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RESEARCH IN THE THEORY OF CONDENSED MATTER AND ELEMENTARY PARTICLES

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PUBLICATIONS

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1. Covariant Methods in Superstring Theory,
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4. Virasoro Algebras and Coset Space Models,
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15. Details of the Non-unitarity Proof for Highest Weight Representations of the Virasoro Algebra,
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Summary of Previous Research (Condensed Matter Physics)

This is a summary of our work supported since 1984 by DOE grant DE-FG02-84ER45144. Our work has played a role both in condensed matter physics and in high energy physics. In this section we describe our work with emphasis on the connections to condensed matter physics. We have tried to set the work in an historical context, but we certainly do not intend this narrative to be a comprehensive history.

Major progress has been made during the last few years in the understanding of two dimensional critical phenomena. The remarkable result of these developments is a complete classification of all possible two dimensional critical phenomena in a certain domain. Our group at Chicago funded in large part by the DOE has played a leading role in this work.

The development of these ideas goes back to the late 1960's and the bootstrap proposals of Kadanoff[1] and Polyakov[2]. They proposed using general principles to classify critical phenomena rather than analyze specific models. Polyakov[3] made the important observation that there is a larger symmetry than ordinary global scale invariance at a critical point. Locality of the coupling between degrees of freedom implies that the system should respond simply to local scale transformations as well. Local scale transformations are called conformal transformations. The constraint of conformal invariance, coupled with the large number of conformal transformations possible in two dimensions (any analytic mapping of the plane onto itself is conformal) is a crucial part of this recent work.

Another part of the story also has its origins in the late 1960's. Dual models and string theories were first studied at this time[4]. In particular, the operator realization of conformal invariance in two dimensions, the Virasoro algebra, was developed.

These developments lay fallow for about a decade. Perhaps the right place to pick up the story is with Polyakov's string theory letters[5]. These renewed interest in the connection between string theory and two dimensional conformal field theory. This work highlighted the role of the trace anomaly, c, of the string world surface. Friedan[6] explored these ideas further, developing analytic operator product expansion techniques and demonstrating that c is just the coefficient of the central extension of the Virasoro algebra and so is crucial in understanding the conformal properties of the system.

Belavin, Polyakov and Zamolodchikov[7] (BPZ) made a major advance in 1983. First, they realized that these ideas would have implications for two dimensional statistical mechanical systems. They also made the deep and beautiful observation that correlation functions for operators with certain special scaling dimensions h (depending on c) would obey linear differential equations, rendering them exactly soluble. This followed from certain properties of the representations of the Virasoro algebra for these h and c values[8]. Furthermore they showed that the Ising model was such a special system, indicating that such systems were physically interesting. Dotsenko[9] soon showed that the three state Potts model was also such a system. Although fascinating, these results did not point the way to any classification of critical phenomena.

The next key step was taken by our group[10]. The bulk of interesting two dimensional critical phenomena, genuine thermal systems with positive Boltzmann weights and sufficient spatial isotropy can be described by effective Landau-Ginsburg hamiltonians that allow an especially straightforward operator interpretation. The representation of the conformal algebra in these systems must be unitary.

This physical constraint is extremely powerful. In fact we proved that the only possibly unitary representations of the Virasoro algebra are $c \geq 1$, $h > 0$ or

$$\begin{array}{ll} c = 1 - \frac{6}{m(m+1)} & m = 2, 3, \ldots \\ h = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)} & p = 1, 2, \ldots, m-1 \\ & q = 1, 2, \ldots, m \end{array} \tag{1}$$

This has become known as the discrete series. All two dimensional critical phenomena obeying the requirements discussed above must, if $c < 1$, be in the discrete series. This result is the crucial ingredient in rendering the classification problem for $c < 1$ tractable.

At about the same time, Andrews, Baxter and Forrester[11] displayed an infinite set of exactly soluble lattice models that Huse[12] showed realized each element of the discrete series. Goddard, Kent and Olive[13] proved that each of these representations was in fact unitary.

Important pieces of the classification problem remained. How many operators of each allowed scaling dimension occur in a given model in the discrete series? How do they couple, i.e., what are the operator product coefficients? The requirement that correlation functions be well behaved constrains this data. This is essentially the conformal bootstrap. The differential equation techniques of BPZ augmented by the powerful Feigin-Fuchs[14] integral representation developed by Dotsenko and Fateev[15] provide a way of implementing this constraint and calculating the operator product coefficients. However, it turned out to be somewhat difficult to provide an exhaustive solution.

The crucial constraint was developed in an important piece of work by Cardy[16]. He invoked the principle of modular invariance, already used to good effect in string theory[17]. This expresses the sensible constraint that a partition function should be well defined on arbitrary two dimensional surfaces, not just the plane. Cardy required that the partition function make sense on the torus.

This simple constraint has powerful consequences. Cardy showed that were only a (small) finite number of allowed sets of multiplicities of scaling dimensions for the first few elements of the discrete series.

Gepner and Witten[18] also explored the constraint of modular invariance in an $SU(2)$ symmetric set of conformal field theories. Gepner[19] noticed a crucial link between the modular properties of the $SU(2)$ and discrete series models enabling him to make progress in the general classification of modular invariants for both systems. Capelli, Itzykson and Zuber[20] made important further progress, allowing them to conjecture a complete solution to the modular invariance constraint. This answer was remarkable — allowed conformal field theories are indexed by the Dynkin diagrams of simply laced Lie algebras. All these models have been given lattice realizations by Pasquier[21] using Dynkin diagrams as a guide to the energetics of generalizations of Andrews-Baxter-Forrester models.

Gepner and Qiu[22] made important progress in understanding modular properties by studying certain Z_n symmetric systems. Recently Capelli, Itzykson and Zuber[23] have given a proof of their conjecture.

The classification of all $c < 1$ conformal field theories is a paradigm for a good answer to a classification problem — a few fundamental assumptions lead to a complete list of all possible critical phenomena in a certain domain. This is one of the few times in physics when a really comprehensive solution to such a problem has been given.

Generalizations of this situation can be obtained by imposing extra symmetries in addition to conformal invariance. Nonabelian continuous symmetries give rise to current algebras. We[10,24] focussed on supersymmetry, a symmetry relating bosons to fermions. This symmetry was originally discovered in the context of string theory. We analyzed the unitary representations of the supersymmetric extensions of the Virasoro algebras, the Neveu-Schwarz and Ramond algebras, and found a very similar pattern, a discrete series followed by a continuum. Remarkably, the first two elements in the super discrete series overlapped the ordinary Virasoro discrete series. The tricritical Ising model and also the ordinary gaussian model at a certain radius are supersymmetric. Physical realizations of these systems — e.g., helium adsorbed on krypton plated graphite[25] — are the first known examples of supersymmetric field theories in nature.

We[26] later showed that the fermionic soliton field of the critical Ising model ($c = 1/2$) is the Nambu-Goldstino of spontaneously broken supersymmetry in the tricritical Ising model ($c = 7/10$). We noted that one of the relevant trajectories leaving the $c = 7/10$ tri-critical Ising model fixed point preserves supersymmety, but that the supersymmetry could be spontaneously broken away from the fixed point, because unitarity allows no supersymmetric states of the Ramond algebra. The resulting massless fermion mode, the nambu-goldstino, becomes the massless free fermion of the Ising model. This is a striking example of how supersymmetry puts rigid constraints on the global topology of the renormalization group flow.

We also completed the study of the supersymmetric $c = 1$ gaussian model by showing that it corresponded to the $c = 1$ critical Ashkin-Teller model at a special value of the coupling[26,27]. This example of supersymmetric critical phenomena is experimentally more accessible than the tri-critical Ising model because there is a supersymmetry index which prevents the supersymmetry from being spontaneously broken away from the critical point.

The constraint of modular invariance in the super case has consequences very similar to its effect in the ordinary case. Our student *Kastor* [28] showed it was a powerful constraint. Capelli[29] formulated a comprehensive classification conjecture for superconformal field theory.

Besides the work on classification of critical phenomena we should mention a number of other important applications of conformal invariance to two dimensional critical phenomena. Cardy[30] has used it to explain finite size scaling results. He, Blote and Nightingale[31] and Affleck[32] explained the universal finite size scaling correction proportional to c. Affleck[33] did beautiful work explaining the behavior of quantum spin chains using Kac-Moody algebras. Cardy also fit the Yang-Lee edge[34] and surface critical phenomena[35] into the picture. He and Ludwig,[36] and Zamolodchikov[37] did an ingenious perturbation theory in $c - 1$, enabling them to study the flow between fixed points in the discrete series.

The main problem now is to find the general classification of conformal field theories for $c \geq 1$. This is important intrinsically and for the application to string theory.

To answer this question, and others, we formulated[38] an abstract mathematical description of conformal field theory. We were motivated by the rigidity of modular invariance that Cardy[16] demonstrated. This immediately led us to the idea that a partition function well defined on moduli space for arbitrary genus Riemann surfaces might be an appropriate starting point for a general abstract description of conformal field theory. This view was bolstered by the observation that correlation functions reconstructed by allowing handles to degenerate would be crossing symmetric as a consequence of modular invariance.

A modular geometry yields a conformal field theory. Conversely, given a conformal field theory, generalized conformal blocks can be constructed to represent the theory as a modular geometry. So classifying conformal field theories is essentially equivalent to classifying modular geometries. As the section Proposed Research describes below, we are actively pursuing the long-term goal of classifying all modular geometries.

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Summary of Previous Research (High Energy Physics)

This is a summary of our work supported since 1984 by DOE grant DE-FG02-84ER45144. Our work has played a role both in condensed matter physics and in high energy physics. In this section we describe our work with emphasis on the connections to high energy physics. We have tried to set the work in an historical context, but we certainly do not intend this narrative to be a comprehensive history.

1. 1979-80

1.1 Nonperturbative renormalization group in two dimensions

Shenker and Tobochnik[1] studied the $O(3)$ nonlinear sigma model in two dimensions. They used Monte Carlo renormalization group techniques to demonstrate the full range of scaling behaviour, from asymptotic freedom at short distance to strong coupling at long distance. This work provided support for the analogous picture for QCD in four dimensions.

1.2 Einstein's equation from two dimensional scale invariance

In his thesis,[2] *Friedan* studied the perturbative renormalization of the generalized nonlinear model in two dimensions. The basic new idea was to see the renormalization group as a flow on the infinite dimensional space of metrics on the target manifold of the nonlinear model. He showed that the renormalization group fixed point equation, the equation for two dimensional scale invariance, is Einstein's equation $R_{\mu\nu} = 0$, with perturbatively calculable corrections away from the long distance limit.

It was soon realized, by *Friedan* and *Shenker* among others, that *Friedan's* derivation of Einstein's equation from the principle of two dimensional scale invariance was the basis for understanding the equation of motion of string theory as the equation of two dimensional conformal invariance, and the possible compactifications of spacetime in string theory as the Ricci-flat (Calabi-Yau) manifolds.

Friedan proved the perturbative renormalizability of the general nonlinear model and explained general covariance as a special case of independence of renormalization scheme, which later was seen to be the true symmetry of the string theory. He calculated the beta function to two loops and studying the possible weak coupling fixed points, in particular the examples given by Calabi-Yau spaces. He explained how one loop fixed points would have to be corrected at higher order to maintain the scale invariance. The two loop calculation was used by Alvarez-Gaumé and Freedman[3] as the basis for their demonstration of cancellation of the two loop term in the supersymmetric nonlinear model, which eventually led to a proof that the Calabi-Yau spaces give conformal nonlinear models to all orders, and thus compactifications of string theory. As in *Friedan's* thesis, the Calabi-Yau metric is corrected order by order in perturbation theory to maintain the scale invariance.

In this model, the field $\phi(\xi)$ takes values in an arbitrary manifold M, and the action is $S(\phi) = \int d^2\xi g_{\mu\nu}(\phi)\partial_a\phi^\mu\partial_a\phi^\nu$ where $g_{\mu\nu}$ is a Riemannian metric on the manifold M. Unlike previously studied field theories, this model admits an infinite number of naively marginal couplings, encoded in the metric $g_{\mu\nu}$ which can vary at each point of M. Under a change of two dimensional scale, $\Lambda \to e^t\Lambda$, the metric coupling $g_{\mu\nu}$ becomes an effective metric coupling $g^t_{\mu\nu}$, given by the renormalization group (rg) equation $dg^t_{\mu\nu}/dt = \beta_{\mu\nu}(g^t)$. The weak coupling limit is the limit of large volume, *i.e.* small curvature, on M, $g_{\mu\nu} \to \infty$. The beta function was calculated perturbatively in terms of the curvature tensor of the metric coupling: $dg_{\mu\nu}/dt = R_{\mu\nu} + \frac{1}{2}R_{\mu\alpha\beta\gamma}R_{\nu}^{\alpha\beta\gamma} + \cdots$.

The rg fixed point equation, the equation for two dimensional scale invariance was thus shown to be Einstein's equation $R_{\mu\nu} = 0$ with perturbatively calculable corrections away from the long distance limit. A discussion was given of the obstructions to the persistence of a fixed point when the linearized beta function has zero modes. It was shown that such an obstruction would always arise at two loops, unless the curvature tensor vanished identically. Any fixed point metric with arbitrarily large volume would thus have to be flat, implying that generic bosonic conformal nonlinear models could only be made from tori.

2. 1980-81

2.1 The naturalness of gauge symmetry at long distance

Shenker,[4] building on work with Fradkin,[5] showed that systems without microscopic gauge invariance could become effectively gauge invariant at short distance. This was also done by Foerster, Nielsen and Ninomiya[6]. It was one of the basic results in the program of deriving realistic long distance physics from generic short distance physics. This program foundered on the impossibility of incorporating chiral fermions (see section 2.2 below).

2.2 Impossibility of weakly coupled cutoff chiral fermions

Nielsen and Ninomiya[7] proved, using algebraic topology, that free fermions on a cubic lattice could not have a chiral low energy spectrum. *Friedan*[8] gave a more complete proof using only differential calculus, which was easily generalized[9] from the cubic lattice to an arbitrary cutoff system, *i.e.* with a finite number of degrees of freedom per unit volume.

This was construed, at least by *Friedan,* as a no-go theorem for fundamental physics based on gauge theory. It eventually became a motivation for interest in a fundamental string theory. The basic formula in *Friedan's* proof can be given a rigorous mathematical interpretation in terms of higher dimensional spectral flow[10].

2.3 The loop expansion in string theory

In 1981 Polyakov[11] revived the covariant approach to string theory[12]. Two dimensional conformal invariance had always played an essential role in the covariant approach[13]. Polyakov expressed this in modern field theory language, stressing covariance and locality on the world-surface. He quantized the Brink-di Vecchia-Howe-Zumino[14] two dimensional action for the string world-surface, taking account of the Fadeev-Popov determinant for the gauge group of world-surface reparametrizations. His purpose was to construct string theories in dimensions lower than the known critical dimensions by showing that new degrees of freedom, the Liouville field, arose. This has not yet born fruit, but Polyakov's approach has turned out to be very useful for studying strings in the critical spacetime dimension, which his calculation can be interpreted as providing a potential explaination of consistency of the theory as cancellation of the local trace anomaly on the world-surface without additional degrees of freedom spoiling unitarity.

In the spring and summer of 1981, *Friedan* extended Polyakov's local approach to take account of the global structure of the string world-surface, writing the string perturbation series as a sum over all surfaces with arbitary numbers of handles,[15] with a measure given by the product of Fadeev-Popov determinant and matter partition function. This was described in a lecture[16] at the Niels

Bohr Institute Workshop on String Theory in October, 1981 and was worked out in detail by O. Alvarez[17]. This has been the foundation for modern treatments of the covariant string perturbation series and one of the roots of recent work on string theory as modular geometry.

3. 1981-82

3.1 String theory and conformal field theory

In the winter of 1981-82, *Friedan* tried to understand the Liouville model and its role in string theory. This approach proved fruitless, but in the process he developed the modern language of conformal field theory, the analytic stress-energy tensor $T(z)$ and the analytic operator product expansion. The work[15] was presented at the Les Houches Summer School in August, 1982 and at the Nordita-Landau Institute Seminar in Copenhagen in September, 1982.

This approach arose out of *Friedan's* frustration with the use of canonical commutation relations to study quantum conformal invariance in two dimensional field theory. He developed a diagrammatic perturbation expansion for the Liouville model (two dimensional gravity) in which the renormalization was performed covariantly, with respect to the two dimensional geometry given by the quantum Liouville field itself. The problem was to verify the conformal invariance of the theory and calculate the central charge of the Virasoro algebra. Canonical commutation relations of the fields gave up manifest covariance and were awkward to use in the interacting theory (though in fact they were used successfully in the Liouville model by Curtright and Thorn[18]). *Shenker* suggested using operator product expansions instead. This idea was very successful. *Friedan* used it to finish the covariant all-orders construction of the quantum Liouville model and confirmed the results of Curtright and Thorn.

Friedan used Ward identitites to demonstrate the connection between the trace anomaly and the central charge of the Virasoro algebra. He gave the contour integral argument relating the singular part of the analytic operator expansion with the commutation relations of the operator modes. In particular, the operator product of two stress-energy tensors was shown equivalent to the commutation relations of the Virasoro operators. This allowed the more general and covariant operator product arguments to replace oscillator algebra.

He showed that the non-spacetime degrees of freedom of the string (*e.g.* Liouville) has to be a conformal field theory with exactly the quantum trace anomaly which combines with the spacetime dimension to give the critical value (*e.g.* 26). This result, which seems not to have been obvious to most workers in the field at the time, implied that the perturbative quantum Liouville model could not be used in spacetime dimensions above one.

He made the key point that the partition function on an arbitrary surface could be calculated from the expectation value of the stress-energy tensor $T(z)$ on the surface, which is the derivative of the logarithm of the partition function with respect to the moduli (parameters) of the surface, and that the expectation value of $T(z)$ could be calculated from the short distance behavior of a two-point function of quantum fields. This is the basic technique for constructing the partition function on arbitrary surfaces from local properties of the conformal field theory. It later became a crucial idea in the modular geometry program, and it remains the only known way to calculate Yukawa couplings in orbifold string compactifications.

4. 1982-83

4.1 Anomalies, index formulas and supersymmetric first quantization

Friedan and Windey,[19] based on a suggestion of E. Witten, calculated the index of the Dirac operator in arbitrary gravitational and gauge fields, and the nonabelian gauge anomaly, by supersymmetric quantum mechanics. This was also done independently by Alvarez-Gaumé[20] and by Witten[21].

Friedan and Windey described the functional integral over the super world lines of a first quantized fermion, giving the super heat kernel of the Dirac operator, and showed that the Greens function of the Dirac operator was obtained by integrating over the super proper time.

This technology served as prototype for superconformal field theory and for the description of loop amplitudes of fermionic string by integrating over the super moduli of super Riemann surfaces (see sections 5.2, 6.5 and 8.3 below).

Friedan and *Shenker* later applied the ideas of this work to formulate the anomaly calculaton for superstrings (see section 6.1 below).

Alvarez-Gaumé and Witten[22] used the technique to calculate the gravitational anomaly in field theory.

4.2 Unitarity in two dimensional (super) conformal field theory

In the fall of 1983, *Friedan, Qiu* and *Shenker*[23] proved a theorem classifying all possible unitary representations of the Virasoro algebra and thus giving the first concrete result towards a classification of all possible two dimensional critical phenomena.

In the late 1960's Kadanoff and Polyakov had proposed that all possible universality classes of critical phenomena could be classified by classifying scale invariant quantum field theories, using the algebraic structure of the operator product expansion. In the early 1970's Polyakov showed that scale invariant local systems would have the additional symmetry of local scale invariance, conformal invariance. He formulated a bootstrap program of classifying all conformally invariant quantum field theories. At the same time, in string theory, the world surfaces of strings were being described by two dimensional conformal field theories.

In the winter of 1982-83, Belavin, Polyakov and Zamolodchikov[24] began to revive the conformal bootstrap program in two dimensions, making use of the results of string theory, in particular the Virasoro algebra, and the analytic stress-energy tensor and the analytic operator product introduced by *Friedan* (see section 3.1 above). They pointed out that conformal field theories made from a certain special class of representations of the Virasoro algebra, the degenerate representations given by vanishings of the determinant formula,[25] which occur for all $c < 1$, were exactly solvable, their correlation functions obeying ordinary differential equations. They showed that the Ising model was made from such representations and suggested that there were other statistical mechanics models in this class. They gave no reason for such representations to occur in conformal field theories describing physical systems. The degenerate representations simply provided a certain solvable class of conformal field theories.

Friedan learned of these ideas during a visit to the Landau Institute in April and May, 1983. On returning from the Soviet Union he began to look for a physical reason for degenerate representations of the Virasoro algebra to occur, and discovered that unitarity imposes significant constraints on the possible representations of the Virasoro algebra with $c < 1$. *Friedan,*Qiu and *Shenker* pursued this investigation to the point of proving the theorem that, if $c < 1$ then unitarity requires c to be one of the discrete series of rational numbers $1 - 6/m(m+1)$, $m = 2, 3, \ldots$, and further that, for each allowed value of c, only a finite number of representations were possible, all degenerate and all giving rational critical indices.

Every universality class of critical system with a local order parameter is described by a Landau-Ginsberg model and thus by a unitary conformal field theory. The theorem of *Friedan, Qiu* and *Shenker* thus was the first step in classifying all two dimensional critical systems with $c < 1$, by listing all possible values of the trace anomaly and all possible values of the critical indices, *i.e.* all possible representations of the Virasoro algebra. In order to complete the classification it remained to determine the possible multiplicities of the representations and the possible operator product coefficients. All of the $c < 1$ representations allowed by unitarity were degenerate, so could be solved by the techniques of Belavin, Polyakov and Zamolodchikov.

Friedan, Qiu and *Shenker* [23] carried out the same classification of possible unitary representations for the Neveu-Schwarz algebra, one of the two supersymmetric extensions of the Virasoro algebra. They found an analogous discrete series with $c < 3/2$, and pointed out that the first two values of c in the super discrete series corresponded to known critical systems, the tri-critical Ising model and the gaussian model. This was the first discovery of supersymmetric critical phenomena and, in fact, the first examples of supersymmetric extended systems observable in nature (at present).

FQS[23] pointed out that these representations of the supersymmetry algebra could be used in string theory. The idea is that, as implied by *Friedan's* 1982 work[15] (see section 3.1 above), any conformal field theory with the correct value of c can be a string ground state. Tensor products of models in the discrete series could be used to obtain the requisite values of c. For example, nine copies of the $c = 1$ superconformal model could be combined with four ordinary flat spacetime dimensions.

Independently, Belavin, Polyakov and Zamolodchikov[24] continued their work on the degenerate representations. They found that if c was in the dense set $1 - 6/m(m+1)$, m any rational number above 2, there was a finite set of degenerate representations on which the operator product expansion could close.

Again, this is only a special class of models which are exactly solvable. The theorem of *Friedan, Qiu* and *Shenker* implies that all local critical systems with $c < 1$ fall within this class of solvable models. Also independently, Andrews, Baxter and Forrester[26] described a class of exactly solvable two dimensional lattice models. Huse[27] found substantial evidence in the work of Andrews *et. al.* that their models at their critical points realize all values of c in the unitary discrete series.

Besides the application to critical phenomena, the main significance of the FQS theorem is that it showed the possibility of carrying out the two dimensional conformal bootstrap program incrementally for $c < 1$. It soon became clear that the classical string ground states are essentially the conformal field theories with the critical value of c, so that classical string theory is essentially a special case of the conformal bootstrap. Thus the FQS result indicated the possibility of using the $c < 1$ classification problem as a practical arena in which to study the more difficult problem of finding the classical ground state of string theory.

5. 1983-84

5.1 Conformal and superconformal invariance in string theory

In the summer of 1983 *Friedan* began to explore the relation between the two dimensional conformal invariance of string theory and the results of his thesis (see section 1.2 above)[28]. This was also being explored at about the same time by Lovelace,[29] Witten[30] and possibly others. The essential step was simply to juxtapose the existing results, by regarding the general nonlinear model of 2 as the functional integral for first quantized strings moving in a background gravitation field. Two dimensional conformal invariance was known to be the condition for unitarity in string theory[12,13]. *Friedan* discussed this in his Les Houches lectures[15] using general language of Ward identities and operator products for the two dimensional quantum stress-energy tensor applicable in arbitary two dimensional conformal field theories. In particular, his analysis of the role of the (hypothetical) quantum Liouville model applied equally well to arbitrary internal degrees of freedom, in particular fixing the Virasoro central charge of the internal system as the difference between the critical value (*e.g.* 26) and the Minkowski spacetime value (4).

By one of the basic results of *Friedan's* thesis,[2] the condition of two dimensional conformal invariance was known to be an equation of motion on the background gravitational field, Einstein's equation plus short distance corrections. The background could be regarded as a description of the string state since the graviton is a mode of the string. Thus two dimensional conformal invariance is the equation of motion of string.

Thus fixed points of the rg of the nonlinear model correspond to ground states of the string, *i.e.* string compactifications of spacetime. *Friedan's* thesis implied that any compactification of bosonic string would have to be flat, at least if only the gravitational mode condensed. A richer class of compactifications were provided by the results of Alvarez-Gaumé and Freedman[3] generalizing *Friedan's* thesis to the general supersymmetric nonlinear model. Extending *Friedan's* two loop calculation for the bosonic model, they found that in the supersymmetric model the two loop term vanishes if the one loop term, the Ricci tensor of spacetime, does. Later the same was found for the three loop term. This suggested the conjecture that spacetimes with zero Ricci tensor give exact classical solutions of string theory. Eventually it turned out[31] that there were nonzero terms in the beta function starting at four loops. But it was shown[32,33] that the higher loop corrections could be cancelled by making small perturbations of the one loop fixed point, the Ricci-flat metric, the pattern for perturbative fixed points originally envisioned in *Friedan's* thesis.

One preliminary step taken by *Friedan* in the summer of 1983 was a unitarity proof for strings in a background given by one ordinary time direction ($c = 1$) plus an arbitrary unitary conformal field theory with $c = 25$. This was equally applicable to compactifications or possible quantum Liouville models.

5.2 Unitarity and spin fields in superconformal field theory

The superstring theories which were discovered in the 1970's were being refined and studied in the early 1980's by Green, Schwarz and Brink. In the fall of 1983 our attention was drawn to the superstring theories, and to the remarkable one loop finiteness results for the type II theories due to Green and Schwarz,[34] by the result of Witten and Alvarez-Gaumé[22] on the cancellation of anomalies in the low energy field theory of the type IIb superstring. The work of Green, Schwarz and Brink was based on the light-cone formalism, which was manifestly unitary but not manifestly Lorentz invariant, a serious deficiency in a theory of gravity. It was clear that the ideas of superconformal field theory should be applied to make a covariant formalism for superstring theory. Our first step was to develop the general theory of superconformal fields, both to understand supersymmetric critical phenomena and to understand the general classical ground states or compactifications of fermionic string.

In the winter of 1983-84 *Friedan* taught a course in string theory to learn the basics of superconformal field theory, including the role of the two superconformal algebras, the Ramond algebra and the Neveu-Schwarz algebra. *Friedan, Qiu* and *Shenker* had already done the unitarity study for the Neveu-Schwarz algebra (see section 4.2 above). *Qiu* realized that we had not done the other algebra, the Ramond algebra and adapted our computer programs to get striking numerical results. The unitary representations of the Ramond algebra began to appear in a discrete series at the same values of c as had already been found for the Neveu-Schwarz algebra. The first nontrivial representations of the Neveu-Schwarz algebra, occurring at $c = 7/10$, had been found to give some of the known critical indices of the tri-critical Ising model (the Z_2-invariant sector).

Now it emerged that the first unitary representations of the Ramond algebra also occurred at $c = 7/10$ and provided the remaining critical indices of the tri-critical Ising model.

We developed this picture into a general theory of two dimensional superconformal fields[35]. We found the determinant formula for the Ramond algebra. This formula was proved by Meurman and Rocha-Caridi[36] and was independently found and proved by Thorn[37]. We proved the non-unitarity theorem for the Ramond algebra, that the possible unitary representations with $c < 3/2$ form a discrete series, and indeed the allowed values of c occurred at exactly the same values of c as for the nontrivial unitary representations in the discrete series of the Neveu-Schwarz algebra. This strongly indicated that the presence of these two sectors was a general characteristic of superconformal field theory.

We explained this by showing that *spin fields,* corresponding to the representations of the Ramond algebra, always occured as geometric disturbances in superconformal field theories, introducing branch cuts in the fermionic components of the superconformal fields.

Two examples of such spin fields had already been seen in free fermion conformal field theories. The Ising spin field introduced exactly this branch cut into the free fermion of the Ising model. And the fermionic vertex operator which was constructed by very complicated oscillator algebra in the 1970's was a spin field for the superconformal field theory of the ten free massless superfields of the covariant superstring.

We also showed that a local bosonic conformal field theory could always be constructed from the superconformal fields and the spin fields by projecting on the subspace of even fermion number. This generalized the Gliozzi-Olive-Scherk projection in the free superfield theory of the covariant superstring. In particular, We showed that the bosonic tricritical Ising model was the spin projection of the superconformal field theory of its soliton excitations.

As technical language for the study of superconformal field theory we worked out the formalism of superconformal operator product expansions in analogy with the analytic operator product expansions of the bosonic conformal field theory. We also worked out the superconformal generalization of the differential equation technique of BPZ. *Qiu*[38] used this technique to calculate the fundamental operator product coefficient of the superfield of the tricritical Ising model.

This work was first reported at the Landau-Nordita Seminar in Moscow, June, 1984. Bershadsky, Knizhnik and Teitelman[39] had studied the Neveu-Schwarz sector of superconformal field theories independently.

6. SUMMER, 1984

By June, 1984 we had identified a number of key problems in string theory. We studied these problems at Aspen in July and August of 1984, and reported on our work in lectures by *Friedan* and *Shenker* at Aspen in August, and by *Friedan* at ITP Santa Barbara in August and September and at the APS-DPF meeting in Santa Fe in November. A summary of this work appeared in reference 40.

6.1 Anomalies in superstring theory

The first and most pressing problem was to find a way to get from weakly coupled superstring theory a low energy four dimensional effective theory with nonabelian gauge symmetry. Otherwise there could be no possibility of string theory being realistic, without completely intractable strongly coupled string effects. The difficulty was that only two kinds of superstring theory were known. The type II theories contained no massless gauge vector modes. The type I theories contained $SO(N)$ massless gauge vector modes, but their low energy limits were thought to be anomalous[22]. The problem of gauge symmetry in string theory was especially noted by Mandelstam[41] and by Witten[42]. Witten had suggested a mechanism for producing gauge symmetry in bosonic string theory, using nonabelian two dimensional current algebra. But it was thought by both Mandelstam and Witten that such a mechanism would not work in superstring theory, because of the shift in the intercept from the bosonic theory.

It seemed to us worth studying the anomaly in the type I string theories as opposed to the low energy field theories, since it seemed conceivable that some special property of string theory would cause the anomaly to vanish. We realized that the anomaly could be calculated in string theory as the unitarity violating one loop six point S-matrix element for the scattering of five physical massless gauge vectors into an unphysical longitudinal massless vector mode. This was something of a departure from field theory anomaly calculations which were then phrased in the off-shell language of current conservation violation. The anomalous S-matrix element could not be calculated in the then prevalent light cone formalism because it gave no vertex for the unphysical mode. But we realized that the covariant formalism did have such a vertex.

We wrote a formal expression for the hexagon diagram giving the one loop anomalous S-matrix element in the type I theories as a multiple integral of the correlation function of the six gauge vector vertex operators on the annulus with respect to the locations of the vertex operators and the modulus of the annulus. A source of this point of view towards calculating the anomaly by first quantized string methods was the work of *Friedan* and Windey[19] and Alvarez-Gaumé[20] on calculating the field theory anomaly using the first quantized particle formalism.

We showed this expression for the string anomaly to Green and Schwarz. They realized that they knew how to calculate it, and carried out the calculation, making the dramatic discovery that the anomaly cancels in the $SO(32)$ type I theory[43]. A well known chain of developments then followed – the Green-Schwarz anomaly cancellation mechanism in field theory,[44] the realization of an $E_8 \times E_8$ anomaly free field theory, the recognition by Freund[45] of the significance of $10 + 16 = 26$, and finally the beautiful synthesis of the heterotic string[46].

6.2 No-go theorem for compactification of type II superstrings

It also was clear to us that a superconformal version of nonabelian current algebra could in fact be used to construct vertex operators for gauge vector particles in the covariant formalism. We described the superconformal current algebras as straightforward generalizations of the bosonic algebras, replacing dimension 1 chiral currents with dimension 1/2 chiral supercurrents. We pointed out that these superconformal current algebras would appear in the supersymmetric extensions of the Wess-Zumino-Witten models. These models were being studied at the same time by Rohm,[47] independently.

These superconformal current algebras gave vertices for massless gauge particles when used in compactifications of type II strings. The shift in the dimension of the current compensated for the shift in the intercept in going from bosonic to fermionic string. We also showed that, conversely, if a type II string theory has massless gauge particles then their vertex operators give a superconformal current algebra.

But we were also able to show that in such compactifications it was impossible for the charged fermions to be massless. The mass operator for the fermions contains the Ramond operator G_0 for the supercurrent algebra. We proved that the zero mode algebra of the supercurrent algebra split into ordinary nonabelian charges and decoupled fermions, which gave an explicit positive lower bound for the G_0 eigenvalue and thus for the charged fermion masses.

This was a no-go theorem for realistic compactifications of weakly coupled type II superstring theory. Our no-go theorem was only for left–right symmetric compactifications of type II strings. Recently Dolan et al.[48] and Kawai et al.[49] have exhibited left–right asymmetric type II compactifications. Dixon, Kaplunovsky and Vafa[50] have extended our no-go result to these compactifications. They combined our arguments for the symmetric case with the classification of unitary representations of current algebra to show that there are no type II compactifications with a gauge group containing $SU(3) \times SU(2) \times U(1)$ and charged massless fermions.

Our decoupling result for the superconformal algebras employed a technique that Goddard and Olive were then using to make currents from bilinear expressions in fermions fields, and were applying to demonstrate that the representations in the discrete series that FQS had shown were the only possible unitaries, were in fact unitary. This proof was completed by Goddard, *Kent* and Olive[51].

6.3 Superconformal field theory and covariant superstring theory

The third problem was to complete the covariant quantization of superstrings by describing the superconformal field theory of the world sheet of the string. This had been only partially carried out in the 1970's. In particular, the vertex operator for fermion emission had not been completely constructed and the spacetime supersymmetry of the covariant theory had never been demonstrated. The spin field of the ten free superfields of the superconformal theory of the worldsheet of the covariant superstring had been constructed in the 1970's by very complicated oscillator algebra. The scaling dimension of this spin field was 10/16. The calculation of its four point functon, in order to find the four point scattering amplitude of the massless fermionic modes of the string, was a tour-de-force of oscillator algebra. The result had actually been obtained first by Mandelstam using light-cone functional integral methods.

By the general superconformal field theory, we knew that the dimensional 10/16 spin field had to exist because there was a corresponding Ramond representation. But a more transparent construction of the spin field was needed in order to make calculation possible. Even more essential was to find the missing part of the fermion vertex which would bring its dimensional up to the value 1 which was needed in the covariant formalism in order that its integral over the world-surface give ghost free scattering amplitudes.

While we were thinking about the problem of the missing dimension 6/16, Goddard and Olive suggested that the superconformal ghosts might provide the missing dimension 6/16 contribution to the fermion vertex operator. We described the superconformal ghost system and calculated the ground state energy in the Ramond sector. It was indeed 6/16, corresponding to a spin field of the needed dimension. We also showed that the integal of a $U(1)$ ghost current gave a scalar field whose exponential was the dimension 6/16 spin field.

6.4 Compactification of superstrings on Calabi-Yau spaces

The third problem was to understand the process of compactification, in order to explain why the low energy limit of string theory should appear four dimensional, if it does, and confront the low energy effective theory with experimental particle physics.

It was clear from *Friedan's* thesis[2] and its supersymmetric generalization[3] that compactifications of superstrings were given by six dimensional manifolds with vanishing Ricci tensor, of which the only known nontrivial examples were the Calabi-Yau manifolds.

Although we pointed out the existence of these compactifications, we did not think it would be fruitful to study them as spacetime manifolds with metrics. Our main reason was that we realized that the spacetime picture of the compactification is only valid in the large radius limit, where the perturbative expansion of the nonlinear model makes sense. We thought it important to realize that the precise description of the classical string ground state is the nonperturbative conformal field theory of the world-sheet. Since the only natural scale in string theory is the Planck length, we expected that the large radius limit for the compactification could not be accurate.

Moreover, the Calabi-Yau metrics were only known via an existence proof; none had ever been constructed and still it is true that no smooth Calabi-Yau metric has ever been constructed. Given such limited knowledge, it seemed unlikely that the nonperturbative nonlinear model could ever be described starting from the Calabi-Yau space.

We adopted what we considered to be the best long-range strategy. Rather than trying to study individual Calabi-Yau spaces to get candidate classical string ground states, we thought it more likely to be fruitful in the long run to attack the general abstract problem of finding the exact conformal field theories representing the string ground states, as a special case of the two dimensional conformal bootstrap program, and possibly eventually deriving the Calabi-Yau metrics from the conformal bootstrap.

6.5 Loop expansion in superstring theory

During the Aspen period, we also began to think about the form of the covariant loop expansion for superstrings, by analogy with the loop expansion for the bosonic string[15,17]. We made the basic observation that the GOS projection would be implemented in at g loops by summing over all 4^g spin structures on the Riemann surfaces with g handles.

7. 1984-85

7.1 Spin fields and the fermion vertex

During the fall of 1984, in collaboration with our graduate students *J. Cohn* and *Z. Qiu,* we[52] worked out a straightforward construction of the spin field of the Ramond-Neveu-Schwarz world-surface fermions $\psi^\mu(z)$, which is the matter part of the spacetime fermion vertex. This was a necessary step in the program of basing superstring theory on superconformal field theory. It was a transparent construction based on the abstract definition of the spin field as the field which introduces a branch cut in the surface fermions $\psi^\mu(z)$. It made evident the superconformal properties of the vertex. It used a technology which made calculations practical. In the 1970's the matter part of the fermion vertex had been constructed and its four point function had been calculated by a *tour-de-force* of oscillator algebra[53].

We attacked the problem from two directions, which were merged together in the final answer. One approach was based on an analogy with condensed matter physics. The critical Ising model in two dimensions is equivalent to a single surface fermion $\psi(z)$. The Ising spin field is exactly the field which produces a branch cut in this fermion field. Thus the spin field of the d surface fermions ψ^μ could be constructed as the product of d Ising spins. Luther and Peschel[54] had constructed the product of two Ising spins in terms of exponentials of a single scalar field. We expected to be able to construct the matter spin field in terms of exponentials of $d/2$ scalar fields.

The second approach, initiated by *J. Cohn,* was to start from the nonabelian algebra of $SO(d)$ currents $j^{\mu\nu} = \psi^\mu\psi^\nu$. The spin fields were defined by their operator products with the currents, determined in turn by their transformation properties as $SO(10)$ spinors. *Cohn* and *Qiu* calculated the four point function by the differential equation technique for current algebras described by Knizhnik and Zamolodchikov[55].

The two approaches merged when we realized that the $d/2$ scalars of the generalized Luther-Peschel construction were also the the $d/2$ scalars of the vertex operator construction of the $SO(10)$ current algebra[56]. The spin fields of the generalized Luther-Peschel construction were exactly the exponential vertex operators corresponding to the spinor weights of $SO(10)$.

This introduction of the vertex operator construction of current algebra into string theory was independent of the heterotic string construction[46] which was being done at the same time. In our application there was a new aspect, because the matter spin field is multi-valued when d is not a multiple of 8. We announced the construction in 40, 23, and 57. But the press of other work delayed publishing the details until 52.

7.2 Covariant quantization of superstrings

During the winter of 1984-85 we worked out the complete spacetime fermion vertex and we demonstrated the spacetime supersymmetry of the covariant fermionic string[23]. E. Martinec collaborated with us in this work.

This work laid the foundations of the covariant superstring theory and has been the basis for a great amount of work on the loop expansion of the superstring theory. It made possible the investigation of string theory as abstract two dimensional superconformal field theory. It provided the technical apparatus for gauge invariant fermionic string field theory[58].

Key elements were the use of local conformal currents to generate surface BRS transformations and spacetime supersymmetry transformations. Once the local properties of these currents were determined, the BRS invariance and supersymmetry of the amplitudes could be demonstrated by contour deformation arguments on arbitrary world-surfaces. Interesting obstructions to these deformations have recently been found and related to expected violations of nonrenormalization theorems[59].

We made a detailed study of the superconformal world-surface ghost system. We bosonized the bilinear ghost number current of the superpartners of the ordinary fermionic ghosts. This bosonization of bosonic fields showed novel features. In particular, additional degrees of freedom remained after bosonization, which were described by a second bosonic field. We showed that there were infinitely many equivalent Hilbert spaces for the spacetime fermion states, generalizing the two equivalent *pictures* found by the early workers on the covariant formalism.

We constructed the local BRS current on the world-sheet and determined the spacetime fermion vertex operators by BRS invariance. There were infinitely many vertex operators corresponding to the infinitely many different pictures. We showed that the different pictures are equivalent in the sense that they give the same string scattering amplitudes. We constructed the picture changing operator which takes one picture to another. The full significance of these different pictures remains to be understood.

We constructed the local world-sheet current which generates spacetime supersymmetry transformations and used it to demonstrate the spacetime supersymmetry of the string theory.

7.3 Conformal invariance and the string equation of motion

In the spring of 1985, *Friedan* collaborated with C. Callan, E. Martinec and M. Perry[60] to extend the results of his thesis[2] to the general background of massless string modes, including the dilaton field and the antisymmetric tensor field and the gauge fields of the heterotic string. The main mystery which was resolved by this work was how two dimensional conformal invariance could give the equation of motion of the dilaton field. This confirmed that two dimensional conformal invariance could be considered as the string equation of motion. Part of this work was done independently by Sen[61] although he did not solve the key problem of the dilaton. The form of the dilaton coupling was suggested by Fradkin-Tseytlin[62]

7.4 Ideas in string field theory

In 1984, W. Siegel began the construction of a gauge invariant field theory of strings in Minkowski space, in which the string fields correspond to the states of a single first quantized string. In the spring of 1985, T. Banks and M. Peskin began the construction of string field theory in terms of the Virasoro algebra of the conformal field theory of the world surface of the first quantized string.

In the spring and summer of 1985, influenced especially by the work of Banks and Peskin and by conversations with Banks, *Friedan*[63,64] formulated a collection of proposals and ideas about the fundamental formulation of string theory.

The ideas of 63 included: the need for background independence of a fundamental string theory; the string field as a half-density; the formal gauge invariance and background independence of the the interaction vertex (overlap integral); the possibility of a background independent trilinear action with the usual kinetic term of the background dependent action provided by the vacuum expectation value of the field; the vacuum expectation value as the ground state of the corresponding two dimensional conformal field theory; the trilinear string field equation as a version of the conformal bootstrap equation or the renormalization group fixed point equation, the solutions being the two dimensional conformal field theories; the crucial role of the string midpoint.

The ideas of 64 included: the need for an abstract algebraic string equation of motion without the need for an *a priori* choice of spacetime; the Virasoro algebra commutation relations or equivalently the operator product relations of the two dimensional stress-energy tensor as an abstract bilinear equation of motion expressing two dimensional conformal invariance; and the role of the local-global principle in string field theory, the string field equations of motion defining a *germ* of a conformal field theory expressing the local properties of the world sheet.

These ideas were presented in the context of string field theory and had some influence on subsequent developments in string field theory[65,66]. They also provided some of the philosophical roots for our formulation of string theory as modular geometry (see section 8.8 below).

7.5 Off-shell string theory and the two dimensional rg

In November 1985, *Friedan*[67] discussed formulating off-shell string theory directly in terms of the two dimensional renormalization group acting in the space of two dimensional supersymmetric field theories.

The classical equation of motion would be the renormalization group fixed point equation. Stochastic quantization would be performed by doing a random walk in the space of quantum field theories with a driving force given by the beta function. This could be construed as doing asymptotically long distance two dimensional physics in the presence of noise.

The absence of tachyons in the classical ground states of the string is equivalent to the absence of ultra-violet instability in the corresponding two dimensional renormalization group fixed points. Thus the classical ground states are completely attractive fixed points describing distinct phases of the two dimensional system. If there are continuous transitions beween these phases, then there unstable fixed points on these boundaries. The unstable fixed points are also supersymmetric conformal field theories. The unstable trajectories leading from an unstable fixed point go to different classical ground states. For example, these would be superconformal field theories interpolating between distinct Calabi-Yau spaces.

When the beta function is written as the gradient of a potential, the unstable fixed points correspond to saddle points. The conformal anomaly coefficient c measures essentially the number of degrees of freedom of the conformal field theory, which should decrease under the renormalization group[68], so the saddle points should be superconformal field theories with larger values of c than in the classical ground states.

Nonperturbative effects would be dominated in the stochastic quantization by trajectories passing from one ground state through the lowest saddle point to another ground state and back again. These effects of these trajectories could be calculated in terms of the non-scale invariant quantum field theories associated with the renormalization group trajectories leaving the saddle points. The trajectories from the lowest saddle point could break degeneracies associated with the classical ground states, for example fixing the string coupling constant and breaking supersymmetry.

The unstable fixed points would break the discrete degeneracy of classical gound states only if all the corresponding stable fixed points are phases of a single two dimensional system. The quantum ground state would be determined by the network of transition probabilities between the classical ground states given by the trajectories through the unstable fixed points.

Distinct string theories would correspond to the different maximal non-scale invariant two dimensional systems all of whose infra-red stable long distance limits are conformal field theories with c equal to the critical value. Such two dimensional systems would be extraordinary, since there would be not a single trajectory with trivial long distance limit (mass gap). A maximal system is one to which no degrees of freedom can be added to give new fixed points without leading to some long distance limit with c less than the critical value. Gauge invariance of the off-shell string theory is only possible if every irrelevant direction at the fixed points comes from some unstable fixed point, which again would be a remarkable property.

In this picture, each string theory would correspond to a single *ur*–two dimensional field theory with infinitely many relevant parameters, all of whose long distance limits would be the classical ground states, for example the various Calabi-Yau space nonlinear models.

These ideas have had some influence on subsequent work on the renormalization group equation as string field equation of motion[69].

8. 1985-86

8.1 Supersymmetry in the twisted $c = 1$ gaussian model

In our work on unitarity, superconformal field theory and two dimensional critical phenomena[23,35] we noted that the second member of the super discrete series, after the tricritical Ising model, had central charge $c = 1$, characteristic of the one component gaussian model (nonlinear model with field taking values in the circle). In the fall and winter of 1985 we completed the study of the supersymmetric $c = 1$ gaussian model by showing that it corresponded to the $c = 1$ critical Ashkin-Teller model at a special value of the coupling[70]. This example of supersymmetric critical phenomena is experimentally more accessible than the tri-critical Ising model because there is a supersymmetry index which prevents the supersymmetry from being spontaneously broken away from the critical point.

It was obvious that the gaussian model at some special values of the coupling has a chiral fermion field of dimension 3/2 which becomes the super-partner of the stress-energy tensor and gives superconformal invariance. Several of the scaling dimensions allowed by unitarity fit the pattern $n^2/24$ of the gaussian model dimensions at these couplings. But two other dimensions, 1/16 and 9/16, did not fit this pattern.

At the Santa Barbara string workshop in August, 1985, we learned of the work of Corrigan and Fairlie[71] on twisted bosonic fields, of the existence of fixed points[72] and of the values 1/16, 9/16 for the scaling dimensions of the twist fields and excited twist fields. It was immediately obvious that the full supersymmetric $c = 1$ model was the twisted version of the gaussian model at the special, supersymmetric coupling. The symmetry of the twisted gaussian model is the dihedral group D_4, which is exactly that of the Ashkin-Teller model, and its operator products also correspond to this spin model. After doing this work we learned that Thorn had earlier obtained a result on the six vertex model which could be interpreted in retrospect as demonstrating the equivalence between the Ashkin-Teller model and the twisted gaussian model, although he did not note the supersymmetry at special values of the coupling. A number of workers have recently obtained similar results independently.[73]

8.2 The Ising model fermion as Nambu-Goldstino

In 70 we also showed that the fermionic soliton field of the critical Ising model ($c = 1/2$) is the Nambu-Goldstino of spontaneously broken supersymmetry in the tri-critical Ising model ($c = 7/10$).

We noted that one of the relevant trajectories leaving the $c = 7/10$ tri-critical Ising model fixed point preserves supersymmety, but that the supersymmetry could be spontaneously broken away from the fixed point, because unitarity allows no supersymmetric states of the Ramond algebra and so the Witten index is zero. Then there would be a massless fermion mode, a Nambu-Goldstino, along this trajectory and the long distance limit would be described by a nontrivial fixed point. Since the number of degrees of freedom, i.e., c, is reduced by the renormalization group[68], that long distance fixed point would have to be the unique conformal field theory with $c < 7/10$, namely the Ising fixed point. By the Ward identity for broken supersymmetry, the Nambu-Goldstino is noninteracting at long distance and thus becomes the free fermion of the Ising model. It is known from analysis of the lattice versions of these models that there is such a flow to an Ising fixed point, and so supersymmetry must be spontaneously broken. This is a striking example of how supersymmetry puts rigid constraints on the global topology of the renormalization group flow.

8.3 Super Riemann surfaces and super moduli space

In the fall of 1985, *Friedan*[57] gave the mathematical definition of super Riemann surfaces and described their basic properties and some of the basic structure of their super moduli spaces. The super Riemann surfaces provide the natural setting for superconformal field theory. They are the world surfaces of fermionic strings. The basic tools for superconformal analysis and algebra on super Riemann surfaces were developed: super power series expansions, super line integrals and contour integrals, the super Cauchy theorem and the superconformal tensor calculus.

Our graduate student, *J. Cohn*,[74] extended this work to describe the $N = 2$ super Riemann surfaces which provide the setting for superstring theory, fermionic strings in spacetime supersymmetric backgrounds (see sections 8.6, 8.4 and 8.5 below). Super Riemann surfaces have provided the mathematical setting for extending the covariant quantization of fermionic string[75] to higher loops[76].

Some aspects of the theory of super Riemann surfaces seem to have been independently described by and by Baranov and Schwarz,[77] in quite different language. Some aspects of supermoduli space and its role in fermionic string theory were earlier discussed by Moore and Nelson[78].

8.4 $N = 2$ superconformal unitarity and superstring compactification

In the winter of 1985-86, Boucher, *Friedan* and *Kent*[79] obtained determinant formulae for the $N = 2$ superconformal algebra and found the constraints on its representations due to unitarity. The results are the first step in classifying the $N = 2$ supersymmetric critical phenomena. They are also the first step in the abstract classification and study of supersymmetric string ground states as $N = 2$ superconformal field theories.

In the spring of 1985 *Qiu* and *Shenker* had obtained some preliminary results on the $N = 2$ algebra, and at the Santa-Barbara workshop Di Vecchia and collaborators[80] reported similar results. At this time *Friedan* was contemplating the idea of constraining or even classifying string ground states by the representation theory of algebras of conformal fields. The $N = 2$ algebra seemed a relatively accessible starting point. Candelas *et al.*[81] had shown that spacetime supersymmetry in the low energy effective field theory of string is equivalent to the Kahler property of spacetime, which is also the property which gives $N = 2$ supersymmetry in the world sheet nonlinear model.

The results of 79 were that, as for the $N = 0, 1$ algebras, unitarity singles out a discrete series (with $c < 3$), which applies to critical phenomena. But it also puts interesting constraints on the representations for $c \geq 3$ which are relevant for string theory. In particular, unitarity forces stability of spacetime supersymmetry under small $N = 2$ perturbations of the background as long as the $U(1)$ charges of the $N = 2$ algebra are quantized. This led to a systematic study of the relation between abstract $N = 2$ superconformal invariance of the world sheet and spacetime supersymmetry (see section 8.5 below).

Di Vecchia, Petersen Yu and Zheng[82] proved for the $N = 2$ discrete series the analog of the Goddard-Olive-*Kent* result on the $N = 0, 1$ discrete series, that all of the representations in the discrete series allowed by the non-unitarity results of Boucher *et al.*[79] are in fact unitary.

8.5 Spectral flow, $N = 2$ and superstring compactification

In a continuation of reference 79 (see section 8.4 above), *Friedan, Kent, Shenker* and Witten made a detailed study of the $N = 2$ algebras with boundary conditions interpolating between the Ramond and Neveu-Schwarz type boundary conditions. It was shown that the SL_2-invariant vacuum of the Neveu-Schwarz sector always flows into a Ramond state whose corresponding spin field is essentially the spacetime supersymmetry world sheet current, as long as the $N = 2$ charges are integers. This is the condition that the partition function of the string can be extended to the $N = 2$ super moduli space[74] (see section 8.6 below). This was the key step in showing that $N = 2$ superconformal invariance of the world sheet with integer $N = 2$ charges is in fact exactly equivalent to spacetime supersymmetry. Parts of this work were also done by Sen[83].

8.6 $N = 2$ super Riemann surfaces and supermoduli space

Our student *Cohn* extended the theory of super Riemann surfaces to $N = 2$ supersymmetry[74]. This provides the appropriate setting for studying string compactifications with spacetime supersymmetry and hence is the right place to phrase general statements about finiteness and nonrenormalization theorems.

8.7 Two Dimensional String

Shenker investigated the string with world sheet $N = 2$ supersymmetry[84]. He pointed out the potential inconsistency involved in dropping the zero mode of the bosonic partners of the spacetime coordinates. This makes it necessary to confront the presence of a second time coordinate. At best it seems that a heterotic version of such a string will have an infinite number of massless particles.

8.8 The modular geometry of string and conformal field theory

In the fall of 1985 Cardy[85] produced an important piece of work, showing that the requirement of one loop modular invariance severely constrained the allowed multiplicities of representations of the Virasoro algebra for models in the discrete series. This immediately led us[86] to the idea that a partition function well defined on moduli space for arbitrary genus Riemann surfaces might be an appropriate starting point for a general abstract description of conformal field theory. This view was bolstered by the observation that correlation functions reconstructed by allowing handles to degenerate would be crossing symmetric as a consequence of modular invariance.

Such a description is well suited to string theory since the specification of the classical ground state already contains the arbitrary genus information necessary for quantum calculation. The ghosts make the partition function into a density to be integrated over the moduli space. Non-perturbative effects have their origin in some completion of the finite genus moduli space, the surfaces of infinite genus.

A key problem was to find a way to encode the necessary degree of smoothness in the partition function to reflect world sheet locality. All conformal field theories have a certain amount of analytic structure, $T(z)$ is analytic, but the general genus one partition function already shows that the partition function is not an analytic object. In certain simple (chiral) theories, though, the partition function is the modulus squared of a section of a holomorphic line bundle on moduli space[87].

The genus one result does suggest interpreting the general partition function as the norm squared of a holomorphic section of a vector bundle with flat metric over moduli space–a modular geometry. Flatness, combined with the holomorphic nature of the section, gives the required real analyticity of the partition function. The flat metric is equivalent to the specification of a representation of the modular group.

An important requirement remains to be imposed. The partition function must factorize correctly[88]. This requirement can be implemented naturally by demanding that the modular geometry be formulated on a new universal moduli space, where as a handle on a surface degenerates to a node the surface joins holomorphically to the surface with nodes erased. This erasure reflects the trivial correlations of the identity operator.

A modular geometry on universal moduli space yields a conformal field theory. Conversely, given a conformal field theory, generalized conformal blocks can be (formally) constructed to represent the theory as modular geometry. So classifying conformal field theories is equivalent to classifying modular geometries.

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