Stochastic Sensitivity Analysis for Computing Greeks

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Abstract: In a risk management of derivative securities, sensitivities are important measures of market risk to analyze the impact of a misspecification of some stochastic model on the expected payoff function. We investigate in this paper an application of Malliavin calculus, which enables the computation of sensitivity derivatives, known as Greeks in finance, without resort to a direct differentiation of the complex payoff functions.

Keywords: Stochastic sensitivity analysis, Malliavin calculus, Greeks in finance

1. INTRODUCTION

We consider a stochastic model or, equivalently, a stochastic differential equation in a well-defined framework of Black-Scholes set-up, which is described by

$$S_t = S_0 + \int_0^t r S_\tau d\tau + \int_0^t \sigma S_\tau dW_\tau , \quad (1)$$

where *S* is the price of underlying asset with S_0 denoting the present (initial) value, *r* denotes the riskless interest rate, σ the volatility, and $(W_t)_{0 \le t \le T}$ is a standard Brownian motion (also known as Wiener process). Note that, in the case of European-type options, we have a closed solution to (1) as follows:

$$S_T = S_0 \exp(\mu T + \sigma W_T), \quad (2)$$

where $\mu = r - \sigma^2 / 2$ for a fixed expiration or maturity time, *T*.

We are, in European options, interested in studying how to evaluate the sensitivity with respect to model parameters, e.g., present price S_0, volatility σ, etc., of the expected payoff

$$E[e^{-rT} \Phi(S_T)], \quad (3)$$

for an exponentially discounted value of the payoff function $\Phi(S_T)$, where $E[\cdot]$ denotes the expectation operator. The sensitivity of more sophisticated payoff functions including path-dependent Asian-type options like

$$E[e^{-rT} \Phi(\frac{1}{T}\int_0^T S_t dt)], \quad (4)$$

may be treated in a similar manner along the lines that will be investigated in the present study. In the Asian option whose payoff functional is defined by (4), we may note that the payoff depends on the average of the asset value in a given period of time.

In finance, this is the so-called model risk problem. Commonly referred to as Greeks, sensitivities in financial market are typically defined as the partial derivatives of the expected payoff function with respect to underlying model parameters. In general, finite difference approximations are heavily used to simulate Greeks by means of Monte Carlo procedures.

However, it is known that the finite difference approximation soon becomes inefficient particularly when payoff functions are complex and discontinuous. This is often the case when we deal with exotic options such as American, lookback, and digital options, etc.

To overcome this difficulty, Broadie and Glasserman [1] proposed a method to put the differential of the payoff function inside the expectation operator required to evaluate the sensitivity. But this idea (i.e., likelihood ratio method) is applicable only when the density of the random variable involved is explicitly known. Recently, Fournie et al. [2] suggested the use of Malliavin calculus, by means of integration by parts, to shift the differential operator from the expected payoff to the underlying diffusion (e.g., Gaussian) kernel, introducing a weighting function.

The real advantage of using Malliavin calculus is that it is applicable when we deal with random variable whose density is not explicitly known as the case of Asian options. Another examples which are similar to the present study and explored by the first author (e.g., Refs. [3,7]) but that are not covered in this paper are models involving a step function and non-smooth objective functions. In these studies, the stochastic sensitivity analysis technique based on the Novikov's identity is used instead of Malliavin calculus.

In this paper, we present a brief introduction of Malliavin calculus, and describe a constructive approach for a stochastic sensitivity analysis for computing Greeks in financial engineering. The present approach enables the simulation of Greeks without resort to direct differentiation of the complex or discontinuous payoff functions.

The remainder of the paper is organized as follows. In Section 2, we briefly review the essence of Malliavin calculus and present integration by parts formula. In Section 3, we describe a constructive approach. Subsection 3.1 presents some explicit formulae for the case of European option. In Subsection 3.2, we investigate the case of Asian option. In Section 4, we present simulation results obtained for the Asian call option. We conclude in Section 5.

2. MALLIAVIN CALCULUS

Following the standard notations that can be found in [6], we present the most concise introduction of Malliavin calculus necessary to our computation.

Let R be the space of random variables of the form $F = f(W_{t_1}, W_{t_2}, \cdots, W_{t_n})$, where f is smooth and W_t denotes the Brownian motion as before. For a smooth random variable $F \in R$, we can define its *derivative* $DF = D_t F$, where the differential operator D is closable. Since D operates on random variables by differentiating functions in the form of partial derivatives, it shares the familiar chain rule property, $D_t(f(F)) = \nabla f(F) \cdot D_t F = f'(F) D_t F$, and other general properties like linearity, etc.

We denote by D^* the Skorohod integral, defined as the adjoint operator of D. If u belongs to Dom(D^*), then $D^*(u)$ is characterized by the following integration by parts formula:

$$E[FD^*(u)] = E[\int_0^T (D_t F) u_t dt]. \quad (5)$$

It is important to note that (5) gives a duality relationship to link operators D and D^*. The adjoint operator D^* behaves like a stochastic integral. In fact, if u_t is an adapted process, then the Skorohod integral coincides with the classical Ito integral: i.e., $D^*(u) = \int_0^T u_t dW_t$. If u_t is non-adapted or generic, one has

$$D^*(Fu) = FD^*(u) - \int_0^T (D_t F) u_t dt \, . \tag{6}$$

The property (6) follows directly from the duality relation (5) and the product rule of the operator *D*. A heuristic derivation of (6) is demonstrated here. Let us assume that *F* and *G* are any two smooth random variables, and u_t a generic process, then by product rule of *D* one has

$$E[GFD^*(u)] = E[\int_0^T D_t (GF) u_t dt] = E[\int_0^T G(D_t F) u_t dt] + E[\int_0^T (D_t G) F u_t dt]$$
$$= E[G\int_0^T (D_t F) u_t dt] + E[GD^*(Fu)]$$

which implies that

$$E[GD^*(Fu)] = E[G\left(FD^*(u) - \int_0^T (D_t F) u_t dt\right)]$$

for *any* random variables *G*. Therefore, (6) must hold almost everywhere.

In the present study, we frequently use the following formal relationship to remove the derivative from a (smooth) random function *f* as follows:

$$E[\nabla f(X)Y] = E[f'(X)Y] = E[f(X)H_{XY}], \tag{7}$$

where *X*, *Y*, and H_{XY} are random variables. It is noted that (7) can be deduced from the integration by parts formula (5), and we have an explicit expression for H_{XY} as

$$H_{XY} = D^*\left(\frac{Y}{\int_0^T D_t X dt}\right). \tag{8}$$

If higher order derivatives are involved then one has to repeat the procedure (7) iteratively. It may be noted that H_{XY} is not unique and other expressions than (8) can be also possible. For more details, the readers are referred to Koda et al. [4] and Montero and Kohatsu-Higa [5].

3. CONSTRUCTIVE APPROACH

In this section, utilizing the technical framework of Malliavin calculus introduced in Section 2, a constructive approach is presented to compute Greeks of European and Asian options, respectively.

3.1. European Option

In the case of European option whose payoff function is defined by (3), the essence of the present method is that the gradient of the expected (discounted) payoff, $\nabla E[e^{-rT}\Phi(S_T)]$, is evaluated by putting the gradient inside the expectation, i.e., $E[e^{-rT}\nabla\Phi(S_T)]$, which involves computations of $\nabla\Phi(S_T) = \Phi'(S_T)$ and ∇S_T . Further, applying Malliavin calculus techniques, the gradient is rewritten as $E[e^{-rT}\Phi(S_T)H]$ for some random variable *H*. It should be noted, however, that there is no uniqueness in this representation since we can add to *H* any random variables that are orthogonal to S_T . In general, *H* involves Ito or Skorohod integrals.

3.1.1. Delta

Now we compute *Delta*, Δ, the first-order partial differential sensitivity coefficient of the expected outcome of the option, i.e., (3), with respect to the initial asset value S_0:

$$\Delta = \frac{\partial}{\partial S_0} E[e^{-rT}\Phi(S_T)] = e^{-rT} E[\Phi'(S_T)\frac{\partial S_T}{\partial S_0}] = \frac{e^{-rT}}{S_0} E[\Phi'(S_T)S_T]$$

Then, with $X = Y = S_T$ in (7), we perform the integration by parts applying (8) to give

$$\Delta = \frac{e^{-rT}}{S_0} E\left[\Phi(S_T)H_{XY}\right] = \frac{e^{-rT}}{S_0} E\left[\Phi(S_T)D^*\left(\frac{S_T}{\int_0^T D_t S_T dt}\right)\right], \tag{9}$$

which removes the derivative of Φ from the expectation as desired.

Since the integral term in the denominator that appears in (9) can be computed as $\int_0^T D_t S_T dt = \sigma T S_T$, we can evaluate the stochastic integral involved in (9) as

$$H_{XY} = D^*\left(\frac{S_T}{\int_0^T D_t S_T dt}\right) = D^*\left(\frac{1}{\sigma T}\right) = \frac{D^*(1)}{\sigma T} = \frac{W_T}{\sigma T}$$

with the help of (6) applied to u=1 (a constant process which is adapted and Ito integral yields $D^*(1) = W_T$). Then the final expression for Δ reads

$$\Delta = \frac{e^{-rT}}{\sigma T S_0} E[\Phi(S_T)W_T]. \tag{10}$$

We may note that when we deal with European options, the present result (10) coincides with the result that is obtained by the explicit computation of the closed formula for the probability density function of S_T.

3.1.2. Vega

Next Greek *Vega*, *V*, is the index that measures sensitivity of the expected payoff (3) with respect to the volatility σ, which can be computed as

$$V = \frac{\partial}{\partial \sigma} E[e^{-rT}\Phi(S_T)] = e^{-rT} E[\Phi'(S_T)\frac{\partial S_T}{\partial \sigma}] = e^{-rT} E[\Phi'(S_T)S_T\{W_T - \sigma T\}],$$

where we have used the solution (2) to evaluate $\partial S_T / \partial \sigma$. Then, utilizing (7) and (8) again with $X = S_T$ and $Y = S_T(W_T - \sigma T)$, we apply the integration by parts to give

$$V = e^{-rT} E\left[\Phi(S_T)H_{XY}\right] e^{-rT} E\left[\Phi(S_T)D^*\left(\frac{S_T(W_T - \sigma T)}{\int_0^T D_t S_T dt}\right)\right] = e^{-rT} E\left[\Phi(S_T)D^*\left(\frac{W_T}{\sigma T} - 1\right)\right].$$

So, we evaluate the stochastic integral as

$$H_{XY} = D^*\left(\frac{W_T}{\sigma T} - 1\right) = \frac{1}{\sigma T} D^*(W_T) - D^*(1) = \frac{1}{\sigma T} D^*(W_T) - W_T \ .$$

With the help of (6) applied to u=1 (adapted process) and $F = W_T$, we have

$$D^*(W_T) = W_T^2 - \int_0^T D_t W_T dt = W_T^2 - \int_0^T 1dt = W_T^2 - T \ .$$

If we bring together the partial results obtained above, we derive the final expression

$$V = e^{-rT} E\left[\Phi(S_T)\left\{\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right\}\right]. \tag{11}$$

3.1.3. Gamma

The last Greek *Gamma*, Γ, involves a second-order derivative,

$$\Gamma = \frac{\partial^2}{\partial S_0^2} E[e^{-rT}\Phi(S_T)] = \frac{e^{-rT}}{S_0^2} E[\Phi''(S_T)S_T^2].$$

Utilizing (7) and (8) with $X = S_T$ and $Y = S_T^2$, we obtain after a first integration by parts

$$\Gamma = \frac{e^{-rT}}{S_0^2} E\left[\Phi'(S_T) D^*\left(\frac{S_T^2}{\int_0^T D_t S_T dt}\right)\right] = \frac{e^{-rT}}{S_0^2} E\left[\Phi'(S_T) D^*\left(\frac{S_T}{\sigma T}\right)\right] .$$

With the help of (6) applied to $u = 1/\sigma T$ (constant adapted process) and $F = S_T$, we have

$$D^*(\frac{S_T}{\sigma T}) = \frac{S_T}{\sigma T} D^*(1) - \frac{1}{\sigma T}\int_0^T D_t S_T dt = S_T\left(\frac{W_T}{\sigma T} - 1\right).$$

Then, repeated application of (7) and (8) with $X = S_T$ and $Y = S_T(W_T/\sigma T - 1)$, the second integration by parts yields

$$\Gamma = \frac{e^{-rT}}{S_0^2} E\left[\Phi'(S_T) S_T\left(\frac{W_T}{\sigma T} - 1\right)\right] = \frac{e^{-rT}}{S_0^2} E\left[\Phi(S_T) D^*\left(\frac{S_T}{\int_0^T D_t S_T dt}\left\{\frac{W_T}{\sigma T} - 1\right\}\right)\right].$$

With the help of (6) as before, we can evaluate the stochastic integral as

$$D^*\left(\frac{S_T}{\int_0^T D_t S_T dt}\left\{\frac{W_T}{\sigma T} - 1\right\}\right) = \frac{1}{\sigma T} D^*\left(\frac{W_T}{\sigma T} - 1\right) = \frac{1}{\sigma T}\left\{\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right\}.$$

If we combine the results obtained above, the final expression becomes

$$\Gamma = \frac{e^{-rT}}{\sigma T S_0^2} E\left[\Phi(S_T)\left\{\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right\}\right]. \tag{12}$$

Comparing (12) with (11), we find the following relationship between V and Γ:

$$\Gamma = \frac{V}{\sigma T S_0^2}\ . \tag{13}$$

Since we have closed solutions for all the Greeks, we can easily check the correctness of the above results.

3.2. Asian Option

In the case of Asian option whose payoff functional is defined by (4), the essence of the present approach is again that the gradient of the expected (discounted) payoff is rewritten as $E[e^{-rT}\nabla\Phi(\frac{1}{T}\int_0^T S_t dt)] = e^{-rT}E[\Phi(\frac{1}{T}\int_0^T S_t dt)H]$, for some random variable *H*. Different from the European options, however, we do not have a known closed formula in this case.

3.2.1. Delta

Delta in this case is given by

$$\Delta = \frac{\partial}{\partial S_0}E[e^{-rT}\nabla\Phi(\frac{1}{T}\int_0^T S_t dt)] = \frac{e^{-rT}}{S_0}E[\Phi'(\frac{1}{T}\int_0^T S_t dt)\frac{1}{T}\int_0^T S_t dt].$$

There are various ways of performing the integration by parts; e.g., the readers are referred to [2]. In the present approach, utilizing (7) and (8) with $X = Y = \int_0^T S_t dt/T$, we may apply the integration by parts to give

$$\Delta = \frac{e^{-rT}}{S_0}E\left[\Phi(\frac{1}{T}\int_0^T S_t dt)D^*\left(\frac{Y}{\int_0^T D_t X dt}\right)\right] = \frac{e^{-rT}}{S_0}E\left[\Phi(\frac{1}{T}\int_0^T S_t dt)D^*\left(\frac{\int_0^T S_t dt}{\sigma\int_0^T tS_t dt}\right)\right].$$

With the help of (6) applied to $u = 1/\sigma$ (constant adapted process) and $F = \int_0^T S_t dt / \int_0^T tS_t dt$, we may obtain

$$\Delta = \frac{e^{-rT}}{S_0}E\left[\Phi(\frac{1}{T}\int_0^T S_t dt)\left(\frac{1}{<T>}\left\{\frac{W_T}{\sigma} + \frac{<T^2>}{<T>}\right\} - 1\right)\right], \tag{14}$$

where

$$<T> = \frac{\int_0^T tS_t dt}{\int_0^T S_t dt} \quad \text{and} \quad <T^2> = \frac{\int_0^T t^2 S_t dt}{\int_0^T S_t dt}$$

are the first two moments of the probability density defined by $p(t) = S_t / \int_0^T S_t dt$.

3.2.2. Vega

Vega in this case becomes

$$V = \frac{\partial}{\partial\sigma}E[e^{-rT}\nabla\Phi(\frac{1}{T}\int_0^T S_t dt)] = e^{-rT}E[\Phi'(\frac{1}{T}\int_0^T S_t dt)\frac{1}{T}\int_0^T \frac{\partial S_t}{\partial\sigma}dt]$$
$$= e^{-rT}E[\Phi'(\frac{1}{T}\int_0^T S_t dt)\frac{1}{T}\int_0^T S_t\{W_t - t\sigma\}dt]$$

As before, with the help of (7) and (8) applied to $X = \int_0^T S_t dt/T$ and $Y = \int_0^T S_t\{W_T - t\sigma)dt/T$, we have

$$V = e^{-rT} E\left[\Phi(\frac{1}{T}\int_0^T S_t dt) D * \left(\frac{Y}{\int_0^T D_t X dt} \right) \right] = e^{-rT} E\left[\Phi(\frac{1}{T}\int_0^T S_t dt) D * \left(\frac{\int_0^T S_t W_t dt}{\sigma \int_0^T t S_t dt} - 1 \right) \right],$$

which, with the help of (6), yields the following expression:

$$V = e^{-rT} E\left[\Phi(\frac{1}{T}\int_0^T S_t dt) \left\{ \frac{\int_0^T \int_0^T S_t W_t dt dW_\tau}{\sigma \int_0^T t S_t dt} + \frac{\int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{(\int_0^T t S_t dt)^2} - W_T \right\} \right]. \quad (15)$$

Using the relation (13), it is straightforward to compute *Gamma* as (15) divided by $\sigma T S_0^2$.

4. MONTE CARLO SIMULATION OF ASIAN OPTION

In order to evaluate the results obtained in Section 3, we present in this section the results of Monte Carlo simulation for computing Delta and Vega in the case of Asian Call option whose payoff functional is defined by (4).

4.1. Delta

In Fig. 1, we present the simulation result of Δ given by (14) with parameters r=0.1, $\sigma = 0.25$, T=0.2 (in years), and $S_0 = K = 100$ (in arbitrary cash units) where K denotes the strike price. We have divided the entire interval of integration into 252 pieces, representing the approximate number of trading days in a year.

Fig. 1 shows how the outcome of the simulation progressively attains its own value. We compare the convergence behavior of the present simulation with the results obtained by Broadie and Glasserman [1] where all the parameters take the same values we have used, and which may provide most extensive and detailed results currently available. The result indicates a fairy good convergence to the steady-state value that is attained at 10,000*th* iteration stage in [1]. The standard deviation of the simulation in this case was 0.005.

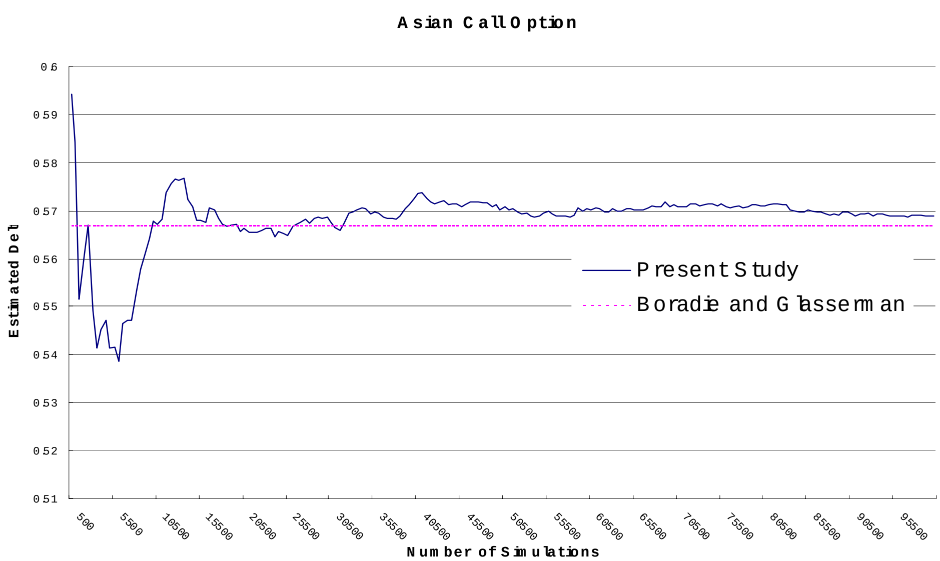

Fig. 1. Estimated *Delta* of Asian Call Option; S_0=K=100, T=0.2, r=0.1, σ=0.25

4.2. Vega

We present in Fig. 2 the result of V given by (15), where all the parameters take the same values we used in the simulation of Δ in Subsection 4.1. Again, we compare the result with the one that is obtained at 10,000*th* iteration stage in [1]. The result indicates that some noticeable bias may remain in the present Monte Carlo simulation, and further study may be necessary to analyze and reduce the bias involved.

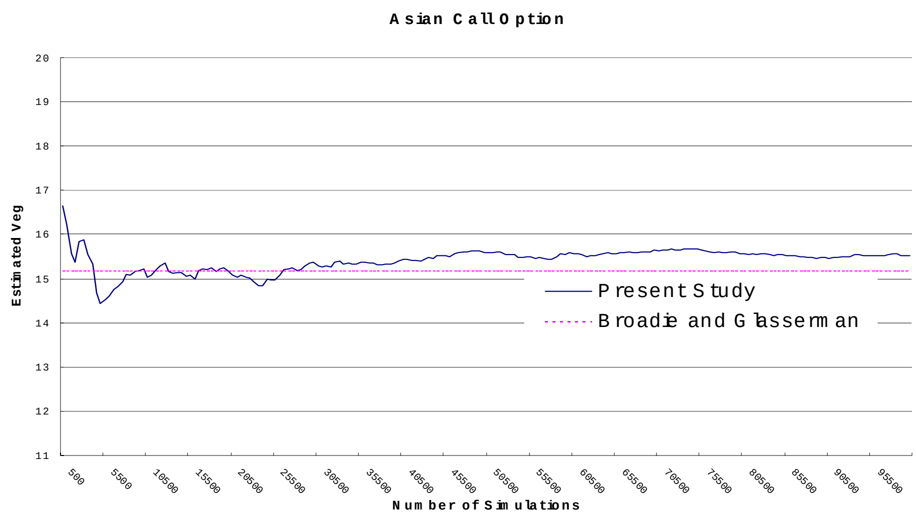

Fig. 2. Estimated *Vega* of Asian Call Option; S_0=K=100, T=0.2, r=0.1, σ=0.25

5. CONCLUSION

We have presented a stochastic sensitivity analysis method, in particular, a constructive approach for computing Greeks in finance using Malliavin calculus. The present approach is useful when the random variables are smooth in the sense of stochastic derivatives. It may be necessary to further investigate and improve Monte Carlo procedures to reduce the bias involved in the simulation of *Vega* in Asian-type options and other sophisticated options.

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