A Method of Objective Analysis for Currents in a Lake with Application to Lake Ontario[1]

DESIRAJU B. RAO[2] AND DAVID J. SCHWAB

Great Lakes Environmental Research Laboratory, Ann Arbor, MI 48104

(Manuscript received 17 September 1980, in final form 12 January 1981)

ABSTRACT

The mean circulation in large lakes is nearly nondivergent in character. This paper takes advantage of this fact to represent the flow field in terms of the transport streamfunction. The horizontal velocity vector (v) and the vertical component of vorticity are then given by $\mathbf{v} = \mathbf{k} \times H^{-1}\nabla\psi$ and $\zeta = \nabla \cdot H^{-1}\nabla\psi$, where ψ is the transport streamfunction, ∇ the horizontal gradient, and $H = H(x, y)$ the equilibrium depth of the lake. If the vorticity field $\zeta(x, y)$ is known, ψ can be determined from the above inhomogeneous equation with $H^{-1}\psi = 0$ on the boundary. The current vector is then obtained from the other equation. In practice, however, currents are measured and not vorticity. Therefore, the proposed objective analysis procedure expands the transport streamfunction in terms of the eigenvectors of the self-adjoint problem $\nabla \cdot H^{-1}\nabla\psi_\alpha = \mu_\alpha\psi_\alpha$ with $H^{-1}\psi_\alpha = 0$ on the boundary. The eigenvalues μ_α and eigenvectors ψ_α are characteristic of the particular lake and are determined numerically by a Lanczos procedure. The expansion coefficients are determined by minimizing the squared error between the calculated v field and available current meter data. Since the ψ_α functions for the entire domain of the basin are known, the currents can be reconstructed at any point. This method has been applied to data gathered in Lake Ontario during the winter months of 1972–73 as part of the International Field Year for the Great Lakes (IFYGL).

1. Introduction

The development and use of methods to analyze data objectively is a fundamental practice in geophysical sciences. Objective analysis in the present context is a systematic procedure that makes it possible to extrapolate data gathered at irregular intervals (in space or time) onto regularly spaced grid points. Objective analysis techniques eliminate the personal biases involved in subjective analysis, help to minimize the effect of instrument errors and remedy sampling limitations.

In meteorology, various methods of objective analysis have been used. The reader may refer to Gandin (1965) for a comprehensive review. A common method is to use weighting factors that are inversely dependent on the distance of a grid point from a set of observations within a region of influence (Cressman, 1959). Other methods of objective analysis for irregularly spaced data use empirical orthogonal functions derived from covariance analysis of data sets (Lorenz, 1956) or orthogonal polynomials generated by the Gram-Schmidt procedure (see e.g., Jalickee *et al.*, 1974). Even though some of the above techniques are amenable to extensions in three- and even four-dimensional analyses, they are restricted to scalar fields.

Objective analysis of vector wind fields has been developed in meteorology for tropical circulation patterns (Hawkins and Rosenthal, 1965), for initialization of numerical models (Brown and Neilon, 1961) and to take account of mesoscale orographic effects in the wind field (Liu and Goodin, 1976). Schwab (1977) developed a simple scheme for analyzing current meter data in Lake Ontario that was based on the Hawkins and Rosenthal (1965) method for winds. The iterative method he developed, however, is not completely objective in that the results depend on the choice of certain parameters for the initial guess at the interpolated current field and the region of influence of each data point.

The method we propose uses orthogonal functions that are characteristic for a given basin and permits a completely objective analysis of vector current fields. These orthogonal functions correspond to streamfunction modes determined by the topography of the basin. The set of streamfunction modes are orthogonal over regularly spaced grid points. In using them for analysis of irregularly spaced data points, one can generate a set of Gram-Schmidt type orthogonal functions from the streamfunction modes so that orthogonality at irregular points is satisfied. We have, however, used an alternative procedure in which the data are represented as a series in the streamfunction modes and the coefficients of expansion are directly determined to satisfy least-square-error requirements through matrix inversion techniques. The advantage of this procedure is that new

[1] GLERL Contribution No. 215.

[2] Present affiliation: Laboratory for Atmospheric Sciences, Goddard Space Flight Center, Greenbelt, MD 20771.

sets of Gram-Schmidt functions need not be generated each time there is a change in the spatial distribution of the observation points.

In the following sections we present a discussion of the basis for the objective analysis technique, comparison of the results from spectral and finite difference methods for the steady-state circulation patterns in Lake Ontario, application of the objective analysis technique to data taken from theoretical simulations and finally the application to real data to examine monthly mean circulation patterns in Lake Ontario during the fall to winter period. During this period the lake is nearly homogeneous and the currents are uniform throughout the depth of the lake. Even though such a condition is not a prerequisite for the application of the proposed objective analysis technique, as will be pointed out later, it offers a convenient simplification in this initial attempt to evaluate the results.

2. Method of analysis

In common with the properties of large-scale flows in the atmosphere and oceans, the large-scale horizontal currents in a lake are also nearly non-divergent. Hence, the rotational component of the currents is dominant over the divergent component. If we represent the transport streamfunction by ψ, then the rotational part of the current is given by

$$\mathbf{v} = H^{-1}\mathbf{k} \times \nabla\psi, \tag{2.1}$$

and the vertical component of vorticity is given by

$$\zeta = \mathbf{k}\cdot\nabla \times \mathbf{v} = \nabla\cdot H^{-1}\nabla\psi, \tag{2.2}$$

where ∇ is the horizontal gradient operator, $H = H(x, y)$ is the equilibrium depth of the lake and $\mathbf{k}$ the vertical unit vector. If the vorticity field $\zeta(x, y)$ is given, Eq. (2.2) represents an inhomogeneous elliptic equation. In the context of closed water bodies such as the Great Lakes, the requirement for a vanishing normal component of the transport on the boundaries yields the condition that

$$H^{-1}\psi = 0 \text{ on the boundary.}$$

Hence, if the ζ field is available from observations at a certain number of points, one can objectively analyze the scalar ζ field inside the domain of the lake with standard techniques and then solve for the transport streamfunction ψ from Eqs. (2.2) and (2.3). The vector current field is then recovered from Eq. (2.1) by use of the computed streamfunction field $\psi(x, y)$. This is the essence of the procedure followed by Schwab (1977) in his objective analysis of Lake Ontario's November mean circulation. Vorticity, however, is not measured directly. It has to be obtained by differentiating the current meter data, which is an undesirable feature.

The objective analysis procedure described below is a more rigorous one and takes advantage of specially constructed orthogonal functions that are specific to a given basin. These orthogonal functions are characteristic solutions associated with the elliptic operator in Eq. (2.2); i.e., they represent the solutions of the eigenvalue problem

$$\left.\begin{aligned} \nabla\cdot H^{-1}\nabla\psi_\alpha &= -\mu_\alpha\psi_\alpha \\ H^{-1}\psi_\alpha &= 0 \text{ on the boundary} \end{aligned}\right\}. \tag{2.3}$$

Eq. (2.3) represents a self-adjoint problem. Consequently, the characteristic values μ_α are real. The characteristic functions ψ_α form an orthogonal and hence a complete set. The scalar index α is used to order the functions ψ_α in some manner. In view of the completeness of the functions ψ_α, they can be used as a basis to represent any arbitrary streamfunction field ψ,

$$\psi = \sum_\alpha a_\alpha\psi_\alpha. \tag{2.4}$$

If the orthonormality condition for ψ_α functions is chosen as $\mu_\alpha \int \psi_\alpha\psi_\beta dA = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker delta, the expansion coefficients a_α are given by

$$a_\alpha = \mu_\alpha \int \psi\psi_\alpha dA, \tag{2.5}$$

where the integration is performed over the area of the basin. When the sum on the right-side of Eq. (2.4) spans the complete spectrum of the elliptic operator in Eq. (2.3), Eq. (2.4) represents a least-square approximation to ψ with the usual restrictions as to quadratic integrability and continuity of ψ and its derivatives.

The streamfunction ψ in Eq. (2.4) determines the transport field in the lake under consideration. The convergence of the series depends on the expansion coefficients a_α. Since the Great Lakes are essentially barotropic from late fall through winter, baroclinic effects are negligible and the forcing function generating the transports is mainly the atmospheric wind stress. The resulting circulation is governed by the nature of the wind stress and topographic properties of the basin. In the spectral representation given in Eq. (2.4), the basin properties are accounted for by the spectral functions ψ_α. The wind stress properties then determine the expansion coefficients a_α. The kinetic energy associated with the currents is given by

$$KE = \tfrac{1}{2}\int H\mathbf{v}\cdot\mathbf{v}dA = \tfrac{1}{2}\sum_\alpha a_\alpha^2.$$

Hence, the distribution of energy among the modes is determined by the wind stress characteristics. The energetically dominant spectral functions change with different winds. Consequently, the choice of spectral functions to be included in the representa-

tion of Eq. (2.4) is an important aspect determining the rate of convergence of the series—a question that will be considered later.

In this investigation, we have applied the objective analysis technique to current meter data obtained during the IFYGL (International Field Year for the Great Lakes—a program conducted during the International Hydrological Decade) experiment of 1972–73. In particular, we examined the mean circulation patterns in Lake Ontario on a monthly basis during November 1972–March 1973. Generally, the lakes exhibit mean circulation patterns on a seasonal cycle due to the variability in the associated atmospheric forcing. Within each season, there are high frequency variabilities in the atmospheric forcing caused by the storm cycles (typically with 5–7 day periods). The transient effects produced by storms in the circulation patterns are complicated. Even though these effects are important in themselves, the mean circulation resulting from the impact of several storms over a given period of time is also of practical importance. Pickett (1977) has chosen one month as a reasonable averaging period to define a mean circulation that is unaffected by transient effects. His analysis revealed that there appear to be significant changes in the mean circulation patterns between different months, which must be related to the corresponding differences in the pattern of atmospheric forcing. In studying these mean circulations, Pickett used a combination of subjective analysis and results from a numerical steady-state circulation model. We have reanalyzed the data using the present objective analysis procedure.

As mentioned earlier, the choice of the appropriate spectral functions is important in ensuring a rapid convergence of the series (2.4). Since there is no *a priori* basis on which such a selection can be made, it is necessary to simulate steady-state circulation patterns for typical wind fields by solving the dynamical equations in the spectral domain. Then the most energetic spectral functions can be tabulated as a function of wind direction for future use.

a. Steady-state spectral model

The dynamical equations governing the steady-state circulation of a homogeneous lake are

$$\left.\begin{aligned} f\mathbf{k}\times\mathbf{M} &= -gH\nabla\eta - \lambda H^{-1}\mathbf{M} + \rho^{-1}\boldsymbol{\tau} \\ \nabla\cdot\mathbf{M} &= 0 \end{aligned}\right\}. \quad (2.6)$$

In the above equations, $\mathbf{M}$ is the transport vector

$$\mathbf{M} = \int_{-H}^{0} \mathbf{v}\,dz.$$

The free surface fluctuation is represented by η and $\boldsymbol{\tau}$ is the wind stress vector. In Eq. (2.6), the bottom friction is assumed to depend linearly on the transport $\mathbf{M}$ and inversely on the depth H. The constant coefficient λ is equal to $c_D\bar{v}$, where c_D is the drag coefficient and $\bar{v}$ is some constant mean current value. The constant of gravitational acceleration is represented by g and f is the Coriolis parameter.

In view of the continuity equation, the transport field can be defined in terms of a streamfunction

$$\mathbf{M} = \mathbf{k}\times\nabla\psi.$$

The vertical component of vorticity is given by

$$\zeta = \mathbf{k}\cdot\nabla\times H^{-1}\mathbf{M} = \nabla\cdot H^{-1}\nabla\psi.$$

The dynamical equation governing the vorticity field obtained from the first equation of (2.6) may be shown to be

$$\nabla\cdot H^{-1}\nabla\psi - H^{-2}\nabla H\cdot\nabla\psi + f(\lambda H)^{-1}J(H,\psi) = \lambda^{-1}(\mathbf{k}\cdot\nabla\times\rho^{-1}\boldsymbol{\tau} - H^{-1}\rho^{-1}\boldsymbol{\tau}\cdot\mathbf{k}\times\nabla H). \quad (2.7)$$

In Eq. (2.7), J represents the Jacobian operator. This equation shows that circulation is governed by two distinct forcing terms. One is due to the curl of the wind stress and is usually negligible for water bodies of the size of the Great Lakes. The second term is due to the interaction between wind stress and bottom slope. It is the latter term that is dominant in determining the circulations in the Great Lakes (see Pickett and Rao, 1977).

Eq. (2.7) has been solved by finite-difference relaxation methods, for example, by Rao and Murty (1970), to examine the circulation patterns in Lake Ontario. We now attempt to solve the vorticity equation by using a spectral expansion procedure. Such methods have been commonly used in meteorology, for example, by Baer and Platzman (1961) and Kasahara (1977). Substitute the expansion (2.4) into Eq. (2.7) and use the orthogonality condition of the ψ_α functions. The result is

$$-a_\alpha + \sum_\beta b_{\alpha\beta}a_\beta = p_\alpha, \quad (2.8)$$

where the inhomogeneous term P_α is given by

$$p_\alpha = \lambda^{-1}\int(\mathbf{k}\cdot\nabla\times\rho^{-1}\boldsymbol{\tau} - H^{-1}\rho^{-1}\boldsymbol{\tau}\cdot\mathbf{k}\times\nabla H)\psi_\alpha dA.$$

The coefficients $b_{\alpha\beta}$ are coupling coefficients given by

$$b_{\alpha\beta} = \int H^{-1}\psi_\alpha[\lambda^{-1}fJ(H,\psi_\beta) - H^{-1}\nabla H\cdot\nabla\psi_\beta]dA. \quad (2.10)$$

If the wind stress $\boldsymbol{\tau}$ and the depth H are constants, the coefficients p_α and $b_{\alpha\beta}$ are zero. Hence all values for a_α are zero, indicating that there is no vertically integrated circulation in a basin of uniform depth under the influence of uniform wind stress. In this

case, the wind stress is simply balanced by a free surface slope. In the case of the Great Lakes, τ may be taken to be uniform, as mentioned earlier, and the circulation features are governed primarily by the topographic characteristics of the basin. Eq. (2.8) leads to a matrix equation

$$(\mathsf{I} + \mathsf{B})\mathbf{a} = \mathbf{p}, \tag{2.11}$$

where I is the identity matrix. The elements of matrix B are

$$\mathsf{B} = |b_{\alpha\beta}|$$

and $\mathbf{a}$ and $\mathbf{p}$ are column vectors with elements

$$\mathbf{a} = \text{col}(a_\alpha), \quad \mathbf{p} = \text{col}(p_\alpha).$$

The matrix equation (2.11) can be solved by standard matrix inversion methods. The coupling elements $b_{\alpha\beta}$ are calculated from finite-difference approximations to Eq. (2.10).

b. Calculation of the spectral functions

The characteristic value problem (2.3) can be solved directly by representing the continuous differential operator in a discrete finite-difference form on a grid covering any lake. The depth can be prescribed from a bathymetric chart of the lake. The size of the resulting matrix is proportional to the number of grid squares. Hence, if a small grid size is chosen in order to resolve the topographic effects accurately, the matrix size will be very large. Even though the matrix is banded in structure, it lacks any symmetric properties and consequently must be stored in the computer to solve for the characteristic values and the functions. Hence, an alternative procedure is used for determining the solutions of Eq. (2.3). This procedure is called the Lanczos procedure. [See Platzman (1975) for a complete discussion.] The procedure consists of first expanding ψ_α in terms of a set of functions W_i that satisfy the same boundary conditions as ψ_α,

$$\psi_\alpha = \sum_{j=1}^{\infty} q_j^\alpha W_j. \tag{2.12}$$

The W_i functions represent an orthonormal set over the area of the basin, i.e.,

$$\int W_i W_j dA = \delta_{ij}.$$

The specification of W_i functions will be considered later. Substitute the expansion (2.12) into Eq. (2.3), multiply by W_i and integrate over the area of the basin. The result, on using the orthonormality condition, is

$$\sum_j r_{ij} q_j^\alpha = \mu_\alpha q_i^\alpha, \quad i = 1, 2, \ldots, \tag{2.13}$$

where the coefficients r_{ij} are given by

$$r_{ij} = \int W_i \nabla \cdot H^{-1} \nabla W_j dA.$$

Eq. (2.13) can be expressed in a matrix form as

$$\mathsf{R}\mathbf{q}^\alpha = \mu_\alpha \mathbf{q}^\alpha, \tag{2.14}$$

which represents a standard matrix eigenvalue problem. In Eq. (2.14), the elements of R are $|r_{ij}|$ and the elements of $\mathbf{q}^\alpha$ are $|q_i^\alpha|$.

In the Lanczos procedure, the W_i functions are chosen so as to make the matrix R a symmetric tridiagonal matrix. Special numerical procedures that involve storing only the main diagonal and the first off-diagonal elements in the computer can then be used to compute the eigenvalues and vectors. In order to make R a symmetric tridiagonal matrix, let

$$\left.\begin{aligned} \gamma_j &= \int W_j \nabla \cdot H^{-1} \nabla W_j dA \\ \hat{W}_{j+1} &= (\nabla \cdot H^{-1} \nabla - \gamma_j) W_j - \theta_j W_{j-1} \\ \theta_{j+1} &= \left(\int \hat{W}_{j+1} \hat{W}_{j+1} dA \right)^{1/2} \\ W_{j+1} &= \hat{W}_{j+1} / \theta_{j+1} \end{aligned}\right\}. \tag{2.15}$$

Given a normalized W_1, Eqs. (2.15) represent a recursion relation for W_j, γ_j and θ_j. The function W_1 is chosen so as to satisfy the necessary boundary condition on ψ_α. In addition, W_0 is assumed to be zero in the above equations. We now have [from Eqs. (2.15)]

$$\nabla \cdot H^{-1} \nabla W_j = \gamma_j W_j + \theta_{j+1} W_{j+1} + \theta_j W_{j-1}. \tag{2.16}$$

Multiplication by W_i and integration over the area of the basin yields

$$\int W_i \nabla \cdot H^{-1} \nabla W_j dA = \gamma_j \int W_i W_j dA + \theta_{j+1} \int W_i W_{j+1} dA + \theta_j \int W_i W_{j-1} dA.$$

Hence, the diagonal elements of R are given by

$$r_{jj} = \int W_j \nabla \cdot H^{-1} \nabla W_j dA = \gamma_j.$$

The first off-diagonal element is

$$r_{j,j-1} = r_{j-1,j} = \int W_{j-1} \nabla \cdot H^{-1} \nabla W_j dA = \theta_j.$$

Thus, the matrix is symmetric and tridiagonal since all other r_{ij}'s are zero.

The Lanczos procedure to obtain the solutions for problem (2.3) then consists of (1) selecting a trial function W_1, (2) recursively applying the relations in

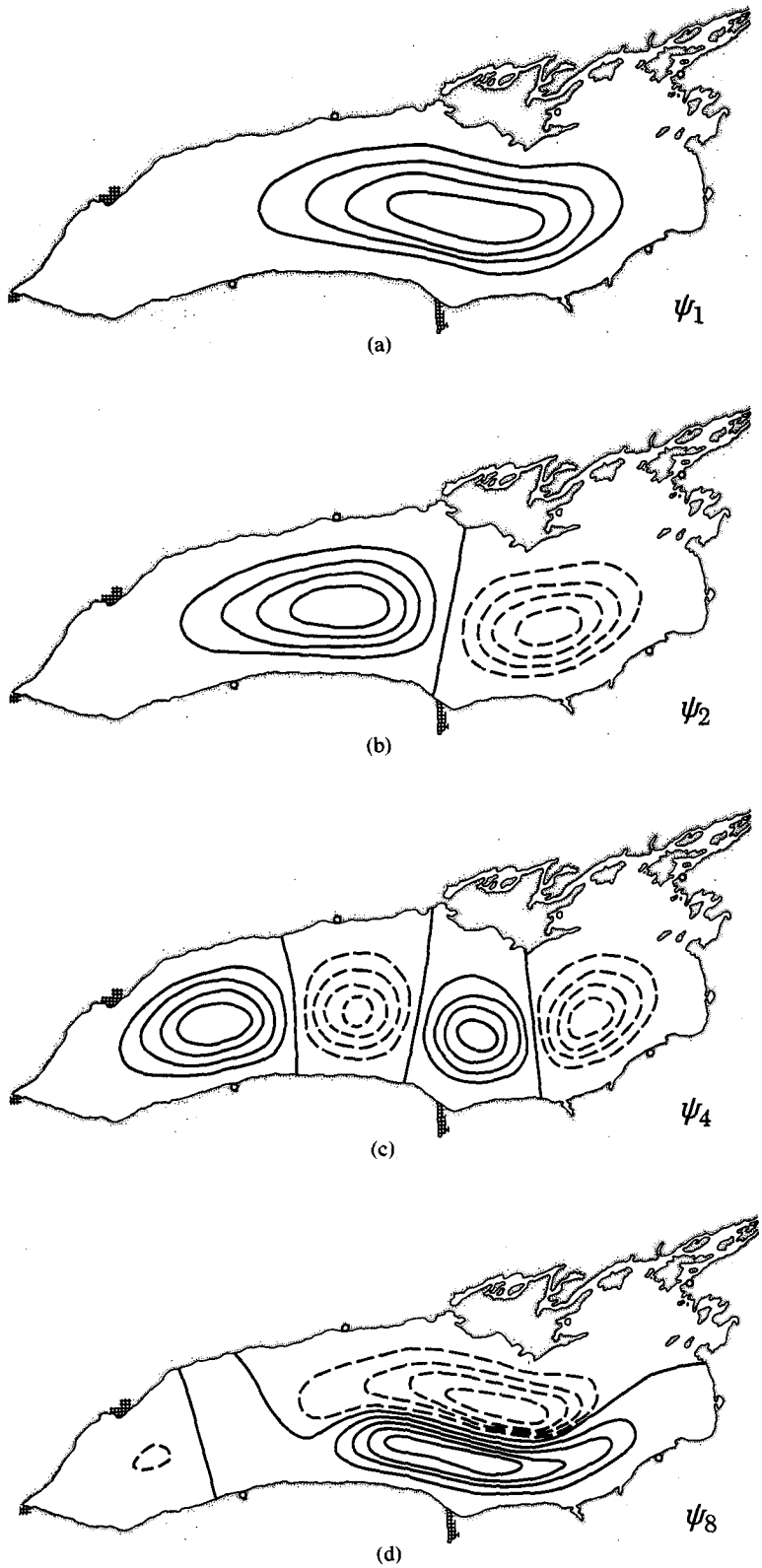

FIG. 1. Structures of four of the basic streamfunction modes used in the objective analysis. Dashed lines indicate negative values and solid lines positive values.

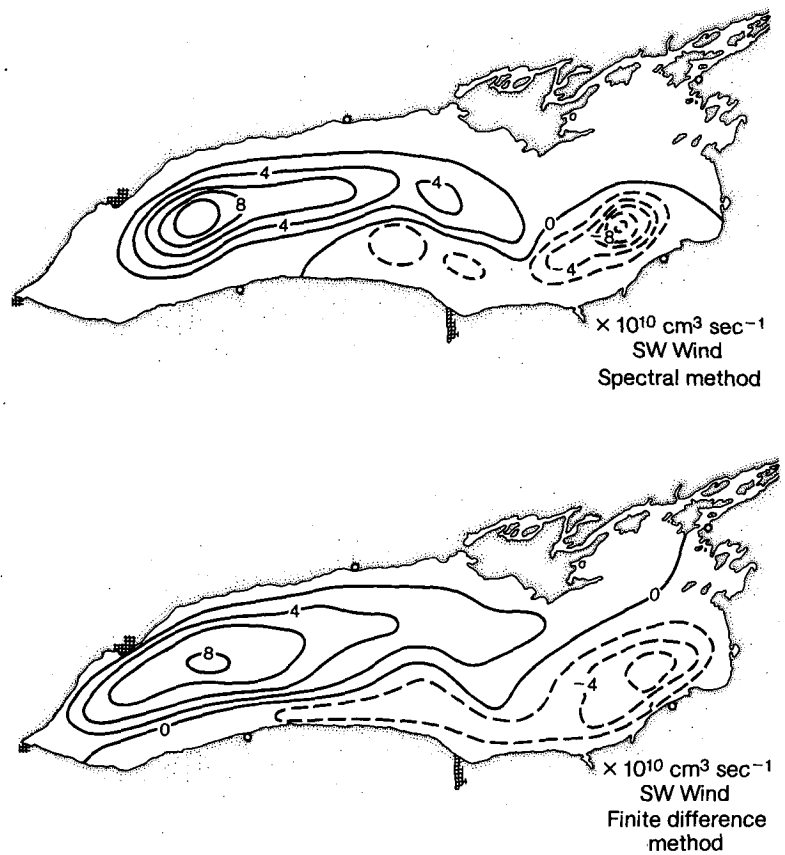

FIG. 2. Steady-state circulation in Lake Ontario using spectral expansion procedure (top panel) and finite-difference relaxation procedure (bottom panel) for a southwesterly wind stress.

Eq. (2.15) to generate γ_j, θ_j and W_j, (3) solving for the eigenvalues μ_α and eigenvectors $\mathbf{q}^\alpha$ and (4) constructing the spectral functions ψ_α from Eq. (2.12). These computations are done with central differences for the gradient operator and summations for the integrals. Even though all W_i functions are orthogonal in theory, truncation errors creep into the calculations after a certain number of iterations, resulting in a W_{n+i} function that is no longer orthogonal to a W_i function. The value for n generally is dependent on the length of W_i. The W_i functions, nevertheless, do exhibit orthogonality in a "local" sense such that any W_i function is orthogonal to all other W_i functions in the range $W_{i-n/2}$ to $W_{i+n/2}(i \geq n)$, and this local orthogonality is found to be sufficient to determine several of the lowest eigenvalues and vectors (Platzman, 1975).

c. *Spectral functions for Lake Ontario*

The 5 km grid employed by Rao and Murty (1970) was used to calculate the spectral functions for Lake Ontario. The grid has a total of 647 squares. In starting the Lanczos procedure, the trial function W_1 is made proportional to the depth of the lake. The calculated spectral functions ψ_α are ordered sequentially with respect to decreasing values of the parameter μ_α. As an illustration, Fig. 1a–1d shows the structures of the functions ψ_1, ψ_2, ψ_4, ψ_8, respectively, corresponding to characteristic values μ_1, μ_2, μ_4 and μ_8 ($\mu_1 < \mu_2 < \mu_4 < \mu_8$). The first function ($\psi_1$) consists of a single gyre. The streamfunction mode ψ_2 consists of two counterrotating gyres, one in the eastern and the other in the western part of the lake. A nodal line, oriented in the north-south direction and located in the eastern part of the lake, separates these gyres. The streamfunction mode ψ_4 has four gyres with three north-south oriented nodal lines separating the gyres. Finally, the streamfunction ψ_8 essentially shows two gyres. In contrast to the gyres of ψ_2, these gyres are elongated in the east-west (longitudinal) direction. A nodal line oriented in the longitudinal direction separates the northern and southern gyres of this mode. We have computed 25 of these characteristic values and functions for use in the simulation of steady-state circulations and objective analysis.

TABLE 1. Energy of the spectral modes for a southwest wind stress.

Spectral function number	Percent of total energy	Cumulative percentage
8	23.6	23.6
4	19.2	42.8
2	12.6	55.4
16	7.8	63.2
12	6.1	69.3
3	4.7	74.0
6	4.7	78.7
22	4.0	82.7
13	3.8	86.5

d. Steady-state circulation in Lake Ontario

During fall and winter, the monthly mean winds over Lake Ontario are primarily from the west, but exhibit a directional variation that ranges between the northwest and southwest sectors, with an occasional shift to the northeast sector. It is necessary to examine steady-state circulation patterns in response to wind stress applied from different compass directions in order to determine the most energetic spectral components. Fig. 2 shows, as an example, the steady-state circulation for a southwesterly wind stress obtained from the spectral expansion method using 25 functions (top panel) and that obtained by a relaxation method applied to the finite-difference equations resulting from Eq. (2.7) (bottom panel). The coefficient of friction in both these calculations is taken to be $\lambda = 5 \times 10^{-2}$ cm s^{-1}. It is seen that the agreement between the two methods is good. The circulation is characterized by two gyres. A large anticyclonic gyre occupies the northwestern part of the Lake Ontario basin. A nodal line stretching down the middle of the lake separates this anticyclonic gyre from a more narrow cyclonic gyre in the southeastern part of the lake. The sense of flow is such that along the north and south shores the flow is in the direction of the wind, while in the middle of the lake there is a return flow against the wind in response to the longitudinal pressure gradient produced by the wind. It is found that as the friction value is reduced, a larger number of spectral functions is required to obtain a closer agreement between the finite difference and spectral procedures. This, of course, is a consequence of the fact that the finite-difference model has a higher degree of resolution than the spectral one. The higher harmonics in the spectral model are less damped in the low-friction case, thereby requiring more functions in the expansion. However, several of the most energetic modes remain the same over a wide range of friction values.

Table 1 shows the convergence of the spectral series in terms of total kinetic energy. In this table, the spectral components are arranged in descending order according to their contribution to the total energy. The series converges fairly rapidly as seen from the fact that the nine most energetic functions account for almost 90% of the total energy. By carrying out similar computations with other wind directions, the most important modes governing the circulation are obtained and tabulated for future use.

3. Objective analysis procedure

If the streamfunction for the lake's circulation is represented spectrally by Eq. (2.4), the transport field can be written as

$$\mathbf{M} = \sum_{\alpha=1}^{n} a_\alpha \mathbf{k} \times \nabla \psi_\alpha. \tag{3.1}$$

The spectral series has been truncated at n modes and

FIG. 3. Circulation pattern given by the objective analysis of currents taken at nine selected locations from the theoretical spectral circulation model (top panel of Fig. 2). In this and the following figures, the arrows indicate the direction and magnitude of the currents at the selected locations.

TABLE 2. Monthly mean wind statistics for Lake Ontario.

Month	Average speed (m s^{-1})	Direction (deg)
Nov 1972	4.9	43
Dec 1972	4.3	241
Jan 1973	4.7	255
Feb 1973	4.3	215
Mar 1973	4.2	150

the n expansion coefficients a_α are unknowns and need to be determined from the data. Now the east-west and north-south components of the transport, M_i and N_i, respectively, can be computed at each station for which current data is available. In Lake Ontario the currents were measured at either 15 or 16 m depth and at 75 m depth at each station. Since the data do not show any significant vertical shear in the currents, the transport at each station is taken to be the product of the mean current and the total depth at the station. Let $\nabla\psi_{i,\alpha}$ represent the gradient of streamfunction mode α at station i. Then Eq. (3.1) becomes

$$\mathbf{M}_i = \sum_{\alpha=1}^{n} a_\alpha \mathbf{k} \times \nabla\psi_{i,\alpha}, \quad i = 1, 2, \ldots m. \quad (3.2)$$

The two algebraic equations represented by Eq. (3.2) can be written for each of the m data stations, resulting in $2m$ equations in n unknowns. If $n = 2m$, a set of $2m$ equations in as many unknowns are obtained. The expansion coefficients a_α determined from the $2m$ equations would exactly fit the data at the observation stations leading to a zero rms error. In general, however, it is preferable to take $n \leq 2m$ so that there are fewer unknowns than equations (an overdetermined system) and determine the coefficients a_α in a least-squares sense. That is, since Eq. (3.2) cannot be satisfied exactly, we try to minimize the rms error, which is defined as

$$E = [m^{-1} \sum_{i=1}^{m} |\mathbf{M}_i - \sum_{\alpha=1}^{n} a_\alpha \mathbf{k} \times \nabla\psi_{i,\alpha}|^2]^{1/2}, \quad (3.3)$$

where $\mathbf{M}_i = (M_i, N_i)$ are the observed data. For convenience, represent the scalar components (M_i, N_i) of the transport vector by a column vector of length $2m \times 1$.

$$\mathbf{M} = \text{col}(M_1, M_2, \ldots M_m, N_1, N_2, \ldots N_m).$$

Eq. (3.2) can be written as

$$\mathbf{M} = \mathsf{B}\mathbf{a},$$

where B is a rectangular matrix of dimensions ($2m \times n$) with coefficients

$$\mathsf{B} = |b_{i\alpha}| = -\frac{\partial\psi_{i,\alpha}}{\partial y}, \quad i = 1, 2, \ldots m; \quad \alpha = 1, 2, \ldots n$$

$$= \frac{\partial\psi_{i,\alpha}}{\partial x}, \quad i = m+1, m+2, \ldots 2m; \quad \alpha = 1, 2, \ldots n$$

and $\mathbf{a}$ is the column vector of length $n \times 1$, i.e.,

$$\mathbf{a} = \text{col}(a_1, a_2, \ldots a_n).$$

The usual procedure of determining the coefficients a_α to minimize the mean-square error given by Eq. (3.3) leads to the solution

$$\mathbf{a} = (\mathsf{B}^T\mathsf{B})^{-1}\mathsf{B}^T\mathbf{M}, \quad (3.4)$$

where B^T is the transpose of B.

TABLE 3. Optimum spectral functions (ψ_α) to yield minimum rms error for various months.

Month	Functions (values of α)	Minimum error (cm s^{-1})
Nov	4, 8, 22, 1, 2, 24	0.45
Dec	8, 4, 22, 2, 1, 24, 6, 12, 14	0.41
Jan	8, 4, 22, 2, 1, 24, 12, 6, 20	0.38
Feb	4, 8, 2, 22, 1, 24, 6	0.47
Mar	8, 22, 4, 2, 24, 12, 20, 1, 25	0.48

TABLE 4. Root-mean-square errors in the currents resulting from objective analysis (cm s^{-1}).

rms error of objective analysis	Months				
	Nov	Dec	Jan	Feb	Mar
At data points from observation	2.68	1.44	2.43	2.11	2.52
At data points from simulation	1.89	0.26	0.38	0.53	0.76
Over the entire lake from simulation	1.97	1.67	2.27	1.43	1.90
Standard deviation of observed speed	6.70	3.86	7.19	4.70	4.53

a. Application to theoretical data

Before applying the above procedure to real data, an analysis of currents taken from the theoretical spectral model shown in Fig. 2 was done in order to test the performance. Transports were taken at the nine grid points closest to current meter stations. Locations of these stations and the magnitudes of the currents are shown in Fig. 3. The nine most energetic spectral functions, which are given in Table 1, are used to compute the objectively analyzed transport field. The result is shown in Fig. 3 in terms of the transport streamfunction. A comparison of the circulation patterns in Figs. 2 and 3 shows that the objective analysis method does indeed give a very good representation of the true circulation. The mag-

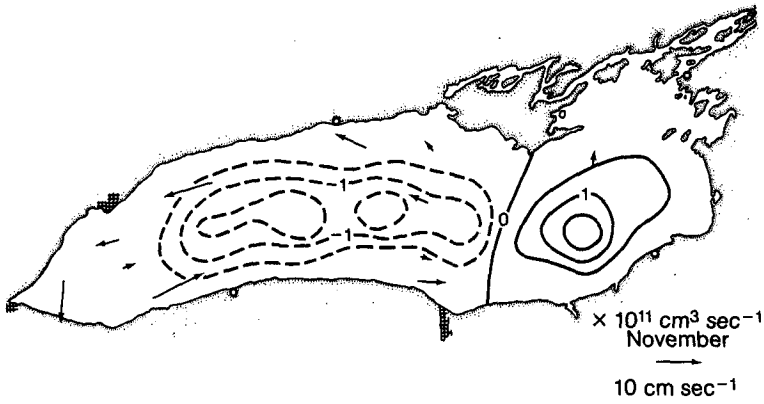

FIG. 4. Circulation of Lake Ontario as given by the objective analysis of observed mean currents for November 1972.

nitudes of the currents and the locations of the gyres are well reproduced. The pattern in Fig. 3 is smoother than the theoretical solution since fewer functions are used in the representation. The rms error between the computed and theoretical currents for the total grid field is 1.47 cm s^{-1} and the standard deviation of the theoretical currents is 3.42 cm s^{-1}.

b. Application to real data

As mentioned earlier, the present objective analysis method is applied to the monthly mean currents in Lake Ontario for the months of November–December 1972 and of January, February and March 1973 as presented by Pickett (1977). The data for May 1972 were not used because many of the current meter records are incomplete. The mean winds for these months are obtained from meteorological towers deployed around the lake during the IFYGL experiment. Table 2 gives the monthly mean wind statistics. In this table, the average speed is the scalar average of the wind speeds and the direction corresponds to the one obtained by a vector average of the wind fields. In the usual meteorological notation, the wind direction corresponds to that from which the wind is blowing, with 0° representing a northerly wind and the angle increasing in a clockwise direction. The data in Table 2 may be considered to represent a wind field whose direction corresponds to the dominant direction of the winds during each period with a speed given by average scalar field.

In analyzing the monthly mean circulations, steady-state circulation patterns are computed for each of the resultant wind directions given in Table 2 and the most energetic spectral components are determined. Then an objective analysis is carried out for each of the months with data taken from the theoretical simulations at points where observations are available in order to determine the optimum number of spectral functions needed for the objective analysis. The optimum number of functions is the number that would result in a minimum error over the entire basin. Normally, as a greater number of functions are taken, the rms error between the observed and predicted values decreases.

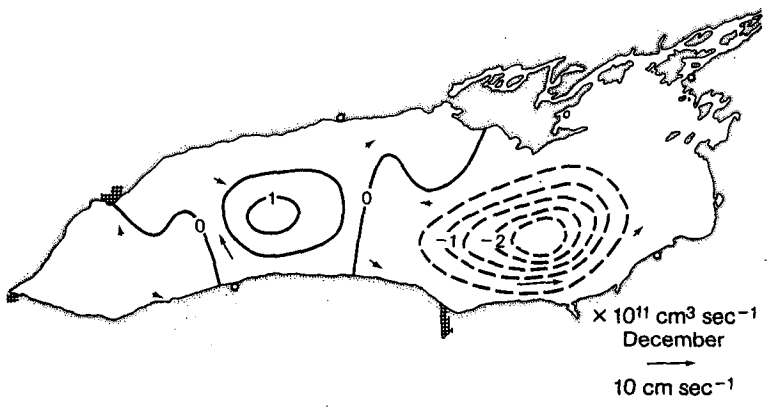

FIG. 5. As in Fig. 4 except for December 1972.

FIG. 6. As in Fig. 4 except for January 1973.

When the number of functions taken is equal to exactly twice the number of observation points, the rms error at the observation points would be zero. However, the resulting analysis would not necessarily correspond to a minimum rms error over the entire basin. The use of theoretically simulated data, where the results are known over the entire basin, helps to establish the optimum number of functions that should be considered to accomplish the above purpose. These considerations have shown that for the various months, the optimum number of the most energetic functions are as given in Table 3. In general, the number of functions is approximately equal to the number of observation stations—that is, about half the total number of observations available, since at each station two data values are given. For all months, six of the spectral components are common even though they may not appear in the same sequence.

Results of the objective analyses applied to the monthly mean currents are shown in Table 4. Table 4 gives the rms errors for the various months resulting from the analyses of the corresponding theoretical steady-state calculations and data. Also, given in the table are the standard deviations (σ) of the observed currents in each of the months. It is seen that the rms errors range from 1.44 to 2.68 cm s^{-1}, while the standard deviations range from 3.86 to 6.70 cm s^{-1}. If the actual variance of data is viewed as the sum of explained variance and unexplained variances, the latter is given by the ratio of $(\text{rms error})^2/\sigma^2$. This ratio has a maximum value of 31% for March and a minimum of 11% for January. Hence the present analysis has been able to account for 69% of the actual variance in the worst case and 89% in the best case. Even though these numbers are obtained from the developmental data set itself and not from an independent data set, in view of the performance of the objective analysis technique in the theoretical simulation cases, it is clear that the proposed technique yields a satisfactory solution.

Figs. 4–8 show the circulation patterns resulting from the objective analysis. The observed currents are shown by arrows on the diagram. The months of November, December and January appear to have

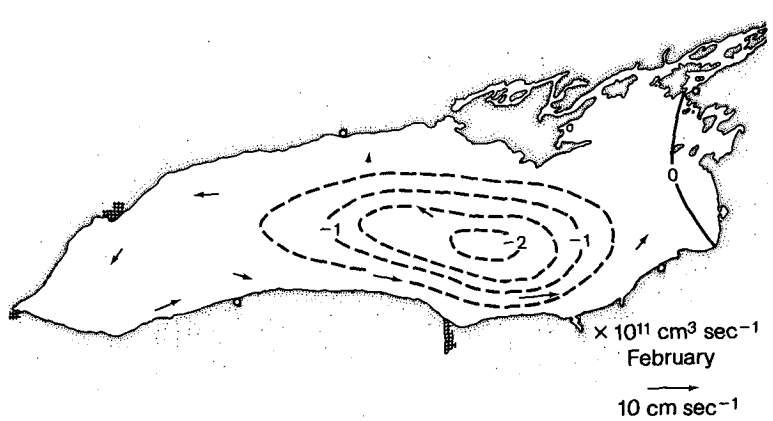

FIG. 7. As in Fig. 4 except for February 1973.

FIG. 8. As in Fig. 4 except for March 1973.

a two-gyre mean circulation in basic agreement with theoretical calculations of the steady-state circulation. The orientation of the gyres differs among the different months as a consequence of the changes in the meteorological forcing. December is the month with a wind direction closest to the southwest wind simulation shown in Fig. 2 and the resemblance between the corresponding patterns is obvious. The months of February and March, on the other hand, appear to produce a single cyclonic gyre in the mean circulation pattern. The reason for such a mean circulation pattern is not obvious. The only theoretical explanation would be that the cyclonic vorticity of the wind stress has become dominant over the bottom slope effect. The likelihood of such an event, however, is very remote from scale considerations as discussed by Pickett and Rao (1977). More likely reasons for the one-gyre circulation patterns during February and March may have to do with the history of transients over the finite interval of time. It should be pointed out that the divergence-reducing objective analysis scheme for currents given by Schwab (1977) resulted in a one-gyre pattern for the November circulation. With the spectral method shown here, the cyclonic structure of the circulation pattern in the relatively data-rich western part of the basin somehow requires a compensating anticyclonic gyre in the eastern basin. The divergence-reducing scheme produces only weak currents in data-poor areas.

c. *Extension to baroclinic currents*

In the analysis reported here, we have considered the simple case of a homogeneous lake in which the currents are essentially the same at all depths. If there are significant vertical variations in the horizontal currents due to baroclinic effects, the present method can be extended to obtain non-divergent circulation patterns in a layered fashion by determining the spectral functions for various layers instead of for a single layer. The spectral functions from the vertical integral over the entire depth will give the barotropic circulation, while the functions from each layer will give the vertical structure of the circulation.

4. Summary

A method of objectively analyzing vector current fields in a homogeneous lake has been presented. The method consists of constructing a set of orthogonal functions that are determined by the geometry and topography of the lake. Once these orthogonal functions are obtained, they are used to simulate steady-state circulation patterns in the lake for different mean wind fields. From these theoretical simulations, the most energetically excited functions for each wind direction are determined and tabulated in a sequence for future use. Even though the entire procedure might appear complicated, the tabulation of the functions to use for objective analysis needs to be done only once for a range of wind directions for each given lake. Once this task is accomplished, the functions are available for future use. When current meter data are given over any period of time, the appropriate orthogonal functions are linearly combined and the expansion coefficients are determined in such a way as to minimize the mean-squared error by a simple matrix inversion. The advantage of this method is that the orthogonal functions used in the analysis are characteristic of each given lake and hence are capable of giving the best fit for the overall circulation pattern of the lake.

The method has been applied to data obtained from selected points of a theoretical simulation model for the circulation of Lake Ontario. The results, when compared to the actual simulation, showed a very good agreement. The method is then used to describe the monthly mean circulation patterns in Lake Ontario during November and December 1972 and January through March 1973. A

comparison of the rms errors and standard deviations showed that the analysis technique accounted for 69% of the total variance of the data in the worst case and 89% in the best case. Hence the proposed analysis technique appears to provide a satisfactory method to analyze and extrapolate objectively the few current meter data that are generally available to determine the large-scale circulation features of lakes.

Acknowledgments. We thank Drs. E. J. Aubert and G. W. Platzman for their helpful comments on the manuscript.

REFERENCES

Baer, F., and G. W. Platzman, 1961: A procedure for numerical integration of the spectral vorticity equation. *J. Meteor.*, **18,** 393–401.

Brown, J., and J. R. Neilon, 1961: Case studies of numerical wind analysis. *Mon. Wea. Rev.*, **89,** 83–90.

Cressman, G. P., 1959: An operational objective analysis system. *Mon. Wea. Rev.*, **87,** 374–376.

Gandin, L. S., 1965: *Objective Analysis of Meteorological Fields.* Gidrometeoizdat, 242 pp. [NTIS N66-18047.].

Hawkins, H. F., and S. L. Rosenthal, 1965: On the computation of streamfunctions from the wind field. *Mon. Wea. Rev.*, **93,** 245–252.

Jalickee, J. B., J. K. S. Ching and J. A. Almazan, 1974: Objective analysis of IFYGL surface meteorological data (IFYGL). *Proc 17th Conf. Great Lakes Res.*, Int. Assoc. Great Lakes Res., McMaster University, Hamilton, Ontario, 733–750.

Kasahara, A., 1977: Numerical integration of the global barotropic primitive equations with Hough harmonic expansions. *J. Atmos. Sci.*, **34,** 687–701.

Liu, C. Y., and W. R. Goodin, 1976: An iterative algorithm for objective wind field analysis. *Mon. Wea. Rev.*, **104,** 784–792.

Lorenz, E. H., 1956: Empirical orthogonal functions and statistical weather prediction. Sci. Rep. No. 1, Dept. of Meteorology, MIT, 49 pp.

Pickett, R. L., 1977: The observed winter circulation of Lake Ontario. *J. Phys. Oceanogr.*, **7,** 152–156.

——, and D. B. Rao, 1977: One- and two-gyre circulations in homogeneous lakes. *IFYGL Bull.*, **19,** 45–49.

Platzman, G. W., 1975: Normal modes of the Atlantic and Indian Oceans. *J. Phys. Oceanogr.*, **5,** 201–221.

Rao, D. B., and T. S. Murty, 1970: Calculation of the steady-state wind-driven circulation in Lake Ontario. *Arch. Meteor. Geophys. Bioklim.*, **A19,** 195–210.

Schwab, D. J., 1977: An objective analysis scheme for lake currents. *IFYGL Bull.*, **19,** 50–52.