Limitations of Balanced Half Sampling

Richard Valliant

U.S. Bureau of Labor Statistics

Room 4925, 2 Massachusetts Avenue NE

Washington DC 20212

June 1995

Abstract

Balanced half-sample (*BHS*) variance estimation is a popular technique among survey statisticians, but it has limitations. These limits are studied theoretically through a model-based approach and illustrated with simulations using artificial and real populations. In the fully balanced case, under a model often used for stratified, clustered populations, *BHS* produces a model-unbiased variance estimator for only one member of a broad class of estimators of totals. Another implementation of *BHS* variance estimation in large, complex surveys is to use partial balancing or grouping of strata to reduce the number of resample estimates that must be calculated. Instead of selecting a fully balanced, orthogonal set of half-samples, strata are combined into groups and a set of half-samples only large enough to be balanced on the groups is selected. For two-stage cluster samples either with or without poststratification this leads to an inconsistent variance estimator.

Key words: balanced repeated replication, inconsistent variance estimator, model-based sampling, partial balancing, poststratification, two-stage cluster sampling.

Acknowledgment

Any opinions expressed are those of the author and do not constitute policy of the Bureau of Labor Statistics. I am grateful to Alan Dorfman and the referees for their careful reviews.

1. Introduction

Balanced half-sample (*BHS*) variance estimators, and resampling estimators generally, are widely used in sample surveys because of their simplicity and flexibility. Properly applied, they can accommodate complex survey designs and complicated estimators without explicit derivations of variance formulae for different types of estimators. Thoughtless application can, however, lead to problems. This paper discusses some of the difficulties associated with *BHS* generally and with the shortcut method known as partial balancing. We consider stratified clustered populations from which two-stage samples are selected. Following the introduction of notation in section 2, section 3 presents a general class of estimators in which the *BHS* variance estimator can be design-unbiased but model-biased. A subclass of estimators is noted where *BHS* is model-unbiased. Section 4 discusses a situation, common in practice, where the partially balanced or grouped *BHS* variance estimator is inconsistent. The inconsistency result is extended to poststratification in section 5. Simulation results using real and artificial populations are given in section 6 in support of the theory.

2. Notation and Model

The population of units is divided into H strata with stratum h containing N_h clusters. Cluster (hi) contains M_{hi} units with the total number of units in stratum h being $M_h = \sum_{i=1}^{N_h} M_{hi}$ and the total in the population being $M = \sum_{h=1}^{H} M_h$. Associated with each unit in the population is a random variable y_{hij} whose finite population total is $T = \sum_h \sum_{i=1}^{N_h} \sum_{j=1}^{M_{hi}} y_{hij}$. The working model is

$$E_M\left(y_{hij}\right)=\mu_h$$

$$\operatorname{cov}_M\left(y_{hij},y_{h'i'j'}\right)=\begin{cases}\sigma^2_{hi} & h=h', i=i', j=j'\\ \sigma^2_{hi}\rho_{hi} & h=h', i=i', j\neq j'\\ 0 & \text{otherwise}\end{cases} \tag{1}$$

A two-stage sample is selected from each stratum consisting of $n_h = 2$ sample clusters and a subsample of m_{hi} sample units is selected within sample cluster (*hi*). The total number of clusters in the sample is $n=\sum_h n_h$. The set of sample clusters from stratum h is denoted by s_h and the subsample of units within sample cluster (*hi*) by s_{hi}. Model (1) is reasonably general in allowing the variance and the covariance among units to be different for every cluster in the population while specifying a common mean within each stratum.

The general estimator of the total T that we will consider in this section has the form:

$$\hat{T}=\sum_h\sum_{i\in s_h}K_{hi}\bar{y}_{hi}\,, \tag{2}$$

where K_{hi} is a coefficient that does not depend on the y's and $\bar{y}_{hi}=\sum_{j\in s_{hi}}y_{hij}\big/m_{hi}$. In order for $\hat{T}$ to be model-unbiased under (1), we must have $\sum_{i\in s_h}K_{hi}=M_h$. Each K_{hi} may also depend on the particular sample selected. A number of examples of estimators that fall in the class defined by (2) were listed in Royall (1986) and Valliant (1987a, 1993). For the ratio estimator, $\hat{T}_R=\sum_h\sum_{i\in s_h}M_{hi}\bar{y}_{hi}\left(M_h\big/\sum_{i\in s_h}M_{hi}\right)$, for example, $K_{hi}=M_{hi}\left(M_h\big/\sum_{i\in s_h}M_{hi}\right)$, a sample dependent quantity. When clusters are selected with probability proportional to M_{hi} and units within clusters are selected with equal probability, the Horvitz-Thompson estimator is unbiased under (1) and has $K_{hi} = M_h / n_h$.

The theory here will cover the situation where H is large. *Lemma* 1 below gives circumstances in which the prediction variance $\text{var}_M\left(\hat{T} - T\right)$ is asymptotically equivalent to $\text{var}_M\left(\hat{T}\right)$. Although we will concentrate on the case of $n_h = 2$, the lemma holds for other bounded sample sizes also. First, define $\text{var}_M\left(\bar{y}_{hi}\right) = \sigma_{hi}^2 \left[1 + \left(m_{hi} - 1\right)\rho_{hi}\right] / m_{hi} \equiv v_{hi}$. The results in Appendix A.1 of Valliant (1993) can be easily modified to obtain

Lemma 1. If, as $H \to \infty$,

(i) $n/M \to 0$,

(ii) $\max_{h,i}\left(M_{hi}\right)$, $\max_{h,i}\left(m_{hi}\right)$, and $\max_{h}\left(N_h\right)$ are $O(1)$

(iii) $\max_{h,i}\left(K_{hi}\right) = O\left(M/n\right)$

(iv) $\left(n/M^2\right)\sum_h \mathbf{K}_h' \mathbf{V}_h \mathbf{K}_h \to G$, a positive constant,

then

$$\begin{aligned} \text{var}_M\left(\hat{T} - T\right) &\approx \text{var}_M\left(\hat{T}\right) \\ &= \sum_h \mathbf{K}_h' \mathbf{V}_h \mathbf{K}_h = \sum_h \sum_{i \in s_h} K_{hi}^2 v_{hi} \end{aligned} \tag{3}$$

where $\mathbf{K}_h = \left(K_{h1}, \ldots, K_{hn_h}\right)'$ and $\mathbf{V}_h = diag\left(v_{hi}\right)$ for $i = 1, \ldots, n_h$.

When the number of strata H is large and the other assumptions in *Lemma* 1 hold, the dominant term of the prediction variance is $\text{var}_M\left(\hat{T}\right)$ just as in the unstratified case studied by Royall (1986) where the sample size of clusters was large and the sampling fraction of clusters was small. The total number of sample clusters actually is large here also, even when $n_h = 2$ in all strata, because $H \to \infty$. Condition (i), $n/M \to 0$, is equivalent to $n/(N\overline{M}) \to 0$ where $\overline{M} = M/N$ is the mean number of units per cluster in the population. Since the maximum cluster sizes are bounded in (ii), the mean $\overline{M}$ is bounded and (i) implies that the overall sampling fraction of clusters n/N is negligible.

In evaluating the performance of the *BHS* variance estimators, our estimation target will be the model variance $\text{var}_M\left(\hat{T}-T\right)$ or its large-sample equivalent $\text{var}_M\left(\hat{T}\right)$. In the presence of a probability sampling plan, another model-related candidate might be $E_pE_M\left(\hat{T}-T\right)^2$ with E_p denoting design-expectation, but, after a sample is selected, various conditionality arguments impel the use of a model, not a random selection plan or a design/model hybrid, for inference (see, e.g. Royall 1988).

3. A Balanced Half Sample Variance Estimator and Limits of its Applicability

Balanced half-sample (*BHS*) variance estimators, proposed by McCarthy (1969), are often used in complex surveys because of their generality and the ease with which they can be programmed. Assume that the population is stratified, as in section 1, and that a sample of $n_h = 2$ primary units is selected from each stratum. There are generalizations of the method to other sample sizes in Gurney and Jewett (1975), Sitter (1993), and Wu (1991), but the $n_h = 2$ case is so common in practice that it deserves special attention. A set of *J* half-samples is defined by the indicators

$$\varsigma_{hi\alpha} = \begin{cases} 1 & \text{if cluster } hi \text{ is in half-sample } \alpha \\ 0 & \text{if not} \end{cases}$$

for *i*=1,2 and α=1,..., *J*. Based on the $\varsigma_{hi\alpha}$, define

$$\begin{aligned}\varsigma_h^{(\alpha)} &= 2\varsigma_{h1\alpha} - 1 \\ &= \begin{cases} 1 & \text{if cluster } h1 \text{ is in half-sample } \alpha \\ -1 & \text{if cluster } h2 \text{ is in half-sample } \alpha \end{cases}\end{aligned}$$

Note also that $-\varsigma_h^{(\alpha)} = 2\varsigma_{h2\alpha} - 1$. A set of half-samples is said to be in full orthogonal balance if

$$\sum_{\alpha=1}^{J} \varsigma_h^{(\alpha)} = 0, \text{ for all } h \text{ and} \tag{4}$$

$$\sum_{\alpha=1}^{J} \varsigma_h^{(\alpha)} \varsigma_{h'}^{(\alpha)} = 0 \ (h \neq h') \tag{5}$$

with a minimal set of half-samples satisfying (4) and (5) having $H + 1 \leq J \leq H + 4$.

Let $\hat{T}^{(\alpha)}$ be the estimator, based on half-sample α, with the same form as the full sample estimator $\hat{T}$. One of several choices of *BHS* variance estimators is

$$v_B\left(\hat{T}\right) = \sum_{\alpha=1}^{J} \left(\hat{T}^{(\alpha)} - \hat{T}\right)^2 \Big/ J .$$

There are other asymptotically equivalent *BHS* estimators, whose large sample properties are the same as those of v_B (Krewski and Rao 1981).

The *BHS* variance estimator is approximately model unbiased under (1) if $E_M\left(v_B\right) = \text{var}_M\left(\hat{T}\right)$ defined by (3). As shown in section 3.1, v_B meets this standard for only one estimator $\hat{T}$ in class (2).

3.1 Model-based Properties

Next, we can evaluate the *BHS* variance estimator and its expectation for the two-stage case. To implement the method, entire clusters are assigned to half-samples, i.e., if a particular cluster is in half-sample α, then all units subsampled from that cluster are assigned to α also. The half-sample estimator of the total is defined as

$$\hat{T}^{(\alpha)} = \sum_h \left(\varsigma_{h1\alpha} K_{h1}^{(\alpha)} \bar{y}_{h1} + \varsigma_{h2\alpha} K_{h2}^{(\alpha)} \bar{y}_{h2} \right)$$

The form of the half-sample term $K_{hi}^{(\alpha)}$ is dictated by the form of $\hat{T}$ and is computed as the full sample coefficient would be if the sample size were $n_h = 1$. The α superscript is attached to $K_{hi}^{(\alpha)}$ since the value will differ from the full sample value. Although we use a superscript α on $K_{hi}^{(\alpha)}$, its value is the same for each half-sample containing unit *hi*. The difference between the half-sample and full-sample estimators is

$$\hat{T}^{(\alpha)} - \hat{T} = \sum_h \sum_{i \in s_h} \left(\varsigma_{hi\alpha} K_{hi}^{(\alpha)} - K_{hi} \right) \bar{y}_{hi} .$$

Using the definitions of $\varsigma_{hi\alpha}$ and $\varsigma_h^{(\alpha)}$, we have $\varsigma_{h1\alpha} = \left[1 + \varsigma_h^{(\alpha)} \right] / 2$ and $\varsigma_{h2\alpha} = \left[1 - \varsigma_h^{(\alpha)} \right] / 2$. The difference $\hat{T}^{(\alpha)} - \hat{T}$ can then be written as

$$\hat{T}^{(\alpha)} - \hat{T} = \sum_h \left\{ \left(\hat{T}_h^{(\alpha)*} - \hat{T}_h \right) + \tfrac{1}{2} \varsigma_h^{(\alpha)} \Delta_{yh}^{(\alpha)} \right\} \qquad (6)$$

where $\hat{T}_h^{(\alpha)*} = \tfrac{1}{2} \left(K_{h1}^{(\alpha)} \bar{y}_{h1} + K_{h2}^{(\alpha)} \bar{y}_{h2} \right)$, $\hat{T}_h = \sum_{i \in s_h} K_{hi} \bar{y}_{hi}$, and $\Delta_{yh}^{(\alpha)} = K_{h1}^{(\alpha)} \bar{y}_{h1} - K_{h2}^{(\alpha)} \bar{y}_{h2}$.

If $\hat{T}^{(\alpha)} - \hat{T}$ is squared out and summed over half-samples, we obtain a tidy reduction, found in McCarthy (1969) and elsewhere, *if* the K_{hi}'s and $K_{hi}^{(\alpha)}$'s have a special form, but *not* in general. In particular, suppose that

(*HS*-1) $K_{hi}^{(\alpha)} = 2 K_{hi}$

holds. This condition corresponds to the standard prescription "double the weights in each half-sample." Not all estimators satisfy *HS*-1; section 3.3 gives examples where that condition is violated. When *HS*-1 does hold, $\hat{T}_h^{(\alpha)*} = \hat{T}_h$, $\Delta_{yh}^{(\alpha)} = 2\Delta_{yh}$ where $\Delta_{yh} = K_{h1}\bar{y}_{h1} - K_{h2}\bar{y}_{h2}$, and

$$\hat{T}^{(\alpha)} - \hat{T} = \sum_h \varsigma_h^{(\alpha)} \Delta_{yh} . \tag{7}$$

Squaring out (7) and summing over an orthogonal set of half-samples gives the *BHS* estimator as

$$v_B = \sum_h \Delta_{yh}^2 .$$

The expectation under model (1) is then easily calculated as

$$E_M\left(v_B\right) = \sum_h \sum_{i \in s_h} K_{hi}^2 v_{hi} + \sum_h \mu_h^2 \left(K_{h1} - K_{h2}\right)^2 , \tag{8}$$

which is the asymptotic variance in (3) plus a positive term. The positive term looks like a bias squared but is present even when $\hat{T}$ is model unbiased. Expression (8) is similar to the result for the separate ratio estimator in single-stage sampling obtained in Valliant (1987b). If the class of estimators is further restricted so that, in addition to *HS*-1,

(*HS*-2) $K_{hi} = K_h$ for all $i \in s_h$

holds, then $\Delta_{yh} = K_h\left(\bar{y}_{h1} - \bar{y}_{h2}\right)$ and $E_M\left(\Delta_{yh}\right) = 0$. With both *HS*-1 and *HS*-2 holding, v_B is approximately model unbiased.

Conditions *HS*-1 and *HS*-2 substantially limit the class of estimators for which *BHS* is appropriate as an estimator of the model variance (3). Because $\sum_{s_h} K_{hi} = M_h$ for model unbiasedness, *HS*-2 implies that $K_h = M_h / n_h = M_h / 2$. In other words, the class of model-unbiased estimators for which *BHS* is appropriate consists of the singleton $\hat{T} = \sum_h M_h \sum_{s_h} \bar{y}_{hi} / n_h$. Section 3.3 gives some examples of other estimators in class (2) where *BHS* does not work because conditions *HS*-1 or *HS*-2 do not hold. In practice, *BHS* is often applied in situations where more elaborate models for $E_M\left(y_{hij}\right)$ than (1) are appropriate. The preceding remarks do not preclude the possibility that *BHS* can successfully be applied to those situations — an area of investigation that will not be pursued further here.

3.2 Design-based Properties

With some sample designs v_B may have desirable design based properties when only *HS*-1 holds, despite the conditional (model) bias in (8). Define π_{hi} to be the selection probability of unit *hi* in a sample of $n_h = 2$. If $K_{hi} = M_{hi}/\pi_{hi}$, (*HS*-1) is satisfied when $K_{hi}^{(\alpha)}$ is calculated by substituting $\pi'_{hi} = \pi_{hi}/2$ for π_{hi}. In that case, $v_B = \sum_{h,s_h}\left(M_{hi}\bar{y}_{hi}/\pi_{hi} - \hat{T}_h\right)^2 \Big/ \left[n_h\left(n_h - 1\right)\right]$ and v_B is design unbiased under with-replacement sampling when $\hat{T}_h$ is design unbiased. When $K_{hi} = M_{hi}/\pi_{hi}$ and the estimator is a differentiable function of totals defined by (2), Krewski and Rao (1981) showed that v_B is design-consistent as $H \to \infty$ and the sampling of clusters is done with replacement. Condition *HS*-2 is not required for these results. When averaged over the design distribution, the second, model-related term in (8) turns into a design variance component, an example of a more general phenomenon pointed out by Smith (1994).

3.3 Examples

Some examples will show the limitations of *BHS* as an estimator of the large-sample model variance $\text{var}_M\left(\hat{T}\right)$. Examples 1-4 each concern estimators of $\hat{T}$ that satisfy the condition $\sum_{i\in s_h} K_{hi} = M_h$ for unbiasedness under (1). In each case below, the half-sample coefficients all reduce to $K_{hi}^{(\alpha)} = M_h$. Thus, the half-sample method tries to estimate the variance of the *BLU* predictor, the expansion estimator, the ratio estimator, and the Horvitz-Thompson estimator all with the same set of half-sample $\hat{T}^{(\alpha)}$'s — a tactic that is obviously incorrect.

Example 1. *BLU* estimator: From Royall (1976) the best linear unbiased (*BLU*) predictor under (1) is $\hat{T}_{BLU} = \sum_{h,s_h} m_{hi}\bar{y}_{hi} + \sum_{h,s_h}(M_{hi} - m_{hi})\left[w_{hi}\bar{y}_{hi} + (1 - w_{hi})\hat{\mu}_h\right] + \sum_{h,r_h} M_{hi}\hat{\mu}_h$ where r_h is the set of nonsample clusters, $w_{hi} = m_{hi}\rho_{hi}/(1 - \rho_{hi} + m_{hi}\rho_{hi})$, $\hat{\mu}_h = \sum_{s_h} u_{hi}\bar{y}_{hi}$, and $u_{hi} = \left[m_{hi}/\sigma_{hi}^2(1 - \rho_{hi} + m_{hi}\rho_{hi})\right]\Big/\left[\sum_{s_h} m_{hi}/\sigma_{hi}^2(1 - \rho_{hi} + m_{hi}\rho_{hi})\right]$. Setting $m_h = \sum_{i\in s_h} m_{hi}$, the coefficient in (2) is $K_{hi} = m_{hi} + \left[M_h - m_h - \sum_{i'\in s_h}(M_{hi'} - m_{hi'})w_{hi'}\right]u_{hi} + w_{hi}(M_{hi} - m_{hi})$ which depends on the particular units in the sample. The half-sample coefficient is simply $K_{hi}^{(\alpha)} = M_h$ and, consequently, the prescription to double the full-sample weights to create half-sample weights does not apply. Therefore, both *HS*-1 and *HS*-2 are violated.

Example 2. Expansion estimator: $\hat{T}_0 = \sum_h (M_h/m_h)\sum_{i\in s_h} m_{hi}\bar{y}_{hi}$. For the full sample $K_{hi} = (M_h/m_h)m_{hi}$. When the $(hi)^{th}$ sample cluster is assigned to half-sample α, the number of sample units in the half-sample, $m_h^{(\alpha)}$, is equal to the number in the $(hi)^{th}$ cluster, m_{hi}. Thus, $K_{hi}^{(\alpha)} = M_h$ and neither *HS*-1 nor *HS*-2 holds. If $m_{hi} = \overline{m}_h$, an allocation that equalizes workload per cluster, then both *HS*-1 and *HS*-2 are satisfied.

Example 3. Ratio estimator: $\hat{T}_R = \sum_h \left(M_h \big/ \sum_{i \in s_h} M_{hi} \right) \sum_{i \in s_h} M_{hi} \bar{y}_{hi}$. $K_{hi} = \left(M_h \big/ \sum_{i \in s_h} M_{hi} \right) M_{hi}$ and $K_{hi}^{(\alpha)} = M_h$. Again, *HS*-1 and *HS*-2 are, in general, violated.

Example 4. Horvitz-Thompson estimator when clusters are sampled with probabilities proportional to M_{hi} and an equal probability subsample is selected within each sample cluster: $\hat{T}_{HT} = \sum_h \left(M_h / n_h \right) \sum_{i \in s_h} \bar{y}_{hi}$. $K_{hi} = M_h / n_h = M_h / 2$ and $K_{hi}^{(\alpha)} = M_h$, so that both *HS*-1 and *HS*-2 hold. In the special case of $\rho_{hi} = \rho_h$, $\sigma_{hi}^2 = \sigma_h^2$, $M_{hi} = \overline{M}_h$, and $m_{hi} = \overline{m}_h$, the *BLU* predictor in example 1 also reduces to $\hat{T}_{HT}$.

It should be noted that standard survey design practices may minimize the effects of violating *HS*-1 and *HS*-2. If clusters are stratified based on size and the sizes M_{hi} and allocations m_{hi} are about the same within a stratum, then each of the estimators in examples 1-4 will be approximately equal to $\hat{T}_{HT}$, the case for which *BHS* works.

4. Partial Balancing

Partial balancing is often used in order to reduce the number of half-sample estimates that must be computed for v_B. Though computationally expedient, partial balancing leads to an inconsistent variance estimator, as will be demonstrated in this section. Suppose again that $n_h = 2$ and that strata are assigned to groups or superstrata. An attempt may be made to assign the same number of strata to each group, but this is not essential. In a particular group all the sample clusters numbered 1 are associated and assigned as a block to a half-sample. Sample clusters numbered 2 are similarly treated as a block. Figure 1 illustrates the grouping of strata and treatment of clusters as blocks. If there are $g = 1,\ldots,G$ groups of strata, then the estimator of the total can be written as

$$\hat{T} = \sum_{g=1}^{G} \left(\hat{T}_{g1} + \hat{T}_{g2} \right)$$

where $\hat{T}_{gi} = \sum_{h \in G_g} K_{hi} \bar{y}_{hi}$, i=1,2 with G_g being the set of strata in group g. The estimator of the total based on half-sample α is

$$\hat{T}^{(\alpha)} = \sum_{g=1}^{G} \left(\varsigma_{g1\alpha} \hat{T}_{g1}^{(\alpha)} + \varsigma_{g2\alpha} \hat{T}_{g2}^{(\alpha)} \right)$$

where $\varsigma_{gi\alpha}$=1 if the units numbered i in group g are in the half-sample and 0 if not, and $\hat{T}_{gi}^{(\alpha)} = \sum_{h \in G_g} K_{hi}^{(\alpha)} \bar{y}_{hi}$ with $K_{hi}^{(\alpha)}$ computed as it would be for the fully balanced case.

The difference between the grouped half-sample estimator and the full sample estimator is

$$\hat{T}^{(\alpha)} - \hat{T} = \sum_{g=1}^{G} \left(\varsigma_{g1\alpha} \hat{T}_{g1}^{(\alpha)} - \hat{T}_{g1} + \varsigma_{g2\alpha} \hat{T}_{g2}^{(\alpha)} - \hat{T}_{g2} \right). \qquad (9)$$

If $K_{hi}^{(\alpha)} = 2K_{hi}$, i.e. *HS*-1 holds, then $\hat{T}_{gi}^{(\alpha)} = 2\hat{T}_{gi}$ and

$$\hat{T}^{(\alpha)} - \hat{T} = \sum_{g} \varsigma_{g}^{(\alpha)} \left(\hat{T}_{g1} - \hat{T}_{g2} \right)$$

where $\varsigma_{g}^{(\alpha)} = 2\varsigma_{g1\alpha} - 1 = -\left(2\varsigma_{g2\alpha} - 1\right)$. With balancing on groups, the grouped *BHS* estimator is

$$v_{GB} = \sum_{g} \left(\hat{T}_{g1} - \hat{T}_{g2} \right)^2 .$$

The expectation of v_{GB} is easily calculated as

$$E_M(v_{GB}) = \sum_g \sum_{h\in G_g} \left(K_{h1}^2 v_{h1} + K_{h2}^2 v_{h2}\right) + \sum_g \left[\sum_{h\in G_g} \mu_h \left(K_{h1} - K_{h2}\right)\right]^2 , \qquad (10)$$

which compares to (8) for the ungrouped case. When *HS*-2 holds, the second term in (10) is zero and the grouped *BHS* estimator is asymptotically model unbiased. Note that v_{GB} is design unbiased if only *HS*-1 holds (Wolter 1985, sec. 3.6).

Even if *HS*-1 and *HS*-2 are satisfied, v_{GB} may perform erratically when the number of groups G is not large. Krewski (1978), in a related case, noted the large variance of a grouped *BHS* estimator compared to the standard variance estimator in stratified simple random sampling when the stratified expansion estimator is used. Lee (1972, 1973) has studied modifications to partial balancing intended to help stabilize the variance of v_{GB}, but those procedures have somewhat limited applicability and have not become part of standard practice. Rao and Shao (1993) have also proposed a repeatedly grouped balanced half-sample (*RGBHS*) procedure that might be adapted to the partially balanced case. The *RGBHS* method applies to a case where a large number of units are selected within a stratum and then assigned at random to two groups for variance estimation.

If, as $H \to \infty$, G is fixed, then v_{GB} can be inconsistent in addition to being unstable. To demonstrate this, we extend an argument given by Rao and Shao (1993) and Shao (1994) for stratified single-stage sampling. Let η_g denote the number of strata assigned to group g and suppose that $\min_g(\eta_g) \to \infty$. Under standard conditions,

$$z_g = \left(\hat{T}_{g1} - \hat{T}_{g2}\right) \Big/ \sqrt{D_g} \xrightarrow{d} N(0,1)$$

where $D_g = \text{var}_M\left(\hat{T}_{g1} - \hat{T}_{g2}\right) = \sum_{h\in G_g} K_h^2 \left(v_{h1} + v_{h2}\right)$. Since $\text{var}_M\left(\hat{T} - T\right) \approx \sum_g D_g$,

$$\frac{v_{GB}}{\mathrm{var}_M\left(\hat{T}-T\right)} \approx \sum_g \frac{D_g}{\sum_{g'} D_{g'}} z_g^2 .$$

If $D_g \big/ \sum_{g'} D_{g'}$ converges to a constant ω_g, it follows that

$$\frac{v_{GB}}{\mathrm{var}_M\left(\hat{T}-T\right)} \to \sum_g \omega_g \chi_g^2 , \tag{11}$$

where χ_g^2 is a central chi-square random variable with 1 degree of freedom. In other words, rather than converging to 1 as would be the case for a consistent variance estimator, the ratio in (11) converges to a weighted sum of chi-square random variables. Note that a result similar to (11) can be obtained if $\eta_g \to \infty$ in only some of the groups. The inconsistency of v_{GB} can manifest itself by $\mathrm{var}_M\left(v_{GB}\right)$ being large and by the length of confidence intervals being excessively variable, as verified in the simulation reported in section 6. A modification of the above formulation that might lead to consistency for v_{GB} would be to somehow let $G \to \infty$ as $H \to \infty$.

The occurrence in practice of this phenomenon may be more frequent than one would at first expect. In household surveys, selection of certainty clusters, i.e., selection with probability 1, is standard practice. The first-stage units in the certainties are usually geographically smaller clusters that are explicitly stratified or implicitly placed in strata through systematic sampling from an ordered list. Frequently, the first-stage sample units from a certainty are divided into two groups and v_{GB} used for variance estimation. This procedure can lead to the inconsistency described above.

5. Poststratification

Poststratification is used in household and other types of surveys to improve the efficiency of estimators and to adjust for undercoverage of the target population due to deficient frames and other reasons. Suppose that the population is divided into design strata, indexed by h, and clusters within strata as in section 2. Each unit is also a member of a class, or poststratum, denoted by c $(c = 1,\ldots,C)$. Each poststratum can cut across design strata, and the set of units in poststratum c is denoted by S_c. The total number of units in poststratum c is $M_c = \sum_h \sum_{i=1}^{N_h} \sum_{j=1}^{M_{hi}} \delta_{hijc}$, where $\delta_{hijc} = 1$ if unit hij is in S_c and 0 if not. Assume that the poststratum sizes M_c are known. Consider the following working model

$$E_M\left(y_{hij}\right) = \mu_c \qquad (hij) \in S_c$$

$$\operatorname{cov}_M\left(y_{hij}, y_{h'i'j'}\right) = \begin{cases} \sigma^2_{hic} & h = h', i = i', j = j', (hij) \in S_c \\ \sigma^2_{hic}\rho_{hic} & h = h', i = i', j \neq j', (hij) \in S_c, (h'i'j') \in S_c \\ \tau_{hicc'} & h = h', i = i', j \neq j', (hij) \in S_c, (h'i'j') \in S_{c'} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Let m_{hic} be the number of sample units in sample cluster hi that are part of poststratum c and $\bar{y}_{hic} = \sum_{j \in s_{hi}} y_{hij}\,\delta_{hijc} \big/ m_{hic}$ be the sample mean of those units. The model for the means $\bar{y}_{hic}$ implied by (12) is

$$E_M\left(\bar{y}_{hic}\right) = \mu_c$$

$$\operatorname{cov}_M\left(\bar{y}_{hic}, \bar{y}_{h'i'c'}\right) = \begin{cases} v_{hic} & h = h', i = i', c = c' \\ \tau_{hicc'} & h = h', i = i', c \neq c' \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where $v_{hic} = \sigma^2_{hic}\left[1 + \left(m_{hic} - 1\right)\rho_{hic}\right] \big/ m_{hic}$. The poststratified estimator is defined as

$$\hat{T}_{ps} = \sum_c \hat{R}_c \hat{T}_c$$

where $\hat{R}_c = M_c / \hat{M}_c$, $\hat{M}_c = \sum_{h,i \in s_h} K_{hic}$, and $\hat{T}_c = \sum_{h,i \in s_h} K_{hic} \bar{y}_{hic}$ with $K_{hic} = K_{hi}\, m_{hic} / m_{hi}$. A simple calculation shows that $\hat{T}_{ps}$ is unbiased under (12). Under the conditions in Valliant (1993, Appendix A.1), $\text{var}_M\left(\hat{T}_{ps} - T\right) \approx \text{var}_M\left(\hat{T}_{ps}\right)$, similar to the non-poststratified case in section 2.

Suppose that strata are grouped as in section 4 and that the *BHS* technique is used on the groups. The estimator $\hat{T}_c$ can be written as $\hat{T}_c = \sum_g \left(\hat{T}_{cg1} + \hat{T}_{cg2}\right)$ with $\hat{T}_{cgi} = \sum_{h \in G_g} K_{hic} \bar{y}_{hic}$ (i=1,2). Similarly, $\hat{M}_c = \sum_g \left(\hat{M}_{cg1} + \hat{M}_{cg2}\right)$ with $\hat{M}_{gci} = \sum_{h \in G_g} K_{hic}$. Let $K_{hic}^{(\alpha)} = K_{hi}^{(\alpha)}\, m_{hic} / m_{hi}$ and let $\hat{R}_c^{(\alpha)} = M_c / \hat{M}_c^{(\alpha)}$ be a half-sample poststratification ratio with $\hat{M}_c^{(\alpha)} = \sum_{g, h \in G_g} \left(\varsigma_{g1\alpha} \hat{M}_{cg1}^{(\alpha)} + \varsigma_{g2\alpha} \hat{M}_{cg2}^{(\alpha)}\right)$ and define $\hat{T}_c^{(\alpha)} = \sum_{g, h \in G_g} \left(\varsigma_{g1\alpha} \hat{T}_{cg1}^{(\alpha)} + \varsigma_{g2\alpha} \hat{T}_{cg2}^{(\alpha)}\right)$. $\hat{M}_{cgi}^{(\alpha)}$ and $\hat{T}_{cgi}^{(\alpha)}$ have the obvious definitions based on $K_{hic}^{(\alpha)}$. The half-sample poststratified estimator is $\hat{T}_{ps}^{(\alpha)} = \sum_c \hat{R}_c^{(\alpha)} \hat{T}_c^{(\alpha)}$.

When the number of strata H is large, $\hat{R}_c^{(\alpha)} \hat{T}_c^{(\alpha)}$ can be expanded around the full sample estimates $\hat{R}_c$ and $\hat{T}_c$ to obtain the approximation

$$\hat{R}_c^{(\alpha)} \hat{T}_c^{(\alpha)} - \hat{R}_c \hat{T}_c \cong \hat{R}_c \left[\hat{T}_c^{(\alpha)} - \hat{T}_c\right] - \hat{R}_c \hat{\mu}_c \left[\hat{M}_c^{(\alpha)} - M_c\right] \tag{14}$$

with $\hat{\mu}_c = \hat{T}_c / \hat{M}_c$. With grouping of strata $\hat{T}_c^{(\alpha)} - \hat{T}_c$ is

$$\hat{T}_c^{(\alpha)} - \hat{T}_c = \sum_{g=1}^{G} \left(\varsigma_{g1\alpha}\left(\hat{T}_{cg1}^{(\alpha)} - \hat{T}_{cg1}\right) + \varsigma_{g2\alpha}\left(\hat{T}_{cg2}^{(\alpha)} - \hat{T}_{cg2}\right)\right). \tag{15}$$

analogous to (9) and a similar expression holds for $\hat{M}_c^{(\alpha)} - \hat{M}_c$. If *HS*-1 holds, (15) reduces to $\hat{T}_c^{(\alpha)} - \hat{T}_c = \sum_g \varsigma_g^{(\alpha)} \left(\hat{T}_{cg1} - \hat{T}_{cg2}\right)$. We also have $\hat{M}_c^{(\alpha)} - \hat{M}_c = \sum_g \varsigma_g^{(\alpha)} \left(\hat{M}_{cg1} - \hat{M}_{cg2}\right)$ and (14) becomes

$$\hat{R}_c^{(\alpha)} \hat{T}_c^{(\alpha)} - \hat{R}_c \hat{T}_c \cong \hat{R}_c \left[\sum_g \varsigma_g^{(\alpha)} e_{gc} \right] \qquad (16)$$

where $e_{gc} = \left(\hat{T}_{cg1} - \hat{T}_{cg2}\right) - \hat{\mu}_c \left(\hat{M}_{cg1} - \hat{M}_{cg2}\right)$. After summing (16) over c, squaring, and using the orthogonality of the $\varsigma_g^{(\alpha)}$'s, the grouped *BHS* estimator is approximately

$$v_{GB} \cong \sum_g \left(\sum_c \hat{R}_c e_{gc} \right)^2 .$$

The expectation of e_{gc} under model (12) is 0. Thus,

$$E_M\left(v_{GB}\right) \cong \sum_g \hat{\mathbf{R}}' \operatorname{var}_M\left(\mathbf{e}_g\right) \hat{\mathbf{R}}$$

where $\hat{\mathbf{R}} = \left(\hat{R}_1, \ldots, \hat{R}_C\right)'$ and $\mathbf{e}_g = \left(e_{g1}, \ldots, e_{gC}\right)'$. By direct calculation this expectation can be shown to be

$$E_M\left(v_{GB}\right) \cong \sum_g \hat{\mathbf{R}}' \Sigma_g \hat{\mathbf{R}} \qquad (17)$$

where Σ_g is the $C \times C$ matrix with $(cc)^{th}$ element $\sum_{h \in G_g} \sum_{i \in s_h} K_{hic}^2 v_{hic}$ and $(cc')^{th}$ element $\sum_{h \in G_g} \sum_{i \in s_h} K_{hic} K_{hic'} \tau_{hicc'}$. Expression (17) is equal to $\operatorname{var}_M\left(\hat{T}_{ps}\right)$ in expression (8) of Valliant (1993) and, consequently, the grouped *BHS* estimator is approximately unbiased. Note that *HS*-2 was not required because the mean μ_c in model (12) does not depend on the stratum *h*.

Unbiasedness notwithstanding, v_{GB} is inconsistent here also. As in section 4, suppose that G is fixed as $H \to \infty$. Again, let η_g denote the number of strata assigned to group g and suppose that $\min_g \left(\eta_g\right) \to \infty$. Since $\hat{\mathbf{R}}'\mathbf{e}_g$ is a linear combination of random variables and each e_{gc} is a sum over a large number η_g of strata, we have, under appropriate conditions,

$$\tilde{z}_g = \hat{\mathbf{R}}'\mathbf{e}_g \Big/ \sqrt{\tilde{D}_g} \xrightarrow{d} N(0,1)$$

where $\tilde{D}_g = \hat{\mathbf{R}}' \operatorname{var}_M\left(\mathbf{e}_g\right) \hat{\mathbf{R}}$. If $\tilde{D}_g \Big/ \sum_{g'} \tilde{D}_{g'} \to \tilde{\omega}_g$, then

$$\frac{v_{GB}}{\operatorname{var}_M\left(\hat{\theta} - T\right)} \xrightarrow{p} \sum_g \tilde{\omega}_g \chi_g^2 , \tag{18}$$

where χ_g^2 is a central chi-square random variable with 1 degree of freedom. Thus, the grouped variance estimator is also inconsistent here.

6. Simulation Results

To illustrate the problems with the grouped *BHS* variance estimator, we conducted two simulation studies. In the first, single-stage cluster sampling was used in artificial populations. In the second study, two-stage cluster samples were selected from a population derived from the U.S. Current Population Survey (CPS) and a poststratified estimator used.

For the first study, two artificial populations having $H = 40$ and $H = 160$ were generated as follows. Constant numbers of clusters per stratum and units per cluster were assigned as $N_h = 100$ and $M_{hi} \equiv \overline{M}_h = 10$. A y variable for each unit in each stratum was generated as $y_{hij} = \mu_h + \varepsilon_{hi} + 2\varepsilon_{hij}$ where both ε_{hi} and ε_{hij} were computed as $\left(x - 6\right)/\sqrt{12}$

with x a chi-square random variable with 6 degrees of freedom. The stratum means μ_h were multiples of 10, assigned in blocks of 20 — $\mu_h = 10$ for the first 20 strata, $\mu_h = 20$ for the next 20 strata, $\mu_h = 30$ for the next 20 strata (for $H = 160$), and so on. Assigning the means in this way was convenient but has no particular effect on results other choices would illustrate the same points. The population with $H = 40$ had a total of $M = 40,000$ units while the $H = 160$ population had 160,000 units. In each stratum a sample of $n_h = 2$ was selected by simple random sampling without replacement and both sample clusters were completely enumerated. The estimator of the total used was $\hat{T} = \sum_h M_h \bar{\bar{y}}_{hs}$ with $\bar{\bar{y}}_{hs} = \sum_{s_h} \bar{y}_{hi} / n_h$. $\hat{T}$ is unbiased with respect to both the model and the stratified simple random sampling design. When the sampling fraction of clusters is small in each stratum, a model-unbiased and design-unbiased estimator of variance is

$$v_B = \sum_h M_h^2 \left(\bar{y}_{h1} - \bar{y}_{h2} \right)^2 / 4,$$

which also equals the *BHS* estimator when a set of half-samples in full orthogonal balance is used. The sampling fraction of clusters in each stratum for both $H = 40$ and $H = 160$ is 2/100. Thus, the lack of a finite population correction (*fpc*) in v_B has a minor effect.

For both artificial populations v_{GB} was computed using $G = 20$ groups and a set of 24 half-samples in full orthogonal balance. When $H = 40$, strata were paired to form the groups. Strata 1 and 2 were paired, strata 3 and 4 were paired and so on. When $H = 160$, strata 1-8 were grouped, strata 9-16, and so on. Note that this type of purposive, as opposed to random, grouping reflects what is typically done in practice.

The second study used a population of 10,841 persons included in the September 1988 CPS. The y variable was weekly wages for each person. The study population contained 2,826 geographic clusters, each composed of about 4 neighboring households. Eight poststrata were formed based on age, race, and sex. Valliant (1993) gives more details about the population and the poststrata definitions. A two-stage sample design was used with clusters as first-stage units and persons as second-stage units. Two sets of 1,000 samples were selected with 100 sample clusters in the first set and 200 sample clusters in the second. In both sets, clusters were selected with probabilities proportional to the number of persons in each cluster. Strata were created in both cases to have about the same total number of households, and $n_h = 2$ sample clusters selected in each stratum using the systematic method described in Hansen, Hurwitz, and Madow (1953, p.343). In each sample cluster, a simple random sample of 4 persons was selected without replacement in clusters with $M_{hi} > 4$; otherwise, the cluster was enumerated completely.

From each sample from the CPS population, the poststratified estimate $\hat{T}_{ps}$, the *BHS* variance estimator v_B based on a set of half-samples in full orthogonal balance, and the grouped *BHS* estimator were calculated. The poststratified estimate $\hat{T}_{ps}$ used $K_{hi} = M_h / n_h$ so that *HS*-1 and *HS*-2 were satisfied. For both sample sizes (*n*=100 and *n*=200), 25 groups of strata were formed in order to compute v_{GB}. For both v_B and v_{GB}, the half-sample totals $\hat{T}_c^{(\alpha)}$ incorporated the factor $\sqrt{1 - n_h / N_h}$, as described in Valliant (1993), to approximately reflect the effect of a non-negligible *fpc*.

Table 1 summarizes results on square root mean square errors (*rmse*'s) and standard error estimates across 1,000 samples from each of the populations. The *rmse* in each simulation was computed as $rmse(\hat{T}) = \left[\sum_{s=1}^{S} \left(\hat{T}_s - T \right)^2 / S \right]^{1/2}$ where *S*=1000 and $\hat{T}_s$ is an estimate of the population total *T* from sample *s*. The average of the square roots of each variance estimate was calculated as $\overline{v^{1/2}} = \sum_s v_s^{1/2} \big/ S$ where v_s is the grouped or ungrouped *BHS* variance estimate from sample *s*. As the ratios, $\overline{v^{1/2}} \big/ rmse$, of average root variance estimate to *rmse* show, neither the grouped *BHS* estimator nor the fully balanced choices have any serious biases in either the artificial or CPS populations.

Table 2 gives coverage percentages over the 1,000 samples of 95% confidence intervals computed using the different variance estimates. Again, no particular defects are observed for the grouped *BHS* estimator. All choices cover at about the nominal level. Calculations were also performed for 90% and 99% intervals with similar results.

Table 3 lists the averages of the half-widths of 95% confidence intervals, i.e. the average over the samples of $1.96\sqrt{v}$ for each variance estimator *v*. The table also shows the variances of those half-widths. Although, for a given simulation, the average length is about the same for both variance estimators, the variances of the half-widths are vastly different. In the (artificial/*H*=40) case, the variance of the v_{GB} half-widths is 1.9 times the variance of the v_B half-widths (3,040/1,591). In the (artificial/*H*=160) case, the ratio is 6.2. The ratios of variances for the (CPS/*H*=50) and (CPS/*H*=100) cases are 2.1 and 4.4.

The relative instability of v_{GB} is further illustrated by Figure 2 which gives density estimates for the two variance estimates from the CPS simulations. The density for the grouped *BHS* estimate is much more heavy-tailed than that of v_B at either sample size. Figure 3 makes related points on confidence interval coverage and length. The standard error estimate $\sqrt{v}$ (v = v_B or v_{GB}) for each sample is plotted versus the estimation error $\hat{T} - T$ for 500 of the samples for (Artificial/*H*=160) and (CPS/*H*=100). Reference lines are drawn at $\sqrt{v} = \left|\hat{T} - T\right| \big/ 1.96$. Points that fall between the two lines correspond to samples where the 95% confidence interval covered the true value. Points outside the reference lines are samples where the confidence intervals did not cover. Circles denote v_B and dots v_{GB}. In both panels v_B has a more narrow range for almost all values of $\hat{T}-T$ than does v_{GB}. The width of confidence intervals based on v_{GB} is erratic in the region where intervals cover *T*. Near $\hat{T}-T=0$ in (CPS/*H*=100), for example, $\sqrt{v_{GB}}$ ranges from about 60 to 160 (in thousands), but the range of $\sqrt{v_B}$ is about 75 to 120.

These results raise the interesting point that despite the sameness of coverage and mean interval length, the fully balanced and partially balanced *BHS* estimators are not equally good. The price paid for partial balancing is wildly fluctuating confidence interval lengths.

7. Conclusion

Though balanced half-sampling can be a flexible and powerful tool in complex sample surveys, the shortcut method of partial balancing should be avoided unless a large number of groups can be formed. The grouped *BHS* variance estimator is at best unstable compared to a fully balanced estimator and at worst inconsistent. Continuing surveys that use partial balancing are likely to observe erratic point estimates of variance over time that do not accurately reflect the precision of estimated means and totals.

REFERENCES

Gurney, M., and Jewett, R.S. (1975). Constructing orthogonal replications for variance estimation. *Journal of the American Statistical Association*, **71**, 819-821.

Hansen, M., Hurwitz, W., and Madow, W. (1953), *Sample Survey Methods and Theory* (Vol.1), New York: John Wiley.

Krewski, D. (1978). On the stability of some replication variance estimators in the linear case. *Journal of Statistical Planning and Inference*, **2**, 45-51.

Krewski, D., and Rao, J.N.K. (1981). Inference from stratified samples: properties of the linearization, jackknife, and balanced repeated replication methods. *Annals of Statistics*, **9**, 1010-1019.

Lee, K.-H. (1972). Partially balanced designs for half sample replication method of variance estimation. *Journal of the American Statistical Association*, **67**, 324-334.

Lee, K.-H. (1973). Using partially balanced designs for the half sample replication method of variance estimation. *Journal of the American Statistical Association*, **68**, 612-614.

McCarthy, P.J. (1969). Pseudo-replication: half-samples. *Review of the International Statistical Institute*, **37**, 239-264.

Rao, J.N.K., and Shao, J. (1993). On balanced half-sample variance estimation in stratified sampling. Carleton University preprint.

Royall, R. (1976). The linear least squares prediction approach to two-stage sampling. *Journal of the American Statistical Association*, **71**, 657-664.

Royall, R. (1986). The prediction approach to robust variance estimation in two-stage cluster sampling. *Journal of the American Statistical Association*, **81**, 119-123.

Royall, R. (1988). The prediction approach to sampling theory. *Handbook of Statistics*, Vol. **6**, eds. P.R. Krishnaiah and C.R. Rao, 399-413, Amsterdam: Elsevier Science Publishers.

Shao, J. (1994). Resampling methods in sample surveys. *Statistics*, to appear.

Sitter, R. (1993). Balanced repeated replications based on orthogonal arrays. *Biometrika*, **80**, 211-221.

Smith, T.M.F. (1994). Sample surveys 1975-1990; an age of reconciliation?" *International Statistical Review*, **62**, 3-34.

Valliant, R. (1987a). Generalized variance functions in stratified two-stage sampling. *Journal of the American Statistical Association*, **82**, 499-508.

Valliant, R. (1987b). Some prediction properties of balanced half-sample variance estimators in single-stage sampling. *Journal of the Royal Statistical Society B*, **49**, 68-81.

Valliant, R. (1993). Poststratification and conditional variance estimation. *Journal of the American Statistical Association*, **88**, 89-96.

Wolter, K. (1985), *Introduction to Variance Estimation*, Springer-Verlag: New York.

Wu, C.J.F. (1991). Balanced repeated replications based on mixed orthogonal arrays. *Biometrika*, **78**, 181-188.

Table 1. Empirical root mean square errors (*mse*) of estimators of totals and ratios of average standard error estimates to the *rmse* in 1,000 samples.

Population	*rmse* (000s)	$\overline{v_B^{1/2}}/rmse$	$\overline{v_{GB}^{1/2}}/rmse$
Artificial populations $\hat{T}_0$			
$H = 40$	5.2	1.002	.997
$H = 160$	9.9	1.051	1.040
CPS population $\hat{T}_{ps}$			
$H = 50$	133.0	1.055	1.049
$H = 100$	97.4	.977	1.022

Table 2. Empirical coverage percentages in 1,000 samples of 95% confidence intervals. L, M, and U are percentages of samples with $\left(\hat{T}-T\right)/\sqrt{v} < -1.96$, $\left|\left(\hat{T}-T\right)/\sqrt{v}\right| \le 1.96$, and $\left(\hat{T}-T\right)/\sqrt{v} > 1.96$, respectively.

Population	v_B			v_{GB}		
Artificial populations	L	M	U	L	M	U
$H = 40$	3.2	94.2	2.6	3.9	93.5	2.6
$H = 160$	2.7	95.7	1.6	3.0	95.0	2.0
CPS population						
$H = 50$	2.5	95.5	2.0	3.2	94.4	2.4
$H = 100$	4.2	93.2	2.6	3.6	94.1	2.3

Table 3. Empirical results for average half-width length and variance of half-width length for 95% confidence intervals over 1,000 samples.

Population	Average half-width (000s)		Variance of half-width (000s)		Ratio of half-width variances (v_{GB}/v_B)
Artificial	v_B	v_{GB}	v_B	v_{GB}	
$H=40$	10.2	10.2	1,591	3,040	1.9
$H=160$	20.3	20.1	1,618	10,076	6.2
CPS					
$H=50$	275.1	273.6	963,597	2,015,957	2.1
$H=100$	186.5	195.1	234,752	1,035,377	4.4

Figure 1. An example of grouping strata and treating sample clusters as blocks when partial balancing is used. Circled units are assigned as a block to a half-sample.

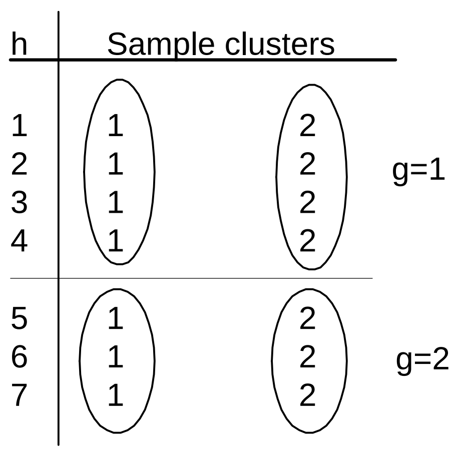

Figure Titles

Figure 1. An example of grouping strata and treating sample clusters as blocks when partial balancing is used. Circled units are assigned as a block to a half-sample.

Figure 2. Nonparametric density estimates for v_B and v_{GB} in the CPS population simulations.

Figure 3. Standard error estimates ($\sqrt{v_B}$ and $\sqrt{v_{GB}}$) plotted versus estimation errors $\left(\hat{T} - T\right)$ for 500 samples from the artificial population (H=160) and the CPS population (H=100). o = $\sqrt{v_B}$; • = $\sqrt{v_{GB}}$.