y\sigma},\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z$$

Horizontal integrations of each term in Eq. (3.8) are presented below (Eq. (3.9) is fully analogous) and are carried out using one of two basic integration rules as noted in the text. These rules are described in APPENDIX - BASIC CALCULATIONS ON LINEAR TRIANGLES:

Horizontal integration of the transient term in Eq. (3.8) utilizes **Rule 1**:

$$\left\langle\left\langle \frac{\partial u}{\partial t},\phi_j\right\rangle_\Omega,\psi_k\right\rangle_Z = \frac{A_{NE_j}}{3}\left\langle \frac{\partial u_j}{\partial t},\psi_k\right\rangle_Z$$

Horizontal integration of the horizontal advection terms in Eq. (3.8) utilizes **Rule 2** and assumes the un-differentiated velocity terms are elementally averaged (i.e., $\overline{u}_n \equiv \frac{1}{3}\sum_{i=1}^{3} u_i$ and $\overline{v}_n \equiv \frac{1}{3}\sum_{i=1}^{3} v_i$):

$$\left\langle \left\langle \left(u\frac{\partial u}{\partial x_\sigma} + v\frac{\partial u}{\partial y_\sigma} \right), \phi_j \right\rangle_\Omega, \psi_k \right\rangle_Z = \left\langle \sum_{n=1}^{NE\,j} \frac{A_n}{3} \left[\overline{u}_n \left(\frac{\partial u}{\partial x_\sigma} \right)_n + \overline{v}_n \left(\frac{\partial u}{\partial y_\sigma} \right)_n \right], \psi_k \right\rangle_Z$$

Horizontal integration of the vertical advection term in Eq. (3.8) utilizes **Rule 1**:

$$\left\langle \left\langle \left(w_\sigma \left(\frac{a-b}{H} \right) \frac{\partial u}{\partial \sigma} \right), \phi_j \right\rangle_\Omega, \psi_k \right\rangle_Z = \frac{A_{NE\,j}}{3} \left(\frac{a-b}{H_j} \right) \left\langle w_{\sigma\,j} \frac{\partial u_j}{\partial \sigma}, \psi_k \right\rangle_Z$$

Horizontal integration of the Coriolis term in Eq. (3.8) utilizes **Rule 1**:

$$\left\langle \left\langle fv, \phi_j \right\rangle_\Omega, \psi_k \right\rangle_Z = \frac{A_{NE\,j}}{3} \left\langle fv_j, \psi_k \right\rangle_Z$$

Horizontal integration of the combined barotropic pressure (i.e., the free surface elevation, atmospheric pressure and tidal potential) gradient term in Eq. (3.8) utilizes **Rule 2:**

$$\left\langle \left\langle \frac{\partial \left[g\zeta + P_s/\rho_o - \alpha g\eta \right]}{\partial x}, \phi_j \right\rangle_\Omega, \psi_k \right\rangle_Z = \left\langle \sum_{n=1}^{NE_j} \frac{A_n}{3} \left(g\frac{\partial \left[\zeta + P_s/g\rho_o - \alpha\eta \right]}{\partial x} \right)_n, \psi_k \right\rangle_Z$$

Horizontal integration of the vertical stress gradient term in Eq. (3.8) utilizes **Rule 1**:

$$\left\langle \left\langle \left(\frac{a-b}{H} \right) \frac{\partial}{\partial \sigma} \left(\frac{\tau_{zx}}{\rho_o} \right), \phi_j \right\rangle_\Omega, \psi_k \right\rangle_Z = \frac{A_{NE\,j}}{3} \left(\frac{a-b}{H_j} \right) \left\langle \frac{\partial}{\partial \sigma} \left(\frac{\tau_{zx\,j}}{\rho_o} \right), \psi_k \right\rangle_Z$$

Horizontal integration of the baroclinic pressure gradient terms in Eq. (3.8) utilizes **Rule 2:**

$$\left\langle \left\langle b_x, \phi_j \right\rangle_\Omega, \psi_k \right\rangle_Z = \left\langle \sum_{n=1}^{NE_j} \frac{A_n}{3} b_{xn}, \psi_k \right\rangle_Z$$

Horizontal integration of the lateral stress gradient term in Eq. (3.8) initially utilizes integration by parts

$$\left\langle \left\langle m_{x\sigma}, \phi_j \right\rangle_\Omega, \psi_k \right\rangle_Z \equiv \left\langle \left\langle \left[\frac{\partial}{\partial x_\sigma}\left(E_\ell \frac{\partial u}{\partial x_\sigma} \right) + \frac{\partial}{\partial y_\sigma}\left(E_\ell \frac{\partial u}{\partial y_\sigma} \right) \right], \phi_j \right\rangle_\Omega, \psi_k \right\rangle_Z$$

$$= \left\langle \left[\int_{\Gamma_n} E_\ell \left(\frac{\partial u}{\partial x_\sigma} + \frac{\partial u}{\partial y_\sigma} \right) \phi_j \, d\Gamma - \sum_{n=1}^{NE_j} \int_{\Omega_n} E_\ell \left(\frac{\partial u}{\partial x_\sigma} \frac{\partial \phi_j}{\partial x} + \frac{\partial u}{\partial y_\sigma} \frac{\partial \phi_j}{\partial y} \right) d\Omega \right], \psi_k \right\rangle_Z$$

$$= \left\langle \left[\int_{\Gamma_n} E_\ell \left(\frac{\partial u}{\partial x_\sigma} + \frac{\partial u}{\partial y_\sigma} \right) \phi_j \, d\Gamma - \sum_{n=1}^{NE_j} \left(\frac{\partial u}{\partial x_\sigma} \frac{\partial \phi_j}{\partial x} + \frac{\partial u}{\partial y_\sigma} \frac{\partial \phi_j}{\partial y} \right)_n \int_{\Omega_n} E_\ell \, d\Omega \right], \psi_k \right\rangle_Z$$

where, $\Gamma_n = \textit{external boundary segment of element } n$. The term is further reduced by assuming that the lateral stresses are zero along all external boundary segments and by lumping the lateral stress coefficient

$$\left\langle \left\langle m_{x\sigma}, \phi_j \right\rangle_\Omega, \psi_k \right\rangle_Z = -\left\langle \left[E_{\ell j} \sum_{n=1}^{NE_j} A_n \left(\frac{\partial u}{\partial x_\sigma} \frac{\partial \phi_j}{\partial x} + \frac{\partial u}{\partial y_\sigma} \frac{\partial \phi_j}{\partial y} \right)_n \right], \psi_k \right\rangle_Z$$

Thus, following horizontal integration and multiplication by $3/A_{NE_j}$, Eqs. (3.8) and (3.9) become:

$$\left\langle \frac{\partial u_j}{\partial t}, \psi_k \right\rangle_Z + \frac{1}{A_{NE_j}} \left\langle \sum_{n=1}^{NE_j} A_n \left[\bar{u}_n \left(\frac{\partial u}{\partial x_\sigma} \right)_n + \bar{v}_n \left(\frac{\partial u}{\partial y_\sigma} \right)_n \right], \psi_k \right\rangle_Z + \left(\frac{a-b}{H_j} \right) \left\langle w_{\sigma j} \frac{\partial u_j}{\partial \sigma}, \psi_k \right\rangle_Z$$
$$- \left\langle f v_j, \psi_k \right\rangle_Z = -\frac{1}{A_{NE_j}} \left\langle \sum_{n=1}^{NE_j} A_n \left(g \frac{\partial \left[\zeta + P_s / g\rho_o - \alpha\eta \right]}{\partial x} \right)_n, \psi_k \right\rangle_Z + \left(\frac{a-b}{H_j} \right) \left\langle \frac{\partial}{\partial \sigma} \left(\frac{\tau_{zx\,j}}{\rho_o} \right), \psi_k \right\rangle_Z \quad (3.10)$$
$$- \frac{1}{A_{NE_j}} \left\langle \sum_{n=1}^{NE_j} A_n b_{xn}, \psi_k \right\rangle_Z - \frac{3}{A_{NE_j}} \left\langle \left[E_{\ell j} \sum_{n=1}^{NE_j} A_n \left(\frac{\partial u}{\partial x_\sigma} \frac{\partial \phi_j}{\partial x} + \frac{\partial u}{\partial y_\sigma} \frac{\partial \phi_j}{\partial y} \right)_n \right], \psi_k \right\rangle_Z$$

$$\left\langle \frac{\partial v_j}{\partial t}, \psi_k \right\rangle_Z + \frac{1}{A_{NE\,j}} \left\langle \sum_{n=1}^{NE\,j} A_n \left[\bar{u}_n \left(\frac{\partial v}{\partial x_\sigma} \right)_n + \bar{v}_n \left(\frac{\partial v}{\partial y_\sigma} \right)_n \right], \psi_k \right\rangle_Z + \left(\frac{a-b}{H_j} \right) \left\langle w_{\sigma j} \frac{\partial v_j}{\partial \sigma}, \psi_k \right\rangle_Z$$

$$+ \left\langle f u_j, \psi_k \right\rangle_Z = -\frac{1}{A_{NE\,j}} \left\langle \sum_{n=1}^{NE_j} A_n \left(g \frac{\partial \left[\zeta + P_s / g \rho_o - \alpha \eta \right]}{\partial y} \right)_n, \psi_k \right\rangle_Z + \left(\frac{a-b}{H_j} \right) \left\langle \frac{\partial}{\partial \sigma} \left(\frac{\tau_{zy\,j}}{\rho_o} \right), \psi_k \right\rangle_Z \quad (3.11)$$

$$- \frac{1}{A_{NE\,j}} \left\langle \sum_{n=1}^{NE_j} A_n b_{y\,n}, \psi_k \right\rangle_Z - \frac{3}{A_{NE\,j}} \left\langle \left[E_{\ell\,j} \sum_{n=1}^{NE_j} A_n \left(\frac{\partial v}{\partial x_\sigma} \frac{\partial \phi_j}{\partial x} + \frac{\partial v}{\partial y_\sigma} \frac{\partial \phi_j}{\partial y} \right)_n \right], \psi_k \right\rangle_Z$$

A standard one-dimensional, Galerkin FEM discretization is used in the vertical, yielding the following integration rule,

$$\left\langle \Upsilon, \psi_k \right\rangle_Z \equiv \begin{cases} \Upsilon_{k-1} \int\limits_{\sigma_{k-1}}^{\sigma_k} \psi_{k-1} \psi_k \, d\sigma + \Upsilon_k \int\limits_{\sigma_{k-1}}^{\sigma_k} \psi_k \psi_k \, d\sigma & k = NV \\ \Upsilon_{k-1} \int\limits_{\sigma_{k-1}}^{\sigma_k} \psi_{k-1} \psi_k \, d\sigma + \Upsilon_k \int\limits_{\sigma_{k-1}}^{\sigma_{k+1}} \psi_k \psi_k \, d\sigma + \Upsilon_{k+1} \int\limits_{\sigma_k}^{\sigma_{k+1}} \psi_{k+1} \psi_k \, d\sigma & 1 < k < NV \\ \Upsilon_k \int\limits_{\sigma_k}^{\sigma_{k+1}} \psi_k \psi_k \, d\sigma + \Upsilon_{k+1} \int\limits_{\sigma_k}^{\sigma_{k+1}} \psi_{k+1} \psi_k \, d\sigma & k = 1 \end{cases}$$

In shorthand notation this can be written as:

$$\left\langle \Upsilon, \psi_k \right\rangle_Z \equiv \Upsilon_{k-1} Inm_{k,1} + \Upsilon_k Inm_{k,2} + \Upsilon_{k+1} Inm_{k,3} = \sum_{m=1}^{3} \Upsilon_{k+m-2} Inm_{k,m}$$

where,

ψ_k = *vertical weighting function, =1 at node k, =0 at all other nodes,*
varies linearly between adjacent nodes
$k = 1$ *at the bottom*
$k = NV$ *at the free surface*
NV = *number of nodes in the vertical*

$$Inm_{k,1} = \begin{cases} \int_{\sigma_{k-1}}^{\sigma_k} \psi_{k-1}\psi_k \, d\sigma = \frac{1}{2}\int_{\sigma_{k-1}}^{\sigma_k} \psi_k \psi_k \, d\sigma = \frac{\sigma_k - \sigma_{k-1}}{6} & for\ k \neq 1 \\ 0 & for\ k = 1 \end{cases}$$

$$Inm_{k,2} = 2\left(Inm_{k,1} + Inm_{k,3}\right) \tag{3.12}$$

$$Inm_{k,3} = \begin{cases} \int_{\sigma_k}^{\sigma_{k+1}} \psi_{k+1}\psi_k \, d\sigma = \frac{1}{2}\int_{\sigma_k}^{\sigma_{k+1}} \psi_k \psi_k \, d\sigma = \frac{\sigma_{k+1} - \sigma_k}{6} & for\ k \neq NV \\ 0 & for\ k = NV \end{cases}$$

Note, that the definition of the weighting/basis function ψ_k reduces integration over the vertical domain Z to integration over only the two vertical elements containing node *k*, i.e., from node $k-1$ to node $k+1$. Also, because the basis functions are linear in space, their derivatives are constant within an element and can be pulled out of elemental integrations.

Vertical integration of the transient term in Eq. (3.10) yields

$$\left\langle \frac{\partial u_j}{\partial t}, \psi_k \right\rangle_Z = \sum_{m=1}^{3} \frac{\partial u_{j,k+m-2}}{\partial t} Inm_{k,m}$$

Vertical integration of the horizontal advection terms in Eq. (3.10) yields

$$\frac{1}{A_{NE_j}} \left\langle \sum_{n=1}^{NE_j} A_n \left[\bar{u}_n \left(\frac{\partial u}{\partial x_\sigma} \right)_n + \bar{v}_n \left(\frac{\partial u}{\partial y_\sigma} \right)_n \right], \psi_k \right\rangle_Z$$

$$= \frac{1}{A_{NE_j}} \sum_{m=1}^{3} \left[\sum_{n=1}^{NE_j} A_n \left[\bar{u}_n \left(\frac{\partial u}{\partial x_\sigma} \right)_n + \bar{v}_n \left(\frac{\partial u}{\partial y_\sigma} \right)_n \right] \right]_{k+m-2} Inm_{k,m}$$

Vertical integration of the vertical advection term in Eq. (3.10) yields

$$\left(\frac{a-b}{H_j}\right)\left\langle w_{\sigma j}\frac{\partial u_j}{\partial \sigma},\psi_k\right\rangle_Z=\left(\frac{a-b}{H_j}\right)\left(\frac{\partial u_j}{\partial \sigma}\right)_{k\text{-}1,k}\left[w_{\sigma j,k-1}\int_{\sigma_{k-1}}^{\sigma_k}\psi_k\psi_{k-1}\,d\sigma+w_{\sigma j,k}\int_{\sigma_{k-1}}^{\sigma_k}\psi_k\psi_k\,d\sigma\right]$$

$$+\left(\frac{a-b}{H_j}\right)\left(\frac{\partial u_j}{\partial \sigma}\right)_{k,k+1}\left[w_{\sigma j,k}\int_{\sigma_k}^{\sigma_{k+1}}\psi_k\psi_k\,d\sigma+w_{\sigma j,k+1}\int_{\sigma_k}^{\sigma_{k+1}}\psi_k\psi_{k+1}\,d\sigma\right]$$

$$=\left(\frac{a-b}{H_j}\right)\left[\left(\frac{\partial u_j}{\partial \sigma}\right)_{k\text{-}1,k}\left(w_{\sigma j,k-1}+2w_{\sigma j,k}\right)Inm_{k,1}+\left(\frac{\partial u_j}{\partial \sigma}\right)_{k,k+1}\left(2w_{\sigma j,k}+w_{\sigma j,k+1}\right)Inm_{k,3}\right]$$

where,

$$\left(\frac{\partial u_j}{\partial \sigma}\right)_{k\text{-}1,k}\equiv u_{j,k-1}\left(\frac{\partial \psi_{k-1}}{\partial \sigma}\right)_{k\text{-}1,k}+u_{j,k}\left(\frac{\partial \psi_k}{\partial \sigma}\right)_{k\text{-}1,k}=\frac{u_{j,k}-u_{j,k-1}}{\sigma_k-\sigma_{k-1}}$$

$$\left(\frac{\partial u_j}{\partial \sigma}\right)_{k,k+1}\equiv u_{j,k}\left(\frac{\partial \psi_k}{\partial \sigma}\right)_{k,k+1}+u_{j,k+1}\left(\frac{\partial \psi_{k+1}}{\partial \sigma}\right)_{k,k+1}=\frac{u_{j,k+1}-u_{j,k}}{\sigma_{k+1}-\sigma_k}$$

Vertical integration of the Coriolis term in Eq. (3.10) yields

$$\left\langle fv_j,\psi_k\right\rangle_Z=\sum_{m=1}^{3}fv_{j,k+m-2}Inm_{k,m}$$

Vertical integration of the barotropic pressure gradient term in Eq. (3.10) yields

$$\frac{1}{A_{NE_j}}\left\langle\sum_{n=1}^{NE_j}A_n\left(g\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial x}\right)_n,\psi_k\right\rangle_Z=\frac{1}{A_{NE_j}}\sum_{n=1}^{NE_j}A_n\left(g\frac{\partial\left[\zeta+P_s/g\rho_o-\alpha\eta\right]}{\partial x}\right)_n LVn_k$$

where

$$LVn_k\equiv\begin{cases}\displaystyle\int_{\sigma_{k-1}}^{\sigma_k}\psi_k\,d\sigma=\frac{\sigma_k-\sigma_{k-1}}{2} & k=NV\\ \displaystyle\int_{\sigma_{k-1}}^{\sigma_{k+1}}\psi_k\,d\sigma=\frac{\sigma_{k+1}-\sigma_{k-1}}{2} & 1<k<NV\\ \displaystyle\int_{\sigma_k}^{\sigma_{k+1}}\psi_k\,d\sigma=\frac{\sigma_{k+1}-\sigma_k}{2} & k=1\end{cases}\tag{3.13}$$

Vertical integration of the baroclinic pressure gradient term in Eq. (3.10) yields

$$\frac{1}{A_{NE\,j}}\left\langle \sum_{n=1}^{NE\,j} A_n b_{xn}, \psi_k \right\rangle_Z = \frac{1}{A_{NE\,j}}\sum_{m=1}^{3}\left[\sum_{n=1}^{NE\,j} A_n b_{xn}\right]_{k+m-2} Inm_{k,m}$$

Vertical integration of the lateral stress terms in Eq. (3.10) yields

$$\frac{3}{A_{NE\,j}}\left\langle \left[E_{\ell\,j}\sum_{n=1}^{NE\,j} A_n\left(\frac{\partial u}{\partial x_\sigma}\frac{\partial \phi_j}{\partial x} + \frac{\partial u}{\partial y_\sigma}\frac{\partial \phi_j}{\partial y}\right)_n\right], \psi_k \right\rangle_Z$$

$$= \frac{3}{A_{NE\,j}}\sum_{m=1}^{3}\left[E_{\ell\,j}\sum_{n=1}^{NE\,j} A_n\left(\frac{\partial u}{\partial x_\sigma}\frac{\partial \phi_j}{\partial x} + \frac{\partial u}{\partial y_\sigma}\frac{\partial \phi_j}{\partial y}\right)_n\right]_{k+m-2} Inm_{k,m}$$

The vertical stress gradient term in Eq. (3.10) is initially integrated by parts, yielding

$$\left(\frac{a-b}{H_j}\right)\left\langle \frac{\partial}{\partial\sigma}\left(\frac{\tau_{zx\,j}}{\rho_o}\right), \psi_k \right\rangle_Z = \left(\frac{a-b}{H_j}\right)\frac{\tau_{sx\,j}}{\rho_o}\bigg|_{k=NV} - \left(\frac{a-b}{H_j}\right)\frac{\tau_{bx\,j}}{\rho_o}\bigg|_{k=1} - \left(\frac{a-b}{H_j}\right)\left\langle \frac{\tau_{zx\,j}}{\rho_o}, \frac{\partial \psi_k}{\partial\sigma} \right\rangle_Z$$

where the free surface stress, $\tau_{sx\,j}$ (applied only for $k=NV$) and bottom stress $\tau_{bx\,j}$ (applied only for k=1) have been introduced. Expressing the vertical stress in terms of the vertical gradient of velocity in the remaining integral term, yields:

$$\left\langle \frac{\tau_{zx\,j}}{\rho_o}, \frac{\partial \psi_k}{\partial\sigma} \right\rangle_Z = \left(\frac{a-b}{H_j}\right)\left\langle E_{z\,j}\frac{\partial u_j}{\partial\sigma}, \frac{\partial \psi_k}{\partial\sigma} \right\rangle_Z =$$

$$\left(\frac{a-b}{H_j}\right)\left[\begin{array}{l}\left(u_{j,k-1}\left(\frac{\partial \psi_{k-1}}{\partial\sigma}\right)_{k-1,k} + u_{j,k}\left(\frac{\partial \psi_k}{\partial\sigma}\right)_{k-1,k}\right)\left(\frac{\partial \psi_k}{\partial\sigma}\right)_{k-1,k}\left(E_{z\,j,k-1}\int_{\sigma_{k-1}}^{\sigma_k}\psi_{k-1}\,d\sigma + E_{z\,j,k}\int_{\sigma_{k-1}}^{\sigma_k}\psi_k\,d\sigma\right) \\ + \left(u_{j,k}\left(\frac{\partial \psi_k}{\partial\sigma}\right)_{k,k+1} + u_{j,k+1}\left(\frac{\partial \psi_{k+1}}{\partial\sigma}\right)_{k,k+1}\right)\left(\frac{\partial \psi_k}{\partial\sigma}\right)_{k,k+1}\left(E_{z\,j,k}\int_{\sigma_k}^{\sigma_{k+1}}\psi_k\,d\sigma + E_{z\,j,k+1}\int_{\sigma_k}^{\sigma_{k+1}}\psi_{k+1}\,d\sigma\right)\end{array}\right]$$

or in shorthand notation

$$\left\langle \frac{\tau_{zx\,j}}{\rho_o}, \frac{\partial \psi_k}{\partial\sigma} \right\rangle_Z = \left(\frac{a-b}{H_j}\right)\sum_{m=1}^{3} u_{j,k+m-2} KVnm_{j,k,m}$$

where

$$KVnm_{j,k,1} = \begin{cases} \left(\frac{\partial \psi_{k-1}}{\partial \sigma}\right)_{k-1,k} \left(\frac{\partial \psi_{k}}{\partial \sigma}\right)_{k-1,k} \left(E_{z\,j,k-1} \int_{\sigma_{k-1}}^{\sigma_k} \psi_{k-1}\, d\sigma + E_{z\,j,k} \int_{\sigma_{k-1}}^{\sigma_k} \psi_{k}\, d\sigma \right) \\ = -\left(\frac{\partial \psi_{k}}{\partial \sigma}\right)_{k-1,k} \left(\frac{\partial \psi_{k}}{\partial \sigma}\right)_{k-1,k} \left(E_{z\,j,k-1} \int_{\sigma_{k-1}}^{\sigma_k} \psi_{k-1}\, d\sigma + E_{z\,j,k} \int_{\sigma_{k-1}}^{\sigma_k} \psi_{k}\, d\sigma \right) \\ = -\frac{E_{z\,j,k} + E_{z\,j,k-1}}{2\left(\sigma_k - \sigma_{k-1}\right)} & \textit{for } k \neq 1 \\ 0 & \textit{for } k = 1 \end{cases}$$

$$KVnm_{j,k,2} = -\left(KVnm_{k,1} + KVnm_{k,3}\right)$$

$$KVnm_{j,k,3} = \begin{cases} \left(\frac{\partial \psi_{k+1}}{\partial \sigma}\right)_{k,k+1} \left(\frac{\partial \psi_{k}}{\partial \sigma}\right)_{k,k+1} \left(E_{z\,j,k} \int_{\sigma_{k}}^{\sigma_{k+1}} \psi_{k}\, d\sigma + E_{z\,j,k+1} \int_{\sigma_{k}}^{\sigma_{k+1}} \psi_{k+1}\, d\sigma \right) \\ = -\left(\frac{\partial \psi_{k}}{\partial \sigma}\right)_{k,k+1} \left(\frac{\partial \psi_{k}}{\partial \sigma}\right)_{k,k+1} \left(E_{z\,j,k} \int_{\sigma_{k}}^{\sigma_{k+1}} \psi_{k}\, d\sigma + E_{z\,j,k+1} \int_{\sigma_{k}}^{\sigma_{k+1}} \psi_{k+1}\, d\sigma \right) \\ = -\frac{E_{z\,j,k+1} + E_{z\,j,k}}{2\left(\sigma_{k+1} - \sigma_{k}\right)} & \textit{for } k \neq NV \\ 0 & \textit{for } k = NV \end{cases} \qquad (3.14)$$

ADCIRC utilizes a generalized slip formulation for the bottom stress term:

$$\frac{\tau_{bx\,j}}{\rho_o} = K_{slip\,j} u_j; \qquad \frac{\tau_{by\,j}}{\rho_o} = K_{slip\,j} v_j$$

where,

$K_{slip\,j} \to \infty$ = *no slip bottom boundary condition*

$K_{slip\,j} = constant$, = *linear slip bottom boundary condition, (* $K_{slip\,j}$ *= linear drag coefficent)*

$K_{slip\,j} = C_d \sqrt{u_j^2 + v_j^2}$, = *quadratic slip bottom boundary condition, (* C_d *= quadratic drag coefficent)*

In final form, the vertical stress gradient term is:

$$\left(\frac{a-b}{H_j}\right)\left\langle \frac{\partial}{\partial\sigma}\left(\frac{\tau_{zx\,j}}{\rho_o}\right),\psi_k \right\rangle_Z = \left(\frac{a-b}{H_j}\right)\frac{\tau_{sx\,j}}{\rho_o}\bigg|_{k=NV} - \left(\frac{a-b}{H_j}\right)u_j\big|_{k=1}K_{slip\,j} - \left(\frac{a-b}{H_j}\right)^2\sum_{m=1}^{3}u_{j,k+m-2}KVnm_{j,k,m}$$

Thus, following vertical integration Eqs. (3.10), (3.11) become:

$$
\begin{aligned}
&\sum_{m=1}^{3}\frac{\partial u_{j,k+m-2}}{\partial t} Inm_{k,m} + \frac{1}{A_{NE\,j}}\sum_{m=1}^{3}\left[\sum_{n=1}^{NE\,j} A_n\left[\bar{u}_n\left(\frac{\partial u}{\partial x_\sigma}\right)_n + \bar{v}_n\left(\frac{\partial u}{\partial y_\sigma}\right)_n\right]\right]_{k+m-2} Inm_{k,m} \\
&\quad + \left(\frac{a-b}{H_j}\right)\left[\left(\frac{\partial u_j}{\partial \sigma}\right)_{k\text{-}1,k}\left(w_{\sigma\, j,k-1} + 2w_{\sigma\, j,k}\right) Inm_{k,1} + \left(\frac{\partial u_j}{\partial \sigma}\right)_{k,k+1}\left(2w_{\sigma\, j,k} + w_{\sigma\, j,k+1}\right) Inm_{k,3}\right] \\
&\quad - \sum_{m=1}^{3} f\, v_{j,k+m-2} Inm_{k,m} = -\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE\,j} A_n\left(g\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial x}\right)_n LVn_k \\
&\quad + \left(\frac{a-b}{H_j}\right)\frac{\tau_{sx\,j}}{\rho_o}\bigg|_{k=NV} - \left(\frac{a-b}{H_j}\right) u_j\big|_{k=1} K_{slip\,j} - \left(\frac{a-b}{H_j}\right)^2 \sum_{m=1}^{3} u_{j,k+m-2} KVnm_{j,k,m} \\
&\quad - \frac{1}{A_{NE\,j}}\sum_{m=1}^{3}\left[\sum_{n=1}^{NE\,j} A_n b_{x\,n}\right]_{k+m-2} Inm_{k,m} - \frac{3}{A_{NE\,j}}\sum_{m=1}^{3}\left[E_{\ell\, j}\sum_{n=1}^{NE\,j} A_n\left(\frac{\partial u}{\partial x_\sigma}\frac{\partial \phi_j}{\partial x} + \frac{\partial u}{\partial y_\sigma}\frac{\partial \phi_j}{\partial y}\right)_n\right]_{k+m-2} Inm_{k,m}
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
&\sum_{m=1}^{3}\frac{\partial v_{j,k+m-2}}{\partial t} Inm_{k,m} + \frac{1}{A_{NE\,j}}\sum_{m=1}^{3}\left[\sum_{n=1}^{NE\,j} A_n\left[\bar{u}_n\left(\frac{\partial v}{\partial x_\sigma}\right)_n + \bar{v}_n\left(\frac{\partial v}{\partial y_\sigma}\right)_n\right]\right]_{k+m-2} Inm_{k,m} \\
&\quad + \left(\frac{a-b}{H_j}\right)\left[\left(\frac{\partial v_j}{\partial \sigma}\right)_{k\text{-}1,k}\left(w_{\sigma\, j,k-1} + 2w_{\sigma\, j,k}\right) Inm_{k,1} + \left(\frac{\partial v_j}{\partial \sigma}\right)_{k,k+1}\left(2w_{\sigma\, j,k} + w_{\sigma\, j,k+1}\right) Inm_{k,3}\right] \\
&\quad + \sum_{m=1}^{3} f\, u_{j,k+m-2} Inm_{k,m} = -\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE\,j} A_n\left(g\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial y}\right)_n LVn_k \\
&\quad + \left(\frac{a-b}{H_j}\right)\frac{\tau_{sy\,j}}{\rho_o}\bigg|_{k=NV} - \left(\frac{a-b}{H_j}\right) v_j\big|_{k=1} K_{slip\,j} - \left(\frac{a-b}{H_j}\right)^2 \sum_{m=1}^{3} v_{j,k+m-2} KVnm_{j,k,m} \\
&\quad - \frac{1}{A_{NE\,j}}\sum_{m=1}^{3}\left[\sum_{n=1}^{NE\,j} A_n b_{y\,n}\right]_{k+m-2} Inm_{k,m} - \frac{3}{A_{NE\,j}}\sum_{m=1}^{3}\left[E_{\ell\, j}\sum_{n=1}^{NE\,j} A_n\left(\frac{\partial v}{\partial x_\sigma}\frac{\partial \phi_j}{\partial x} + \frac{\partial v}{\partial y_\sigma}\frac{\partial \phi_j}{\partial y}\right)_n\right]_{k+m-2} Inm_{k,m}
\end{aligned}
\tag{3.16}
$$

Equations (3.15), (3.16) present the spatially discretized solution for velocity at horizontal node *j* and vertical node *k* used by ADCIRC 3D. These equations are discretized in time using a two time level scheme at the present (*s*) and future (*s*+1) time levels as described below:

Transient term: $\displaystyle\sum_{m=1}^{3}\frac{u^{s+1}_{j,k+m-2} - u^{s}_{j,k+m-2}}{\Delta t} Inm_{k,m}$

Horizontal advection: $\dfrac{1}{A_{NE\,j}}\sum_{m=1}^{3}\left[\sum_{n=1}^{NE\,j} A_n\left[\overline{u}_n^s\left(\dfrac{\partial u^s}{\partial x_\sigma}\right)_n + \overline{v}_n^s\left(\dfrac{\partial u^s}{\partial y_\sigma}\right)_n\right]\right]_{k+m-2} Inm_{k,m}$

Vertical advection:

$$\left(\frac{a-b}{H_j^s}\right)\left[\left(\frac{\partial u_j^s}{\partial \sigma}\right)_{k\text{-}1,k}\left(w_{\sigma\,j,k-1}^s + 2w_{\sigma\,j,k}^s\right) Inm_{k,1} + \left(\frac{\partial u_j^s}{\partial \sigma}\right)_{k,k+1}\left(2w_{\sigma\,j,k}^s + w_{\sigma\,j,k+1}^s\right) Inm_{k,3}\right]$$

Coriolis: $\sum_{m=1}^{3} f\left[\alpha_1 v_{j,k+m-2}^{s+1} + (1-\alpha_1) v_{j,k+m-2}^s\right] Inm_{k,m}$

Free surface stress: $\dfrac{a-b}{2}\left[\dfrac{\tau_{sx\,j}^{s+1}}{H_j^{s+1}\rho_o} + \dfrac{\tau_{sx\,j}^{s}}{H_j^{s}\rho_o}\right]_{k=NV}$

Bottom stress: $(a-b)K_{slip\,j}^s\left[\dfrac{\alpha_2 u_j^{s+1}}{H_j^{s+1}} + \dfrac{(1-\alpha_2)u_j^s}{H_j^s}\right]_{k=1}$

Barotropic pressure gradient:

$$\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE_j}\frac{A_n}{2}\left(g\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^s}{\partial x} + g\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s+1}}{\partial x}\right)_n LVn_k$$

Vertical stress: $(a-b)^2\sum_{m=1}^{3}\left[\alpha_3\dfrac{u_{j,k+m-2}^{s+1}}{\left(H_j^{s+1}\right)^2} + (1-\alpha_3)\dfrac{u_{j,k+m-2}^{s}}{\left(H_j^{s}\right)^2}\right] KVnm_{j,k,m}^s$

Baroclinic pressure gradient: $\dfrac{1}{A_{NE\,j}}\sum_{m=1}^{3}\left[\sum_{n=1}^{NE_j} A_n b_{xn}^s\right]_{k+m-2} Inm_{k,m}$

Lateral stress: $\dfrac{3}{A_{NE\,j}}\sum_{m=1}^{3}\left[E_{\ell\,j}^s\sum_{n=1}^{NE_j} A_n\left(\dfrac{\partial u^s}{\partial x_\sigma}\dfrac{\partial\phi_j}{\partial x} + \dfrac{\partial u^s}{\partial y_\sigma}\dfrac{\partial\phi_j}{\partial y}\right)_n\right]_{k+m-2} Inm_{k,m}$

Substituting these into Eqs. (3.15) and (3.16), multiplying by Δt and grouping velocities at time levels s+1 and s yields:

$$
\begin{aligned}
&\sum_{m=1}^{3} u_{j,k+m-2}^{s+1} Inm_{k,m} - \Delta t \sum_{m=1}^{3} f\alpha_1 v_{j,k+m-2}^{s+1} Inm_{k,m} + \left(\frac{(a-b)\Delta t \alpha_2 K_{slip\,j}^{s}}{H_j^{s+1}} \right) u_j^{s+1}\Big|_{k=1} \\
&\quad + \alpha_3 \Delta t \left(\frac{a-b}{H_j^{s+1}} \right)^2 \sum_{m=1}^{3} u_{j,k+m-2}^{s+1} KVnm_{j,k,m}^{s} = \sum_{m=1}^{3} u_{j,k+m-2}^{s} Inm_{k,m} + \Delta t \sum_{m=1}^{3} f(1-\alpha_1) v_{j,k+m-2}^{s} Inm_{k,m} \\
&\quad - \left(\frac{(a-b)\Delta t (1-\alpha_2) K_{slip\,j}^{s}}{H_j^{s}} \right) u_j^{s}\Big|_{k=1} - (1-\alpha_3)\Delta t \left(\frac{a-b}{H_j^{s}} \right)^2 \sum_{m=1}^{3} u_{j,k+m-2}^{s} KVnm_{j,k,m}^{s} \\
&\quad - \frac{\Delta t}{A_{NE\,j}} \sum_{m=1}^{3} \left[\sum_{n=1}^{NE_j} A_n \left[\bar{u}_n^{s} \left(\frac{\partial u^{s}}{\partial x_\sigma} \right)_n + \bar{v}_n^{s} \left(\frac{\partial u^{s}}{\partial y_\sigma} \right)_n \right] \right]_{k+m-2} Inm_{k,m} \\
&\quad - \Delta t \left(\frac{a-b}{H_j^{s}} \right) \left[\left(\frac{\partial u_j^{s}}{\partial \sigma} \right)_{k\text{-}1,k} \left(w_{\sigma\,j,k-1}^{s} + 2 w_{\sigma\,j,k}^{s} \right) Inm_{k,1} + \left(\frac{\partial u_j^{s}}{\partial \sigma} \right)_{k,k+1} \left(2 w_{\sigma\,j,k}^{s} + w_{\sigma\,j,k+1}^{s} \right) Inm_{k,3} \right] \\
&\quad + \frac{\Delta t (a-b)}{2} \left[\frac{\tau_{sx\,j}^{s+1}}{H_j^{s+1}\rho_o} + \frac{\tau_{sx\,j}^{s}}{H_j^{s}\rho_o} \right]_{k=NV} \\
&\quad - \frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} \frac{A_n}{2} \left(g \frac{\partial \left[\zeta + P_s / g\rho_o - \alpha\eta \right]^{s}}{\partial x} + g \frac{\partial \left[\zeta + P_s / g\rho_o - \alpha\eta \right]^{s+1}}{\partial x} \right)_n LVn_k \qquad (3.17) \\
&\quad - \frac{\Delta t}{A_{NE\,j}} \sum_{m=1}^{3} \left[\sum_{n=1}^{NE_j} A_n b_{x_n}^{s} \right]_{k+m-2} Inm_{k,m} - \frac{3\Delta t}{A_{NE\,j}} \sum_{m=1}^{3} \left[E_{\ell\,j}^{s} \sum_{n=1}^{NE_j} A_n \left(\frac{\partial u^{s}}{\partial x_\sigma} \frac{\partial \phi_j}{\partial x} + \frac{\partial u^{s}}{\partial y_\sigma} \frac{\partial \phi_j}{\partial y} \right)_n \right]_{k+m-2} Inm_{k,m}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{m=1}^{3} v_{j,k+m-2}^{s+1} Inm_{k,m} + \Delta t \sum_{m=1}^{3} f\,\alpha_1 u_{j,k+m-2}^{s+1} Inm_{k,m} + \left(\frac{(a-b)\,\Delta t\,\alpha_2 K_{slip\,j}^{s}}{H_j^{s+1}} \right) v_j^{s+1}\Big|_{k=1} \\
&\quad + \alpha_3\,\Delta t \left(\frac{a-b}{H_j^{s+1}} \right)^2 \sum_{m=1}^{3} v_{j,k+m-2}^{s+1} KVnm_{j,k,m}^{s} = \sum_{m=1}^{3} v_{j,k+m-2}^{s} Inm_{k,m} - \Delta t \sum_{m=1}^{3} f\,(1-\alpha_1)\, u_{j,k+m-2}^{s} Inm_{k,m} \\
&\quad - \left(\frac{(a-b)\,\Delta t\,(1-\alpha_2)\, K_{slip\,j}^{s}}{H_j^{s}} \right) v_j^{s}\Big|_{k=1} - (1-\alpha_3)\,\Delta t \left(\frac{a-b}{H_j^{s}} \right)^2 \sum_{m=1}^{3} v_{j,k+m-2}^{s} KVnm_{j,k,m}^{s} \\
&\quad - \frac{\Delta t}{A_{NE\,j}} \sum_{m=1}^{3} \left[\sum_{n=1}^{NE\,j} A_n \left[\bar{u}_n^{s} \left(\frac{\partial v^{s}}{\partial x_\sigma} \right)_n + \bar{v}_n^{s} \left(\frac{\partial v^{s}}{\partial y_\sigma} \right)_n \right] \right]_{k+m-2} Inm_{k,m} \\
&\quad - \Delta t \left(\frac{a-b}{H_j^{s}} \right) \left[\left(\frac{\partial v_j^{s}}{\partial \sigma} \right)_{k\text{-}1,k} \left(w_{\sigma\,j,k-1}^{s} + 2 w_{\sigma\,j,k}^{s} \right) Inm_{k,1} + \left(\frac{\partial v_j^{s}}{\partial \sigma} \right)_{k,k+1} \left(2 w_{\sigma\,j,k}^{s} + w_{\sigma\,j,k+1}^{s} \right) Inm_{k,3} \right] \\
&\quad + \frac{\Delta t\,(a-b)}{2} \left[\frac{\tau_{sy\,j}^{s+1}}{H_j^{s+1} \rho_o} + \frac{\tau_{sy\,j}^{s}}{H_j^{s} \rho_o} \right]_{k=NV} \\
&\quad - \frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE\,j} \frac{A_n}{2} \left(g \frac{\partial \left[\zeta + P_s/g\rho_o - \alpha\eta \right]^{s}}{\partial y} + g \frac{\partial \left[\zeta + P_s/g\rho_o - \alpha\eta \right]^{s+1}}{\partial y} \right)_n LVn_k \\
&\quad - \frac{\Delta t}{A_{NE\,j}} \sum_{m=1}^{3} \left[\sum_{n=1}^{NE\,j} A_n b_{y_n}^{s} \right]_{k+m-2} Inm_{k,m} - \frac{3\Delta t}{A_{NE\,j}} \sum_{m=1}^{3} \left[E_{\ell\,j}^{s} \sum_{n=1}^{NE\,j} A_n \left(\frac{\partial v^{s}}{\partial x_\sigma} \frac{\partial \phi_j}{\partial x} + \frac{\partial v^{s}}{\partial y_\sigma} \frac{\partial \phi_j}{\partial y} \right)_n \right]_{k+m-2} Inm_{k,m}
\end{aligned}
\tag{3.18}
$$

Prior to obtaining the 3D velocity solution in ADCIRC, a complex velocity, q, is defined as

$$q \equiv u + iv \quad where \quad i \equiv \sqrt{-1}$$

and Eqs. (3.17) and (3.18) are rewritten so that the x momentum equation is the real part and the y momentum equation is the imaginary part of a single complex equation:

$$\begin{aligned}
&\sum_{m=1}^{3} q_{j,k+m-2}^{s+1} Inm_{k,m} + i\Delta t\, f \alpha_1 \sum_{m=1}^{3} q_{j,k+m-2}^{s+1} Inm_{k,m} + \left(\frac{(a-b)\Delta t \alpha_2 K_{slip\,j}^{s}}{H_j^{s+1}} \right) q_j^{s+1} \Big|_{k=1} \\
&\quad + \alpha_3 \Delta t \left(\frac{a-b}{H_j^{s+1}} \right)^2 \sum_{m=1}^{3} q_{j,k+m-2}^{s+1} KVnm_{j,k,m}^{s} = \sum_{m=1}^{3} q_{j,k+m-2}^{s} Inm_{k,m} - i\Delta t\, f (1-\alpha_1) \sum_{m=1}^{3} q_{j,k+m-2}^{s} Inm_{k,m} \\
&\quad - \left(\frac{(a-b)\Delta t (1-\alpha_2) K_{slip\,j}^{s}}{H_j^{s}} \right) q_j^{s} \Big|_{k=1} - (1-\alpha_3)\Delta t \left(\frac{a-b}{H_j^{s}} \right)^2 \sum_{m=1}^{3} q_{j,k+m-2}^{s} KVnm_{j,k,m}^{s} \\
&\quad - \frac{\Delta t}{A_{NE\,j}} \sum_{m=1}^{3} \left[\sum_{n=1}^{NE\,j} A_n \left[\overline{u}_n^{s} \left(\frac{\partial q^{s}}{\partial x_\sigma} \right)_n + \overline{v}_n^{s} \left(\frac{\partial q^{s}}{\partial y_\sigma} \right)_n \right] \right]_{k+m-2} Inm_{k,m} \\
&\quad - \Delta t \left(\frac{a-b}{H_j^{s}} \right) \left[\left(\frac{\partial q_j^{s}}{\partial \sigma} \right)_{k-1,k} \left(w_{\sigma\,j,k-1}^{s} + 2 w_{\sigma\,j,k}^{s} \right) Inm_{k,1} + \left(\frac{\partial q_j^{s}}{\partial \sigma} \right)_{k,k+1} \left(2 w_{\sigma\,j,k}^{s} + w_{\sigma\,j,k+1}^{s} \right) Inm_{k,3} \right] \\
&\quad + \frac{\Delta t (a-b)}{2} \left[\frac{\tau_{sx\,j}^{s+1}}{H_j^{s+1}\rho_o} + \frac{\tau_{sx\,j}^{s}}{H_j^{s}\rho_o} + i \left(\frac{\tau_{sy\,j}^{s+1}}{H_j^{s+1}\rho_o} + \frac{\tau_{sy\,j}^{s}}{H_j^{s}\rho_o} \right) \right]_{k=NV} \\
&\quad - \frac{\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} \frac{A_n}{2} \left(g \frac{\partial [\zeta + P_s/g\rho_o - \alpha\eta]^{s}}{\partial x} + g \frac{\partial [\zeta + P_s/g\rho_o - \alpha\eta]^{s+1}}{\partial x} \right)_n LVn_k \\
&\quad - \frac{i\Delta t}{A_{NE\,j}} \sum_{n=1}^{NE_j} \frac{A_n}{2} \left(g \frac{\partial [\zeta + P_s/g\rho_o - \alpha\eta]^{s}}{\partial y} + g \frac{\partial [\zeta + P_s/g\rho_o - \alpha\eta]^{s+1}}{\partial y} \right)_n LVn_k \\
&\quad - \frac{\Delta t}{A_{NE\,j}} \sum_{m=1}^{3} \left[\sum_{n=1}^{NE_j} A_n \left(b_{x_n}^{s} + i b_{y_n}^{s} \right) \right]_{k+m-2} Inm_{k,m} \\
&\quad - \frac{3\Delta t}{A_{NE\,j}} \sum_{m=1}^{3} \left[E_{\ell\,j}^{s} \sum_{n=1}^{NE_j} A_n \left(\frac{\partial q^{s}}{\partial x_\sigma} \frac{\partial \phi_j}{\partial x} + \frac{\partial q^{s}}{\partial y_\sigma} \frac{\partial \phi_j}{\partial y} \right)_n \right]_{k+m-2} Inm_{k,m}
\end{aligned} \tag{3.19}$$

Re-arranging and consolidating terms yields the form of the 3D momentum equations solved in ADCIRC:

$$\begin{aligned}
&\left(1+i\Delta t\, f\,\alpha_1\right)\sum_{m=1}^{3} q_{j,k+m-2}^{s+1} Inm_{k,m} + \alpha_3\,\Delta t\left(\frac{a-b}{H_j^{s+1}}\right)^2 \sum_{m=1}^{3} q_{j,k+m-2}^{s+1} KVnm_{j,k,m}^{s} \\
&\quad + \left(\frac{(a-b)\,\Delta t\alpha_2\, K_{slip\,j}^{s}}{H_j^{s+1}}\right) q_j^{s+1}\Big|_{k=1} \\
&= \sum_{m=1}^{3}\left\{\left(1-i\Delta t\, f\left(1-\alpha_1\right)\right) q_{j,k+m-2}^{s} - \Delta t\left[ladvec + lstress + bcpg\right]_{j,k+m-2}\right\} Inm_{k,m} \qquad (3.20)\\
&\quad - \Delta t\, vadvec_{j,k} - \Delta t\, btpg_j\, LVn_k - \left(1-\alpha_3\right)\Delta t\, vstress_{j,k}^{s} - \left(\frac{(a-b)\,\Delta t\left(1-\alpha_2\right) K_{slip\,j}^{s}}{H_j^{s}}\right) q_j^{s}\Big|_{k=1} \\
&\quad + \frac{\Delta t\,(a-b)}{2}\left[\frac{\tau_{sx\,j}^{s+1}}{H_j^{s+1}\rho_o} + \frac{\tau_{sx\,j}^{s}}{H_j^{s}\rho_o} + i\left(\frac{\tau_{sy\,j}^{s+1}}{H_j^{s+1}\rho_o} + \frac{\tau_{sy\,j}^{s}}{H_j^{s}\rho_o}\right)\right]_{k=NV}
\end{aligned}$$

where,

$$ladvec_{j,k} \equiv \left[\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE\,j} A_n\left[\bar{u}_n^{s}\left(\frac{\partial q^{s}}{\partial x_\sigma}\right)_n + \bar{v}_n^{s}\left(\frac{\partial q^{s}}{\partial y_\sigma}\right)_n\right]\right]_k$$

$$vadvec_{j,k} \equiv \left(\frac{a-b}{H_j^{s}}\right)\left[\left(\frac{\partial q_j^{s}}{\partial \sigma}\right)_{k\text{-}1,k}\left(w_{\sigma\,j,k-1}^{s} + 2w_{\sigma\,j,k}^{s}\right) Inm_{k,1} + \left(\frac{\partial q_j^{s}}{\partial \sigma}\right)_{k,k+1}\left(2w_{\sigma\,j,k}^{s} + w_{\sigma\,j,k+1}^{s}\right) Inm_{k,3}\right]$$

$$bcpg_{j,k} \equiv \left[\frac{1}{A_{NE\,j}}\sum_{n=1}^{NE\,j} A_n\left(b_{x_n}^{s} + i b_{y_n}^{s}\right)\right]_k$$

$$btpg_j \equiv \frac{1}{A_{NE\,j}}\sum_{n=1}^{NE\,j}\frac{g\,A_n}{2}\left[\begin{aligned}&\left(\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s}}{\partial x} + \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s+1}}{\partial x}\right)\\ &\quad + i\left(\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s}}{\partial y} + \frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]^{s+1}}{\partial y}\right)\end{aligned}\right]_n$$

$$lstress_{j,k} \equiv \left[\frac{3E_{\ell j}^{s}}{A_{NE j}} \sum_{n=1}^{NE j} A_n \left(\frac{\partial q^s}{\partial x_\sigma} \frac{\partial \phi_j}{\partial x} + \frac{\partial q^s}{\partial y_\sigma} \frac{\partial \phi_j}{\partial y} \right)_n \right]_k$$

$$vstress_{j,k}^{s} \equiv \left(\frac{a-b}{H_j^s} \right)^2 \sum_{m=1}^{3} q_{j,k+m-2}^{s} KVnm_{j,k,m}^{s}$$

Eq. (3.20) has a matrix structure, although due to the specific formulation that was used to obtain this equation, the matrix is uncoupled in the horizontal direction and is tri-diagonal in the vertical direction. Thus, Eq. (3.20) is solved separately for each horizontal node *j*. Symbolically, Eq. (3.20) can be written as:

$$Mq = F_r$$

where,

M = *complex tridiagonal matrix*
q = *complex solution vector for velocity*
F_r = *complex forcing vector*

M consists of:

$$M(k,k-1) = \begin{cases} (1+i\Delta t\alpha_1 f)\, Inm_{k,1} + \alpha_3 \Delta t \left(\dfrac{a-b}{H_j^{s+1}} \right)^2 KVnm_{j,k,1}^{s} & \textit{for } k \neq 1 \\ 0 & \textit{for } k = 1 \end{cases}$$

$$M(k,k) = \begin{cases} (1+i\Delta t\alpha_1 f)\, Inm_{k,2} + \alpha_3 \Delta t \left(\dfrac{a-b}{H_j^{s+1}} \right)^2 KVnm_{j,k,2}^{s} & \textit{for } k \neq 1 \\ (1+i\Delta t\alpha_1 f)\, Inm_{k,2} + \alpha_3 \Delta t \left(\dfrac{a-b}{H_j^{s+1}} \right)^2 KVnm_{j,k,2}^{s} + \left(\dfrac{(a-b)\Delta t\alpha_2 K_{slip\, j}^{s}}{H_j^{s+1}} \right) & \textit{for } k = 1 \end{cases}$$

$$M(k,k+1) = \begin{cases} (1+i\Delta t\alpha_1 f)\, Inm_{k,3} + \alpha_3 \Delta t \left(\dfrac{a-b}{H_j^{s+1}} \right)^2 KVnm_{j,k,3}^{s} & \textit{for } k \neq NV \\ 0 & \textit{for } k = NV \end{cases}$$

and F_r consists of:

$$
F_r(k) = \begin{cases}
= \begin{cases}
\sum_{m=2}^{3} \left\{ \left(1 - i\Delta t\, f\left(1-\alpha_1\right)\right) q^{s}_{j,k+m-2} - \Delta t\left[ladvec + lstress + bcpg \right]_{j,k+m-2} \right\} Inm_{k,m} \\
-\Delta t\, vadvec_{j,k} - \Delta t\, btpg_{j}\, LVn_k - \left(1-\alpha_3\right) \Delta t\, vstress^{s}_{j,k} \\
-\left(\dfrac{\left(a-b\right)\Delta t\left(1-\alpha_2\right) K^{s}_{slip\, j}}{H^{s}_{j}} \right) q^{s}_{j,k}
\end{cases} & for\ k = 1 \\
\begin{cases}
\sum_{m=1}^{3} \left\{ \left(1 - i\Delta t\, f\left(1-\alpha_1\right)\right) q^{s}_{j,k+m-2} - \Delta t\left[ladvec + lstress + bcpg \right]_{j,k+m-2} \right\} Inm_{k,m} \\
-\Delta t\, vadvec_{j,k} - \Delta t\, btpg_{j}\, LVn_k - \left(1-\alpha_3\right) \Delta t\, vstress^{s}_{j,k}
\end{cases} & for\ k \neq 1, NV \\
= \begin{cases}
\sum_{m=1}^{2} \left\{ \left(1 - i\Delta t\, f\left(1-\alpha_1\right)\right) q^{s}_{j,k+m-2} - \Delta t\left[ladvec + lstress + bcpg \right]_{j,k+m-2} \right\} Inm_{k,m} \\
-\Delta t\, vadvec_{j,k} - \Delta t\, btpg_{j}\, LVn_k - \left(1-\alpha_3\right) \Delta t\, vstress^{s}_{j,k} \\
+ \dfrac{\Delta t\left(a-b\right)}{2} \left[\dfrac{\tau^{s+1}_{sx_j}}{H^{s+1}_{j}\rho_o} + \dfrac{\tau^{s}_{sx_j}}{H^{s}_{j}\rho_o} + i\left(\dfrac{\tau^{s+1}_{sy_j}}{H^{s+1}_{j}\rho_o} + \dfrac{\tau^{s}_{sy_j}}{H^{s}_{j}\rho_o} \right) \right]
\end{cases} & for\ k = NV
\end{cases}
$$

4.0 VERTICAL VELOCITY

The vertical component of velocity is obtained in ADCIRC by solving the 3D continuity equation

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right) \tag{4.1}$$

for w after u and v have been determined from the solution of the 3D momentum equations. In Eq. (4.1), the subscript "z" has been added to the horizontal derivatives to emphasize that these derivatives are evaluated along level coordinate surfaces. Eq. (4.1) is solved subject to the free-surface and bottom kinematic boundary conditions:

$$w_s = \frac{\partial \zeta}{\partial t} + u_s \frac{\partial \zeta}{\partial x_z} + v_s \frac{\partial \zeta}{\partial y_z} \qquad at \;\; z = \zeta \tag{4.2}$$

$$w_b = -u_b \frac{\partial h}{\partial x_z} - v_b \frac{\partial h}{\partial y_z} \qquad at \;\; z = \text{-}h \tag{4.3}$$

where u_s, v_s, w_s are the velocity components at the free surface (z=ζ) and u_b, v_b, w_b are the velocity components at the bottom (z=-h) assuming a slip condition is applied there.

Eq. (4.1) is discretized in horizontal space as:

$$\left\langle \frac{\partial w}{\partial z}, \phi_j \right\rangle_\Omega = -\left\langle \frac{\partial u}{\partial x_z}, \phi_j \right\rangle_\Omega - \left\langle \frac{\partial v}{\partial y_z}, \phi_j \right\rangle_\Omega \tag{4.4}$$

The horizontal integration utilizes **Rule 1** for the left side and **Rule 2** for the right side (these rules are described in APPENDIX - BASIC CALCULATIONS ON LINEAR TRIANGLES). After multiplication by $3/A_{NE\,j}$, Eq. (4.4) becomes:

$$\frac{\partial w_j}{\partial z} = -\frac{1}{A_{NE\,j}} \sum_{n=1}^{NE\,j} A_n \left[\left(\frac{\partial u}{\partial x_z} \right)_n + \left(\frac{\partial v}{\partial y_z} \right)_n \right] \tag{4.5}$$

Eq. (4.5) is discretized over the vertical using a simple finite-difference for the left side and centering the right side:

$$\frac{w_{j,k}-w_{j,k-1}}{z_k-z_{k-1}}=-\frac{1}{2A_{NE_j}}\sum_{n=1}^{NE_j}A_n\left\{\left[\left(\frac{\partial u}{\partial x_z}\right)_n+\left(\frac{\partial v}{\partial y_z}\right)_n\right]_k+\left[\left(\frac{\partial u}{\partial x_z}\right)_n+\left(\frac{\partial v}{\partial y_z}\right)_n\right]_{k-1}\right\} \tag{4.6}$$

Eq. (4.6) can be written in terms of the stretched coordinate system as:

$$\frac{w_{j,k}-w_{j,k-1}}{\sigma_k-\sigma_{k-1}}\left(\frac{a-b}{H_j}\right)=\frac{1}{2A_{NE_j}}\sum_{n=1}^{NE_j}A_n\left\{\begin{aligned}&-\left[\left(\frac{\partial u}{\partial x_\sigma}\right)_n+\left(\frac{\partial v}{\partial y_\sigma}\right)_n\right]_k\\&+\left[\left(\frac{\partial \zeta}{\partial x}\right)_n+\left(\frac{\sigma_k-a}{a-b}\right)\left(\frac{\partial H}{\partial x}\right)_n\right]\left(\frac{a-b}{H_j}\right)\left(\frac{\partial u_j}{\partial \sigma}\right)_k\\&+\left[\left(\frac{\partial \zeta}{\partial y}\right)_n+\left(\frac{\sigma_k-a}{a-b}\right)\left(\frac{\partial H}{\partial y}\right)_n\right]\left(\frac{a-b}{H_j}\right)\left(\frac{\partial v_j}{\partial \sigma}\right)_k\\&-\left[\left(\frac{\partial u}{\partial x_\sigma}\right)_n+\left(\frac{\partial v}{\partial y_\sigma}\right)_n\right]_{k-1}\\&+\left[\left(\frac{\partial \zeta}{\partial x}\right)_n+\left(\frac{\sigma_{k-1}-a}{a-b}\right)\left(\frac{\partial H}{\partial x}\right)_n\right]\left(\frac{a-b}{H_j}\right)\left(\frac{\partial u_j}{\partial \sigma}\right)_{k-1}\\&+\left[\left(\frac{\partial \zeta}{\partial y}\right)_n+\left(\frac{\sigma_{k-1}-a}{a-b}\right)\left(\frac{\partial H}{\partial y}\right)_n\right]\left(\frac{a-b}{H_j}\right)\left(\frac{\partial v_j}{\partial \sigma}\right)_{k-1}\end{aligned}\right\} \tag{4.7}$$

The subscript "σ" indicates that the horizontal derivatives in (4.7) are evaluated along stretched coordinate surfaces. Discretizing the vertical derivative of horizontal velocity as:

$$\left(\frac{\partial u_j}{\partial \sigma}\right)_k=\left(\frac{\partial u_j}{\partial \sigma}\right)_{k-1}=\left(\frac{u_{j,k}-u_{j,k-1}}{\sigma_k-\sigma_{k-1}}\right)$$

and re-arranging yields:

$$
w_{j,k} = w_{j,k-1} + \frac{1}{A_{NE_j}} \sum_{n=1}^{NE_j} A_n \left\{ \begin{array}{l} -\dfrac{(\sigma_k - \sigma_{k-1}) H_j}{2(a-b)} \left\{ \left[\left(\dfrac{\partial u}{\partial x_\sigma} \right)_n + \left(\dfrac{\partial v}{\partial y_\sigma} \right)_n \right]_k + \left[\left(\dfrac{\partial u}{\partial x_\sigma} \right)_n + \left(\dfrac{\partial v}{\partial y_\sigma} \right)_n \right]_{k-1} \right\} \\ + \left[\left(\dfrac{\partial \zeta}{\partial x} \right)_n + \left(\dfrac{\dfrac{\sigma_k + \sigma_{k-1}}{2} - a}{a-b} \right) \left(\dfrac{\partial H}{\partial x} \right)_n \right] (u_{j,k} - u_{j,k-1}) \\ + \left[\left(\dfrac{\partial \zeta}{\partial y} \right)_n + \left(\dfrac{\dfrac{\sigma_k + \sigma_{k-1}}{2} - a}{a-b} \right) \left(\dfrac{\partial H}{\partial y} \right)_n \right] (v_{j,k} - v_{j,k-1}) \end{array} \right\} \quad (4.8)
$$

Eq. (4.3) is used to determine $w_{j,1}$; Eq. (4.8) can then be solved recursively for k=2, 3, … up to the surface. (Notice that the vertical differences of horizontal velocity in Eq. (4.8) are evaluated at node j only.)

As discussed by Luettich et al. (2002) and Muccino et al. (1997), the result obtained for the vertical velocity at the free surface from Eq. (4.8), $w_{j,k=surface}$, may not match the free surface boundary condition, $w_{j,s}$, as specified in Eq. (4.2). This discrepancy is due to error in local fluid mass conservation, (Luettich et al., 2002). ADCIRC attempts to optimally correct the vertical velocity obtained from Eq. (4.8) using an adjoint approach. This results in a correction to Eq. (4.8) that is linear over the depth:

$$
w_{j,k}^{ajoint\ corrected} = w_{j,k} + \left(w_{j,s} - w_{j,k=surface} \right) \left[\frac{\dfrac{W_f}{H^2} + \dfrac{\sigma - b}{a-b}}{\dfrac{2W_f}{H^2} + 1} \right] \quad (4.9)
$$

In Eq. (4.9), W_f weights the relative importance of satisfying continuity in the interior of the fluid vs satisfying the free surface boundary condition in the adjoint equation. Setting W_f =0 forces the corrected vertical velocity to exactly satisfy the free surface and bottom boundary conditions. Setting W_f to be large (e.g., W_f ~100) adds a uniform correction to the vertical velocity solution that is equal to half the surface boundary error. ADCIRC uses a default value of W_f =0.

5.0 SPERHICAL COORDINATE FORMULATION

ADCIRC solves the spherical governing equations by transforming these equations into an equivalent set of equations in Cartesian coordinates using a standard cylindrical projection. Applying the hydrostatic and Bousinesque approximations and assuming the radius of the Earth is much greater than the thickness of the ocean, the 3D equations in spherical coordinates (λ, ϕ, z) are: (e.g., Haidvogel and Beckmann, 1999)

Continuity

$$\nabla \cdot \vec{u} = \frac{1}{R\cos\phi}\frac{\partial u}{\partial \lambda} + \frac{1}{R\cos\phi}\frac{\partial(v\cos\phi)}{\partial \phi} + \frac{\partial w}{\partial z} + \frac{2w}{R} = 0 \tag{5.1}$$

Horizontal Momentum

$$\frac{du}{dt} = fv - \frac{g}{R\cos\phi}\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial\lambda} + \frac{\partial}{\partial z}\left(\frac{\tau_{z\lambda}}{\rho_o}\right) - b_\lambda + m_\lambda + \frac{uv\tan\phi}{R} \tag{5.2}$$

$$\frac{dv}{dt} = -fu - \frac{g}{R}\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial\phi} + \frac{\partial}{\partial z}\left(\frac{\tau_{z\phi}}{\rho_o}\right) - b_\phi + m_\phi - \frac{u^2\tan\phi}{R} \tag{5.3}$$

where

$R = 6.3782064x10^6 m$, *mean radius of the Earth*

$\lambda, \phi =$ *longitude, latitude*

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{R\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{R}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}$$

$$\frac{\tau_{z\lambda}}{\rho_o}, \frac{\tau_{z\phi}}{\rho_o} = E_z\frac{\partial u}{\partial z},\ E_z\frac{\partial v}{\partial z}$$

$$m_\lambda, m_\phi \equiv \frac{1}{(R\cos\phi)^2}\frac{\partial}{\partial\lambda}\left(E_\ell\frac{\partial u}{\partial\lambda}\right) + \frac{1}{R^2}\frac{\partial}{\partial\phi}\left(E_\ell\frac{\partial u}{\partial\phi}\right),\quad \frac{1}{(R\cos\phi)^2}\frac{\partial}{\partial\lambda}\left(E_\ell\frac{\partial v}{\partial\lambda}\right) + \frac{1}{R^2}\frac{\partial}{\partial\phi}\left(E_\ell\frac{\partial v}{\partial\phi}\right)$$

$$b_\lambda, b_\phi \equiv \frac{g}{R\cos\phi}\frac{\partial}{\partial\lambda}\int_z^\zeta\frac{(\rho-\rho_o)}{\rho_o}dz,\ \frac{g}{R}\frac{\partial}{\partial\phi}\int_z^\zeta\frac{(\rho-\rho_o)}{\rho_o}dz$$

and other variables are as defined in Section 3.

Using a standard, orthogonal cylindrical projection centered at (λ_o, ϕ_o):

$$x = R\left(\lambda - \lambda_o\right)\cos\phi_o$$
$$y = R\,\phi \tag{5.4}$$

Derivatives are evaluated using the chain rule:

$$\frac{\partial}{\partial\lambda} = \frac{\partial}{\partial x}\frac{\partial x}{\partial\lambda} + \frac{\partial}{\partial y}\frac{\partial y}{\partial\lambda} = R\cos\phi_o\frac{\partial}{\partial x}$$
$$\frac{\partial}{\partial\phi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial\phi} = R\frac{\partial}{\partial y}$$

Substituting for the derivatives in the spherical coordinate system with those in the Cartesian system gives a transformed set of spherical equations:

Continuity

$$\nabla\cdot\vec{u} = S_p\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{v\tan\phi}{R} + \frac{2w}{R} = 0 \tag{5.5}$$

Horizontal Momentum

$$\frac{du}{dt} = fv - g\,S_p\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\tau_{zx}}{\rho_o}\right) - b_x + m_x + \frac{uv\tan\phi}{R} \tag{5.6}$$

$$\frac{dv}{dt} = -fu - g\frac{\partial\left[\zeta + P_s/g\rho_o - \alpha\eta\right]}{\partial y} + \frac{\partial}{\partial z}\left(\frac{\tau_{zy}}{\rho_o}\right) - b_y + m_y - \frac{u^2\tan\phi}{R} \tag{5.7}$$

where

$$S_p \equiv \frac{\cos\phi_o}{\cos\phi} = \textit{spherical coordinate correction factor} \tag{5.8}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\, S_p \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

$$\frac{\tau_{zx}}{\rho_o}, \frac{\tau_{zy}}{\rho_o} = E_z \frac{\partial u}{\partial z},\ E_z \frac{\partial v}{\partial z}$$

$$m_x, m_y \equiv S_p^{\ 2} \frac{\partial}{\partial x}\left(E_\ell \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(E_\ell \frac{\partial u}{\partial y} \right),\quad S_p^{\ 2} \frac{\partial}{\partial x}\left(E_\ell \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y}\left(E_\ell \frac{\partial v}{\partial y} \right)$$

$$b_x, b_y \equiv g\, S_p \frac{\partial}{\partial x} \int_z^\zeta \frac{(\rho - \rho_o)}{\rho_o} dz,\ g \frac{\partial}{\partial y} \int_z^\zeta \frac{(\rho - \rho_o)}{\rho_o} dz$$

A simple scaling analysis suggests that it may be permissible to drop the final two terms in Eq. (5.5) and the final terms in Eqs. (5.6) and (5.7), provided we avoid the regions near the poles where tan ϕ approaches infinity. This limitation is consistent with the singularity at the poles inherent in the original equations, Eqs. (5.1) - (5.3), and in the cylindrical mapping. Thus it does not appear to impose significant additional restrictions on the applicability of the governing equations.

Dropping these terms simplifies the transformed spherical governing equations to:

Continuity

$$\nabla \cdot \vec{u} = S_p \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.9}$$

Horizontal Momentum

$$\frac{\partial u}{\partial t} + u\, S_p \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fv = -g\, S_p \frac{\partial \left[\zeta + P_s / g \rho_o - \alpha \eta \right]}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\tau_{zx}}{\rho_o} \right) - b_x + m_x \tag{5.10}$$

$$\frac{\partial v}{\partial t} + u\, S_p \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu = -g \frac{\partial \left[\zeta + P_s / g \rho_o - \alpha \eta \right]}{\partial y} + \frac{\partial}{\partial z}\left(\frac{\tau_{zy}}{\rho_o} \right) - b_y + m_y \tag{5.11}$$

These equations are identical to their Cartesian counterparts with the exception that spatial derivatives with respect to the *x* coordinate direction are multiplied by the spherical coordinate correction factor, S_p, defined in Eq. (5.8). Consequently, Eqs. (5.9) - (5.11) comprise a generalized set of equations that allows ADCIRC to function using either a Cartesian horizontal grid (by setting $S_p = 1$) or a longitude, latitude horizontal grid (by converting the longitude and latitude values into equivalent linear coordinate values, Eq. (5.4), and evaluating S_p, Eq. (5.8)).

The velocity in Eqs. (5.9) - (5.11) is aligned with the original coordinate reference frame (e.g., for the spherical coordinates *(u, v, w)* are aligned with (λ, ϕ, z)) and therefore it is not necessary to transform the velocities to a different coordinate system.

Vertical integration of Eqs. (5.9) - (5.11) is identical to integration of the Cartesian equations. Thus, the two-dimensional, vertically-integrated equations are identical to their Cartesian counterparts, except that spatial derivatives with respect to the *x* coordinate direction are multiplied by S_p.

6.0 LATERAL BOUNDARY CONDITIONS

Elevation specified boundary condition –ADCIRC 2DDI and 3D

An elevation specified boundary condition is implemented by zeroing out all off diagonal terms in the row corresponding to each elevation boundary node in the GWCE, Eq. **(1.17)**, and setting the on diagonal term in that row equal to the root mean square value of all of the other diagonal terms in the GWCE matrix (to maintain matrix conditioning). The right side vector entry corresponding to each elevation specified boundary node is set equal to the specified elevation multiplied by the root mean square value mentioned previously. Symmetry is maintained in the left side matrix by zeroing out the off diagonal terms in the column corresponding to each elevation boundary node. To allow this, each off diagonal term in the column corresponding to an elevation boundary node is multiplied by the elevation boundary value and then subtracted from the right side vector of the corresponding equation.

Specified flux boundary condition –ADCIRC 2DDI

ADCIRC allows the specification of boundary conditions consisting of normal flux per unit width (e.g., zero flux across land boundary segments and nonzero flux across river boundary segments). These normal fluxes can be applied as either natural or essential boundary conditions and the user may specify whether the tangential velocity along these boundaries is set to zero or computed assuming free slip along the boundary. The specified normal flux per unit width is inserted into the boundary integral term that appears in the right side of the GWCE, Eq. **(1.17)**, at each normal flux boundary node. (The convention used in ADCIRC for inputting normal flux per unit width is that flux into the domain is positive and flux out of the domain is negative. Therefore, the sign must be changed on the normal flux prior to using it in the GWCE since the derivation of this equation assumes that a positive flux is in the direction of the outward pointing normal.) If the normal flux is applied as a natural boundary condition, no modifications are made to the momentum equations. If the normal flux is applied as an essential boundary condition, the depth-average normal velocity, U_N, is forced to be equal to the normal flux per unit width divided by the local depth and multipled by –1 (to maintain the convention that U_N is positive in the direction of the outward pointing normal). Further details of the implementation of the essential normal flux boundary condition in ADCIRC are presented below.

At any node in the horizontal, the momentum equations solved in ADCIRC 2DDI have the structure:

$$\begin{bmatrix} AUV_1 & -AUV_2 \\ AUV_2 & AUV_1 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{6.1}$$

where AUV_1, AUV_2 are the matrix entries computed from the finite element assembly process, and F_x, F_y comprise the right side forcing vector. At flux specified boundary nodes, the equations are rotated into a normal - tangential coordinate system. The normal and tangential velocities, U_N and U_T, are defined as the dot product of the velocity vector and the normal and tangential unit vectors, $\hat{N} = (N_x, N_y)$ and $\hat{T} = (T_x, T_y)$:

$$
\begin{aligned}
&UT_x + VT_y = U_T \\
&UN_x + VN_y = U_N
\end{aligned}
\tag{6.2}
$$

($\hat{N}$ and $\hat{T}$ are defined in the APPENDIX.) At specified normal flux boundary nodes the *y*-momentum equation in (6.1) is replaced by the expression for the normal velocity in (6.2) and the *x*-momentum equation is replaced by the tangential momentum equation formed by multiplying the *x*-momentum equation (6.1) by T_x and adding the *y*-momentum equation (6.1) multiplied by T_y. Since $T_x = N_y$ and $T_y = -N_x$ (see APPENDIX), the resulting system is:

$$
\begin{bmatrix} AUV_1 N_y - AUV_2 N_x & -AUV_2 N_y - AUV_1 N_x \\ N_x & N_y \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} F_x N_y - F_y N_x \\ U_N \end{bmatrix}
\tag{6.3}
$$

The left side matrix in (6.3) does not have the symmetry of the original equations, (6.1). This can be recovered by adding the tangential momentum equation to the normal equation multiplied by AUV_2 and dividing the result by AUV_1:

$$
\begin{bmatrix} N_y & -N_x \\ N_x & N_y \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \dfrac{F_x N_y - F_y N_x + AUV_2 U_N}{AUV_1} \\ U_N \end{bmatrix}
\tag{6.4}
$$

For the case that the tangential flux is also specified (e.g., equal to zero), the right side of the first equation in (6.4) is replaced by U_T.

Zero normal velocity gradient boundary condition – ADCIRC 2DDI (version 42.05)

A zero normal velocity gradient boundary condition is implemented by replacing the momentum equations at specified boundary nodes with equations that enforce the no-normal velocity gradient condition. The computed velocity field is then used to determine a normal flux across the boundary and this normal flux is used in the boundary flux integral in the GWCE at the next time step.

The no normal velocity gradient is enforced in ADCIRC using two different approaches. The first approach (boundary condition type 40) defines a fictitious node inside the domain for each boundary node. Each fictitious node is located on the inward pointing normal to the boundary a distance away from the corresponding boundary node equal to the distance of the furthest neighbor from that boundary node (see Figure 6.1). At each time step the velocity is computed at each node in the domain other than the zero normal velocity gradient boundary nodes. Next, the velocity is interpolated in space to each fictitious node. Finally, the velocity at each zero normal gradient boundary node is set equal to the velocity at the corresponding fictitious node. The distance to the fictitious node (d in Figure 6.1) is selected so that the fictitious node lies outside of the layer of elements immediately adjacent to the boundary. This way the velocity at the fictitious node can be interpolated without knowing the velocity at any adjacent zero normal velocity gradient nodes.

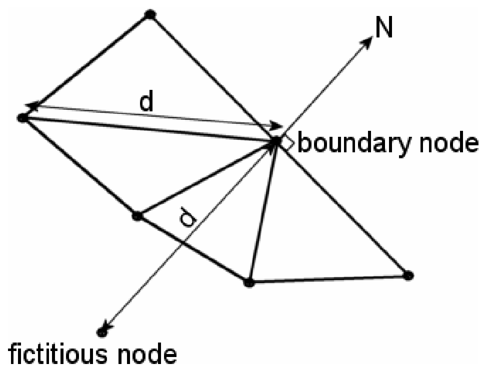


Figure 6.1 Schematic of the configuration used to determine the velocity at a zero normal velocity gradient boundary node. The velocity at the boundary node is set equal to the interpolated velocity at a fictitious node that is a distance d away from the boundary node where d is the distance to the furthest neighbor from that boundary node.

The second approach (boundary condition type 41) for enforcing the no normal velocity gradient in ADCIRC is by imposing the conditions

$$\frac{\partial U}{\partial N} = 0$$
$$\frac{\partial V}{\partial N} = 0 \tag{6.5}$$

at the boundary nodes. This can be expressed in terms of U, V, and $\hat{N}$ as:

$$\frac{\partial U}{\partial N} = N_x \frac{\partial U}{\partial x} + N_y \frac{\partial U}{\partial y} = 0$$
$$\frac{\partial V}{\partial N} = N_x \frac{\partial V}{\partial x} + N_y \frac{\partial V}{\partial y} = 0 \qquad (6.6)$$

Applying a Galerkin weighted residual formulation with linear basis functions to Eq. (6.6) yields:

$$\left\langle N_x \frac{\partial U}{\partial x}, \phi_j \right\rangle + \left\langle N_y \frac{\partial U}{\partial y}, \phi_j \right\rangle$$
$$= \sum_{n=1}^{NE_j} \left[\int_{\Omega_n} N_x \frac{\partial U}{\partial x} \phi_j \, d\Omega + \int_{\Omega_n} N_y \frac{\partial U}{\partial y} \phi_j \, d\Omega \right] = \sum_{n=1}^{NE_j} \left[\left(N_x \frac{\partial U}{\partial x} + N_y \frac{\partial U}{\partial y} \right) \int_{\Omega_n} \phi_j \, d\Omega \right]$$
$$= \sum_{n=1}^{NE_j} \frac{A_n}{3} \left[N_x \sum_{i=1}^{3} U_i \frac{\partial \phi_i}{\partial x} + N_y \sum_{i=1}^{3} U_i \frac{\partial \phi_i}{\partial y} \right] = \sum_{n=1}^{NE_j} \frac{1}{6} \left[N_x \sum_{i=1}^{3} U_i b_i + N_y \sum_{i=1}^{3} U_i a_i \right] = 0$$
$$\left\langle N_x \frac{\partial V}{\partial x}, \phi_j \right\rangle + \left\langle N_y \frac{\partial V}{\partial y}, \phi_j \right\rangle \qquad (6.7)$$
$$= \sum_{n=1}^{NE_j} \left[\int_{\Omega_n} N_x \frac{\partial V}{\partial x} \phi_j \, d\Omega + \int_{\Omega_n} N_y \frac{\partial V}{\partial y} \phi_j \, d\Omega \right] = \sum_{n=1}^{NE_j} \left[\left(N_x \frac{\partial V}{\partial x} + N_y \frac{\partial V}{\partial y} \right) \int_{\Omega_n} \phi_j \, d\Omega \right]$$
$$= \sum_{n=1}^{NE_j} \frac{A_n}{3} \left[N_x \sum_{i=1}^{3} V_i \frac{\partial \phi_i}{\partial x} + N_y \sum_{i=1}^{3} V_i \frac{\partial \phi_i}{\partial y} \right] = \sum_{n=1}^{NE_j} \frac{1}{6} \left[N_x \sum_{i=1}^{3} V_i b_i + N_y \sum_{i=1}^{3} V_i a_i \right] = 0$$

Multiplying Eq. (6.7) by the constant 6 and rearranging, gives the final, spatially discretized version of Eq. (6.6)used in ADCIRC:

$$\sum_{n=1}^{NE_j} \left[\sum_{i=1}^{3} \left(N_x b_i + N_y a_i \right) U_i \right] = 0$$
$$\sum_{n=1}^{NE_j} \left[\sum_{i=1}^{3} \left(N_x b_i + N_y a_i \right) V_i \right] = 0 \qquad (6.8)$$

If Eq. (6.8) is solved at only time level *s*+1, it requires the construction and solution of matrix problems for the U and V velocity components. To avoid this Eq. (6.8) is split between time levels *s*+1 and *s*. Assuming that each element attached to a boundary node is numbered so that node 1 corresponds to the boundary node and nodes 2 and 3 correspond to the remaining nodes in the element, Eq. (6.8) can be written as:

$$U_1^{s+1} = -\frac{\sum_{n=1}^{NE_j}\left[\sum_{i=2}^{3}\left(N_x b_i + N_y a_i\right)U_i^s\right]}{\sum_{n=1}^{NE_j}\left[N_x b_1 + N_y a_1\right]}$$

$$V_1^{s+1} = -\frac{\sum_{n=1}^{NE_j}\left[\sum_{i=2}^{3}\left(N_x b_i + N_y a_i\right)V_i^s\right]}{\sum_{n=1}^{NE_j}\left[N_x b_1 + N_y a_1\right]} \quad (6.9)$$

Eq. (6.9) allows the velocity at each boundary node to be determined independently of the velocity at the adjacent boundary nodes, although it introduces a potentially undesirable time lag into the solution.

Radiation boundary condition on velocity – ADCIRC 2DDI

A radiation boundary condition on velocity (boundary condition type 30) is implemented by specifying a relationship between the normal velocity and the elevation field along the boundary. The most common of this type of boundary condition is a Sommerfield radiation condition. Normal velocities computed at a radiation boundary and the corresponding normal fluxes are then inserted into the boundary integral term that appears in the right side of the GWCE, Eq. (1.17).

Specified flux boundary condition –ADCIRC 3D

ADCIRC allows the specification of boundary conditions consisting of normal flux per unit width (e.g., zero flux across land boundary segments and nonzero flux across river boundary segments). These normal fluxes can either be applied as natural or essential boundary conditions and the user may specify whether the tangential velocity along these boundaries is set to zero or computed assuming free slip along the boundary. The specified normal flux per unit width is inserted into the boundary integral term that appears in the right side of the GWCE, Eq. **(1.17)**, at each normal flux boundary node. (The convention used in ADCIRC for inputting normal flux per unit width is that flux into the domain is positive and flux out of the domain is negative. Therefore, the sign must be changed on the normal flux prior to using it in the GWCE since the derivation of this equation assumes that a positive flux is in the direction of the outward pointing normal.) If the normal flux is applied as a natural boundary condition, no modifications are made to the momentum equations. In this case the momentum equations will try to generate an appropriate vertical distribution of velocity over the depth, although vertical integration of this velocity may not exactly match the specified normal boundary flux. If the normal flux is applied as an essential boundary condition, the depth-average normal velocity, U_N, is forced to be equal to the normal flux per unit width divided by the local depth and multipled by –1 (to maintain the convention that U_N is positive in the direction of the outward pointing normal). In this case

ADCIRC assumes the normal velocity is distributed uniformly over the depth. This is probably not a good assumption if the normal velocity is nonzero! If a free slip tangential boundary condition is used, ADCIRC will attempt to compute a tangential velocity that is consistent with the specified normal velocity. Implementation of the essential normal flux boundary condition in ADCIRC is described below.

At any node in the horizontal, the momentum equations solved in the 3D version of ADCIRC have the structure:

$$Mq = F_r \tag{6.10}$$

where, M is a complex tridiagonal matrix, q (= u + iv) is the complex solution vector for velocity, F_r is the complex forcing vector and recall that the real and imaginary parts of (6.10) correspond to the x and y momentum equations, respectively. Row k in matrix M consists of:

$$\begin{aligned} M(k,k-1) &= Auv_{1,k-1} + i\,Auv_{2,k-1} \\ M(k,k) &= Auv_{1,k} + i\,Auv_{2,k} \\ M(k,k+1) &= Auv_{1,k+1} + i\,Auv_{2,k+1} \end{aligned} \tag{6.11}$$

where:

$$Auv_{1,k-1} = \begin{cases} Inm_{k,1} + \alpha_3 \Delta t \left(\dfrac{a-b}{H_j^{s+1}} \right)^2 KVnm_{j,k,1}^{s} & \textit{for } k \neq 1 \\ 0 & \textit{for } k = 1 \end{cases}$$

$$Auv_{2,k-1} = \begin{cases} \Delta t \alpha_1 f\, Inm_{k,1} & \textit{for } k \neq 1 \\ 0 & \textit{for } k = 1 \end{cases}$$

$$Auv_{1,k} = \begin{cases} Inm_{k,2} + \alpha_3 \Delta t \left(\dfrac{a-b}{H_j^{s+1}} \right)^2 KVnm_{j,k,2}^{s} & \textit{for } k \neq 1 \\ Inm_{k,2} + \alpha_3 \Delta t \left(\dfrac{a-b}{H_j^{s+1}} \right)^2 KVnm_{j,k,2}^{s} + \left(\dfrac{(a-b)\Delta t \alpha_2 K_{slip\,j}^{s}}{H_j^{s+1}} \right) & \textit{for } k = 1 \end{cases}$$

$$Auv_{2,k} = \Delta t \alpha_1 f\, Inm_{k,2} \qquad \textit{for all } k$$

$$Auv_{1,k+1} = \begin{cases} Inm_{k,3} + \alpha_3 \Delta t \left(\dfrac{a-b}{H_j^{s+1}} \right)^2 KVnm_{j,k,3}^{s} & \text{for } k \neq NV \\ 0 & \text{for } k = NV \end{cases}$$

$$Auv_{2,k+1} = \begin{cases} \Delta t \alpha_1 f \, Inm_{k,3} & \text{for } k \neq NV \\ 0 & \text{for } k = NV \end{cases}$$

At boundary nodes where normal flux is specified, the y-momentum equation is replaced by the equation for the normal velocity:

$$u_k N_x + v_k N_y = U_N \tag{6.12}$$

Because the vertical distribution of normal velocity is uniform, this applies locally at each node in the vertical. The x-momentum equation is replaced by the tangential momentum equation formed by multiplying the original x-momentum equation by T_x and adding the original y-momentum equation multiplied by T_y. Since $T_x = N_y$ and $T_y = -N_x$ (see APPENDIX), the resulting system is:

$$\begin{aligned} &\left(Auv_{1,k-1} N_y - Auv_{2,k-1} N_x \right) u_{k-1} - \left(Auv_{2,k-1} N_y + Auv_{1,k-1} N_x \right) v_{k-1} \\ &+ \left(Auv_{1,k} N_y - Auv_{2,k} N_x \right) u_k - \left(Auv_{2,k} N_y + Auv_{1,k} N_x \right) v_k \\ &+ \left(Auv_{1,k+1} N_y - Auv_{2,k+1} N_x \right) u_{k+1} - \left(Auv_{2,k+1} N_y + Auv_{1,k+1} N_x \right) v_{k+1} \\ &= \mathrm{Re}\{F_r\} N_y - \mathrm{Im}\{F_r\} N_x \end{aligned} \tag{6.13}$$

The left sides of (6.12) and (6.13) do not have the symmetry of the original momentum equations. This can be recovered by multiplying (6.12) by Auv_2 at levels k-1, k, and k+1 and adding these to (6.13):

$$\begin{aligned} &Auv_{1,k-1} N_y u_{k-1} - Auv_{1,k-1} N_x v_{k-1} + Auv_{1,k} N_y u_k - Auv_{1,k} N_x v_k \\ &+ Auv_{1,k+1} N_y u_{k+1} - Auv_{1,k+1} N_x v_{k+1} \\ &= \mathrm{Re}\{F_r\} N_y - \mathrm{Im}\{F_r\} N_x + \left(Auv_{2,k-1} + Auv_{2,k} + Auv_{2,k+1} \right) U_N \end{aligned} \tag{6.14}$$

Multiplying (6.12) by Auv_1 at levels k-1, k, and k+1 and adding these together gives:

$$\begin{aligned} &Auv_{1,k-1} N_x u_{k-1} + Auv_{1,k-1} N_y v_{k-1} + Auv_{1,k} N_x u_k + Auv_{1,k} N_y v_{k-1} \\ &+ Auv_{1,k+1} N_x u_{k+1} + Auv_{1,k+1} N_y v_{k+1} = \left(Auv_{1,k-1} + Auv_{1,k} + Auv_{1,k+1} \right) U_N \end{aligned} \tag{6.15}$$

Equations (6.14) and (6.15) can now be written in the form of (6.10):

$$M^* q = F_r^* \tag{6.16}$$

where

$$M^*\left(k,k-1\right) = Auv^*_{1,k-1} + i\, Auv^*_{2,k-1}$$
$$M^*\left(k,k\right) \quad = Auv^*_{1,k} \; + i\, Auv^*_{2,k}$$
$$M^*\left(k,k+1\right) = Auv^*_{1,k+1} + i\, Auv^*_{2,k+1}$$

$$Auv^*_{1,k-1} = Auv_{1,k-1} N_y$$
$$Auv^*_{2,k-1} = Auv_{1,k-1} N_x$$
$$Auv^*_{1,k} \quad = Auv_{1,k} N_y$$
$$Auv^*_{2,k} \quad = Auv_{1,k} N_x$$
$$Auv^*_{1,k+1} = Auv_{1,k+1} N_y$$
$$Auv^*_{2,k+1} = Auv_{1,k+1} N_x$$

$$F_r^* = \left[\mathrm{Re}\left\{F_r\right\} N_y - \mathrm{Im}\left\{F_r\right\} N_x + \left(Auv_{2,k-1} + Auv_{2,k} + Auv_{2,k+1}\right) U_N\right]$$
$$+ i\left[\left(Auv_{1,k-1} + Auv_{1,k} + Auv_{1,k+1}\right) U_N\right]$$

For the case that the tangential flux is also specified (e.g., equal to zero), the *x*-momentum equation is replaced by.

$$u_k N_y - v_k N_x = U_T \tag{6.17}$$

and the *y*-momentum equation is replaced by (6.12). This also generates a system of equations of the form of (6.16) where:

$$M^*\left(k,k-1\right) = 0$$
$$M^*\left(k,k\right) \quad = Auv^*_{1,k} \; + i\, Auv^*_{2,k}$$
$$M^*\left(k,k+1\right) = 0$$

$$Auv^*_{1,k} = N_y$$
$$Auv^*_{2,k} = N_x$$

$F_r^* = U_T + iU_N$

Zero normal elevation gradient boundary condition – ADCIRC 2DDI and 3D

A zero normal elevation gradient boundary condition could be implemented by replacing the GWCE equation corresponding to each zero normal elevation gradient boundary node with the equation:

$$\frac{\partial \zeta}{\partial n} \equiv N_x \frac{\partial \zeta}{\partial x} + N_y \frac{\partial \zeta}{\partial y} = 0 \tag{6.18}$$

where the normal unit vector, $\hat{N} = (N_x, N_y)$ is defined in the APPENDIX. Applying the Galerkin spatial discretization to Eq. (6.18) gives:

$$\begin{aligned}
\left\langle \frac{\partial \zeta}{\partial n}, \phi_j \right\rangle &\equiv \left\langle N_x \frac{\partial \zeta}{\partial x}, \phi_j \right\rangle + \left\langle N_y \frac{\partial \zeta}{\partial y}, \phi_j \right\rangle \\
&= \sum_{n=1}^{NE_j} \left[\int_{\Omega_n} N_x \frac{\partial \zeta}{\partial x} \phi_j \, d\Omega + \int_{\Omega_n} N_y \frac{\partial \zeta}{\partial y} \phi_j \, d\Omega \right] \\
&= \sum_{n=1}^{NE_j} \left[N_x \left(\frac{\partial \zeta}{\partial x} \right)_n \int_{\Omega_n} \phi_j \, d\Omega + N_y \left(\frac{\partial \zeta}{\partial y} \right)_n \int_{\Omega_n} \phi_j \, d\Omega \right] \\
&= \sum_{n=1}^{NE_j} \frac{A_n}{3} \left[N_x \left(\frac{\partial \zeta}{\partial x} \right)_n + N_y \left(\frac{\partial \zeta}{\partial y} \right)_n \right] \\
&= \sum_{n=1}^{NE_j} \frac{1}{6} \left[N_x \sum_{i=1}^{3} \zeta_i b_i + N_y \sum_{i=1}^{3} \zeta_i a_i \right] = 0
\end{aligned} \tag{6.19}$$

Eq. (6.19) can be evaluated implicitly in time using Eq. **(1.18)**. Multiplying through by the constant 6, yields a final set of discrete equations for the zero normal elevation gradient boundary condition.

$$\sum_{n=1}^{NE_j} \left[N_x \sum_{i=1}^{3} \zeta_i^{*s+1} b_i + N_y \sum_{i=1}^{3} \zeta_i^{*s+1} a_i \right] = 0 \tag{6.20}$$

One problem with applying this boundary condition is that it renders the GWCE left side matrix nonsymmetric. In contrast to the case of an elevation specified boundary condition where a straightforward manipulation restores matrix symmetry, there is no exact method for restoring the symmetry of this system. Consequently, this boundary condition has not been implemented in ADCIRC at the present.

Radiation boundary condition on elevation – ADCIRC 2DDI/3D

A radiation boundary condition on elevation could be implemented by specifying a relationship between the normal flux and the elevation field along the boundary. The most common of this type of boundary condition is a Sommerfield radiation condition. A difficulty with applying any type of boundary condition imposed on the GWCE, is that it renders the left side matrix nonsymmetric and therefore is not supported in the present version of ADCIRC.

7.0 BAROCLINIC PRESSURE GRADIENT CALCULATION NOTES

As presented previously, the baroclinic pressure gradient is defined by

$$b_x \equiv g\frac{\partial}{\partial x}\int_z^{\zeta}\frac{(\rho-\rho_o)}{\rho_o}dz, \qquad b_y \equiv g\frac{\partial}{\partial y}\int_z^{\zeta}\frac{(\rho-\rho_o)}{\rho_o}dz$$

These terms are evaluated in ADCIRC in two steps. In the initial step, the 3D baroclinic pressure field is computed as:

$$BPress(z) \equiv \int_z^{\zeta}\frac{g(\rho-\rho_o)}{\rho_o}dz = \frac{gH}{\rho_o(a-b)}\int_{\sigma}^{a}(\rho-\rho_o)d\sigma = \frac{gH}{\rho_o(a-b)}\int_{\sigma}^{a}(\sigma_T-\sigma_{To})d\sigma$$

where, density has been replace by the standard oceanographic "sigma t" variable

$$\sigma_T \equiv \rho - 1000, \qquad \sigma_{To} \equiv \rho_o - 1000$$

This should not be confused with the variable, σ, representing the dimensionless vertical coordinate system. In the second step, the horizontal baroclinic pressure gradients are computed as horizontal derivatives (in level or z coordinates) of the baroclinic pressure field.

$$b_x \equiv \frac{\partial}{\partial x}BPress, \qquad b_y \equiv \frac{\partial}{\partial y}BPress$$

For any horizontal node *j* and vertical node *k*, ADCIRC computes these gradients at the vertical position of node *k*. This is accomplished for each element containing node *j* by vertically interpolating the baroclinic pressure field at the element vertices to the vertical position of node *k* and then computing the horizontal gradients directly.

8.0 APPENDIX - BASIC CALCULATIONS ON LINEAR TRIANGLES

Consider the triangular finite element with vertices numbered 1-3, counter-clockwise around the element. Any variable, Υ, can be expanded linearly within the element based on nodal values as:

$$\Upsilon = \Upsilon_1\phi_1 + \Upsilon_2\phi_2 + \Upsilon_3\phi_3 = \sum_{i=1}^{3} \Upsilon_i\phi_i$$

where,

$$\Upsilon_1, \Upsilon_2, \Upsilon_3 = \textit{nodal values of } \Upsilon \textit{ at elemental verticies 1, 2, 3}$$

$$\phi_1, \phi_2, \phi_3 = \textit{linear basis functions defined as:}$$

$$\phi_1 = \frac{x_2 y_3 - x_3 y_2 + b_1 x + a_1 y}{2A}; \ \phi_2 = \frac{x_3 y_1 - x_1 y_3 + b_2 x + a_2 y}{2A}; \ \phi_3 = \frac{x_1 y_2 - x_2 y_1 + b_3 x + a_3 y}{2A}$$

$$a_1 = x_3 - x_2; \ a_2 = x_1 - x_3; \ a_3 = x_2 - x_1$$

$$b_1 = y_2 - y_3; \ b_2 = y_3 - y_1; \ b_3 = y_1 - y_2$$

$$A = \frac{b_1 a_2 - b_2 a_1}{2} = \textit{elemental area}$$

Spatial derivatives are computed as:

$$\frac{\partial \Upsilon}{\partial x} = \sum_{i=1}^{3} \Upsilon_i \frac{\partial \phi_i}{\partial x}; \qquad \frac{\partial \Upsilon}{\partial y} = \sum_{i=1}^{3} \Upsilon_i \frac{\partial \phi_i}{\partial y}$$

where,

$$\frac{\partial \phi_i}{\partial x} = \frac{b_i}{2A}; \qquad \frac{\partial \phi_i}{\partial y} = \frac{a_i}{2A}$$

Spatial integrations are computed using:

$$\int_A \phi_i^{e_i} \phi_j^{e_j} dA = 2A \frac{(e_i)!(e_j)!}{(e_i + e_j + 2)!}$$

If $i \neq j$ and $e_i = e_j = 1$, $\int_A \phi_i \phi_j dA = \frac{A}{12}$. If $i = j$ and $e_i = e_j = 1$, $\int_A \phi_j \phi_j dA = \int_A \phi_j^2 dA = \frac{A}{6}$.

Because linear basis and expansion functions are used, their derivatives are constant across an element and spatial integrations involving derivatives become:

$$\int_A \phi_i \frac{\partial \phi_j}{\partial x, \partial y} dA = \frac{\partial \phi_j}{\partial x, \partial y} \int_A \phi_i dA = \frac{A}{3} \frac{\partial \phi_j}{\partial x, \partial y}$$

ADCIRC uses both exact and lower order integrations (i.e., lumping). Horizontal spatial integrations used in the GWCE are presented in full in SECTION 1.0. Horizontal spatial integrations used in either the 2DDI (SECTION 2.0) or 3D (SECTION 3.0) momentum equations are summarized by the following integration rules:

Rule 1: (nodal lumping, applied to terms that do not contain spatial gradients)

$$\left\langle \Upsilon, \phi_j \right\rangle_\Omega \equiv \sum_{n=1}^{NE_j} \int_{\Omega_n} \Upsilon \, \phi_j \, d\Omega = \Upsilon_j \sum_{n=1}^{NE_j} \int_{\Omega_n} \phi_j \, d\Omega = \frac{A_{NE\,j}}{3} \Upsilon_j$$

Rule 2: (fully consistent, applied only to spatial gradient terms)

$$\left\langle \frac{\partial \Upsilon}{\partial x, \partial y}, \phi_j \right\rangle_\Omega \equiv \sum_{n=1}^{NE_j} \int_{\Omega_n} \frac{\partial \Upsilon}{\partial x, \partial y} \phi_j \, d\Omega = \sum_{n=1}^{NE_j} \left(\frac{\partial \Upsilon}{\partial x, \partial y} \right)_n \int_{\Omega_n} \phi_j \, d\Omega = \sum_{n=1}^{NE_j} \frac{A_n}{3} \left(\frac{\partial \Upsilon}{\partial x, \partial y} \right)_n$$

Rule 2a: (approximation to Rule 2, used in older versions of ADCIRC)

$$\left\langle \frac{\partial \Upsilon}{\partial x, \partial y}, \phi_j \right\rangle_\Omega = \sum_{n=1}^{NE_j} \frac{A_n}{3} \left(\frac{\partial \Upsilon}{\partial x, \partial y} \right)_n \approx \frac{A_{NE_j}}{3NE_j} \sum_{n=1}^{NE_j} \left(\frac{\partial \Upsilon}{\partial x, \partial y} \right)_n$$

where,

$A_n =$ *area of element n*

$A_{NE\,j} \equiv \sum_{n=1}^{NE_j} A_n =$ *area of all elements containing node j*

$NE_j =$ *number of elements containing node j*

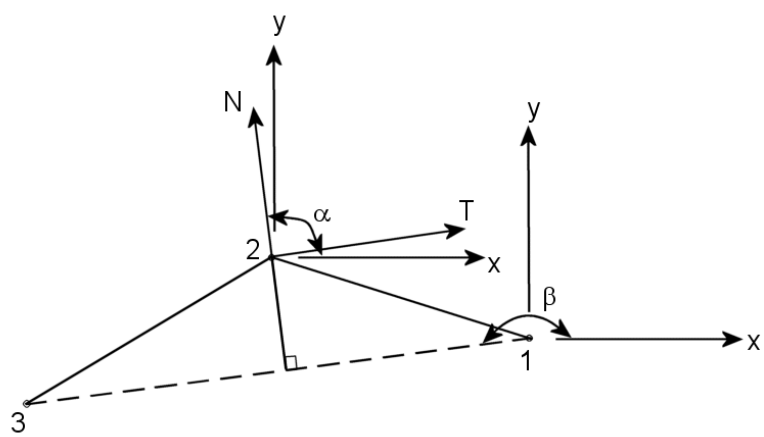
$\phi_j =$ *horizontal weighting function, =1 at node j, =0 at all other nodes,*
varies linearly between adjacent nodes

Note, that the definition of the weighting function ϕ_j reduces integration over the horizontal domain Ω to integration over only the NE_j elements containing node *j*. Also, **Rule 2** assumes a

Galerkin finite element formulation in which the quantity being differentiated (Υ in the integration rules described above) varies linearly within an element. Therefore, the spatial derivative is constant within an element and can be pulled out of the elemental integrations. Finally, **Rule 2a** is equal to **Rule 2** for uniformly sized elements. It was implemented in early versions of ADCIRC-2DDI and is included in this document for posterity sake. It was removed from the code as of version XX.XX.

The component of a vector quantity, $\vec{\Upsilon}$, $\left(\vec{\Upsilon} \equiv \Upsilon_x, \Upsilon_y\right)$, in any direction can be computed as the dot product of the vector quantity and the unit vector in the specified direction. In ADCIRC this is done at boundary nodes, where the horizontal velocity field may be rotated into components that are normal and tangential to each boundary node or where the elevation gradient normal to the boundary may be specified.

If a node is on the interior of a boundary (i.e., it is not the end node where two *different types* of boundaries meet), unique normal and tangential directions are defined as shown in the figure below.



Definition figure for normal and tangential directions at boundary node 2, provided that this node does not mark the end of one boundary type and the beginning of another. In this situation the normal direction is defined to be perpendicular to the line connecting nodes 1 and 3. The ADCIRC grid file requires boundary nodes to be specified with the domain interior on the left as one progresses along the boundary.

The normal and tangential components $\left(\Upsilon_N, \Upsilon_T\right)$ of vector $\vec{\Upsilon}$ are:

$$\Upsilon_N = \vec{\Upsilon} \cdot \hat{N} \equiv \Upsilon_x N_x + \Upsilon_y N_y$$

$$\Upsilon_T = \vec{\Upsilon} \cdot \hat{T} \equiv \Upsilon_x T_x + \Upsilon_y T_y$$

and the normal and tangential spatial derivatives of a scalar function, Υ, are:

$$\frac{\partial \Upsilon}{\partial N} = \frac{\partial \Upsilon}{\partial x} N_x + \frac{\partial \Upsilon}{\partial y} N_y$$
$$\frac{\partial \Upsilon}{\partial T} = \frac{\partial \Upsilon}{\partial x} T_x + \frac{\partial \Upsilon}{\partial y} T_y$$

The unit vectors, $\hat{N}$ and $\hat{T}$, are:

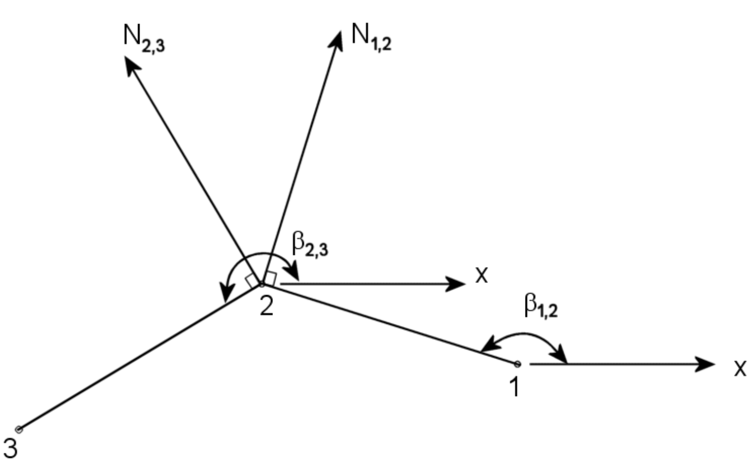
$$N_x = \cos\alpha \qquad N_y = \sin\alpha$$
$$T_x = \cos(\alpha - 90) \qquad T_y = \sin(\alpha - 90)$$

Since $\beta = \alpha + 90$, the unit vectors can be written more conveniently as:

$$N_x = \cos(\beta - 90) = \sin\beta = \frac{y_3 - y_1}{L_{31}} \qquad N_y = \sin(\beta - 90) = -\cos\beta = \frac{x_1 - x_3}{L_{31}}$$
$$T_x = \cos(\beta - 180) = -\cos\beta = N_y \qquad T_y = \sin(\beta - 180) = -\sin\beta = -N_x$$

where (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the horizontal coordinates of nodes 1, 2 and 3 and $L_{31} \equiv \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$ is the horizontal distance between nodes 1 and 3.

If a node is located where two *different types* of boundaries meet, two normal and tangential directions are defined for the node, one for each boundary, as shown in the figure below.



Definition figure for normal and tangential directions at boundary node 2, when this node marks the end of one boundary type and the beginning of another. In this situation two normal and two tangential directions are defined for node 2, one for computations pertaining to the boundary type to the right of node 2 (i.e., for

In this case

$$\Upsilon_{N\,1,2} = \vec{\Upsilon} \cdot \hat{N}_{1,2} \equiv \Upsilon_x N_{1,2_x} + \Upsilon_y N_{1,2_y}$$
$$\Upsilon_{T\,1,2} = \vec{\Upsilon} \cdot \hat{T}_{1,2} \equiv \Upsilon_x T_{1,2_x} + \Upsilon_y T_{1,2_y}$$
$$\Upsilon_{N\,2,3} = \vec{\Upsilon} \cdot \hat{N}_{2,3} \equiv \Upsilon_x N_{2,3_x} + \Upsilon_y N_{2,3_y}$$
$$\Upsilon_{T\,2,3} = \vec{\Upsilon} \cdot \hat{T}_{2,3} \equiv \Upsilon_x T_{2,3_x} + \Upsilon_y T_{2,3_y}$$

and spatial derivatives are handled in an analogous way.

The unit vectors are:

$$N_{1,2_x} = \sin\beta_{1,2} = \frac{y_2 - y_1}{L_{21}} \qquad N_{1,2_y} = -\cos\beta_{1,2} = \frac{x_1 - x_2}{L_{21}}$$
$$T_{1,2_x} = N_{1,2_y} \qquad T_{1,2_y} = -N_{1,2_x}$$
$$N_{2,3_x} = \sin\beta_{2,3} = \frac{y_3 - y_2}{L_{32}} \qquad N_{2,3_y} = -\cos\beta_{2,3} = \frac{x_2 - x_3}{L_{32}}$$
$$T_{2,3_x} = N_{2,3_y} \qquad T_{2,3_y} = -N_{2,3_x}$$

where $L_{21} \equiv \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and $L_{32} \equiv \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$ are the distances between nodes 1 and 2 and nodes 2 and 3, respectively.

9.0 REFERENCES

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