COMPUTER NOTE: SIMULATION OF CYLINDRICAL FLOW TO A WELL USING THE U.S. GEOLOGICAL SURVEY MODULAR FINITE-DIFFERENCE GROUND-WATER FLOW MODEL[1]

by

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ABSTRACT

Cylindrical (axisymmetric) flow to a well is an important specialized topic of ground-water hydraulics and has been applied by many investigators to determine aquifer properties and determine heads and flows in the vicinity of the well. A recent modification to the U.S. Geological Survey Modular Three-Dimensional Finite-Difference Ground-Water Flow Model provides the opportunity to simulate axisymmetric flow to a well. The theory involves the conceptualization of a system of concentric shells that are capable of reproducing the large variations in gradient in the vicinity of the well by decreasing their area in the direction of the well. The computer program presented serves as a preprocessor to the U.S. Geological Survey model by creating the input data file needed to implement the axisymmetric conceptualization. Data input requirements to this preprocessor are described and a comparison with a known analytical solution indicates that the model functions appropriately.

INTRODUCTION

The simulation of radial or cylindrical flow to a well has been used by many investigators to determine aquifer properties and determine heads and flows in the vicinity of the well (Stallman, 1963; Mundorff and others, 1972; Lindner and Reilly, 1983; and Reilly, 1984). Simulation of the flow near the well requires that the changes in the head gradient due to the radial convergence of flow lines are accurately represented. These large variations in gradient in the vicinity of the well can be accurately simulated by using an axisymmetrical two-dimensional representation of the flow field.

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A recent modification to the U.S. Geological Survey (USGS) Modular Three-Dimensional Finite-Difference Ground-Water Flow Model (McDonald and Harbaugh, 1988, and Harbaugh, 1992) provides the opportunity to use the axisymmetric representation. Thus, problems in well hydraulics can now be simulated using the widely used USGS model.

GENERALIZED FINITE-DIFFERENCE PACKAGE

The original USGS Modular Ground-Water Flow Model (McDonald and Harbaugh, 1988), known as MODFLOW, assumed that model nodes are in the center of rectangular cells and that transmissivity is constant within a cell. Based on these assumptions, the model calculates coefficients, called conductance, that are multiplied by head difference to determine flow between cells. A generalized finite-difference capability was developed for the USGS model (Harbaugh, 1992) to allow the model program to read conductance as input instead of calculating it internally under the assumption of block-centered finite-difference discretization.

This generalized finite-difference capability allows the conductances to be calculated externally using a node centered radial discretization (as described below) for input to the flow model. This note documents a preprocessor computer program that calculates conductances and storage capacities for a radial axisymmetric node-centered finite-difference grid. The calculated conductances and storage capacities can then be included as the input to MODFLOW to simulate axisymmetric flow to a well.

CALCULATION OF CYLINDRICAL CONDUCTANCES AND STORAGE CAPACITIES

The discretization approach used to simulate axisymmetric flow in this preprocessor is described in Bennett and others (1990). The conceptualization and calculation of the conductances and storage capacities are summarized as follows:

In axisymmetric flow to a well, the flow occurs through concentric shells that decrease in area in the direction of the well (fig. 1). The head gradient increases approaching the well, because the cross-sectional area for flow to pass through is decreasing. Thus, finer discretization (or closer spacing of nodes) is required near the well to accurately represent this increasing gradient. The conductance that represents the porous media between nodes must account for the changing grid spacing and the radial geometry.

The shells used for the calculation of the lateral (radial) conductance, vertical conductance, and storage capacity are shown in figure 2. In the formulation used in this preprocessor, as explained in Bennett and others (1990), the nodes in the radial direction are located on an expanding mesh spacing. Each node is located at a distance from the center of the well that is a multiple of the distance of the node interior to it. This is given by

$$r_{i+1} = \alpha r_i \ . \qquad (1)$$

The innermost (first) node is located at the well radius r_1; the next (second) node is located at the radius r_2, where $r_2 = \alpha r_1$; the outermost (n^{th}) node is located at the radius r_n, where $r_n = \alpha^{n-1} r$. The modeled area ends at the last lateral node. As shown in figure 2, the lateral conductance shells that account for lateral (radial) flow incorporate the thickness of a layer and connect adjacent nodes within the layer. Except for the top and bottom layers, these nodes are represented as centered in a layer. In the top and bottom layers, nodes are located vertically on the outermost edge of the layer (the boundary of the flow system). The vertical conductance shells that account for vertical flow represent the porous media between vertically adjacent nodes, with cylindrical shell walls located at radii $r_{i-1/2}$ and $r_{i+1/2}$. At interior nodes, radii $r_{i-1/2}$ and $r_{i+1/2}$ are chosen so that the vertical conductance is equally distributed to either side of the node i. However, the formulation differs for the innermost and outermost vertical conductance shells. The innermost shell extends from the well radius, r_1, to the first intermediate radius $r_{1+1/2}$; the outermost shell extends from the intermediate radius, $r_{n-1/2}$, to the last node, r_n. The storage capacity shells that account for water released from storage surround a node and represent the layer thickness in the vertical direction; in the radial direction storage shells extend between cylindrical walls located at radii $r_{i-1/2}$ and $r_{i+1/2}$. However, the innermost and outermost shells have the same radial boundaries as the innermost and outermost vertical conductance shells.

Hydraulic conductance, C, is defined as the hydraulic conductivity, K, multiplied by the area through which the flow occurs, A, divided by the length of the flow path, L as given by

$$C = KA/L. \tag{2}$$

In rectangular finite-difference grids, the area, A, is a constant between two nodes. However, in axisymmetric (or cylindrical) coordinates the cross-sectional flow area in the lateral direction is not constant between two nodes. The lateral (radial) conductance can be derived as the limit of many radial conductances in series (see Bennett and others, 1990), and is given as

$$C_r = \frac{2\pi K_r \Delta z}{ln\left(\frac{r_{i+1}}{r_i}\right)} \tag{3}$$

because $r_{i+1} = \alpha r_i$, eqn. (3) can be written as

$$C_r = \frac{2\pi K_r \Delta z}{ln\,(\alpha)} \tag{4}$$

where:

C_r = lateral (radial) conductance (L^2/T),

K_r = lateral (radial) hydraulic conductivity for the model layer (L/T),

Δz = the thickness of the model layer (L),

r_i = radial distance of node in column i (L), and

α = the radial nodal spacing multiplier.

Because α is a constant in the formulation described here, the lateral (radial) conductance is a constant for all shells in the same layer.

The vertical conductance changes laterally as a function of the increasing area through which the vertical flow is occurring. The cross-sectional area for vertical flow in an internal vertical conductance shell is given as

$$A_i = \pi\left(r^2_{i+\frac{1}{2}} - r^2_{i-\frac{1}{2}}\right) . \qquad (5)$$

The shell-wall radii ($r_{i+1/2}$ and $r_{i-1/2}$) are specified such that the area on each side of the node is equal; therefore,

$$r^2_{i+\frac{1}{2}} - r^2_i = r^2_i - r^2_{i-\frac{1}{2}} \qquad (6)$$

and, with the additional specification that,

$$r_{i+\frac{1}{2}} = \alpha r_{i-\frac{1}{2}} \qquad (7)$$

the intermediate radii (or shell-wall radii) are determined such that the interior blocks defining vertical conductance extend from $r_i\sqrt{\frac{2}{(\alpha^2+1)}}$ to $\alpha r_i\sqrt{\frac{2}{(\alpha^2+1)}}$.

The resulting area through which vertical flow occurs is then determined to be

$$A_i = 2\pi r_i^2 \frac{(\alpha^2 - 1)}{(\alpha^2 + 1)} . \qquad (8)$$

The vertical cross-sectional flow area for the innermost and outermost vertical conductance shells is half of the value from equation 8, because the modeled area starts at the first lateral node (the well) and ends at the last lateral node.

In this axisymmetrical representation, the hydraulic conductivity can be different for each layer. If it is assumed that the vertical conductivity changes discretely between layers, then the net vertical conductance between two layers is the result of two conductances in series. It can be shown (McDonald and Harbaugh, 1988, p. 5-6) that the resultant conductance is given by

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{9}$$

where C_{eq} is the equivalent conductance, and C_1 and C_2 are the two conductances that are in series.

The vertical conductance between nodes j and j+1 at lateral coordinate i is given as

$$C_{z_{j+\frac{1}{2},i}} = \frac{1.}{\frac{1.}{C_{z_{j,i}}} + \frac{1.}{C_{z_{j+1,i}}}} \tag{10}$$

where

$$C_{z_{j,i}} = \frac{K_{z_j} A_i}{\frac{1}{2}\Delta z_j} , \tag{11}$$

$$C_{z_{j+1,i}} = \frac{K_{z_{j+1}} A_i}{\frac{1}{2}\Delta z_{j+1}} , \tag{12}$$

$C_{z_{j+\frac{1}{2},i}}$ = vertical conductance for column i between layers j and j+1 (L^2/T), and

K_{z_j} = vertical hydraulic conductivity for layer j (L/T).

In the discretization scheme used in this preprocessor, nodes in the top and bottom layers are treated differently because the mesh ends at the nodal location. Thus vertical flow between the top or bottom layer and the adjacent interior layer involves the full thickness of the top or bottom layer. Accordingly, the full Δz is used in the denominator of the vertical conductance calculation for layer 1 or the bottom layer, instead of 1/2 Δz.

The storage capacity is the product of the volume represented by the node and the specific storage. This is given by

$$S_{c_{j,i}} = A_i S_{s_j} \Delta z \tag{13}$$

where

S_{s_j} = the specific storage for layer j (L^{-1}), and

$S_{c_{j,i}}$ = the storage capacity for node j,i (L^2).

Although the formulation used in this processor does not allow the conductance to change as saturated thickness changes under water-table conditions, water-table storage can be represented by setting $S_s\Delta z$ equal to the specific yield, S_y for the top layer. (This is accomplished in the preprocessor by setting the specific storage, Ss, of layer 1 in the input data equal to Sy/Δz).

IMPLEMENTATION IN THE MODULAR MODEL

The formulation for the calculation of storage capacity and conductance has been implemented for use in the modular model. In order to facilitate preparation of input data and the presentation of output, the radial grid is represented by a single layer in MODFLOW. In this case, the rows in the modular model represent vertical layers and columns represent the radial shells in our axisymmetric simulation.

A preprocessor was written that calculates the conductances and storage capacity as presented above. This program then outputs these coefficients as an input data file for the General Finite-Difference (GFD) Package (Harbaugh, 1992). Lateral node locations, although not needed for model input, are printed in a separate file for use in interpreting results. The input required for this preprocessor is minimal and is listed in appendix 1. The input consists of two data sets. The first data set contains the radial grid multiplier, α, the radius of the well, r_w, (which is taken to be the location of r_1), the number of radial shells, the number of layers, and the unit number that MODFLOW will use to read the GFD Package data. The second data set contains the thickness, horizontal hydraulic conductivity, vertical hydraulic conductivity, and the specific storage for each layer (within MODFLOW, each radial layer will be considered a row).

The GFD Package requires that grid spacing be specified for possible use in other packages (Harbaugh, 1992, p. 9). The recharge package uses the grid spacing to calculate the cell area in the calculation of volumetric recharge rate from the recharge flux. With the model ‘turned on its side’, the layer thickness would become the spacing along columns (DELC in MODFLOW); however, vertical grid spacing is not meaningful with regard to the calculation of areal recharge rates to the top of the system -- that is, recharge is not dependent on layer thickness. Accordingly, the cell spacing specified in the GFD input by this radial preprocessor is such that DELC is set to 1.0 for row 1 (which is the row that represents the top layer of the modeled system) and to 0.0 for all other rows, and DELR (the spacing along rows) is specified as the lateral (areal) cell area. Thus, for row 1, cell area is correctly calculated as the product of DELR and DELC to represent the surface area upon which recharge (if specified in the model) would occur. For all other rows, the cell area calculates as 0.0. This facilitates the use of the recharge package to distribute areal recharge to the top model layer (row 1). Recharge flux specified for other model layers (any row other than row 1) will result in no recharge because the cell area will be zero. This calculation of areas for recharge enables the radial flow model to simulate circular recharge basins.

The simulation of the axisymmetric model requires that all other data sets necessary for MODFLOW must be prepared by the user. These would include as a minimum the Basic Package, a solver package, and the Well Package. For the Basic Package, the number of layers (NLAY), rows (NROW), and columns(NCOL), would be specified as: one layer (NLAY), the number of layers simulated in the axisymmetrical model (NROW), and the number of radial shells (NCOL). Any constant-head boundary conditions (which must be radially symmetric) would be specified in MODFLOW’s IBOUND array. The slice-successive over relaxation (SSOR) solver is not recommended for this application, because it would function only in a line-successive mode. The strongly-implicit procedure (SIP), however, handles the variation in coefficient values effectively. The only stress packages recommended for use with this preprocessor are the Well and Recharge Packages. The Well Package would be used to specify the discharge rate at nodes representing the well along the inner radial boundary. The well nodes will always be located in column one when using the axisymmetrical conceptualization. If a well is screened in more than one layer, the user must specify the discharge for each layer.

The implementation of the axisymmetric discretization in the preprocessor presented here for use with MODFLOW assumes a constant saturated thickness. This assumption impacts the utility of the model for the simulation of water-table aquifers. The model does have the capability to simulate the water released from storage at the water-table boundary, but this boundary does not move in the simulation and shells do not go dry. Thus, when applying this implementation to water-table problems, the user must evaluate the drawdowns at the top layer to check if any changes in saturated thickness are significant enough to invalidate the assumption used in the preprocessor.

Copies of the source code for the radial-flow preprocessor (RADMOD) and the General Finite-Difference (GFD) Package are available for a nominal charge from the U.S. Geological Survey, National Water Information System Office, 437 National Center, Reston, VA 22092, Phone: (703) 648-5695. Questions concerning the program may be addressed to either author.

EXAMPLE PROBLEM

To illustrate the axisymmetrical representation using the GFD Package, a sample simulation is undertaken and the results compared to the analytical solution of Neumann (1974). A 100 ft thick unconfined aquifer has a horizontal hydraulic conductivity of 100 ft/d, a vertical hydraulic conductivity of 10 ft/d, a specific yield of 0.2 and a specific storage of 0.000005/ft. To determine the time drawdown distribution over a 1-day period, the continuous system will be discretized into a system of nodes with 11 layers, and 40 columns radially spaced with a multiplier (α) of 1.5, as shown in figure 3. The preprocessor input data for this discretization is given in appendix 2.

The well is partially penetrating and is screened in the bottom 25 ft of aquifer with a radius of 0.936 ft and discharges at a total rate of 125,670 ft^3/d. The well is simulated by the Well Package of the Modular Model and the nodes in layers 11, 10, and 9 of column 1 are defined to have a discharge of 25130, 50270 and 50270 ft^3/d respectively. A constant zero drawdown is simulated at the far radial boundary (column 40), which is at a distance of $6.9x10^6$ ft.

A comparison of the model results and the drawdown calculated from the Neuman analytical solution (Neuman, 1974) for this case is shown in figure 4 for a point 50 ft deep (half the thickness of the aquifer) at a radius of 16 ft. Because the Neuman solution also assumes that the saturated thickness of the aquifer does not change, the solutions should be the same. The comparison uses dimensionless variables, and results from the two methods are in good agreement. The model can match the continuous analytical solution more closely by using finer discretization in time and space, which is shown by a second comparison that uses 19 layers, with the time discretization and the vertical discretization near the observation well and at the top of the screened zone more detailed than the previous simulation.

SUMMARY

Axisymmetric flow to a well can be simulated with the widely used ground-water flow model MODFLOW by using the General Finite-Difference Package (Harbaugh, 1992). The preprocessor presented in this paper employs radial discretization to enhance the simulation of head gradients near the well and generates the input data required for the General Finite-Difference Package of MODFLOW. The simulation has the ability to represent layers of different hydraulic conductivity, partial penetration of the well, and the release of water from storage at the water table. A comparison with the analytical solution of Neuman (1974) illustrates the ability of the model to simulate radial systems.

REFERENCES

Bennett, G.D., Reilly, T.E., and Hill, M.C., 1990, Technical training note in ground-water hydrology: Radial flow to a well: U.S. Geological Survey Water-Resources Investigations Report 89-4134, 83 p.

Harbaugh, A.W., 1992, A generalized finite-difference formulation for the U.S. Geological Survey modular three-dimensional finite-difference ground-water flow model: U. S. Geological Survey Open-File Report 91-494, 60 p.

Lindner, J.B., and Reilly, T.E., 1983, Analysis of three tests of the unconfined aquifer in southern Nassau County, Long Island, New York: U.S. Geological Survey Water-Resources Investigations Report 82-4021, 51 p.

McDonald, M.G., and Harbaugh, A.W., 1988, A modular three-dimensional finite-difference ground-water flow model: Techniques of Water-Resources Investigations of the U.S. Geological Survey, book 6, chapt. A1, 586 p.

Mundorff, M.J., Bennett, G.D., and Ahmad, Masood, 1972, Electric analog studies of flow to wells in the Punjab aquifer of West Pakistan: U.S. Geological Survey Water-Supply Paper 1608-N, 28 p.

Neuman, S.P., 1974, Effect of partial penetration on flow in unconfined aquifers considering delayed gravity response: Water Resources Research, v. 10, no. 2, p. 303-312.

Reilly, T.E., 1984, A Galerkin finite element flow model to predict the transient response of a radially symmetric aquifer: U.S. Geological Survey Water-Supply Paper 2198, 33 p.

Stallman, R.W., 1963, Electric analog of three-dimensional flow to wells and its application to unconfined aquifers: U.S. Geological Survey Water-Supply Paper 1536-H, 37 p.

APPENDICES

Appendix 1 --Preprocessor input instructions

INPUT TO PROGRAM:

Data set 1:

ALPHA,RW,NCOL,NROW,LOCAT

Format (2F10.0,3I5)

Data set 2 (one record for each layer in the model):

IR,DELZ(IR),KR(IR),KZ(IR),SS(IR)

Format (I5,4F10.0)

Where:

ALPHA=R(I+1)/R(I), This is the ratio of the lateral distance between neighboring nodes {usually between 1.1 and 2.0},

RW=Radius of the well

NCOL=Number of nodes laterally (variable NCOL in MODFLOW) {must be greater than 4 and is usually greater than 15}

NROW=Number of layers (variable NROW in MODFLOW)

LOCAT = Unit number for reading GFD data in MODFLOW

IR=Layer number

DELZ=Thickness of layer IR

KR=Radial hydraulic conductivity in layer IR

KZ=Vertical hydraulic conductivity in layer IR

SS=Specific storage of material in layer IR (L^{-1}), can be set to $S_y/\Delta z$ for the top layer to represent the water table.

Note: The output from this program is all of the required input data for the General Finite-Difference Package (GFD). The value for IGFDCB (Harbaugh, 1992, p.13) is specified as zero. This can be changed, if desired, with an editor after the GFD data file has been created by the preprocessor.

Appendix 2 -- Preprocessor input data for sample problem

```
   1.50      .936   40   11   12
 1        5.     100.0      10.0 .04
 2       10.     100.0      10.0 .000005
 3       10.     100.0      10.0 .000005
 4       10.     100.0      10.0 .000005
 5       10.     100.0      10.0 .000005
 6       10.     100.0      10.0 .000005
 7       10.     100.0      10.0 .000005
 8       10.     100.0      10.0 .000005
 9       10.     100.0      10.0 .000005
10       10.     100.0      10.0 .000005
11        5.     100.0      10.0 .000005
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LIST OF FIGURES

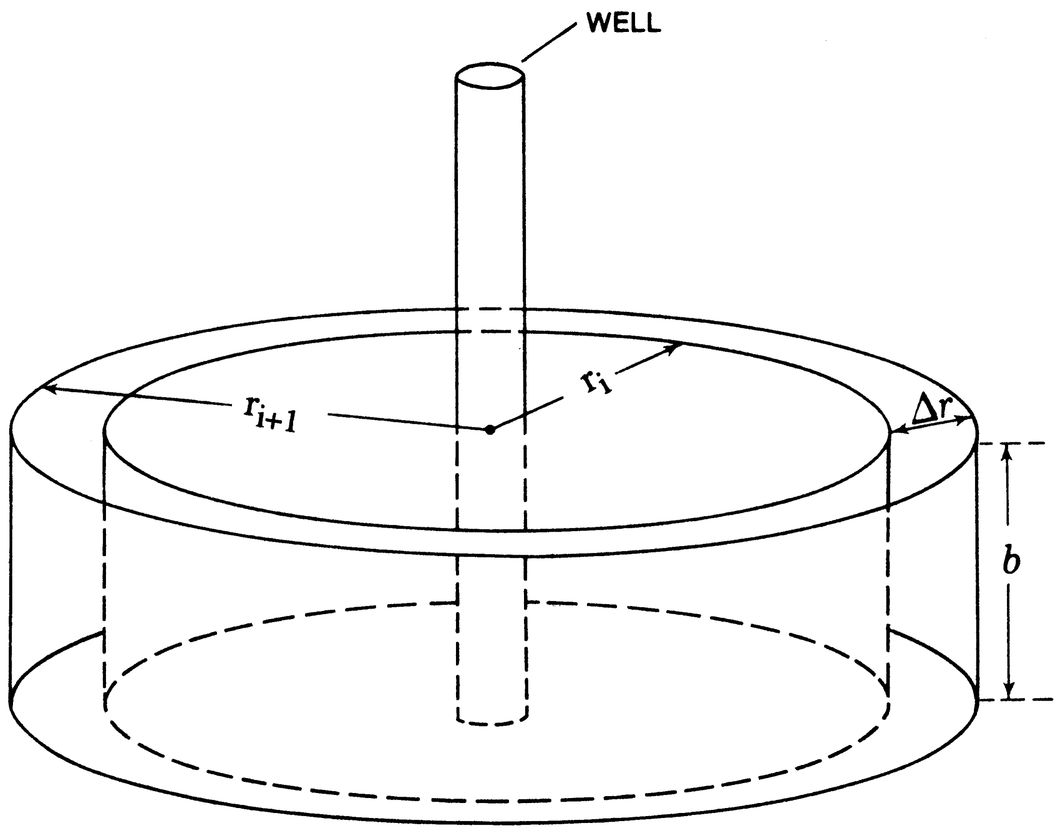

Figure 1. Decrease of cross-sectional area with decreasing radius

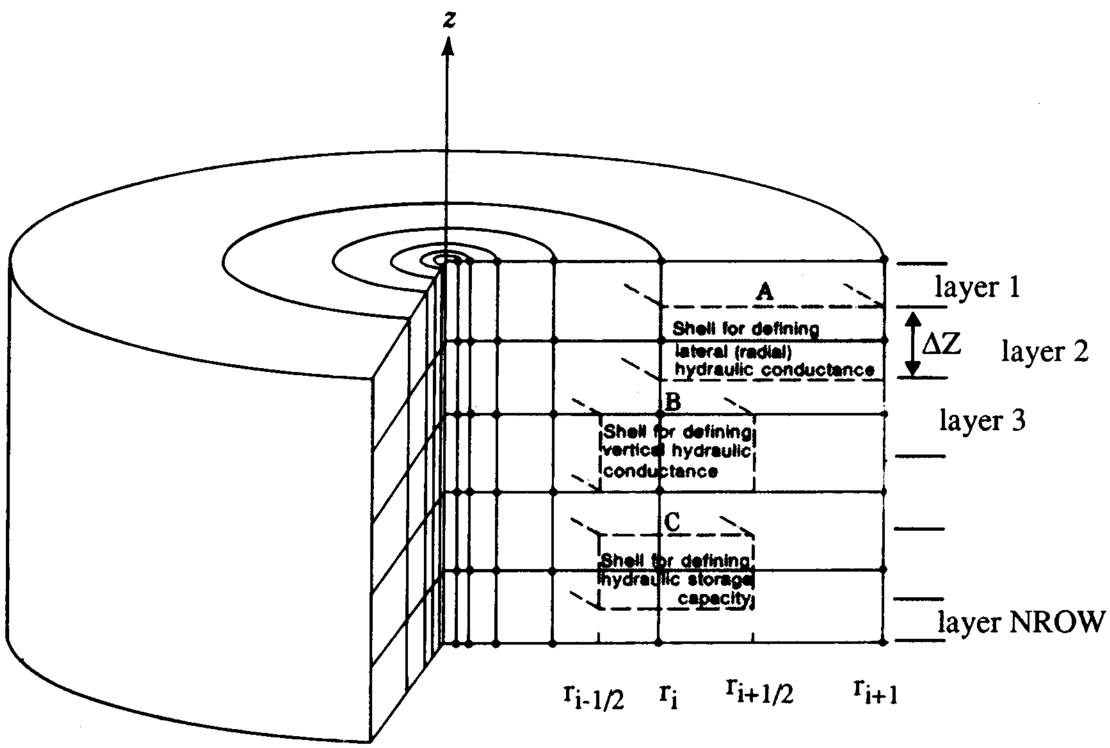

Figure 2. Shells used for definition of hydraulic conductance and storage capacity in axially symmetric finite-difference grid.

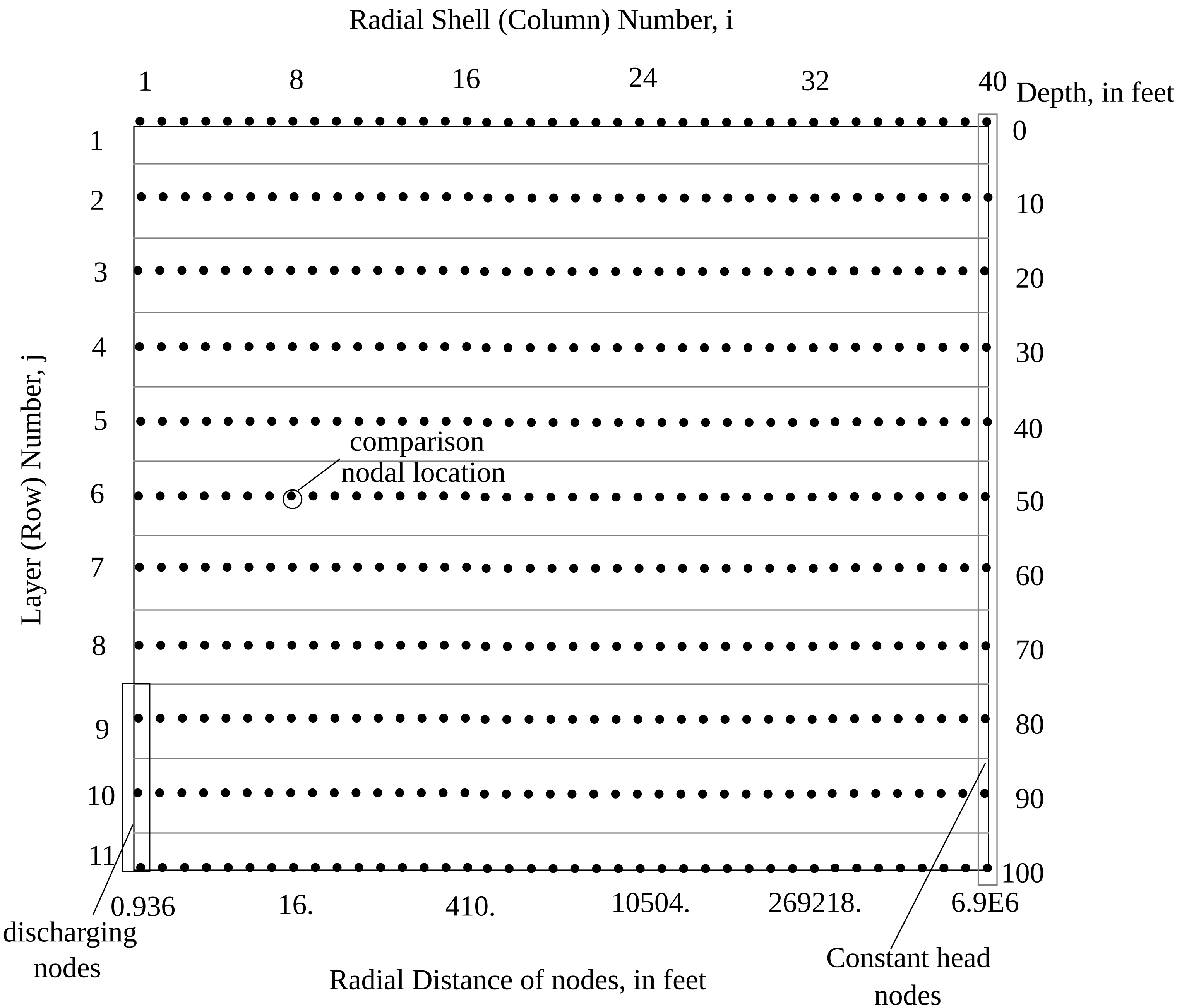

Figure 3. Location of nodes for the example problem for comparison with the Neuman analytical solution..

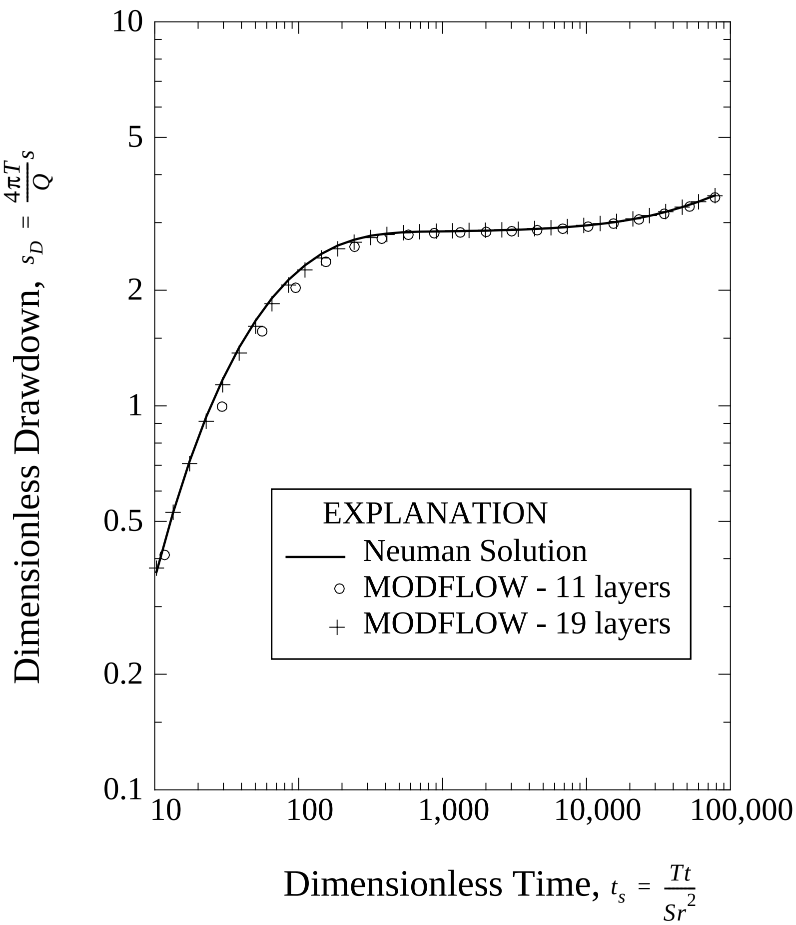

Figure 4. Comparison of dimensionless drawdown calculated using MODFLOW and the Neuman analytical solution.