Evaluating and Adjusting for Chain Drift in National Economic Accounts

Christian Ehemman

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by

Christian Ehemann[1]

acehemann@verizon.net

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It has long been known that a chained index will give different estimates of the change in an economic aggregate over a given period if the linking is done at different frequencies (Frisch [1936]) Frisch called this property chain drift. Although some index number formulas do not display chain drift given certain hypothetical data sets, in practice chain drift is ubiquitous. Nevertheless, little is known about how to select the best interval for linking. The economic theory of index numbers is based on assumptions that are likely to be most nearly fulfilled when the linking interval is short. However, significant chain drift has been observed when the data are volatile, so monthly and quarterly linking intervals are often avoided in favor of smoother, annual data. Yet annual linking is a compromise that lacks a firm theoretical or empirical basis. The effects of specific instances of volatility, such as business cycles and temporary price spikes (e.g., for petroleum), on the accuracy of chained estimates for national economic aggregates remain conjectures. This paper attempts to measure the effects of chain drift on the accuracy of selected real aggregates in the U.S. national income and product accounts (NIPA's).

Real economic aggregates in the NIPA's are constructed using the Fisher quantity index. Quarterly and annual data are for each series are chained separately. Inconsistencies between the quarterly and annual estimates caused by chain drift are reconciled by adjusting the quarterly estimates at annual rate to average to the annual estimates [U. S. Bureau of Economic Analysis, 2001, p. M-15]. From 1992 to 1995, the Bureau of Economic Analysis supplemented its estimates of national economic aggregates obtained by quarterly and annual linking with estimates based on a longer linking interval. These estimates, known as benchmark-years weighted chains, were based on Fisher indexes linked primarily at five-year intervals. The estimates, which were presented as experimental, were intended to provide superior measures of growth for longer periods [Young, 1992]. In this paper, we reconsider whether, and when, such longer linking intervals may be appropriate.

Our approach to evaluating chain drift in real NIPA aggregates is to construct estimates of selected aggregates for which the uncertainty caused by potential chain drift is minimized. Series were chosen either because they are among the most important or because there is an a priori reason for suspecting that they might be particularly sensitive to chain drift. Our finding is that the official estimates are usually very close to the drift-adjusted estimates. For real GDP over the period 1967-2002, the maximum difference from the official BEA estimate in any quarter is 0.31 percentage point at annual rate and the average annual growth rate over the period is virtually unchanged. However, the difference over the twelve year period 1930-1942 averages 0.46 percentage point *per year* lower than the official estimate and is about one percentage point lower in three of these years. Moreover, there is an even greater effect on the calculation of some GDP components. For imports of goods over the 29-quarter period 1979Q1-1986Q2, the drift-adjusted estimates of growth average 1.93 percentage points higher than the official estimates; in one quarter the estimate is 5.46 percentage points higher (at annual rate).

In constructing estimates that minimize the effects of chain drift, we pursue two suggestions by Forsyth and Fowler [1981]. The first stems from their observation that, while some chain drift is unavoidable, index number formulas differ in their propensity for chain drift.

Hence an index number formula can be selected for which the ambiguity caused by conflicting results from different linking intervals is as small as possible. The second suggestion, which was repeated by Szulc [1983], is that the linking interval for any given series should vary from period to period so that atypical observations that can degrade the accuracy of the index are avoided. However, neither of these papers suggested a way to implement this idea. We suggest a solution to this problem and will refer to the resulting estimates as optimally-linked index numbers.

Forsyth and Fowler [1981] investigated chain drift in the Laspeyres, Paasche, geometric, and Fisher indexes. They found the Fisher index to be the most successful in avoiding chain drift, with a chain drift intermediate between the Laspeyres and the Paasche. Lent [2000] investigated chain drift in the geometric and Tornqvist price indexes. She found that the sources of chain drift in the Tornqvist index tended to be offsetting, so that chain drift could be expected to be small with no presumption as to sign. The Tornqvist quantity index, however, is not an option for the calculation of real GDP because it is undefined when a detailed quantity component is zero or changes sign. Both of these events occur in the detailed data used by the U.S. Bureau of Economic Analysis to calculate real GDP, with sign changes occurring, in particular, in the data for change in private inventories.

The implicit Tornqvist quantity index is defined as the relative change in total expenditures divided by the (direct) Tornqvist price index. Like the (direct) Tornqvist and the Fisher, the implicit Tornqvist quantity index is superlative. Unlike the (direct) Tornquist, it can be used to calculate the change in real GDP, because detailed prices are always positive. Ehemann [2005] compared the propensity for chain drift in the Fisher and implicit Tornqvist quantity indexes. Using data for real GDP and real gross private domestic investment, he found that although the Fisher and implicit Tornqvist gave almost identical results when the linking interval was quarterly or annual, they diverged as the linking interval lengthened. For both series, using the Fisher, the average annual growth rate over the period 1965-1995 increased monotonically with longer linking intervals, while for the implicit Tornqvist, the divergence from the annually linked estimates was smaller and had no trend. Moreover, for the Fisher quantity index, he obtained sufficient conditions under which the observed increase in the calculated growth rate as the linking interval lengthens should be expected. The implicit Tornqvist index thus appears to provide more accurate rates of change than does the Fisher index when longer linking intervals are used and does equally well as the Fisher when linking intervals are short. In this paper, all calculations will use the implicit Tornqvist quantity index.

In the economic theory of index numbers, the Fisher and Tornqvist index number formulas, which yield the relative change in an aggregate between two periods, are derived from consumer choice theory on the basis of strong assumptions: (i) that consumers optimize successfully and so are in equilibrium in both periods and (ii) that consumers have homothetic utility functions (see Diewert [1976]). Samuelson and Swamy [1974] showed that if these assumptions are correct, and only then, chain drift will not occur. Although empirical evidence rejects the homotheticity assumption, the force of this assumption is reduced by our adoption of the implicit Tornqvist index. The force of the assumption that consumers optimize successfully in all periods, which is also empirically dubious, can be reduced by attempting to identify and exclude periods in which consumer equilibrium is less likely. This is what we attempt to do in our "optimal linking" procedure, which combines an economic criterion with a result from graph theory.

Not unexpectedly, the benefits of optimally-linked estimates come with their own costs. Optimally-linked estimates entail more complex calculation but in most time periods give estimates that are the same or nearly the same as existing superlative indexes. When differences occur, a judgement must be made as to whether they are due to the inappropriateness of the assumptions underlying the traditional superlative index or to those underlying optimal linking. Finally, the calculation of the optimal linking interval is dependent on the length of the series examined. The latest data point might indicate that recent history should be revised. However, the unpredictability of such an occurrence might present difficulties for a national statistical agency.

The format of this paper is as follows. Section 1 investigates the potential of random disturbances to bias the Tornqvist index. The conclusions of this section lead, in Section 2, to the proposed optimal linking method. In Section 3, growth rates based on the implicit Tornqvist with optimal linking are presented for real GDP and several other economic aggregates; these results are compared with other estimates. Section 4 concludes. An appendix provides an example of optimal linking intervals with three commodities.

1. The effect of stochastic shocks

The question of the interval over which index number formulas should be applied was discussed by Forsyth and Fowler [1981] and Szulc [1983]. Chaining is required if the proper interval is shorter that the period over which the rate of change in an aggregate needs to be measured. Tastes change over time as does the set of goods available in the marketplace, so utility maximization as a basis for index number theory is most realistic when the interval over which change is measured is short. Consequently, in an environment of steady change, price change over longer intervals is best measured by cumulating price change measured over short intervals. However, both papers distinguish between situations in which steady change predominates and those in which oscillations predominate. When prices or quantities oscillate, the intransitivity of chain indexes–their failure to return to a previous value when prices and quantities so return–becomes evident. If all prices (for a price index) or quantities (for a quantity index) return to earlier levels, a counterintuitive result (that the index does not return to its earlier level) can be avoided by chaining across the entire period. It is their view that steady change is the better approximation to most economic data than oscillations that leads these authors to a strong preference for chain indexes. This preference is qualified, however, by the suggestion that longer linking intervals are appropriate when cyclical, or bouncing, data are encountered.

When does this qualification apply? It might be unwise to recognize the shortcomings of high frequency linking in the presence of oscillations only when the oscillations are obvious. One interpretation of prices is that, like much other economic data, they are stochastic in all periods. The "bouncing" of prices, to which the aforementioned authors refer, becomes apparent when the trend is sufficiently small. If this interpretation is correct, one should be concerned about the possible effects of random shocks in all periods, regardless of trend. There is no shortage of sources from which shocks can originate: small samples in data sources, interruptions to the supply of inputs (such as petroleum), severe weather, swings in the value of financial assets, etc. In this section, we investigate the effect of stochastic shocks on the value of a Tornqvist price index. The goal is to determine, qualitatively, the conditions that would make linking across multi-period intervals desirable.

The effect of randomness on the expected values of geometric or Tornqvist price indexes has been investigated by Prasada Rao and Selvanathan [1992], Erickson [1998], and Greenlees [2001]. All these authors find that stochastic behavior leads to bias, but the direction of bias is highly sensitive to the assumptions made. With nonstochastic expenditure shares and multiplicative errors in relative prices, Prasada Rao and Selvanathan find positive bias while Erickson and Greenlees find negative bias. The source of this difference is that while Prasada Rao and Selvanathan assume that the logarithms of the errors are normally distributed with mean zero, Erickson and Greenlees assume that the errors themselves have mean one. Thus, the difference in results turns on the familiar fact that for a random variable $x \sim N(0, \sigma^2)$, $E[\exp(x)] = \exp(\sigma^2/2)$.

One might conjecture that, if prices and quantities are both subject to random shocks, the effects on the Tornqvist index would tend to be offsetting. In this case, the implications for real GDP measurement would be quite limited. On the other hand, the Tornqvist price index contains prices as ratios but quantities only as levels. This suggests the possibility that shocks to prices will have a much greater effect than those to quantities. We investigate these two possibilities in terms of some examples.

Our approach, following Greenlees, is to approximate the Tornqvist price index by a Taylor's series and then take expectations. The (binary) Tornqvist price index, giving the price level in period 1 relative to that in period 0, is defined as

$$P^{TB} = \prod_{i=1}^{i=n} T_i \qquad where \qquad T_i = x_i^{w_{0i}+w_{1i}},$$

$x_i = p_{1i}/p_{0i}$ is the price of the ith good in period 1 relative to its price in period 0, w_{0i} is the share of expenditures contributed by the ith good in period 0, and w_{1i} is its expenditure share in period 1. There are n goods. Here the x_i, w_{0i}, and w_{1i} are all random variates and the Taylor's series expansion is taken around their expected values.

Denoting the expected values of the random variates by overbars and letting $\overline{T_i}$ denote T_i evaluated at these values, we obtain the quadratic approximation

(1) $$\frac{T_i}{\overline{T}_i} \approx 1 + \frac{\overline{w_0}+\overline{w_1}}{2\bar{x}}(x-\bar{x}) + \frac{\ln\bar{x}}{2}(w_0-\overline{w_0}) + \frac{\ln\bar{x}}{2}(w_1-\overline{w_1})$$

$$+ \frac{(\overline{w_0}+\overline{w_1})(\overline{w_0}+\overline{w_1}-2)}{8\bar{x}}(x-\bar{x})^2 + \frac{(\ln\bar{x})^2}{8}(w_0-\overline{w_0})^2 + \frac{(\ln\bar{x})^2}{8}(w_1-\overline{w_1})^2$$

$$+ \frac{(\overline{w_0}+\overline{w_1})\ln\bar{x}}{4\bar{x}}(x-\bar{x})(w_0-\overline{w_0}) + \frac{(\overline{w_0}+\overline{w_1})\ln\bar{x}}{4\bar{x}}(x-\bar{x})(w_1-\overline{w_1}) + \frac{(\ln\bar{x})^2}{4}(w_0-\overline{w_0})(w_1-\overline{w_1}),$$

where the subscripts i for x_i, w_{0i}, w_{1i}, and their respective means have been omitted.

The bias, in percentage points, of the Tornqvist price index due to stochastic disturbances can be calculated as

(2) $$bias(P^{TB}) = 100\times E\left[\prod_{i=1}^{i=n} T_i - \prod_{i=1}^{i=n}\overline{T}_i\right].$$

If the variates for each good are distributed independently of those for all other goods, the expected value of the index can be approximated by the product of the expected values of expressions, one for each good, having the form of the right-hand side of (1) times $\overline{T}_i$. When expectations are taken in (1), the linear terms in (containing deviations from means) vanish, while the quadratic terms become a weighted sum of variances or covariances. If, in addition, the joint distribution of x, w_0, and w_1 is the same for all i, combining equations (1) and (2) yields

(3) $$bias(P^{TB}) = 100\times\left[\left(\sum_{h=1}^{h=6} a_h\sigma_h^2\right)^n - 1\right]\bar{x}^{.5n(\overline{w_0}+\overline{w_1})},$$

where the a_h are elements of the vector of coefficients

$$A = \left(\frac{(\overline{w_0}+\overline{w_1})(\overline{w_0}+\overline{w_1}-2)}{8\bar{x}}, \frac{(\ln\bar{x})^2}{8}, \frac{(\ln\bar{x})^2}{8}, \frac{(\overline{w_0}+\overline{w_1})\ln\bar{x}}{4\bar{x}}(x-\bar{x}), \frac{(\overline{w_0}+\overline{w_1})\ln\bar{x}}{4\bar{x}}(x-\bar{x}), \frac{(\ln\bar{x})^2}{4}\right)$$

and the s_h are elements of the vector of variances and covariances

$$S = (\sigma_x^2, \sigma_{w_0}^2, \sigma_{w_1}^2, \sigma_{xw_0}, \sigma_{xw_1}, \sigma_{w_0w_1}).$$

A similar calculation yields the standard deviation of the errors in the Tornqvist index due to stochastic disturbances. The Taylor's series quadratic approximation of the ratio $T_i^2/\overline{T}_i^2$ is

$$(4) \qquad \frac{T_i^2}{\overline{T_i^2}} \approx 1 + \frac{\overline{w_0}+\overline{w_1}}{\bar{x}}(x-\bar{x}) + \ln\bar{x}(w_0-\overline{w_0}) + \ln\bar{x}(w_1-\overline{w_1})$$

$$+ \frac{(\overline{w_0}+\overline{w_1})(\overline{w_0}+\overline{w_1}-1)}{2\overline{x^2}}(x-\bar{x})^2 + \frac{(\ln\bar{x})^2}{2}(w_0-\overline{w_0})^2 + \frac{(\ln\bar{x})^2}{2}(w_1-\overline{w_1})^2$$

$$+ \frac{(\overline{w_0}+\overline{w_1})\ln\bar{x}}{\bar{x}}(x-\bar{x})(w_0-\overline{w_0}) + \frac{(\overline{w_0}+\overline{w_1})\ln\bar{x}}{\bar{x}}(x-\bar{x})(w_1-\overline{w_1}) + (\ln\bar{x})^2(w_0-\overline{w_0})(w_1-\overline{w_1}).$$

After forming the product, for all i, of $\overline{T_i^2}$ times the right-hand side of this equation and taking expectations, the standard deviation, in percentage points, can be calculated as

$$(5) \qquad 100\times\sigma(P^{TB}) = 100\times\left(E\left[\prod_{i=1}^{i=n}T_i^2\right] - \left(E\left[\prod_{i=1}^{i=n}\overline{T_i}\right]\right)^2\right)^{1/2}.$$

Again assuming that the joint distribution of x_i, w_{0i}, and w_{1i} is the same for all i, equations (4) and (5) can be combined to give

$$(6) \qquad 100\times\sigma(P^{TB}) = 100\times\left[\left(\sum_{h=1}^{h=6}b_h\sigma_h^2\right)^n - 1\right]^{.5}\bar{x}^{.5n(\overline{w_0}+\overline{w_1})},$$

where the b_h are elements of the vector of coefficients

$$B = \left(\frac{(\overline{w_0}+\overline{w_1})(\overline{w_0}+\overline{w_1}-1)}{2\overline{x^2}}, \frac{(\ln\bar{x})^2}{2}, \frac{(\ln\bar{x})^2}{2}, \frac{(\overline{w_0}+\overline{w_1})\ln\bar{x}}{\bar{x}}, \frac{(\overline{w_0}+\overline{w_1})\ln\bar{x}}{\bar{x}}, (\ln\bar{x})^2\right).$$

Implementing this procedure requires statistical assumptions about how the prices and quantities, and thus how x, w_0, and w_1, are generated. We assume, as in equations (3) and (6), that the processes generating the observations for each good are identical. In the first model that we consider, prices and quantities exhibit non-serially correlated deviations from constant rates of growth. Let

$$ln\, p_t = \alpha + E(ln\, p_{t-1}) + u_t \quad \text{and} \quad ln\, q_t = \beta + E(ln\, q_{t-1}) + v_t,$$

where α and β are growth rates while u_t and v_t are normally and independently distributed disturbances with zero means and variances σ_u^2 and σ_v^2, respectively. It follows that $x = p_t/p_{t-1}$

has the lognormal distribution with mean $exp(\alpha + \sigma_u^2)$. The distributions of the expenditure shares are more complicated. The numerator of each share is the product of the *i*th price and quantity, while the corresponding denominator is the sum of price-quantity products for all goods. We assume that the number of goods is sufficiently large that the denominators are essentially non-stochastic. Thus, the denominator is approximated by summing the expected values of each term. Because all goods are assumed to behave alike, the expenditure shares have approximately the lognormal distribution with mean $1/n$. The variances and covariances are found to be:

$$var(x) = exp[2(\alpha + \sigma_u^2)][exp(2\sigma_u^2) - 1],$$

$$var(w_0) = var(w_1) = n^{-2}[exp(\sigma_u^2 + \sigma_v^2) - 1],$$

$$cov(x, w_0) = n^{-1}exp(\alpha + \sigma_u^2)\ [exp(.5\sigma_u^2) - 1)],$$

$$cov(x, w_1) = n^{-1}exp(\alpha + \sigma_u^2)\ [exp(-.5\sigma_u^2) - 1)],$$

$$cov(w_0, w_1) = 0.$$

Suppose the parameter values are: $\alpha = .03$, $\beta = .03$, $\sigma_u^2 = .0002$, $\sigma_v^2 = .0002$, and $n = 100$. The variances of the disturbances correspond to standard deviations of 1.41 percentage points on the assumed 3 percent growth rates for prices and quantities. With these values, the bias in the Tornqvist index due to stochastic disturbances is found to be 0.0002 percentage point, a positive but trivial amount. The standard deviation of the index is 0.21 percentage point, so errors in the rate of price change of one percentage point or more would be rare.

However, under alternative, equally plausible assumptions, the standard deviation of the Tornqvist index is markedly higher. Consider the autoregressive model

$$ln\, p_t = \alpha + \varphi\, ln\, p_{t-1} + u_t\,, \quad ln\, q_t = \beta + \theta\, ln\, q_{t-1} + v_t,$$

where φ and θ are constants between zero and one, and u_t and v_t are distributed normally and independently as before. The expected value of x is now $exp[\alpha + \sigma_u^2/(1 - \varphi^2)]$, while the expected values of w_0 and w_1 are unchanged. The variances and covariances are:

$$var(x) = exp[2(\alpha + \sigma_u^2/(1 - \varphi^2))][exp(2\sigma_u^2/(1 - \varphi^2)) - 1],$$

$$var(w_0) = var(w_1) = n^{-2}[exp(\sigma_u^2/(1 - \varphi^2) + \sigma_v^2/(1 - \theta^2)) - 1],$$

$$cov(x, w_0) = n^{-1}exp(\alpha + \sigma_u^2/(1 - \varphi^2))\ [exp(.5\sigma_u^2) - 1)],$$

$$cov(x, w_1) = n^{-1}exp(\alpha + \sigma_u^2/(1 - \varphi^2))\ [exp(-.5\sigma_u^2) - 1)],$$

$$cov(w_0, w_1) = n^{-2}\, exp(.5\ (\varphi^2\, \sigma_u^2/(1 - \varphi^2) + \theta^2\sigma_v^2/(1 - \theta^2))) - 1).$$

Using the same parameter values as before plus $\varphi = 0.8$ and $\theta = 0.8$, we find the bias in the Tornqvist index to be - .036 percentage point, negative but still small, while its standard error is .34 percentage points, about 50 percent larger than before.

Again using the autoregressive model, but doubling the variances of the disturbances to $\sigma_u^2 = \sigma_v^2 = .0004$ and keeping other parameters the same yields a bias of - .072 and a standard deviation of .49 percentage points. If instead, the disturbance variances have their original value but the number of goods in the index is reduced from 100 to 10, the bias is - .031 percentage point but the standard deviation balloons to 1.09 percentage points. Doubling the value of the (decimal) growth rate for prices, α, to .06 was found to have little effect of the bias or standard deviation, and changing the value of β had no effect.

To determine the relative importance of the price and quantity disturbances in these results, they were recalculated with the quantity disturbance variance, σ_v^2, set equal to zero. The results were striking in that the quantity disturbances had virtually no effect. The largest difference was for the standard deviation of the Tornqvist index when the number of goods was 10. Setting $\sigma_v^2 = 0$ reduced this standard deviation by .0001 percentage point!

Although the parameters in these examples are only illustrative, several conclusions can be drawn about the likely effect of stochastic disturbances on real economic aggregates calculated using the implicit Tornqvist index. Significant bias in such estimates as a result of stochastic disturbances may be infrequent. Nevertheless, because the potential number of estimated index values is large, stochastically induced bias in some series and time periods should be expected. Most likely to be affected are indexes for aggregates consisting of few goods, those that include goods with highly volatile prices, and those for which price shocks tend to persist over time. Quantity shocks, on the other hand, should not be a concern.

2. The optimal linking interval

The results of the preceding section imply that to avoid measurement errors in multi-period growth rates caused by stochastic disturbances in intervening periods, it is necessary is to identify any large, temporary changes in relative prices. One could then eliminate the effect of these large disturbances by chaining over them, introducing multi-period linking on a case by case basis. The number of consecutive observations eliminated would depend on the statistical characteristics of the price series. Quarterly linking would be avoided if the quarterly data exhibited a great deal of volatility compared with, say, annual data. Alternatively, the volatility associated with energy or food prices might indicate a need for chaining across several years. Forsyth and Fowler, and Szulc, both suggest the possibility of chaining with a variable linking interval, but they do not suggest a systematic way of implementing this idea. More recently, Hill [2001] has considered linking nonconsecutive periods based on a spanning tree.

The rationale for chaining is that point-to-point indexes are most consistent with economic theory and therefore the most reliable when the periods being compared are similar. The dimensions for which similarity is of greatest concern are the preferences of consumers and the economic choices open to them. However, for neither of these will the most similar period necessarily be the adjacent period. Seasonal variation is an obvious example. Statistical methods can smooth out the regular part of these fluctuations, but the irregular extremes remain in the data. Fads may persist over several years. Particular goods may be temporarily in short supply. Employment fluctuations have major effects on both the quantities and mix of goods purchased. It follows that the goal of chaining might be better realized if linking were done between periods that were most alike, in some appropriate sense, rather than always between periods that are adjacent in time. Our finding that price fluctuations dominate quantity fluctuations in the calculation of the Tornqvist price index suggests similar relative prices as a criterion for selecting periods as endpoints for linking. Consider an example. If there are three periods with slowly changing prices except that, in period 2, there is a non-recurring spike in an important good, one would chain directly from period 1 to period 3. In the absence of the spike, one would chain from period 1 to 2 and then from 2 to 3. The economic theory of index numbers requires that the link points in a chain index be a series of equilibrium points; the assumption that the consumer has succeeded in purchasing his utility-maximizing market basket is much more likely to be a good approximation in the periods in which the price spike is absent.

There are many ways in which relative weights can be assigned to avoid observations that are atypical while retaining those that form a relatively smooth sequence. We shall adopt an approach based on quadratic loss functions, which have gained acceptance in a wide range of economic contexts. We shall first define the distance between the relative prices in any two periods. Then we shall choose a subset of observations, including the first and last of the available observations, such that, as one traverses from the first selected observation to the last, the sum of squared distances is a minimum. We expect that, most of the time, observations will be selected that come from adjacent time periods. However, observations in which there are important temporary price changes will be skipped.

The measure of distance between two price vectors we shall use is

Relative-price distance: The relative-price distance, $D(j, k)$, between two price vectors $P_j = (p_{ij})$ and $P_k = (p_{ik})$ is

$$D(j,k) = \min_{\lambda_{jk}} \left(\sum_{i=1}^{i=n} \frac{s_{ij} + s_{ik}}{2} (\ln p_{ik} - \ln p_{ij} - \lambda_{jk})^2 \right)^{1/2}, \tag{7}$$

where s_{ij} and s_{ik} are expenditure shares in periods j and k, respectively.[2]

[2] Another measure of dissimilarity in relative prices, used by Hill [2001], is the Paasche-Laspeyres spread, defined as the absolute value of the difference between the logarithms of the Paasche and Laspeyres indexes. However, the meaning of the unit of distance it provides is less intuitive. The interpretation of (7) derives from its close relation to Euclidean distance.

It is assumed in this section that all expenditure shares are positive. When $k > j$ and all prices change in the same proportion ζ, $p_{ik} = \zeta p_{ij}$ for all i, $\lambda_{jk} = ln\ \zeta$, and $D(j,k) = 0$. Otherwise, $D(j,k)$ depends on the sum over goods of the share-weighted squared differences between the rate of change in each good's price and the minimizing constant λ_{jk}.

Performing the minimization in (7), we obtain

(8) $$\lambda_{jk} = \sum_i \frac{s_{ij} + s_{ik}}{2} (\ln p_{ik} - \ln p_{ij}),$$

which is just the logarithm of the corresponding Tornqvist price index. Thus, the relative-price distance D between j and k is equal to the (share-weighted) standard deviation of relative price changes over the interval, taken around a measure of the average change in prices. The squared distance is the variance of these price changes.

Let $\mathbf{T} = \{\tau_j\}$, $j = 1, 2, ..., t$, be the ordered set of periods for which data are available and let $\mathbf{T}^* = \{\tau_{j^*}\}$, $j^* = 1^*, 2^*, ..., t^*$, be the subset of $\mathbf{T}$ selected for linking. (Here $1^* = 1$ and $t^* = t$, but, in general, $j^* \neq j$.) The elements of $\mathbf{T}^*$ are those that minimize the loss function

$$L = \sum_{j^*=2^*}^{j^*=t^*} \sum_{i=1}^{i=n} \frac{s_{i,(j-1)^*} + s_{ij^*}}{2} (\ln p_{ij^*} - \ln p_{i,(j-1)^*} - \lambda_{(j-1)^*,j^*})^2,$$

where $\lambda_{(j-1)^*,j^*}$ is computed from (8) for each pair $((j-1)^*, j^*)$. In the language of graph theory, a graph is defined by a set of points, called nodes, and the set of allowed transitions from one node to another, called edges. In the present case, the nodes are the set of price vectors P_j and all transitions are allowed except that the transition from any node must be to a node later in time. This last requirement makes ours a directed graph. Using squared relative-price distance as the measure of distance between nodes, we seek the shortest path through the graph from P_0 to P_t. A well-known solution to this optimization problem has been provided by Dijkstra [1959].[3]

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[3] The problem considered here is different from the spanning tree problem considered by Hill [2005] and thus has a different solution. Hill is interested in finding, for a given data set, the linking sequences for which different superlative indexes will yield the most similar index values. His criterion function thus includes all observations, some of which are found to provide a path through the graph while the others reside on branches. Our goal is the best estimates of cumulative rates of growth. We need the shortest path through the graph; observations not on this path are not represented in the minimizing value of L.

(Information about the Dijkstra algorithm is available on many web sites.)[4] The selection of the $\mathbf{T}^*$ can be interpreted as providing the sequence of subintervals that best approximate the constant relative prices assumption of the Hicks aggregation theorem.

Table 1 shows an example of the intransitivity of an annually chained index. There are two goods and four years. In Year 3, all prices and quantities return to their values in Year 0. However, the annually linked Tornqvist price index, which is 1.00 in Year 0, is 0.85 in Year 3, or 15 percent lower. When the Dijkstra algorithm is applied to the same data, the result is mathematically trivial but economically significant. Because the data in the starting and ending periods are the same, the distance between these nodes is zero and all intermediate observations are discarded. The price index in the final period takes on its reasonable value of one. A dramatic improvement in the transitivity of the index results from linking in accordance with the Dijkstra algorithm rather than with consecutive observations. Another example, optimal linking when there are three goods, is presented in the Appendix.

It remains to describe a method for filling in estimates for the years (or quarters) that linking with the Dijkstra algorithm omits. Suppose there are m years between two nodes and let g_j, $j = 1, \ldots, m$ be annual Tornqvist rates of price change (inflation rates, expressed as decimals). Let G be the corresponding cumulative (i.e., annually chained) inflation rate over m years and let Γ be the binary Tornqvist inflation rate. Suppose that all the g_j are positive. We would like to adjust the g_j so that the cumulative growth rate equals Γ. Proportional adjustment of the g_j would give the revised annual inflation rates

$$(9) \qquad g_j^* = g_j + (\Gamma - G)\frac{g_j}{G}.$$

However, equation (9) is not exact because cumulative growth satisfies

$$(1 + g_1)(1 + g_2) \ldots (1 + g_m) = 1 + G.$$

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[4] This footnote provides an outline of the algorithm. Initially, the distance from the terminal node to the terminal node, d_s, is set equal to zero and all other distances to the terminal node, d_i, not yet determined, are set equal to infinity. The set of nodes T is initialized to the set of all nodes. The following process is repeated until T is empty. (1) The node $v_m \in T$ with smallest distance to the terminal node is identified. (2) The set of edges (v_m, v_t) with $v_t \in T$ is identified. (3) Each distance d_t is updated to $d_t = d_m +$ length (v_m, v_t) if that distance is smaller than the current value of d_t. (4) v_m is deleted from T.

The result is the length of the minimum-distance path from each node to the terminal node. Some additional bookkeeping can provide the nodes that comprise this path. Note that "distance" in this footnote corresponds to "relative-price distance squared" in the text.

Proportional adjustment of the expressions $ln\ (1 + g_j)$, so that these values cumulate to $ln\ (1 + \Gamma)$, is seen to require

$$(10) \qquad \ln(1 + g_j^{*}) \;=\; \ln(1 + g_j) \;+\; [\ln(1 + \Gamma) \;-\; \ln(1 + G)]\frac{\ln(1 + g_j)}{\ln(1 + G)}.$$

Equation (10) simplifies to

$$(11) \qquad g_j^{*} \;=\; (1 + g_j)^{\{ln\ (1\ \cdot\ \Gamma)\}\ \{ln\ (1\ \cdot\ G)\}} - 1.$$

For GDP, price deflation has not occurred since the 1930's, but major components of GDP experienced price declines more recently.[5] If some of the g_j's are negative, the adjustment task is more complicated. Proportional adjustment will move some rates of change in the opposite direction from the overall adjustment, requiring the other adjustments to be larger than necessary. Specifically, if $\Gamma > G$ (directly measured growth greater than cumulative growth), negative one-period changes will be made more negative and the positive adjustments must be made more positive to compensate, while if $\Gamma < G$, it is the positive rates of change that will be adjusted in the wrong direction. However, all adjustments would be in the same direction if the logarithms of the absolute values of $ln\ (1 + g_j)$, $j = 1, ..., m$, were adjusted equally.

To implement such adjustment, let a denote the sum of the $ln\ (1 + g_j)$ that are positive and let b denote the sum of the $ln\ (1 + g_j)$ that are negative. Let $\underline{a}$ and $\underline{b}$ denote these sums, respectively, after equal logarithmic adjustment of their absolute values. The relative adjustment to a will be the reciprocal of the relative adjustment to b, so

$$a/\underline{a} \;=\; \underline{b}/b.$$

The adjusted sums $\underline{a}$ and $\underline{b}$ must also satisfy

$$\underline{a} + \underline{b} \;=\; ln\ (1 + \Gamma).$$

Substituting $ab/\underline{a}$ for $\underline{b}$ in the latter equation gives

$$\underline{a}^2 \;-\; ln\ (1 + \Gamma)\ \underline{a} + ab = 0.$$

From the quadratic formula,

$$\underline{a} \;=\; [\tfrac{1}{2}\ ln\ (1 + \Gamma)] \;\pm\; \tfrac{1}{2}\ \{[ln\ (1 + \Gamma)]^2 \;-\; 4ab\}^{1\ 2},$$

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[5] For example, in the year 2002, prices of the following aggregates were lower than in the preceding year: personal consumption expenditures on durable goods, nonresidential gross private domestic investment, total exports, and total imports.

where, of course, $ab < 0$. The positive square root applies. Adjusted positive growth rates are calculated from

$$ln\ (1 + g_j^*) \ = \ (\underline{a}/a)\ ln\ (1 + g_j)$$

or

(12) $$g_j^* \ = \ (1 + g_j)^{\underline{a}\ a} \ - 1,$$

while adjusted negative growth rates are calculated from

$$g_j^* \ = \ (1 + g_j)^{\underline{b}\ b} \ - 1.$$

Note that if no growth rates are negative, $a = ln\ (1 + G)$ and $\underline{a} \ = \ ln\ (1 + \Gamma)$, so (12) gives the same result as (11).[6]

3. Real growth estimates with optimal linking intervals

Growth rates for real economic aggregates in the U.S. national income and product accounts were estimated using the minimum squared distance linking rule developed in the preceding section. The implicit Tornqvist index was used. Where the optimal linking interval is short, the Fisher index would give virtually the same result. However, where long intervals are found to be optimal, the choice of the implicit Tornqvist index reduces chain drift. As anticipated, the results for many periods are close to the published values. This section presents estimates of rates of growth for the selected series for time periods for which use of an optimal linking interval gives significantly different results.

Most of our results use quarterly, seasonally adjusted data. The quarterly series were GDP, gross private domestic investment, private fixed investment, imports of goods, and imports of food, feed, and beverages. Data were for the period 1967-2002 for all series except imports of food, feed, and beverages, for which component detail is not available prior to 1985. For series beginning in 1967, the level of detail was the deflation level for that year. Where additional detail became available in subsequent years, the data were extended at the original level of detail using the corresponding Fisher price and quantity indexes. The periods preceding, during, and immediately following World War II were more volatile than subsequent U.S. economic experience, and so the linking rule of the preceding section might be expected to have a large impact on estimates for those periods. Quarterly real data are not available for the NIPA's prior to 1947. The minimum squared distance linking rule was applied to annual data for the period 1929-1959, but only for GDP. A data set was constructed using the deflation level of detail for

[6] The adjustment procedure described in this paragraph was suggested in another context by Marshall Reinsdorf.

1935. The additional detail posited for the years 1929-1934 did not affect the calculated values of real GDP.

To carry out the calculations reported here, the procedure for identifying linking intervals described in the preceding section had to be modified slightly. In contrast to our previous assumption, not all expenditure shares in the NIPA's are positive. Imports enter GDP with negative sign and change in private inventories, part of both GDP and GPDI, can be either a positive or negative. It thus becomes possible for squared relative-price difference to be negative, which would leave relative-price distance (upon taking the square root) undefined. To avoid this possibility, the definition of relative-price distance was modified by replacing the sum in (7) by its absolute value before taking the square root. Were this adjustment not made, relative-price distances for GDP would be undefined in the early 1970's as a result of the effect of the oil-price shocks of that period on imports prices. In addition, extending the analysis through 2002 (rather than through 1997) requires certain adjustments to avoid data discontinuities caused by the recent switch in the NIPA's industrial classification system from SIC to NAICS beginning with the year 1997. In most cases, data discontinuities were avoided by reverting to a higher level of aggregation. However, data discontinuities in change in private inventories required special treatment.[7]

When optimal linking intervals are calculated for real GDP over the thirty-five year period 1967-2002, one-quarter intervals dominate with two-quarter intervals the next most common. The average linking interval is 1.7 quarters. The longer linking intervals include three that are 4-quarter, two 5-quarter, one 7-quarter, and one 17-quarter. Similar patterns emerge for the optimal linking intervals for the other series tested.

For most linking intervals of more than one quarter, the difference in growth rates between using quarterly linking and optimal linking intervals was small. For the five series tested, there were a total of 11 periods for which the average growth rate over the period for the optimally-linked implicit Tornqvist estimate differed from the quarterly Fisher estimate by more than 0.15 percentage point. Of these, one was for GDP, three were for GPDI, four were for imports of goods, and three were for imports of foods, feeds, and beverages. There were no large

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[7] Most of the 1997 discontinuity in the data for inventory change in nominal dollars can be avoided by aggregating to ten major components. We constructed nominal-dollar series for the entire period by using gradually changing weighted averages of SIC-based and NAICS-based estimates for the quarters of 1997. Choosing appropriate price indexes for these series is a more difficult problem. Price indexes for inventory change cannot be constructed using the Fisher or Tornqvist formulas because of sign change problems. Following Ehemann [2004], price indexes for inventories were based on gross flows (additions or reductions to inventories) when such indexes were available. Thus, the price indexes used to deflate to the major components of inventory change in nominal dollars were those for farm marketings, manufacturing shipments, and (with some disaggregation) wholesale and retail trade sales. For the remaining components, which are relatively small, implicit price deflators for inventory stocks were used.

differences for private fixed investment. This reflects the fact that although there were large changes in relative prices within private fixed investment over the period as a result of the huge decline in computer prices, these changes were quite steady, with few price reversals.

Table 2 summarizes the results for the 11 periods for which the optimally-linked implicit Tornqvist has a large effect. Column (6) shows the difference in the average rate of real growth, in percent change at annual rate, between the optimally-linked implicit Tornqvist and the quarterly Fisher over the period shown in column (2). Table 3 shows separately the amounts of this difference attributable to using the implicit Tornqvist index rather than the Fisher and of optimal rather than quarterly linking. That table shows that almost all the differences are the result of optimal linking.

Table 2 shows that in three periods the differences in average growth rates are particularly large, almost two percentage points for imports of goods (1979Q1-1986Q2) and more than half a percentage point for two periods for imports of foods, feeds and beverages. For imports of food, feeds, and beverages, the largest contributor to the difference is large fluctuations in coffee prices. For imports of goods, the most important contributor is fluctuations in oil prices. The effect of the optimally-linked implicit Tornqvist is also seen in the persistence of the difference in estimated growth rates. The large average difference in estimated growth rate for imports of goods extends for 29 quarters (more than seven years), while the two large differences for foods, feeds, and beverages, which are consecutive in time, persist for 15 and 33 quarters, respectively, a total of twelve years. The difference in estimated growth for GDP (1972Q4-1977Q1) at annual rate is much smaller, 0.19 percentage point. However, this difference persists for 17 quarters (more than four years), so this effect might be considered significant in some contexts, given the importance of this aggregate.

The 29-quarter linking interval for imports of goods provides a good illustration of the importance of fluctuating relative prices compared with changes in relative prices that do not return to earlier levels. In the first quarter of 1979, the beginning of this period, the price of imports of petroleum relative to the price of imports (using the official price indexes based to the year 2000 = 100) was 0.698. In the second quarter of 1986, the end of this period, the relative price of petroleum, after a huge run-up in intervening quarters, returned to the value of 0.696, a local minimum. In contrast, the relative price of petroleum in 1972Q4, near the beginning of the petroleum price increases of the 1970's, was 0.323. However, this relative price does not initiate a long linking interval because the price never returned to this level.

The average differences in growth rates over periods found to be optimal linking intervals conceal even larger differences for many individual quarters. Some of these differences are shown in Table 2, column (8). Imports of goods in 1983Q3 provides the largest difference in growth rates (at annual rate) found for any quarter, 5.46 percentage points. The largest difference for an individual quarter for foods, feeds, and beverages (1992Q4) is close behind at 3.61 percentage points. The largest difference for a quarter for gross private domestic investment was 0.62

percentage point (1981Q3) and the largest for GDP is 0.38 percentage point (1973Q1). Note that the differences in estimates can take on either sign.

When the Dijkstra algorithm is applied to the GDP price data for 1929-59, the minimum-distance path is found to include two long periods: the twelve year period 1930-1942 and the eleven year period 1943-1954. All the other links were for consecutive years. Although the published (Fisher) price index and the annually chained Tornqvist price index both show a GDP price level that is slightly lower in 1942 than in 1930, the binary Tornqvist price index for this period shows a price increase. The corresponding binary implicit Tornqvist quantity index increases an average of 0.46 percentage point less per year than the chained implicit Tornqvist. The largest differences in annual growth rate occurred in the three years of greatest price change, 1931 and 1932, which were years of sharp price declines, and 1942, which was a year of sharp price increases. The annual growth rate shown by the optimally-linked implicit Tornqvist quantity index was 0.93 percentage point lower than the annually-linked implicit Tornqvist quantity index in 1931 and 1.00 percentage point lower in 1932, both years in which real GDP declined sharply. The annual growth rate shown by the optimally-linked implicit Tornqvist quantity was 1.06 percentage points lower in 1942, a year of strong increase in GDP. Thus, the optimally-linked estimates show a sharper decline from the 1929 peak and a slower recovery from the Great Depression than do the official (Fisher) estimates, which are similar those obtained from the annually-linked implicit Tornqvist.

Over the period 1943-1954, the optimally-linked implicit Tornqvist quantity index again shows a smaller increase in real GDP than does the annually-linked index. In this case, the difference is a less dramatic 0.19 percentage point per year. (The cumulative difference over the period is 2.51 percentage points.) The largest differences between annually-linked and optimally-linked implicit Tornqvist growth rates, - 0.29 and -0.31 percentage point in 1946 and 1947, respectively, occurred during the postwar recession. The optimally-linked estimates imply a steeper decline in this period than do the official estimates.

4. Summary and conclusions

As Frisch [1936] observed, finding that chained and binary indexes give different estimates over a specified period is very different from determining which of these estimates is more nearly correct. In this paper, we have attempted to assess the effects of chain drift on the national income and product accounts of the U.S. by comparing them with alternative estimates in which bias caused by chain drift has been reduced. The alternative estimates differ from the official estimates in three ways. First, the index number formula employed is the implicit Tornqvist index rather than the Fisher index used in the official estimates. Although these two index number formulas give very similar estimates with annual linking, a companion paper [Ehemann 2005] shows that lengthening the linking interval has substantially less effect on estimates obtained using the implicit Tornqvist index than on those using the Fisher. Thus, the implicit Tornqvist provides more flexibility to consider other linking intervals. Second, instead of annual linking, a procedure is employed for choosing the optimal linking interval for each particular aggregate and period.

This optimal linking interval was most often found to be a quarter, but was occasionally several years in length. It was these long intervals that provided differences from the official estimates. Third, and least important, levels of detail were held constant to simplify calculations and so that changes in detail would not affect results.

The implicit Tornqvist quantity index requires calculation of the (direct) Tornqvist price index. An analysis of the effects of stochastic shocks on the Tornqvist price index showed that large shocks to prices could bias the estimated aggregated price change. However, significant effects required rather large price shocks and such large shocks should not be expected to be frequent. The effects of quantity shocks were found to be very small. Thus, a criterion was sought for identifying large, transitory shocks to prices. The procedure adopted was to choose, for a specified aggregate and period, the subset of observations, in original sequence, for which a measure of relative price change from one observation to the next was minimized. This procedure filtered out the most important temporary price changes for individual goods as well as more general changes in relative prices that were subsequently reversed. In addition to the purely statistical benefit of the resulting smoother changes in relative prices for the component goods of the aggregate, the smoother price changes had two other advantages: (1) the omitted observations will often be ones in which consumers are less likely to be in equilibrium, so that the assumptions of the economic theory of index numbers are more nearly fulfilled, and (2) aggregate price change is measured between those pairs of observations that most nearly fulfill the requirement of proportional price change of the Hicks aggregation theorem.

When index numbers are calculated with different linking intervals, the rate of change in the aggregate is ambiguous due to chain drift. A rule is required for choosing the best linking interval. The optimally-linked implicit Tornqvist quantity index provides a rule that is an alternative to an arbitrarily selected constant interval. The suggested rule, together with the implicit Tornqvist index number formula, reduces potential permanent shocks to the estimated aggregate resulting from the combination of nonhomothetic preferences and temporary price shocks. However, it is based on some additional assumptions: (1) A quadratic criterion function is adopted for the minimization of relative price change. A different functional form could give different results. (2) Estimates for recent periods must be regarded as provisional because of the possibility that relative prices will return to those observed earlier, thus changing the optimal linking interval. (3) By focusing on pairs of observations with similar relative prices, no weight is given to the evolution of consumer preferences over time. In time-distant pairs with similar relative prices, consumers may have somewhat different preferences, in contrast to the assumption of the economic theory of index numbers.

The empirical examples using the optimally-linking implicit Tornqvist indicate that, most of the time, the national income and product accounts of the United States are not adversely affected by chain drift. Some significant differences in growth rates were found for real imports of goods in the 1980's and for real imports of foods, feeds, and beverages in the 1980's and 1990's, but these series have exceptionally high price volatility. Significant differences were also found for real GDP during the Great Depression and the transition to World War II, but this period was among the most turbulent, economically, in U. S. history. Smaller differences should be expected, on average, if the procedures of this paper were extended to other U.S. series. On the other hand, for countries with small open economies subject to large fluctuations in relative prices, the optimally-linked implicit Tornqvist quantity index might produce larger differences from conventional measures than it does for the United States.

Appendix:
Selecting Linking Intervals When There are Three Goods

As an example of the process for selecting intervals for linking, suppose that there are three goods with relative prices p_{1j}, p_{2j}, and p_{3j}, respectively, in period j. Let s_{1j}, s_{2j}, and s_{3j} be the corresponding expenditure shares. From equations (7) and (8) in the text, the squared relative-price distance between the prices prevailing at times j' and j'' is

$$D^2(j',j'') = \sum_{i=1}^{i=3} \frac{s_{ij'} + s_{ij''}}{2} \left[\ln p_{ij''} - \ln p_{ij'} - \sum_{h=1}^{h=3} \frac{s_{hj'} + s_{hj''}}{2} (\ln p_{hj''} - \ln p_{hj'}) \right]^2 .$$

To represent the relative-price distance graphically, assume, provisionally, that all elasticities of substitution equal one and preferences are homothetic. Then the relative shares (denoted by s_1, s_2, and s_3, respectively) are constant and the Tornqvist index collapses to the geometric index. Define relative prices in period j as

$$r_{ij} = \frac{p_{ij}}{\prod_{h=1}^{h=3} p_{hj}^{s_h}} \qquad \text{for } i = 1,\ 2,\ 3.$$

Then

$$D^2(j',j'') = \sum_{i=1}^{i=3} s_i (\ln r_{ij''} - \ln r_{ij'})^2 . \tag{A.1}$$

Because the logarithms of the relative prices are linearly dependent, the squared distance $D^2(j', j'')$ can be expressed in two dimensions rather than three. Define

$$y_j = M_1\, s_1 \ln r_{1j} + M_2\, s_2 \ln r_{2j} \tag{A.2}$$

and

$$z_j = N_1\, s_1 \ln r_{1j} + N_2\, s_2 \ln r_{2j}, \tag{A.3}$$

where M_1, M_2, N_1, and N_2 are chosen so that

(A.4) $$(y_{j''} - y_{j'})^2 \;+\; (z_{j''} - z_{j'})^2 \;=\; D^2(j',j'').$$

Solutions for M_1, M_2, N_1, and N_2 are derived below and are found to be functions of the expenditure shares.

Figure 1 illustrates the geometry of relative-price distance and provides a graphical example of the proposed criterion for eliminating misleading information. Initially, in period $t = 0$, relative prices are equal and are located at the origin, point O. In successive periods they are at points A, B, C, and D. In the directed graph of these points, a particular point can be reached from any point that occurs earlier in time. The Euclidean distance between any two points is their relative-price distance. We seek the path between points O and D such that the sum of the squared relative-price distances are a minimum. Because the angle OAB is greater than 90 degrees, the path from O to B is shorter going from O to A and then from A to B than directly from O to B. Similarly, because the angle BCD is less than 90 degrees, the direct path from B to D is shorter than the path including C. Point C might yet be included in the selected set of points $\mathbf{T}^*$ because the path O to C and C to D is shorter than the direct path from O to D. However, the path O to A, A to B, and B to D is shorter still, so $\mathbf{T}^*$ contains O, A, B, and D.

Finally, consider the more realistic case in which elasticities of substitution are between zero and one and preferences are not homothetic. The graph now exists in four dimensions, two representing relative prices and two the corresponding expenditure shares. Figure 1 can be interpreted as showing the projection of this graph onto the two dimensions representing relative prices. In the projection, the topology of the graph is unchanged, but the apparent distance between two points no longer corresponds to the actual distance. This problem can be easily handled by labeling each edge of the graph with the correct distance.

The remainder of this Appendix provides a solution for equations (A.2) and (A.3), which yield coordinates in two dimensions of the three relative prices. Specifically, we obtain distance-preserving values for the coefficients M_1, M_2, N_1, and N_2 of the functions y and z. Combining (A.1) and (A.4), the distance-preserving values of y and z are seen to satisfy

(A.5) $$\sum_{i=1}^{i=3} s_i(\Delta \ln r_i)^2 \;=\; \Delta y^2 \;+\; \Delta z^2,$$

where the operator Δ denotes the difference between values at times $j = j''$ and $j = j'$.

We begin by representing each side of (A.5) by a quadratic form in the expressions $s_1\Delta\, r_1$ and $s_2\Delta\, r_2$. On the left-hand side, because

$$\sum_{i=1}^{i=3} s_i \ln r_i \;=\; 0 \qquad \text{for} \quad j = j'', j',$$

we can make the substitution $s_3(\Delta \ln r_3)^2 = (1/s_3)(s_1\Delta r_1 + s_2\Delta r_{2\cdot})^2$. Rearranging terms then yields

$$\text{(A.6)}\quad \sum_{i=1}^{i=3} s_i(\Delta \ln r_i)^2 = \left(\frac{1}{s_1}+\frac{1}{s_3}\right)(s_1\Delta \ln r_1)^2 + \left(\frac{1}{s_2}+\frac{1}{s_3}\right)(s_2\Delta \ln r_2)^2 + \frac{2}{s_3}(s_1\Delta \ln r_1)(s_2\Delta \ln r_2).$$

To transform the right-hand side of (A.5) into a quadratic form in the same expressions, we take differences in (A.2) and (A.3), square both sides of each equation, and add the results to obtain

$$\begin{aligned} \Delta y^2 + \Delta z^2 &= (M_1 s_1\Delta \ln r_1 + M_2 s_2\Delta \ln r_2)^2 + (N_1 s_1\Delta \ln r_1 + N_2 s_2\Delta \ln r_2)^2 \\ \text{(A.7)}\qquad &= (M_1^2 + N_1^2)(s_1\Delta \ln r_1)^2 + (M_2^2 + N_2^2)(s_2\Delta \ln r_2)^2 \\ &\quad + 2(M_1M_2 + N_1N_2)(s_1\Delta \ln r_1)(s_2\Delta \ln r_2). \end{aligned}$$

Define

$$h_1 = \left(\frac{1}{s_1}+\frac{1}{s_3}\right), \quad h_2 = \left(\frac{1}{s_2}+\frac{1}{s_3}\right), \quad \textit{and} \quad h_3 = \frac{1}{s_3}.$$

Then equating coefficients in (A.6) and (A.7) yields three equations:

$$\text{(A.8)}\qquad \begin{aligned} h_1 &= M_1^2 + N_1^2 \\ h_2 &= M_2^2 + N_2^2 \\ h_3 &= M_1M_2 + N_1N_2. \end{aligned}$$

Because there are four unknowns on the right-hand side, there are many solutions. We shall choose a solution such that

$$\text{(A.9)}\qquad M_2 = \gamma M_1 \qquad \text{and} \qquad N_2 = -\gamma N_1$$

for some γ. Squaring both sides of each equation in (A.9) and adding the results gives

$$\text{(A.10)}\qquad M_2^2 + N_2^2 = \gamma^2(M_1^2 + N_1^2).$$

Substituting the first two equations of (A. 4) into (A.10), solving for γ, and choosing the positive square root yields

$$\text{(A.11)}\qquad \gamma = (h_1/h_2)^{1\ 2}.$$

To solve for M_1, we substitute the two equations of (A.9) into the third equation of (A.8), obtaining

(A.12) $$h_3 = \gamma(M_1^2 - N_1^2).$$

Dividing both sides of (A.12) by γ and adding the result to the first equation of (A.8) yields

$$h_1 + \frac{h_3}{\gamma} = 2M_1^2.$$

It follows, on substituting from (A.11) and choosing the positive square root, that

(A.13) $$M_1 = \left[\frac{1}{2}\left(h_1 + h_3\sqrt{\frac{h_1}{h_2}}\right)\right]^{1/2}.$$

To obtain M_2, we substitute (A.11) and (A.13) into the first equation of (A.9), obtaining

(A.14) $$M_2 = \left[\frac{1}{2}\left(h_2 + h_3\sqrt{\frac{h_2}{h_1}}\right)\right]^{1/2}.$$

To obtain N_1, we substitute (A.9) squared into the first equation of (A.8). Choosing the positive square root, we have

(A.l5) $$N_1 = \left[\frac{1}{2}\left(h_1 - h_3\sqrt{\frac{h_1}{h_2}}\right)\right]^{1/2}.$$

To obtain N_2, we substitute (A. 11) squared into the second equation of (A.9), obtaining

(A.16) $$N_2 = \left[\frac{1}{2}\left(h_2 - h_3\sqrt{\frac{h_2}{h_1}}\right)\right]^{1/2}.$$

It remains to show that the solutions for N_1 and N_2 are real. In both cases, the solution will be real if $h_3 < \sqrt{h_1 h_2}$. Squaring both sides of this inequality and substituting for the h_i's gives the requirement

$$\left(\frac{1}{1-s_1-s_2}\right)^2 < \left(\frac{1}{s_1}+\frac{1}{1-s_1-s_2}\right)\left(\frac{1}{s_2}+\frac{1}{1-s_1-s_2}\right).$$

Taking reciprocals gives

$$(1-s_1-s_2)^2 < \left(\frac{1}{\frac{1}{s_1}+\frac{1}{1-s_1-s_2}}\right)\left(\frac{1}{\frac{1}{s_2}+\frac{1}{1-s_1-s_2}}\right)$$

$$= \frac{s_1}{1-s_1}\frac{s_2}{1-s_2}(1-s_1-s_2)^2.$$

Because s_1, s_2, and s_3 must each be between zero and one, the expression $\frac{s_1}{1-s_1}\frac{s_2}{1-s_2}$ is always between zero and 0.5. Thus, the inequality holds and the solution is real.

Substituting (A.9), (A.14), (A.15), and (A.16) into (A.2) and (A.3) gives a pair of equations for y_t and z_t:

$$y_t = \left[\frac{1}{2}\left(h_1 + h_3\sqrt{\frac{h_1}{h_2}}\right)\right]^{1/2} s_1 \ln r_{1t} + \left[\frac{1}{2}\left(h_2 + h_3\sqrt{\frac{h_2}{h_1}}\right)\right]^{1/2} s_2 \ln r_{2t}$$

$$z_t = \left[\frac{1}{2}\left(h_1 - h_3\sqrt{\frac{h_1}{h_2}}\right)\right]^{1/2} s_1 \ln r_{1t} + \left[\frac{1}{2}\left(h_2 - h_3\sqrt{\frac{h_2}{h_1}}\right)\right]^{1/2} s_2 \ln r_{2t}.$$

References

Diewert, W. Erwin, (1976) "Exact and Superlative Index Numbers," *Journal of Econometrics* 4(2), pp. 115-145.

Dijkstra, Edsger W., (1959) "A Note on Two Problems in Connection with Graphs," *Numerische Mathematik* 1, pp. 269-271.

Ehemann, Christian, (2005) "Chain Drift in Leading Superlative Indexes," Working paper 2005-09, Bureau of Economic Analysis, U. S. Department of Commerce, Washington, D.C.

Ehemann, Christian, (2004) "An Alternative Estimate of Real Inventory Change for National Economic Accounts," *International Journal of Production Economics*, 93-94C, pp. 101-110.

Erickson, Timothy, (1998) "Expected Values of Laspeyres and Geometric Mean Indexes When Price Relatives and Expenditure Weights are Replaced by Well-Behaved Estimates," DPINR Working Leaflet #1, U. S. Bureau of Labor Statistics, Washington, D.C.

Forsyth, F. G., and R. F. Fowler, (1981) "The Theory and Practice of Chain Price Index Numbers," *Journal of the Royal Statistical Society,* Series A 144 Part 2, pp. 224-246.

Frisch, Ragnar, (1936) "Annual Survey of Economic Theory: The Problem of Index Numbers," *Econometrica* 4 , pp. 1-39.

Greenlees, John S., (2001) "Random Errors and Superlative Indexes," *BLS Working Paper 343*. Washington, Bureau of Labor Statistics.

Hill, Robert J., (2001) "Measuring Inflation and Growth Using Spanning Trees," *International Economic Review,* 42(1), pp. 167-186.

Lent, Janice, (2000) "Chain Drift in Some Price Index Estimators," Proceedings of the Survey Research Methods Section, American Statistical Association. On the Internet at www.amstatorg/srms/Proceedings/y2000.html.

Prasada Rao, D. S., and E. A. Selvanathan (1992) "Uniformly Minimum Variance Unbiased Estimators of Theil-Tornqvist Index Numbers," *Economics Letters,* 39, pp. 123-127.

Samuelson, Paul A., and S. Swamy, (1974) "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis, *American Economic Review* 64(4), pp. 566-593.

Szulc, Bohdan J., (1983) "Linking Price Index Numbers," in W. E. Diewert and C. Montmarquette, eds., *Price Level Measurement: Proceedings of a Conference Sponsored by Statistics Canada,* Ottawa: Minister of Supply and Services Canada, pp. 567-598.

U. S. Bureau of Economic Analysis, (2001) *National Income and Product Accounts of the United States, 1929-97: Volume 1.* Washington, D.C.: U.S. Government Printing Office.

Young, Allan R., (1992), "Alternative Measures of Change in Real Output and Prices," *Survey of Current Business* 72(4), pp. 32-48.

Table 1.–Example of Chain Index Intransitivity

	Year 0	Year 1	Year 2	Year 3
Price of good 1	1.00	2.00	1.50	1.00
Quantity of good 1	3.00	1.40	2.20	3.00
Price of good 2	1.00	0.30	0.50	1.00
Quantity of good 2	2.00	5.00	1.70	2.00
Tornqvist price relative	--	0.98	0.94	0.93
Tornqvist chain price index	1.00	0.98	0.92	0.85

Table 2.–Growth Rate Differences for Selected Series and Periods, Quarterly Fisher vs. Optimally Linked Implicit (OLI) Tornqvist

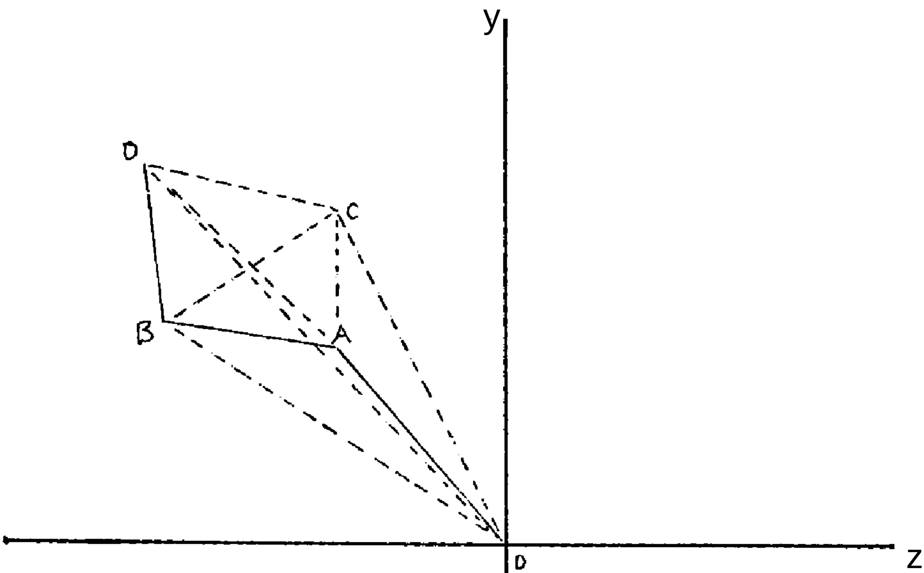
(1) Series (real)	(2) Period	(3) Quarters	(4) Percent change at annual rate: Quarterly Fisher	(5) Percent change at annual rate: OLI Tornqvist	(6) Percent change at annual rate: Difference (5) - (4)	(7) Maximum difference in percent change (a.r.): Quarter	(8) Maximum difference in percent change (a.r.): Difference
Gross domestic product	1972Q4-1977Q1	17	2.28	2.47	0.19	1974Q4	0.31
Gross private domestic investment	1972Q3-1973Q1	2	14.65	14.43	- .22	1973Q1	- .38
Gross private domestic investment	1973Q1-1973Q3	2	-.23	.23	.46	1973Q2	.51
Gross private domestic investment	1981Q2-1983Q1	7	-9.08	-9.30	-.22	1981Q3	- .62
Imports of goods	1979Q1-1986Q2	29	5.84	7.77	1.93	1983Q3	5.46
Imports of goods	1989Q4-1991Q2	6	- .19	.08	.27	1990Q4	.46
Imports of goods	1996Q3-1997Q2	3	14.41	14.62	.21	1997Q2	.29
Imports of goods	1997Q4-1999Q4	8	12.19	12.04	- .15	1999Q2	- .30
Imports of foods, feeds, and beverages	1985Q2-1989Q1	15	.21	-.32	-.54	1988Q2	- 1.14
Imports of foods, feeds, and beverages	1989Q1-1997Q2	33	4.23	3.65	-.58	1992Q4	- 3.61
Imports of foods, feeds, and beverages	1997Q2-1998Q1	3	10.82	10.63	-.19	1997Q2	- .84

Table 3.–Growth Rate Differences by Source

(1) Series (real)	(2) Period	(3) Quarterly Implicit Tornqvist	(4) Effect of Implicit Tornqvist[a]	(5) Effect of Optimal Linking[b]
		Percent change at annual rate		
Gross domestic product	1972Q4-1977Q1	2.28	0.00	0.19
Gross private domestic investment	1972Q3-1973Q1	14.63	-.02	-.20
Gross private domestic investment	1973Q1-1973Q3	-.23	.00	.46
Gross private domestic investment	1981Q2-1983Q1	-9.19	-.11	-.11
Imports of goods	1979Q1-1986Q2	5.84	.00	1.93
Imports of goods	1989Q4-1991Q2	- .18	.01	.26
Imports of goods	1996Q3-1997Q2	14.42	.01	.20
Imports of goods	1997Q4-1999Q4	12.22	.03	-.18
Imports of foods, feeds, and beverages	1985Q2-1989Q1	.19	-.02	-.52
Imports of foods, feeds, and beverages	1989Q1-1997Q2	4.20	-.03	-.55
Imports of foods, feeds, and beverages	1997Q2-1998Q1	10.80	-.02	-.17

[a] Column (3), this table, less column (4) of Table 2.
[b] Column (5) of Table 2 less column (3) of this table.

Figure 1--Relative-Price Distances with Three Goods



Segment-lengths squared

OA = 2.89	AC = 0.81
OB = 7.29	AD = 2.89
OC = 6.25	BC = 1.96
OD = 12.25	BD = 1.00
AB = 1.44	CD = 1.69

Sums of squared lengths

OA + AB + BC + CD = 7.98

OA + AB + BD = 5.33

OA + AC + CD = 5.39

OA + AD = 5.78

OB + BC + CD = 9.25

OB + BD = 8.29

OC + CD = 7.94

OD = 12.25