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The Three-Body Problem in Nuclear Matter

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### Errata

Page 3, lines 2 and 12 read "Coulomb".

Page 35, line 10 read  $\frac{\zeta_{kP}^0(r_{12})}{j_0(kc)} = \zeta^D(r_{12})$

Page 43, Equation (4.33) reads:

$$\begin{aligned} z^{(1)}(r_{12}, r_{23}, r_{31}) &= \eta_{12}(1 - \zeta_{13} + \frac{1}{2} \zeta_{13} \zeta_{23}) \\ &+ \eta_{13}(1 - \zeta_{12} + \frac{1}{2} \zeta_{12} \zeta_{23}) - \eta_{23}(\zeta_{12} + \zeta_{13} + \zeta_{12} \zeta_{13}) \\ &+ [\zeta_{12} \zeta_{13} + \frac{1}{2} (\zeta_{12} + \zeta_{13}) \zeta_{23} - \frac{4}{3} \zeta_{12} \zeta_{13} \zeta_{23}] \end{aligned}$$

Page 55, Equation (5.31),  $\omega(x) = \eta(x) + x \left( \frac{\partial \eta}{\partial x} \right)$

Page 75, Figure 1 caption, read "defect" instead of "defend"

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## 1. Introduction

Nuclear matter is a uniform system of infinite identical nucleons, stripped of their Coulomb interactions. A study of the properties of this system and in particular the evaluation of its binding energy as a function of its density is a valuable first step towards a theory of realistic finite nuclei starting from first principles. For instance, the binding energy per particle of nuclear matter should give the "volume term" in the well known semi-empirical mass formula for nuclei, which is experimentally deduced to be about 16 MeV. Further, the experience gained from the treatment of the energy and density of this many-body system should be of great help in evaluating the surface, Coulomb and other terms in the mass formula.

There are several mathematical formalisms developed for studying nuclear matter. A comparison of these and their relative merits is discussed by Brandow<sup>(1)</sup>. The most successful, and in many ways the simplest, theory is based on the Brueckner-Goldstone<sup>(2,3)</sup> formalism. The Hamiltonian is split into  $H_0 = \sum_i T_i + U_i$  and  $H_1 = \sum_{i \neq j} v_{ij} - \sum_i U_i$ , where  $T_i$  are the kinetic energies of the nucleons,  $v_{ij}$  are the inter nucleon potentials and  $U_i$  are single particle potentials to be chosen conveniently. It is assumed that there are no intrinsic many-body forces. The ground state of  $H_0$  is just that of a zero temperature Fermi gas (it is non-degenerate) and the perturbed

wave function and energy are expanded in a perturbation series in powers of  $H_1$ . The potentials  $v_{ij}$  are used in the form of their reaction matrices, and the resulting series is represented by a set of diagrams similar to the Feynman diagrams. This formalism and the concept of the reaction matrix are explained in the preceding article by Ben Day.

Subsequent to the development of this well-defined formalism, considerable effort has gone into evaluating the lower order terms and treating higher order terms in a consistent way. It appears now that the theory is approaching a satisfactory conclusion, both theoretically and in terms of agreement with experiment. One of the significant steps in this process was the development of the Reference Spectrum Method by Bethe, Brandow and Petschek<sup>(4)</sup> which provided a relatively simple and analytic method for evaluating the reaction matrix. This also revealed several qualitative features with which higher order diagrams could be studied. Such a study resulted in another significant step, namely the realization that the Brueckner-Goldstone series does not converge in powers of the interaction or the reaction matrix, and that it should be rearranged in powers of the number of nucleons involved.<sup>(5,6)</sup> The evaluation by Bethe of the three-body energy to all orders in perturbation lends support to the idea that the above rearrangement in powers of the density should converge for nuclear matter.

Complete calculations using realistic potentials with tensor forces etc. have not yet been fully done. But the work done so far reveals that the sum of the two- and three-body contributions gives a binding energy of about 13 to 18 MeV per particle at the observed density of  $k_F = 1.36 F^{-1}$  or  $\rho = 0.178 F^{-3}$ , with the higher binding corresponding to "soft-core" potentials.<sup>7,8</sup> There are reasons to hope that the corrections to this, such as four-body terms etc. should not be more than a couple of MeV, and methods have been suggested for absorbing these. The experimental value for the binding energy per particle is about 16 MeV. This is a fairly good agreement for a theory beginning from first principles, namely an evaluation of the energy of the many-body system starting from the interparticle potential. In estimating the agreement with experiment one must remember that the theory really calculates the potential energy which is about -40 MeV for a typical nucleon in the Fermi sea, and that the binding energy is a difference between the large potential and kinetic energies. It is hoped that the uncertainty (and possible discrepancy) of about 3 MeV in a total of about 40 MeV will be reduced by the correction terms and by more precise calculations with tensor forces etc.

Since the Brueckner-Goldstone formalism and the Reference Spectrum Method for evaluating the reaction matrix have been described in detail in the preceding article by Day,<sup>9</sup> we will proceed from where Day has left off. Thus Sec. 2 will briefly

gather together some features of the two-body wave function and the reaction matrix needed for subsequent use. Section 3 will discuss the convergence problems of the expansion and bring out the need to rearrange terms in powers of the density. Sections 4 and 5 will deal with a method for evaluating the three-body energy which gives the  $\rho^2$  term in the new rearrangement. Section 6 involves the treatment of tensor forces, and finally Sec. 7 will be devoted to the choice of the single-particle potential energies to be used in the theory. It is strongly recommended to the reader to familiarize himself with the ideas in the preceding article before embarking on this one.

## 2. Some Properties of the Two-body Problem

The Reference Spectrum Method for evaluating the  $g$  matrix, described in detail in the preceding article (henceforth referred to as A) can be summarized very briefly as follows:

The  $g$  matrix is defined in terms of the internucleon potential  $v$  by

$$g = v - v \frac{Q}{e} g \quad (2.1)$$

where  $Q$  and  $e$  refer respectively to the Pauli exclusion operator and the energy denominator in the Goldstone diagram. This is essentially a two-body operator, and involves non-trivially the relative coordinate  $r$ , the momenta of the two particles and a parameter which depends on the other particles excited. Let us define a two-body wavefunction  $\psi(\vec{r})$  and a defect function  $\zeta(\vec{r})$  corresponding to an initial plane wave state  $\phi(\vec{r})$  by

$$\psi(\vec{r}) = \left(1 - \frac{Q}{e} g\right) \phi(\vec{r}) \quad (2.2)$$

and

$$\begin{aligned} \zeta(\vec{r}) &= \phi(\vec{r}) - \psi(\vec{r}) \\ &= \frac{Q}{e} g \phi(\vec{r}) = \frac{Q}{e} v \psi(\vec{r}) \end{aligned} \quad (2.3)$$

The Reference Spectrum approximation,<sup>4</sup> which uses a quadratic form for the single-particle potential energies, amounts to dropping the  $Q$  operator, and replacing the energy denominator by  $\frac{1}{m^*}(-\nabla^2 + \gamma^2)$  in coordinate space. These approximations and corrections to them are explained in the original paper<sup>4</sup> and in A. Here  $m^*$  refers to the reduced mass in the Reference Spectrum energy,  $\gamma^2$ , a positive quantity, is the parameter involving



off-energy-shell contributions from other excited particles, and the factor  $\frac{\hbar^2}{M}$  is suppressed. It then follows from (2.3) that

$$(\nabla^2 - \gamma^2) \varphi(\vec{r}) = -m^* v(\vec{r}) [\phi(\vec{r}) - \xi(\vec{r})] \quad (2.4)$$

and the matrix element

$$\begin{aligned} \langle \phi' | g | \phi \rangle &= \langle \vec{k} | g | \vec{k}_0 \rangle \\ &= \langle \vec{k} | \xi \rangle \left( \frac{k^2 + \gamma^2}{m^*} \right) \end{aligned} \quad (2.5)$$

The g matrix element can thus be obtained by solving the differential equation (2.4) as exactly as possible and using the solution for  $\xi(\vec{r})$  in (2.5). This has been done quite accurately<sup>4,10</sup> and the first order energy, which is the g matrix itself, has been evaluated. However, for the purposes of studying higher order diagrams, involving large numbers of g matrices, it is useful to extract some qualitative features of the function  $\xi$  and the operator g. The great advantage of the Reference Spectrum Method, as compared to directly solving the integral equation (2.1) for g, is the ease with which it lends itself to such qualitative understanding.

In this connection, it is useful to separate the potential into a short range part  $v_s$  and a long range part  $v_l$ , as originally suggested by Moszkowski and Scott.<sup>11</sup> The separation distance d was so chosen by these workers that the defect function  $\xi$  had zero slope and value at  $r = d$ , i.e. the "wound" in the wave function  $\psi$  due to the repulsive core got "healed"

at  $r = d$  due to the attractive part in  $v_s$ . This separation distance is, of course, a function of the initial momenta  $\vec{k}_0, \vec{P}_0$ . But, for a standard hard core potential, with a core radius  $c = 0.4 F$ , the distance  $d$  is about  $1 F$  for a wide range of  $k_0, P_0$  up to about  $2 F^{-1}$ .<sup>12</sup> Moszkowski and Scott show that the reaction matrix  $g_s$  corresponding to this  $v_s$  is zero for free nucleons. Although this is not so for a general off-energy-shell  $g_s$  matrix in nuclear matter, it is still useful to make such a separation, since the two parts  $v_s$  and  $v_l$  have quite different properties and have to be treated differently.  $v_s$  contains the strong repulsive core and is best treated in terms of its reaction matrix  $g_s$  to get finite matrix elements. Further, because of its short range,  $g_s$  has high Fourier components, and as we will see, a strong momentum dependence. On the other hand,  $v_l$  is the relatively weak tail of the attractive part and consequently has a rapidly convergent Born series. Further, although its matrix elements depend strongly on the momentum transfer, the diagonal element is relatively independent of momentum. All these properties will be discussed and used in detail at various places in later sections. It should be noted, however, that although such a separation is often useful, it is possible to evaluate the  $g$  matrix for the full  $v(r)$ , by solving the differential equation (2.4) for  $\vec{\chi}(r)$ , and this is usually more convenient for accurate numerical work.

The function  $\chi_s(r)$  corresponding to  $v_s$  has some useful features. Consider for instance its s-wave part, written as

usual in the form  $\frac{\chi_s^0(r)}{r}$ . Then from (2.4),  $\chi_s^0(r)$  obeys

$$\left[ \frac{d^2}{dr^2} - \gamma^2 \right] \chi_s^0(r) = m^* v_s(r) \left[ \chi_s^0(r) - r j_0(k_0 r) \right] \quad (2.6)$$

Inside the hard core,  $\psi(r) = 0$  and hence  $\chi_s^0(r) = r j_0(k_0 r)$ , the free S-wave function. Outside the core,  $\chi_s^0$  falls off to zero slope and value at  $r = d$ , by definition of  $d$ . If there had been no attraction outside the core, then  $\chi_s^0(r) = e^{-\gamma r}$  for  $r > c$ . The attractive part makes the decay of  $\chi_s^0(r)$  only faster. Thus, a graph of  $\chi_s^0(r)$  has an approximate shape shown in Fig. 1. Inside the core radius, the function  $\chi_s^0 = r j_0(k_0 r) \simeq r$  for small  $k_0$ . Deviation from this arises only for large enough  $k_0$  so that  $k_0 c \sim 1$ . Further, outside the core,  $\chi_s^0$  does not vary much with  $k_0$ , especially for large  $\gamma$ . The largest  $k_0$  dependence outside the core arises from the core boundary value  $\chi_s^0(c) = c j_0(k_0 c)$ . Therefore the outer function, when suitably normalized at the core radius, i.e. either  $\frac{\chi_s^0(r)}{c j_0(k_0 c)}$  or  $\frac{\chi_s^0(r)}{c j_0(k_0 c)}$ , should be relatively independent of  $k_0$ .

Further, since  $\chi_s^0$  is roughly triangular with a peak at  $r = c$ , its fourier transform should be peaked around  $kc = \pi/2$  i.e.  $k \sim 4 F^{-1}$ . Thus, when the  $g_s$  matrix acts on the filled Fermi sea, it tends to excite intermediate states of momenta typically around  $4 F^{-1}$ . All these features of the defect function will be useful in subsequent discussion.

We can also make similar estimates for the reaction matrix  $g_s$ . We have,

$$\begin{aligned} \langle \vec{k} | \frac{g_s}{e} | \vec{k}_0 \rangle &= \langle \vec{k} | \phi_s \rangle \\ &= \frac{1}{\Omega} \int e^{-i\vec{k} \cdot \vec{r}} \phi_s(\vec{r}) d^3r \end{aligned} \quad (2.7)$$

where  $\Omega$  is the volume of integration. Consider the diagonal case  $|\vec{k}\rangle = |\vec{k}_0\rangle$ . The integral in (2.7) can be split into two parts, inside and outside the core radius respectively. Inside the core,  $\phi_s(\vec{r}) = \phi(\vec{r}) = e^{i\vec{k}_0 \cdot \vec{r}}$ , so that

$$\int_{\text{Core}} e^{-i\vec{k}_0 \cdot \vec{r}} \phi_s(\vec{r}) d^3r = \int_{\text{Core}} d^3r = V_c \quad \text{the core volume} \quad (2.8)$$

Outside the core, the contribution is not so trivial, but the shape of  $\chi_s^0$  in Fig. 1 shows that the result should be of the same order as the core contribution. An estimate of about  $2V_c$  to  $3V_c$  is reasonable. Thus,

$$\langle \vec{k}_0 | \frac{g_s}{e} | \vec{k}_0 \rangle = \frac{1}{\Omega} 3V_c \quad (2.9)$$

As an application, consider the first order direct diagram in  $g_s$ . Its contribution, using an average value  $k_0$ , is approximately,

$$E_s^{(1)} = \frac{1}{2} \sum_{\substack{m, n \\ < k_F}} \langle \vec{k}_0 | \frac{g_s}{e} | \vec{k}_0 \rangle \simeq \frac{N^2}{2\Omega} \langle \vec{k}_0 | \frac{g_s}{e} | \vec{k}_0 \rangle \frac{k_0^2 + \gamma^2}{m^*}$$

where  $N$  is the number of particles

$$\simeq \frac{1}{2} \rho \cdot 3V_c \cdot \frac{k_0^2 + \gamma^2}{m^*} \quad \text{per particle.} \quad (2.10)$$

For purposes of estimation, we can use

$$\rho = 0.170 F^{-3} \simeq \frac{1}{6} F^{-3} \quad \text{for } k_F = 1.36 F^{-1}$$

$$V_c = 0.27 F^3 \quad \text{for } c = 0.4$$

$$m^* = 1$$

$$\text{and } k_0^2 + \gamma^2 = 1.5 k_F^2$$

(see preceding article)

This gives the first order  $g_s$  contribution from the "direct" diagram,

$$E_s^{(1)} = \frac{1}{2} (0.170) (0.81) (1.5 k_F^2) \simeq 0.10 k_F^2 \simeq 8 \text{ MeV per particle}$$

This is, of course, only an estimate, and is given only to illustrate the essential simplicity of the BBP method. The exact results can always be obtained by solving (2.4) and evaluating (2.5). The estimate in eq. (2.9) for  $\langle \vec{k}_0 | \frac{g_s}{2} | \vec{k}_0 \rangle$  is very valuable in studying higher order diagrams, as we will see in the next section.

It should be noted here that the first order contribution of  $g_s$  is positive. However, the long range part  $v_l$  makes a large negative contribution (about -59 MeV)<sup>13</sup> through its first Born approximation, so that the full  $g$  matrix  $g \simeq g_s + v_l$  is negative for small  $k_0$ . This gives, then, a positive contribution to the binding energy from the first order direct diagram.

We further note that since  $\mathcal{G}_s$  is largely  $k_0$  independent, the diagonal  $g_s$  matrix element

$$\langle \vec{k}_0 | g_s | \vec{k}_0 \rangle \simeq \frac{k_0^2 + \sigma^2}{m^*} \langle \vec{k}_0 | \mathcal{G} \rangle$$

has a strong quadratic dependence on  $k_0$ . We will see in Sec. 5 that this proves to be a troublesome feature in evaluating the three-body energy and a full section will be devoted to incorporate this momentum dependence of  $g_s$ .

We have described above some selected properties of the

two-body wave function and reaction matrix, which are needed for the study of higher orders that follows. The reader is referred to the preceding article for fuller details of the two-body problem.

### 3. The Convergence of the Brueckner-Goldstone Series

We will study in this section the convergence of the Brueckner-Goldstone expansion order by order. In general, the expansion will contain diagrams involving the single-particle potential energy  $U$  (See A). Obviously, their contribution will depend on the choice of  $U$ , which in turn depends on what pure  $g$  matrix diagrams one is trying to cancel out by these "U diagrams". Therefore let us first concentrate on diagrams that do not involve  $U$ .

The first order diagrams are shown in Fig. 2. Here, Fig. 2b is the "exchange" of 2a. In the previous section, we made a crude estimate of the direct diagram 2a as  $\frac{1}{2} \rho \sum_{\vec{m}, \vec{n}} (g_s + v_g)$   $\simeq +8 - 59 \text{ MeV} = -51 \text{ MeV}$ . The exchange diagram can be similarly evaluated. However, one can see that the exchange diagram is "almost" diagonal, inasmuch as the momentum transfer  $\vec{m} - \vec{n}$  is quite small, so that for a given  $\vec{m}$  and  $\vec{n}$ , it should not be very different from the direct term. But, for a spin-isospin independent potential, the states  $\vec{m}$  and  $\vec{n}$  must have the same spin-isospin values in the exchange diagram. Consequently, the exchange diagram contribution is multiplied by an additional factor of  $-\frac{1}{4}$ , the minus sign arising from the Goldstone rule mentioned in A. As a result, the total first order contribution should be about  $3/4$  the direct term, i.e. about  $-39 \text{ MeV}$ . We wish to emphasize again, that whereas the ease with which such estimates can be made is the great advantage of the BBP

method, it is always possible to get more exact answers by solving the differential equation for  $\vec{\chi}(r)$ . A recent such calculation by Kirson<sup>12</sup> gives a value of -38.35 MeV for the first order energy, using the Standard Hard Core potential, in very close agreement with our estimate.

There are no second order diagrams in the Bureckner-Goldstone expansion. The third order direct diagrams are only a handful, as shown in Fig. 3. The remaining third order diagrams can be obtained by simply "exchanging" one or more of the  $g$  matrices in Fig. 3. Of the third order diagrams, Fig. 3a and Fig. 3c caught early attention, in as much as they seemed to represent self energy effects. The "bubble interaction"  $\langle bn|g|bn \rangle$  in Fig. 3a, for example, when summed over the state  $n$ , may be considered as part of the single-particle energy of the state  $b$ , and might therefore be counteracted by a suitable  $U(b)$ . A similar statement would be valid for Fig. 3c, where the "bubble interaction"  $\langle mn|g|mn \rangle$  may be included as part of the hole energy  $U(m)$ . This method, which corresponds to the Hartree method in atomic physics, is explained in Goldstone's paper<sup>3</sup>, and forms the basis of the BBP choice of their single-particle energies. The BBP choice, which will be discussed in Sec. 7, absorbs, on the average, diagrams 3a, 3c and their exchanges, as part of the single-particle energies.

Concurrently, it was shown by Rajaraman<sup>14</sup> that the remaining third order diagrams are comparable in size to the



bubble diagrams, and should be taken into account to make a consistent approximation. However, he also showed that even though these are not obviously self-energy type diagrams, more than 90% of their contribution could still be absorbed into the single-particle energies. These additional diagrams were then included by BBP in determining their reference spectrum parameters. The spirit of the efforts at that time was still based on the hope that the Brueckner-Goldstone series converges as you go to higher order diagrams in  $g$ , and that if you have accounted for all third order diagrams in the above manner, this should leave only small errors from fourth and higher order terms.

However, when calculations were performed by Razavy<sup>10</sup> for the first order energy with such a single-particle spectrum, the resulting binding energy was only about 8 MeV per particle. Razavy used the Hamada-Johnston potential.<sup>15</sup> A similar result was obtained by Brueckner and Masterson<sup>16</sup> using the (very similar) Breit potential. However, Brown, Schappert and Wong<sup>17</sup> showed that both the Razavy result and the Brueckner-Masterson (Br-M) result needed corrections, which coincidentally reduced both values down to about 4 MeV or so. In particular, the Br-M calculations did not include the off-energy-shell effects on the single-particle energies. More recently Coon and Dabrowski<sup>18</sup> have incorporated the off-energy-shell corrections into the Br-M energies. It should be noted in this connection that use of both the Coon-Dabrowski and the Brown et al

corrections to the Br-M calculations is wrong since these corrections duplicate one another. It should also be remembered that Coon-Dabrowski inclusion of off-energy-shell effects into the single-particle energies had already been achieved earlier by the BBP spectrum.

These remarks are a digression from our main point that the sum of the first and third order energies after these corrections give only about 4 MeV of binding per particle as compared to the experimental value of 16 MeV.

This can be improved upon by using a "soft-core" repulsion, such as the exponential core of Wong<sup>7</sup>, instead of the infinite "hard core". This would clearly decrease the repulsion, and was estimated by Wong to add about 4-5 MeV to the binding energy. This still leaves a discrepancy of over 7 MeV, the cause of which turned out to be connected with the underlying hope that the Brueckner-Goldstone series converged order by order.

This hope turned out to be quite false, as a closer inspection of higher order diagrams revealed. It was shown by Rajaraman<sup>5</sup> that there exist, in the expansion, sub-sets of diagrams characterized by the number of hole-lines in them, where higher and higher order terms in each sub-set do not become smaller, so that evaluating the series order by order is not the proper procedure. It was also suggested that each of these sub-sets should be summed in coordinate space and that the

resulting sequence corresponding to an increasing number of hole lines will converge. We will now outline these arguments.

Consider the two fourth order diagrams in Fig. 4. Compare both of these to the third order diagram in Fig. 3b. Fig. 4a has one more factor  $g/e$  (the "bubble" interaction) and has to be integrated over one more independent state, namely  $n$ , as compared to Fig. 3b. Of course, the contribution of a diagram involves integrals over all the intermediate state momenta, and although the integrand  $g \frac{1}{e} g \frac{1}{e} g \frac{1}{e} g$  factors into contributions of the individual  $g$  matrices, the integrated result will not. Nevertheless, for purposes of making estimates, we can write

$$\frac{\text{diagram 4a}}{\text{diagram 3b}} \approx \sum \langle \vec{b}_n | \frac{g}{e} | \vec{b}_n \rangle \text{ with a typical } b \sim 4 F^{-1}$$

$$\approx \rho \langle \vec{k}_0 | \frac{g}{e} | \vec{k}_0 \rangle \text{ with } \vec{k}_0 = \frac{\vec{b} - \vec{n}}{2} \quad (3.1)$$

This is where our estimates for  $\langle \vec{k}_0 | \frac{g}{e} | \vec{k}_0 \rangle$  at the end of the last section are very useful. There we showed that

$$\langle \vec{k}_0 | \frac{g}{e} | \vec{k}_0 \rangle = \langle \vec{k}_0 | \psi \rangle$$

gives a core contribution of  $V_c$ , and something of the same order ( $2V_c$  to  $3V_c$ ) from outside the core. Thus,

$$\frac{\text{diagram 4a}}{\text{diagram 3b}} = 3.5 \rho V_c = 3.5 \left( \frac{c}{2r_0} \right)^3 \approx \frac{1}{7} \quad (3.2)$$

where  $2r_0 =$  inter-particle distance  $= 2.24 F$ . Consider, on the other hand, the diagram 4b. Here again we have an additional factor of  $g/e$  as compared to Fig. 3b, but the additional independent momentum to be integrated over, whether it be  $p$  or  $q$ , is above the Fermi sea. Thus,

$$\frac{\text{diagram 4b}}{\text{diagram 3b}} \approx \sum_{p > k_F} \frac{g}{e}$$

As we showed in the last section, the typical particle momentum  $p$  excited by the  $g$  matrix is about  $\frac{\pi}{2c} \approx 4F^{-1}$ . This we justified by considering the Fourier transform of  $\zeta(r)$ , and the large value of the momentum was seen to arise because of the hard core. Thus, even though the matrix element  $g/e$  is of the same order as before, the phase space over which this is integrated is much larger than the Fermi sea, and we would get a result

$$\frac{\text{diagram 4b}}{\text{diagram 3b}} \approx 3.5 \left( c \frac{\pi}{2c} \right)^3 \quad (3.3)$$

On detailed consideration, this factor turns out to be an over estimate in as much as  $g/e$  is smaller than  $3.5 V_c$  for high momenta, and the typical momentum  $p$  is somewhat smaller than  $\frac{\pi}{2c}$ . However, this ratio is at least of the order of unity, as compared to the ratio of 1/7th in eq. (3.2). A better way of estimating this ratio is given by Ben Day.<sup>9</sup> We have

$$\begin{aligned} \frac{\text{diagram 4b}}{\text{diagram 3b}} &\approx \sum_{\vec{k}} \langle \vec{k} | \frac{g}{e} | \vec{k}_0 \rangle = \sum_{\vec{k}} \langle \phi_{\vec{k}} | \zeta_0 \rangle \\ &\quad \text{where } |\vec{k}\rangle = |\phi_{\vec{k}}\rangle = e^{i\vec{k}\cdot\vec{r}} \\ &= \sum_{\vec{k}} (\phi_{\vec{k}} \text{ at } r=0) \langle \phi_{\vec{k}} | \zeta_0 \rangle \text{ since } \phi_{\vec{k}}(0)=1 \\ &= \zeta_0(r=0) = \phi_{\vec{k}}(r=0) = 1 \end{aligned}$$

Once again we see that this ratio arises because  $\zeta(r=0) = \phi_{\vec{k}}(r=0)$ , which in turn is because of the hard core. The argument used here in going from third to fourth order, is clearly

valid at all orders. We conclude then, that in going to a diagram of next higher order, if the extra independent intermediate state introduced is a particle, the higher order diagram remains of the same size, whereas if the intermediate state is a hole, then there is a reduction by about a seventh.

The above convergence behavior becomes more transparent when summarised in direct physical terms thus:

With an infinite hard core in the potential, we must of course anticipate convergence problems in powers of the potential. In fact, if we use the potential as it is, even the matrix elements diverge. The situation is improved by the use of the reaction matrix  $g$ , which is at least finite. This renders every diagram in powers of  $g$  finite. However, this does not imply that such a sequence of finite diagrams will converge order by order. Now, if a diagram contains  $n$  hole lines, it corresponds to an interaction between  $n$  particles, since it is easy to note that every hole line corresponds to one particle being excited out of the Fermi sea. Since it is the hard core which leads to convergence problems, and since the probability of a large number of particles being within each other's core radius is small ( $\rho c^3 < 1$ ), we would expect a diagram to get smaller as the number of hole lines increases. But if the number of hole lines is kept constant and the number of  $g$  matrices is increased by adding particle lines only, then there is not likely to be good convergence. Our semi-quantitative

arguments earlier simply corroborate this conjecture. This possibility of convergence in powers of the density had been suggested as early as in 1957 by Hugenholtz.<sup>19</sup>

At this juncture, it is again helpful to separate the potential into  $v_s$  and  $v_\ell$ . An arbitrary diagram would then consist of an arbitrary sequence of  $v_s$  and  $v_\ell$  interactions, and if uninterrupted two-body ladders of  $v_s$  are summed into  $g_s$  as usual, then we have diagrams with arbitrary combinations of  $g_s$  and  $v_\ell$ . These may be separated into three classes:

(a) Diagrams involving  $v_\ell$  alone: These will be very small inasmuch as the second Born term in  $v_\ell$  is seen to be 2% of the first.<sup>11,12</sup> The reason<sup>13</sup> behind this large diminution will be clear in Sec. 7. (b) Diagrams mixed in  $v_\ell$  and  $g_s$ : We will show in Sec. 7 that these can be absorbed into the single-particle energies. (c) Diagrams involving  $g_s$  alone: It is these diagrams which, owing to the hard core, lead to a non-convergent sequence for a fixed number of hole lines, for the reasons outlined.

Nevertheless, if you consider one such sequence, say, of all diagrams with three hole lines, i.e. the three-body clusters, then the contributions of successive orders alternate in sign. That is, a seventh order diagram as compared to a sixth order one, would contain an extra factor  $\frac{Q}{E_0 - H_0} g_s = \frac{g_s}{-e}$  which is negative since  $g_s$  is positive. Thus, it is possible that the sequence may have a finite sum.

This was in fact shown to be the case. We will use now a convention for drawing diagrams introduced by Rajaraman for handling n-body cluster diagrams. Every nucleon is represented by a vertical line, with interactions represented by horizontal lines or wiggles as before. "Particle" and "hole" states are distinguished only by arrows. A Goldstone diagram and its representation in the new convention are shown in Fig. 5. The disadvantage of the new convention is that it does not distinguish between particle and hole states very clearly. On the other hand, it brings out the unity of all n-body diagrams of all orders. Thus, all three-body diagrams are "ladders" with these vertical lines. This already suggests that all these three-body ladders may be summed in a manner similar to Brueckner's summation of all two-body v-ladders into a g-matrix. It was shown<sup>5</sup> that the sum of such three-body ladders, called the T matrix, is finite and can be evaluated in a manner similar to the reference spectrum method for the two-body g-matrix. A three-body wave function analogous to  $\Psi^{\text{reference}}$  can be defined, which obeys a three-body Schrodinger type differential equation. If this equation can be solved, then the T matrix, or the three-body energy, can be easily obtained. The same procedure can be adopted for four-body and higher cluster diagrams. Whereas the corresponding n-body Schrodinger type equations would be harder and harder to solve as n increases, the solution nevertheless exists, and leads to finite energies for the contribution of all the n-body diagrams. Furthermore,

in view of the short range nature of  $g_s$ , these contributions should converge as  $n$  increases.

Although Rajaraman's work suggested the above possibilities, it did not attempt the actual solution of the 3-body differential equation to get the 3-body energy. This was done a year later by Bethe,<sup>6</sup> who, with the help of Faddeyev's technique<sup>20</sup> obtained a satisfactory solution to the three-body wave function and energy. This work also revealed in greater detail the above convergence difficulties, and the effect of going from third order to all three-body terms. We will describe this work in detail in the next section. We only need to mention here that Bethe's work, subsequently further improved by Day<sup>21</sup> and Kirson,<sup>12</sup> leads to a three-body energy of about -5 MeV, which, compared to the -38.3 MeV for the two-body energy, indicates a good rate of convergence in the cluster expansion, thus giving us hope that the four-body energy would be less than 1 MeV.



#### 4. The Three-Body Energy

We will now discuss a method for determining the three-body wave function and energy in nuclear matter, developed by Bethe in 1964.<sup>6</sup> The wave function and energy correspond to the sum of all Goldstone diagrams with three hole lines. We will use the convention<sup>5</sup> mentioned in the last section for drawing cluster diagrams, whereby all three-hole-line diagrams will consist of three upgoing lines. The interactions  $g$  are wiggly horizontal lines as before, and energy denominators are just the energies of the intermediate states shown minus the starting energies, as can be verified by comparing the two diagrams in fig. 5. Subject to a handful of exceptions, the set of all allowed three-hole-line Goldstone diagrams is just the set of all possible three-body ladders one can draw, in the new convention. The exceptions arise because there are some "three-hole" Goldstone diagrams, such as the "hole-bubble" diagram in fig. 3c which cannot be represented as part of the ladder sequence, and conversely there are a few ladder diagrams that have no Goldstone analogues and should be subtracted away. These anomalies will be taken into account later on, but let us for the moment consider the sum of all three-body ladder diagrams.

Let us denote by  $T$ , the matrix denoting the sum of all three-body ladders, analogous to the  $g$  matrix for the two-body ladders. In other words,  $\langle \vec{l} \vec{m} \vec{n} | T | \vec{l} \vec{m} \vec{n} \rangle$  is the sum of all three-body diagrams that begin and end on states  $l$ ,  $m$ , and  $n$  respectively. Since binding energy diagrams are "vacuum to vacuum" in the second quantised language, the states  $\vec{l}$ ,  $\vec{m}$  and  $\vec{n}$  are below the sea. We will start by evaluating only direct diagrams i. e. where

the three particles are restored to the same respective state in which they started. Exchange diagrams, belonging for instance to  $\langle \vec{m} \vec{l} \vec{n} | T | \vec{l} \vec{m} \vec{n} \rangle$  are related in a fairly simple way to the direct diagrams, as in the two-body case and will be dealt with later. Needless to say, except for the initial and final states  $\vec{l}$ ,  $\vec{m}$ ,  $\vec{n}$ , all other states in the ladders should be above the Fermi sea, and no two successive  $g$  matrices should refer to the same pair of intermediate states. These rules are directly carried over from Goldstone diagrams. Fig. 6 shows a typical diagram belonging to  $\langle \vec{l} \vec{m} \vec{n} | T | \vec{l} \vec{m} \vec{n} \rangle$ . Clearly this set of diagrams can be divided into three distinct groups depending on which pair of particles is involved in the last  $g$  interaction. Let us define  $T^{(1)}$ ,  $T^{(2)}$  and  $T^{(3)}$  to be the sum of the group of diagrams in whose last interaction the particle 1, 2, and 3 respectively is a spectator.

Clearly,

$$T = T^{(1)} + T^{(2)} + T^{(3)} \quad (4.1)$$

such a separation was suggested by Faddeyev<sup>20</sup> in connection with the 3-body scattering matrix. Now, a diagram belonging to  $T^{(3)}$ , such as the one

in fig. 6, must have  $g_{12}$  as its last interaction. The next lower interaction must therefore be either  $g_{13}$  or  $g_{23}$ . Thus, the part of the diagram below the  $g_{12}$ , itself corresponds to a term in either  $T^{(1)}$  or  $T^{(2)}$ . The only exception to this is if there is no interaction at all below  $g_{12}$  i. e. if the entire diagram corresponds only to  $g_{12}$ . Thus,

$$T^{(3)} = g_{12} - g_{12} \frac{1}{e} \{ T^{(1)} + T^{(2)} \} \quad (4.2)$$

and cyclic permutations

The first term on the right hand side,  $g_{12}$ , which corresponds to the third particle not interacting at all, is really part of the two-body energy and is an example of the unwanted exceptions we spoke of earlier. We will eventually subtract away its effect from  $T^{(3)}$ .

Equation (4.2) and its two cyclic permutations, form a set of three coupled integral equations for  $T^{(1)}$ ,  $T^{(2)}$  and  $T^{(3)}$ . As in the case of the two-body  $g$  matrix, these integral equations are best solved by transforming the problem into coordinate space and solving for suitably defined wave functions. It should be noted that in Eq. (4.2) we have dropped the exclusion operator  $Q$  which would ensure that the intermediate states of particles 1 and 2 remain above the sea. This approximation is similar to the one made in A for the two-body  $g$  matrix and is justified by the same arguments. The single-particle energies that go into the denominator  $e$ , are given by the same reference spectrum as in the two-body case.

Let us now define three-body wave functions  $\Psi^{(i)}$  by,

$$T^{(3)} \Phi = g_{12} \Psi^{(3)}, \text{ and cyclic permutations,} \quad (4.3)$$

where  $\Phi$  is the unperturbed plane wave three particle state. It follows from Eq. (4.2) that

$$\begin{aligned} \Psi^{(3)} &= \left[ 1 - \frac{1}{e} \{ T^{(1)} + T^{(2)} \} \right] \Phi \\ &= \Phi - \frac{1}{e} \{ g_{23} \Psi^{(1)} + g_{13} \Psi^{(2)} \}, \text{ and cyclic permutations} \end{aligned} \quad (4.4)$$

the wave functions  $\Psi^{(i)}$  and the above equation (4.4) are the three-body analogues of  $\Psi(\vec{r})$  and its equation (2.2) for the two-body case. As in the two-body case, if we convert  $\frac{1}{e} g_{ij}$  into differential operators in the variables  $\vec{r}_{ij}$ , then (4.4) would give a set of coupled differential equations for  $\Psi^{(i)}$ .

The operator  $e$ , deceptive in its abbreviated form, is more complicated than in the two-body case. In every three-body diagram, the first and last energy denominators correspond to two nucleons being excited and the third below the sea. Thus, in fig. 6, at the level C, nucleons 1 and 2 are in excited states  $\vec{a}$  and  $\vec{p}$  whereas 3 has returned to the state  $\vec{n}$  below the sea. Every energy denominator except the first and the last, such as at level D in fig. 6, corresponds to all three nucleons in excited states. Using the reference spectrum<sup>22</sup> (see A) both types of energy denominators can be expressed in the form  $\frac{1}{m^*} (-\nabla_{ij}^2 + \gamma^2)$ , but the value of  $\gamma$ , using typical values for all the momenta involved, is higher when all three nucleons are excited. To be more explicit, at level D,

$$e = E(a) + E(b) + E(d) - E(l) - E(m) - E(n)$$

$$= \frac{1}{2m^*} \left[ a^2 + b^2 + d^2 - l^2 - m^2 - n^2 + 6\Delta k_F^2 \right]$$

$$= \frac{1}{m^*} \left[ k_{ab}^2 + P_{ab}^2 + \frac{d^2 - l^2 - m^2 - n^2}{2} + 3\Delta k_F^2 \right]$$

But  $\vec{P}_{ab} = \frac{\vec{a} + \vec{b}}{2} = \frac{\vec{l} + \vec{m} + \vec{n} - \vec{d}}{2}$  by momentum conservation

$$P_{ab}^2 = \frac{l^2 + m^2 + n^2 + d^2}{4} \quad \text{averaging over angles}$$

$$e = \frac{1}{m^*} \left[ k_{ab}^2 + \frac{3d^2}{4} - \frac{l^2 + m^2 + n^2}{4} + 3\Delta k_F^2 \right]$$

$$= \frac{1}{m^*} \left[ -\nabla_{12}^2 + \frac{3d^2}{4} + (3\Delta - 0.45) k_F^2 \right] \quad (4.5)$$

using  $\langle l^2 \rangle = \langle m^2 \rangle = \langle n^2 \rangle = 0.6 k_F^2$

thus  $e = \frac{1}{m^*} (-\nabla_{12}^2 + \gamma_1^2)$

with  $\gamma_1^2 = \frac{3d^2}{4} + (3\Delta - 0.45) k_F^2$  (4.6)

The value of the particle momentum  $d$  can be typically taken as  $\sim \frac{\pi}{2c}$  from the arguments in Sec. 2 for the most probable momenta excited.

On the other hand, at the level C in fig. 6, where only two particles are excited, the energy denominator is on the energy shell, i. e.

$$e = E(a) + E(p) - E(l) - E(m)$$

$$= \frac{1}{m^*} \left[ k_{ap}^2 - k_{lm}^2 + 2\Delta k_F^2 \right] = \frac{1}{m^*} (-\nabla_{12}^2 + \gamma_2^2) \quad (4.7)$$

$$\text{where } \gamma_2^2 = 2\Delta k_F^2 - k_{lm}^2$$

$$= (2\Delta - 0.3) k_F^2 \ll \gamma_1^2. \quad (4.8)$$

Therefore, even using average values for the momenta involved, there are two distinct energy denominators, similar in form, but with different  $\gamma$ 's. In either case, we know from Sec. 2 how  $\frac{1}{e} g_{ij}$  operates on the two-body plane wave state. It effects only the relative coordinate  $r_{ij}$  and gives

$$\frac{1}{e} g_{ij} \left\{ e^{i(\vec{k} \cdot \vec{r}_{ij} + 2\vec{P} \cdot \vec{R}_{ij})} \right\} = \zeta_{\vec{k}, \vec{P}}(\vec{r}_{ij}) e^{2i\vec{P} \cdot \vec{R}_{ij}} \quad (4.9)$$

where  $\zeta_{\vec{k}, \vec{P}}(\vec{r}_{ij})$  can be obtained from (2.4) and its general features such as "healing" etc. were discussed. Clearly  $\zeta$  depends on the value of  $\gamma$ ; let us use the symbol  $\eta$  to denote this function when  $\gamma = \gamma_2$  and  $\zeta$  itself when  $\gamma = \gamma_1$ . To incorporate the distinction between  $\eta(\vec{r}_{ij})$  and  $\zeta(\vec{r}_{ij})$ , the former occurring when two particles are excited and the latter when all three are, we split the  $\psi^{(i)}$  in Eq. (4.4) into  $(\psi^{(i)} - \phi)$  and  $\phi$ . When  $\frac{1}{e} g_{ij}$  acts on  $\phi$ , clearly only the

$i^{\text{th}}$  and  $j^{\text{th}}$  particle are excited. Hence

$$\frac{1}{e} g_{ij} \phi = \eta(\vec{r}_{ij}) \quad (4.10)$$

Here we have used the limit of zero momenta for all hole states, so that  $\phi = 1$ . This approximation, reasonable inasmuch as the average hole momentum in  $\sqrt{3} k_f \approx 1/3c$ , will be used in all subsequent discussion. Thus,

Eq. (4.4) becomes

$$\phi - \psi^{(3)} = \frac{1}{e} g_{23} \phi + \frac{1}{e} g_{13} \phi - \frac{1}{e} g_{23} (\phi - \psi^{(1)}) - \frac{1}{e} g_{13} (\phi - \psi^{(2)})$$

or,

$$Z^{(3)} = \eta(\vec{r}_{23}) + \eta(\vec{r}_{13}) - \frac{1}{e} g_{23} Z^{(1)} - \frac{1}{e} g_{13} Z^{(2)} \quad (4.11)$$

where  $Z^{(i)} = \phi - \psi^{(i)}$ .

We thus have to solve the three coupled differential equations implied in (4.11), for the  $Z^{(i)}$ . Since each  $Z^{(i)} = \phi - \psi^{(i)}$  represents an excited wave function with no initial state in it, the operation of  $\frac{1}{e} g_{ij}$  on it will involve only the function  $\zeta(\vec{r}_{ij})$ , and not  $\eta(\vec{r}_{ij})$ .

However, the  $Z^{(i)}$  are functions of all three coordinates. Therefore, to find the result of operating  $\frac{1}{e} g_{12}$  on  $Z^{(3)}(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ , one must first Fourier analyse  $Z^{(3)}$ , replace  $e^{i\vec{k} \cdot \vec{r}_{12}}$  by  $\zeta_{\vec{k}, \vec{P}}(\vec{r}_{12})$  in each Fourier component and then perform the inverse Fourier transformation. The resulting function is of course not related in any simple way to  $Z^{(3)}(\vec{r}_1, \vec{r}_2, \vec{r}_3)$  and the coupled equations (4.11) are not easy to solve. Consequently, two approximations

a simpler one due to Bethe, and an improved version due to Day have been suggested for obtaining the  $Z^{(i)}$ . We will present both of them here. Once the  $Z^{(i)}$  are known, the three-body energy is easily obtained in terms of these functions. Thus  $\langle lmn | T^{(3)} | lmn \rangle$  is just the sum of all diagrams of the type in fig. 6, which end with  $g_{12}$  and have all possible combinations of  $g_{ij}$  underneath with one exception. The diagrams shown in fig 7 have no analogues in the Goldstone series since they don't conserve momentum. Noting that the state at the level D in fig. 6 is  $\psi^{(1)}$ , whereas it is  $\phi$  at level D in fig. 7, we get, on removing the term of fig. 7,

$$\begin{aligned}
 \langle lmn | T^{(3)} | lmn \rangle &= \left\langle \phi \left| g_{12} \frac{1}{(-e)} \right| g_{23} \psi^{(1)} + g_{13} \psi^{(2)} \right\rangle \\
 &- \left\langle \phi \left| g_{12} \frac{1}{(-e)} (g_{23} + g_{13}) \right| \phi \right\rangle \\
 &= \left\langle \phi \left| g_{12} \frac{1}{e} \right| g_{23} Z^{(1)} + g_{13} Z^{(2)} \right\rangle \\
 &= \frac{1}{\Omega^3} \int_{\Omega} \eta(\vec{r}_{12}) \left\{ g(\vec{r}_{23}) Z^{(1)}(\vec{r}_1, \vec{r}_2, \vec{r}_3) \right. \\
 &\quad \left. + g(\vec{r}_{13}) Z^{(2)}(\vec{r}_1, \vec{r}_2, \vec{r}_3) \right\} d\tau_1 d\tau_2 d\tau_3
 \end{aligned}$$

where  $\Omega$  is the volume of integration. (4.12)

Some comments are due concerning equation (4.12). In deriving it, we have explicitly written out two powers of  $g$ , i. e. the top two "rungs" of the three-body ladder, and left the rest in the wave functions  $Z^{(1)}$  and  $Z^{(2)}$ . To be more precise, we have written



$$\langle \ell mn | T^{(3)} | \ell mn \rangle = \langle \phi | g_{12} \frac{1}{e} | g_{23} Z^{(1)} + g_{13} Z^{(2)} \rangle \quad (4.13)$$

instead of the original definition

$$\langle \ell mn | T^{(3)} | \ell mn \rangle = \langle \phi | g_{12} | \psi^{(3)} \rangle \quad (4.14)$$

There are many reasons for this. First of all, this explicitly brings out the last energy denominator which corresponds to two excited particles, and leads to

$\langle \phi | g_{12} \frac{1}{e} = \langle \eta_{12} |$ . Secondly, the pure two-body term  $\langle \phi | g_{12} | \phi \rangle$  present in

(4.13) has been removed in (4.14). Thirdly, of course, the unwanted diagrams in fig. 7 have also been removed by the use of  $Z^{(1)}$  and  $Z^{(2)}$  instead of  $\psi^{(1)}$  and

$\psi^{(2)}$ .

It should also be noted that the  $g$  matrix is not a function of the coordinates alone, as assumed in the last line of (4.12). Whereas at large distances  $\vec{g}(\vec{r}) \rightarrow v(\vec{r})$ , at short distances  $g$  is highly momentum dependent. As shown in Sec. 2, and in Day's preceding article, one may write

$$\frac{1}{e} g \phi = \zeta,$$

where, inside the core,  $\zeta = \phi$ . Hence, inside the core  $g = e = \frac{1}{m^+} (k^2 + \gamma^2)$  in the reference approximation, where  $k$  is the momentum corresponding to the relative coordinate  $r$  in  $g(r)$ . Incorporating this momentum dependence is complicated by the fact that the integral (4.12) involves  $g(\vec{r}_{23})$  which depends on  $k_{23}$ , along with functions  $\eta(\vec{r}_{12})$  and  $Z^{(1)}(\vec{r}_1, \vec{r}_2, \vec{r}_3)$  which involve other coordinates.

This problem will be discussed in Section 5. For the time being let us consider the  $g(\vec{r}_{23})$  and the  $g(\vec{r}_{13})$  in (4.12) to be independent of momentum, but evaluated at a suitable average value of the relevant momentum. Finally, as mentioned at the beginning of this section, there are some Goldstone diagrams which are not present in such "ladder" sequences. For the three-body case, the sum  $T^{(1)} + T^{(2)} + T^{(3)}$  defined above does not include the "hole-bubble" diagram and the "hole-hole" diagram shown in fig. 8(a) and 8(b) respectively. However, fig. 8a is a component of the standard expression for the potential energy of hole states (see also Sec. 7). Diagram 8b can be explicitly calculated by integrating the product of the three  $g$  matrices over the independent momenta. This diagram has been shown by Rajaraman<sup>(14)</sup> to be smaller by a factor of about  $\frac{1}{32}$  compared with 8a; it could probably be absorbed into  $U(m)$  and  $U(n)$ .

Subject to these remarks,  $T^{(3)}$  and similarly  $T^{(1)}$  and  $T^{(2)}$  can be evaluated from Eq. (4.12), once the functions  $Z^{(i)}(\vec{r}_1, \vec{r}_2, \vec{r}_3)$  are known. Let us now proceed to evaluate these functions from the coupled equations (4.11)

### Three-Body Wave Function

It is useful to change the coordinates in (4.11) from  $\vec{r}_1, \vec{r}_2, \vec{r}_3$  to  $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2, \vec{p}_3 = \frac{\vec{r}_1 + \vec{r}_2}{2} - \vec{r}_3$ . We then have,

$$\begin{aligned} \frac{1}{e} g_{12} Z^{(3)}(\vec{r}_{12}, \vec{p}_3, \vec{r}_3) &= Z_g^{(3)}(\vec{r}_{12}, \vec{p}_3, \vec{r}_3) \\ &= 8(2\pi)^{-6} \int d^3 P \int d^3 k \zeta_{\vec{k}, \vec{P}}(\vec{r}_{12}) e^{i2\vec{P} \cdot \vec{p}_3} Z_T^{(3)}(\vec{P}, \vec{k}, \vec{r}_3) \end{aligned} \quad (4.15)$$

where  $Z_T^{(3)}(\vec{P}, \vec{k}, \vec{r}_3)$  is the Fourier transform of  $Z$  given by

$$Z_T^{(3)}(\vec{P}, \vec{k}, \vec{r}_3) = \int d^3 \rho_3' \int d^3 r_{12}' e^{-i 2\vec{P} \cdot \vec{\rho}_3'} e^{-i \vec{k} \cdot \vec{r}_{12}'} Z^{(3)}(\vec{r}_{12}', \vec{\rho}_3', \vec{r}_3) \quad (4.16)$$

The coordinate  $\vec{r}_3$  is unaffected and this corresponds to keeping the particle 3 fixed instead of the center-of-mass. It should be noted that  $\zeta_{\vec{k}, \vec{P}}(\vec{r}_{12})$  actually depends also on the momentum of the third particle through the factor  $\gamma_1$  (see eq. 4.6), but as can be seen from fig. 1, the function  $\zeta$  does not vary much with  $\gamma$ , for large  $\gamma_1$ , so that an average value may be used.

Subject to this, eq. (4.15) is still an exact representation of the operator  $\frac{1}{e} g_{12}$ . Now, for  $r_{12} < c$ , we have

$$\zeta_{\vec{k}, \vec{P}}(\vec{r}_{12}) = e^{i \vec{k} \cdot \vec{r}_{12}},$$

so that,

$$Z_g^{(3)}(\vec{r}_{12}, \vec{\rho}_3, \vec{r}_3) = Z^{(3)}(\vec{r}_{12}, \vec{\rho}_3, \vec{r}_3) \text{ for } r_{12} < c \quad (4.17)$$

For  $r_{12} > c$ , both the Day and Bethe approximations involve nulling the function  $\zeta(\vec{r}_{12})$  outside the integral in eq. (4.15), in some average sense.

To justify this, we first note that in the reference approximation, the strongest dependence of  $\zeta_{\vec{k}, \vec{P}}(\vec{r}_{12})$  on the angles of  $\vec{P}$ ,  $\vec{k}$  and  $\vec{r}_{12}$  is contained in

$P_L(\hat{k} \cdot \hat{r}_{12})$  so that upon integration over  $d^3 k$ , only the S-wave part

$\zeta_{\vec{k}, \vec{P}}^0(\vec{r}_{12})$  survives. Of course, the integrand also contains the  $Z$  function

which depends on  $k$  as well. However, Kirson has shown that the conclusion, viz. that only the S-wave part of  $\zeta$  matters, is nevertheless justified.

One reason for this is that the components  $\zeta^L$  for  $L = 0$  are relatively small because the core does not have a strong effect. There is only a very small dependence on the angle between  $P$  and  $k$  due to Pauli corrections, etc. Further, as shown in sec. 2, for  $r_{12} > c$ , the S-wave function  $\zeta_{k,P}^0(r_{12})$  is nearly independent of  $k$  and  $P$ . The dependence on  $P$  arises through  $\gamma_1$ , to which  $\zeta_{k,P}^0(r_{12})$  is not very sensitive, and the largest dependence on  $k$  arises because of matching the function at the core radius  $c$ , with the interior solution  $j_0(kr)$ . Thus it is reasonable to treat  $\frac{\zeta_{k,P}^0(r_{12})}{j_0(kc)} = \zeta^D(r_{12})$

as essentially independent of  $k$ , and  $P$ , as long as these momenta are not large compared to  $\frac{1}{c}$ . Hence, we may write

$$\begin{aligned}
 & Z_g^{(3)}(\vec{r}_{12}, \vec{\rho}_3, \vec{r}_3) \\
 &= \frac{8}{(2\pi)^6} \int d^3 P \int d^3 k \zeta_{\vec{k}, \vec{P}}^0(r_{12}) e^{2i \vec{P} \cdot \vec{\rho}_3} Z_T^{(3)}(\vec{P}, \vec{k}, \vec{r}_3) \\
 &= \frac{8}{(2\pi)^6} \zeta^D(r_{12}) \int d^3 P \int d^3 k e^{i \vec{k} \cdot \vec{c}} e^{2i \vec{P} \cdot \vec{\rho}_3} Z_T^{(3)}(\vec{P}, \vec{k}, \vec{r}_3) \\
 &= \zeta^D(r_{12}) Z^{(3)}(\vec{c}, \vec{\rho}_3, \vec{r}_3) \tag{4.18}
 \end{aligned}$$

This is the Day approximation. The vector  $\vec{c}$  is directed along  $\vec{r}_{12}$ , and  $\zeta^D(r_{12})$  which is normalised to unity at the core radius, has to be

evaluated for values of  $k$  and  $P$  at which the Fourier transform  $Z^{(3)}_{\text{T}}$  is peaked. Kirson<sup>12</sup>, who explains the above approximation with greater care, shows that  $Z^{(3)}_{\text{T}}$  is peaked around  $k \approx P \approx \frac{0.6}{c}$ . This will be elaborated on later.

The Bethe approximation treats  $\frac{\zeta^0_{k,P}(\vec{r}_{12})}{J_d^{(k)}(\vec{r}_{12})} = \zeta^B(\vec{r}_{12})$  as independent of  $k, P$  rather than  $\zeta^D(\vec{r}_{12})$ . In this approximation,

$$Z^{(3)}_g(\vec{r}_{12}, \vec{p}_3, \vec{r}_3) \approx \frac{8}{(2\pi)^6} \zeta^B(\vec{r}_{12}) \int d^3P \int d^3k e^{i\vec{k} \cdot \vec{r}_{12}} e^{2i\vec{P} \cdot \vec{p}_3} Z^{(3)}_{\text{T}}(\vec{P}, \vec{k}, \vec{r}_3)$$

$$= \zeta^B(\vec{r}_{12}) Z^{(3)}(\vec{r}_{12}, \vec{p}_3, \vec{r}_3) \quad (4.19).$$

This is the Bethe approximation, which replaces  $\frac{1}{e} g_{12}$  simply by a multiplicative factor  $\zeta^B(\vec{r}_{12})$ . We will first use the Bethe approximation which, because of the above simplicity, leads to an algebraic solution to the coupled equations (4.11), and reveals transparently some of the features of the three-body problem involved. We will then use the more accurate but more complicated Day approximation and discuss its merits over the Bethe approximation.

In the approximation of eq. (4.19), the coupled equations (4.11) become,

$$Z^{(3)} = \eta(\vec{r}_{13}) + \eta(\vec{r}_{23}) - \zeta^B(\vec{r}_{23}) Z^{(1)} - \zeta^B(\vec{r}_{13}) Z^{(2)}$$

and cyclic permutations

$$(4.20)$$

Equations (4.20) can be solved algebraically for the  $Z$ 's in terms of the  $\eta$ 's and  $\zeta$ 's, which we know by solving the reference equation (2.4). We get

$$Z^{(3)} = \frac{\eta_{23} u_{23} + \eta_{13} u_{13} - \eta_{12} (u_{13} + u_{23} - 2u_{13} u_{23})}{u_{23} u_{13} + u_{13} u_{12} + u_{12} u_{23} - 2u_{12} u_{23} u_{13}} \quad (4.21)$$

where  $u_{ij} = 1 - \zeta_{ij}^B = 1 - \zeta^B(\vec{r}_{ij})$   
and  $\eta_{ij} = \eta(\vec{r}_{ij})$

We can insert these  $Z^{(i)}$  into the integral (4.12) for the three-body energy.

Take, for instance, diagrams of the type shown in fig. 6. These diagrams, summed over  $l, m, n$ , give,

$$W = \rho^2 \int \eta(\vec{r}_{12}) g(\vec{r}_{23}) Z^{(1)}(\vec{r}_1, \vec{r}_2, \vec{r}_3) d\tau_1 d\tau_2 \text{ per particle} \quad (4.22)$$

where particle 3, instead of the center of mass, is kept fixed. Note that one of the  $\Omega$ 's in eq. (4.12) is cancelled by integration over the coordinate of the fixed particle 3. The result (4.22) is proportional to  $\rho^2$ , as expected on physical grounds for three-body clusters.

Whereas we can evaluate  $Z^{(1)}$  from eq. (4.21) and integrate (4.22) for  $W$ , it is useful as shown by Bethe, to study the solution in certain limiting cases using simplifying approximations, to gain some insight into what is going on. Let us, as before replace  $\Phi = |lmn\rangle$  by unity since the hole state momenta are small. Let us also take  $\eta(\vec{r}_{ij})$  and  $\zeta^B(\vec{r}_{ij})$  to depend only on the magnitude of  $r_{ij}$ . This is reasonable as explained before since we are using averages over the momenta anyway and the angular averaging will pick out only the S-wave parts. (BBP show, <sup>4</sup> for instance, that  $\eta$  and  $\zeta$

for  $L \neq 0$  can be represented by a function of the type  $\frac{e^{-\gamma r}}{r}$  outside the core with a somewhat larger value of  $\gamma$  than the true value). In this approximation and in uniform nuclear matter, the functions  $g$ ,  $x^{(i)}$  and  $Z^{(i)}$  all depend only on the relative separation distances  $r_{ij}$ , since no directions have been picked out in space.

Now take the case when all three particles are far apart, i. e.  $r_{ij} \rightarrow \infty$ . Then  $\zeta^B(r_{ij}) \rightarrow 0$  and  $u(r_{ij}) \rightarrow 1$ . Thus,

$$Z^{(3)} \rightarrow \eta_{23} + \eta_{13} \quad (4.23)$$

and

$$W \rightarrow W_{\text{III}} = \rho^2 \int \eta(r_{12}) g(r_{23}) \left\{ \eta(r_{23}) + \eta(r_{13}) \right\} d\tau_1 d\tau_2 \quad (4.24)$$

It is easy to see that eq. (4.24) is just the sum of the third order direct diagrams in fig. 3a and 3b. Kohler<sup>23</sup> had obtained a similar form for the diagram 3b. The fact that  $W$  becomes just the third order energy for large  $r_{ij}$  is reasonable since at these distances the potential is weak enough to give a good convergence order by order and the third order clearly dominates the three-body energy. To see the behavior for smaller values of  $r_{ij}$ , let us look at the case

$$r_{12} = r_{23} = r_{13}$$

then  $u_{12} = u_{13} = u_{23} = u = 1 - \zeta$  and  $\eta_{12} = \eta_{23} = \eta_{13} = \eta$

Thus,

$$Z^{(1)} = \frac{2\eta}{3 - 2u} = \frac{2\eta}{1 + 2\zeta} \quad (4.25)$$

If the distances  $r_{ij}$  are very small (within the core radii) then  $\zeta = 1$ , and

$$Z^{(1)} = \frac{2\eta}{3} \quad (4.26)$$

This clearly corresponds to the very strong potential in the core region, and hence represents all orders of perturbation. This result is a third of eq. (4.23) which represents only the third order contribution. This diminution by a factor of 3 in the core region, in going from third order to all orders, may be partly understood thus: When two nucleons are within a core radius apart, the infinite core completely destroys the wave function in that region and leads to a large  $\zeta$  and large energy. When three nucleons are all within each other's cores, an uncorrelated treatment will give three times this energy since three such pairs are involved. This is what happens in third order. In actuality however, we can do no more than destroy the wavefunction once when all the nucleons are close together, so that a fully correlated treatment (full three-body energy to all orders) will give only  $\frac{1}{3}$  of the third order energy. Actually, when all  $r_{ij} < c$ , this solution is exact, since under these conditions,  $\zeta_{ij} = \eta_{ij} = \frac{1}{e} g_{ij} = 1$  so that the original coupled equations (4.11) become

$$Z^{(3)} = Z + Z^{(2)} + Z^{(1)} .$$

However, when all  $r_{ij} < c$ , the three particles are in identical situations and hence by symmetry  $Z^{(1)} = Z^{(2)} = Z^{(3)} = \frac{2}{3}$ . This solution, simple and exact for all  $r_{ij} < c$ , should be remembered in the context of the Day approximation as well.

Finally, for small  $\zeta$ , one can expand (4.25) as

$$Z^{(1)} = \frac{2\eta}{1+2\zeta} = 2\eta - 4\eta\zeta + 8\eta\zeta^2 - 16\eta\zeta^3 + \dots \quad (4.27)$$



This series represents the contribution of three-body diagrams order by order. It is clearly not convergent for  $\zeta > \frac{1}{2}$ . For large inter-particle separations,  $\zeta \rightarrow 0$ , and hence the series converges rapidly, so that a few lower order diagrams suffice for the pure long range part of the force. Within the hard core, however,  $\zeta = 1$  and the terms in the expansion (4.27) keep increasing although the whole series has a closed sum of  $\frac{2\eta}{3}$ . Even for any reasonable "soft" repulsive core,  $\zeta$  is still a little greater than  $\frac{1}{2}$  and the perturbation expansion above will not converge. (See for instance Sprung and Bhargava's work<sup>8</sup> using the Bessel potential<sup>7</sup> with only a finite core of about 650MeV))

Thus we see many of the anticipated features of the three-body problem in this simplified discussion. More exact results can, of course, be obtained from eq. (4.21) and (4.22). We note at this point that eq. (4.22) may be

written as<sup>25</sup>

$$W_{III} = \rho^2 \int g(r_{23}) F_a(r_{23}) d\tau_{23}$$

where

$$F_a(r_{23}) = \int \eta(r_{12}) \left\{ \eta(r_{23}) + \eta(r_{13}) \right\} d\tau_1 \quad (4.28)$$

Similarly, for the full three-body energy

$$W = \rho^2 \int g(r_{23}) F_1(r_{23}) d\tau_{23}$$

where

$$F_1(r_{23}) = \int \eta(r_{12}) Z^{(1)}(r_{12}, r_{23}, r_{13}) d\tau_1 \quad (4.29)$$

We will give graphs of the functions  $F_a$ ,  $F_1$  and their ratio  $f_b = \frac{F_1}{F_a}$ , using the more accurate Day approximation for the  $Z^{(i)}$ , which we will outline

now.

### Day's Approximation

Day approximates the action of the operator  $\frac{1}{e} g_{ij}$  by eq. (4.18) as compared to the simpler multiplicative approximation by Bethe in eq. (4.19). It is clear that Day retains more of the operative character of  $\frac{1}{e} g_{ij}$ , and as we shall see, his resulting solutions for the  $Z^{(1)}$  are considerably better. As the  $Z^{(1)}$  are functions only of the interparticle distances, we can rewrite the Day approximation (4.18) as

$$\begin{aligned} \frac{1}{e} g_{12} Z^{(3)}(r_{12}, r_{23}, r_{13}) &= Z^{(3)}_g(r_{12}, r_{23}, r_{13}) \\ &= \zeta^D(r_{12}) Z^{(3)}(c, r'_{23}, r'_{13}) \end{aligned} \quad (4.30)$$

where  $r'_{23}$  and  $r'_{13}$  correspond to the coordinates  $\vec{c}$ ,  $\vec{\rho}_3$ , and  $\vec{r}_3$  in the eq. (4.18), and their meaning can be read from fig. 9. The Day approximation thus differs from that of Bethe, in that it "shrinks" the triangle of the particles in addition to multiplying by  $\zeta^D(r_{12})$ . Note that for  $r_{12} < c$ ,  $Z^{(3)}_g = Z^{(3)}$  (eq. 4.17), and  $\zeta^D(r_{12}) \approx 1$ .

Substituting (4.30) into the coupled equations (4.11), we get,

$$\begin{aligned} Z^{(3)}(r_{12}, r_{23}, r_{13}) &= \eta(r_{23}) + \eta(r_{13}) - \zeta^D(r_{23}) \left\{ \begin{array}{l} Z^{(1)}(r_{12}, r_{23}, r_{13}) \text{ if } r_{23} < c \\ Z^{(1)}(r_{12}, c, r'_{13}) \text{ if } r_{23} > c \end{array} \right\} \\ &\quad - \zeta^D(r_{13}) \left\{ \begin{array}{l} Z^{(2)}(r_{12}, r_{23}, r_{13}) \text{ if } r_{13} < c \\ Z^{(2)}(r'_{12}, r'_{23}, c) \text{ if } r_{13} > c \end{array} \right\} \end{aligned} \quad (4.31)$$

These equations correspond to eq. (4.20) of the Bethe approximation, and unlike the latter, cannot be solved algebraically, since  $Z^{(3)}$  at one set of points

is coupled to  $Z^{(2)}$  and  $Z^{(1)}$  at other points on the "reduced triangle." Of course, if all  $r_{ij} < c$ , then the solution is  $Z^{(1)} = \frac{2}{3} \eta$ . Similarly, if only  $r_{12} > c$ , then  $r'_{13}$  and  $r'_{23}$  are necessarily less than  $c$ , so that  $Z^{(i)}(c, r'_{23}, r'_{13}) = \frac{2}{3} \eta$ , and this again leads to an analytical solution of (4.31).

But, for the general case where at least two of the  $r_{ij}$  are greater than  $c$ , one has to resort to a numerical method. The method involves substituting for the  $Z^{(1)}(r'_{12}, c, r'_{13})$  and  $Z^{(2)}(r'_{12}, r'_{23}, c)$  in eq. (4.31) from the other two coupled equations, which leads to points on a smaller triangle.

If this is repeated successively, you eventually reach (in general) a triangle where at least two of  $r_{ij} < c$ , when the solution is known. Day has calculated this numerical solution for several values of  $r_{ij}$ . However, the method does not work for some instances such as when all three particles are collinear, in which cases you never reach a stage when two of the distances are less than  $c$ . In the more general case, the numerical solution is laborious.

Consequently, Day has suggested that equations (4.31) be replaced by an approximate form, which is amenable to an analytic solution. The wave function defects  $Z^{(i)}$  are large only when the  $r_{ij} \sim c$ , and all the  $\zeta$ 's and  $\eta$ 's drop off rapidly for  $r_{ij} > c$ . But if  $r_{12}$  is not much larger than  $c$ , then  $r'_{13} \sim r_{13}$  and  $r'_{23} \sim r_{23}$ . Thus, one might try to replace eq. (4.31) by

$$Z^{(3)}(r_{12}, r_{23}, r_{13}) = \eta(r_{23}) + \eta(r_{13}) - \zeta^D(r_{23}) \left\{ \begin{array}{l} Z^{(1)}(r_{12}, r_{23}, r_{13}) \text{ if } r_{23} < c \\ Z^{(1)}(r_{12}, c, r_{13}) \text{ if } r_{23} > c \end{array} \right\} \\ - \zeta^D(r_{13}) \left\{ \begin{array}{l} Z^{(2)}(r_{12}, r_{23}, r_{13}) \text{ if } r_{13} < c \\ Z^{(2)}(r_{12}, r_{23}, c) \text{ if } r_{13} > c \end{array} \right\} \quad (4.32)$$

These equations are amenable to an analytic solution. This simply involves substituting for  $Z^{(1)}$  and  $Z^{(2)}$  from the other two coupled equations when you get two of the  $r_{ij}$  to be equal to. One can then exploit symmetry properties such as  $Z^{(3)}(c, r_{23}, c) = Z^{(2)}(c, r_{23}, c)$  and  $Z^{(1)}(c, c, r) = Z^{(1)}(r, c, c)$  to solve the set of three equations implied in (4.32). The result is

$$Z^{(1)}(r_{12}, r_{23}, r_{13}) = \eta_{12} \left( 1 - \zeta_{13} + \frac{1}{2} \zeta_{13} \zeta_{23} \right) + \eta_{13} \left( 1 - \zeta_{12} + \frac{1}{2} \zeta_{12} \zeta_{23} \right) - \eta_{23} \left( \zeta_{12} + \zeta_{13} + \zeta_{12} \zeta_{13} \right) + \left[ \zeta_{12} \zeta_{13} + \frac{1}{2} (\zeta_{12} + \zeta_{13}) \zeta_{23} - \frac{4}{3} \zeta_{12} \zeta_{13} \zeta_{23} \right] \quad (4.33)$$

with  $\eta_{ij} = \eta(r_{ij})$  and  $\zeta_{ij} = \zeta^D(r_{ij})$

This is Day's analytic solution to (4.32). Day finds<sup>21</sup> that it agrees very well with the numerical solution to his original equations (4.31). Further, Kirson<sup>12</sup> finds that when the analytic solution (4.33) is substituted into the right hand side of (4.31), the resulting iterated solution for  $Z^{(3)}(r_{12}, r_{23}, r_{13})$  is very close to the analytic solution. Two of Kirson's graphs comparing the analytic solution (4.33) and its first iteration are given in fig. 10. Since the analytic solution agrees very closely with its first iterate and with the numerical solution to the more exact equations (4.31) evaluated by Day for several sets of values of  $r_{ij}$ , and since (4.31) itself is only an approximation, we will use eq. (4.33) for the  $Z^{(i)}$ .

This analytic solution by Day is considerably better than the simpler solution (4.21) for the  $Z^{(i)}$ . First of all, the underlying approximation for the operator  $\frac{1}{e} g_{ij}$ , in eq. (4.18) is more accurate than the corresponding

Bethe approximation in eq. (4.19). Secondly, the Bethe solution has drastic discontinuities which occur sometimes when one of the  $r_{ij}$  equals  $c$ . This may be partly understood by considering the solution (4.21) when all  $r_{ij} \rightarrow c^+$ . Then the  $u_{ij} = 1 - \zeta_{ij}^B$  tend to vanish and the solution approaches the indeterminate form  $\frac{0}{0}$ , whereas when all  $r_{ij} < c$ , we know the result to be  $Z^{(1)} = \frac{2}{3}$ . When  $r_{23}$  and  $r_{13}$  are less than  $c$ , the Bethe solution gives

$$Z^{(3)} = \frac{1 - \eta_{12}}{Z_{12}^{(3)}} \quad (4.34)$$

If now  $r_{12}$  approaches  $c$  from outside, the above solution is highly sensitive to the exact behavior of  $\eta_{12}$  and  $\zeta_{12}^B$  near the core, whereas for  $r_{12} < c$ , it is equal to  $\frac{2}{3}$ . There are no such discontinuities in the Day solution

A comparison of the Day and Bethe solutions is given in fig. 11.

Of course, the ~~three-body~~ energy and the function  $F_a$  and  $F_1$  involve integrals of the  $Z^{(i)}$  and consequently the two solutions give comparable results since the discontinuities do not matter here. But Dahlblom<sup>26</sup>, doing the corresponding calculation for tensor forces (see sec. 6), found that the Bethe procedure led to great difficulties while the Day method was straight-forward. Also with central forces, the Day solution is clearly more accurate. This is especially clear from Kirson's work: If the solution (4.21) is used on the right hand side of (4.31), the resulting iterated solution is completely different, in contrast to the behavior of the Day solution, Fig. 10.

It should be mentioned at this stage that having obtained the  $Z^{(i)}$  by the above methods, Kirson evaluates their momentum transform and finds it peaked at  $k$ ,  $P \approx \frac{0.6}{c}$ . This was mentioned earlier in connection with

eq. (4. 18) and (4. 19), where the function  $\zeta$  had to be factored out of the integral for the  $Z_g^{(1)}$ . This factoring was justified only if the  $\zeta$  was relatively independent of  $k$ ,  $P$ , which is true only if they are less than  $\frac{1}{c}$ . It is therefore gratifying that the momentum dependence of the  $Z^{(1)}$  conforms to this requirement. Further Kirson finds that if  $k$ ,  $P \approx \frac{0.6}{c}$  are used as input in eq. (4. 19), the resulting  $Z^{(1)}$  give an output momentum dependence which agrees well with the input.

This completes our discussion of the evaluation of the three-body wave-function defects  $Z^{(1)}$ . The Day solution (4. 33) may be used for evaluating the function  $F_a$ , or the more complicated expressions derived in Sec. 5 and 6, and the three-body energy as defined by equations (4. 29) and (4. 12) respectively. We will quote the results after discussing exchange diagrams.

### Exchange Diagrams

The problem of exchange diagrams may seem complicated inasmuch as for every direct diagram you can exchange any of the interactions, or "rungs of the ladder" in the diagram. However, as is illustrated in the example in fig. 12, when you exchange an intermediate interaction, the resulting diagram can be redrawn so that it looks like a direct diagram, with two of the final momenta  $\vec{l}$  and  $\vec{m}$  exchanged, as compared to the initial ordering of the three momenta. Thus fig. 12a and 12b are equivalent. It is clear therefore that no matter how many of the interactions are exchanged, the result would simply amount to permuting the final momenta  $\vec{l}$ ,  $\vec{m}$  and  $\vec{n}$ . Thus, all direct and exchange diagrams would be contained in  $\langle lmn | T | lmn \rangle$ ,  $\langle mln | T | lmn \rangle$ ,  $\langle lnm | T | lmn \rangle$ ,  $\langle nml | T | lmn \rangle$ ,  $\langle mnl | T | lmn \rangle$  and  $\langle nlm | T | lmn \rangle$ .

The exchange diagrams, however, carry additional statistical weights as compared to the direct diagrams, if one uses spin-isospin independent forces<sup>14</sup>. For such forces, each particle retains its spin-isospin values and hence all states on a given vertical line in the direct diagram have the same value of  $s_z$  and  $\tau_z$ , although any two vertical lines may have different spin and isospin components. However, for an exchange diagram such as the ones in fig. 12, both the states  $\vec{l}$  and  $\vec{m}$  must have the same spin-isospin since a particle from each of these states goes into the other. Thus, the number of allowed states, and hence the contribution to the energy is reduced by a factor of 4. In addition, if these diagrams are drawn in the Goldstone Convention, it can be seen that fig. 12 corresponds to only two nucleon loops, unlike the three in a direct diagram. Consequently, according to the Goldstone rule, there is an additional minus sign associated with fig. 12, and altogether, therefore, it is multiplied by  $-\frac{1}{4}$  due to the above arguments. Similarly, when all three nucleons interchange momenta, as in  $\langle mn | T | lm \rangle$ , there is a multiplicative factor of  $\frac{+1}{16}$  since now all three nucleons must have the same  $s_z$  and  $\tau_z$ , with only one nucleon loop altogether. These arguments clearly have to be modified for tensor forces.

But for these statistical factors, the matrix elements  $\langle lmn | T | lmn \rangle$  and any one of its exchanges, say  $\langle lnm | T | lmn \rangle$ , are not very different. Compare, for instance, the two diagrams in fig. 13 which are corresponding terms belonging to  $\langle lmn | T | lmn \rangle$  and  $\langle lnm | T | lmn \rangle$  respectively. For every set of intermediate momenta in fig. 13a there is a term with the same set in fig. 13b. The only difference is in the final interaction. The

final interaction has different matrix elements for the two cases, since the momentum transfer in the two cases is different by  $\vec{q} = \vec{m} - \vec{n}$ . But since  $\vec{m}$  and  $\vec{n}$  are holes, the average value of  $qc = \sqrt{1.2} k_F c \approx 0.6$ , and it was shown<sup>14</sup> in the context of third order diagrams, that the resulting difference is only a factor of about  $\frac{1}{6} q^2 c^2$  i.e. about 6% . . . Therefore, up to a few percent, the direct and exchange matrix elements of T are equal

It was also shown by Bethe<sup>6</sup>, that in the approximation of neglecting  $q^2 c^2$ , the inclusion of all exchange terms, amounts to using even angular momentum contributions only. This has no effect for a Serber type attractive force which acts only on even L states anyway, but it cuts down the repulsive core effect to only the even L states, supporting such an assumption that had been made by Brueckner and Gammel<sup>27</sup> in their g matrix calculations.

Several calculations have been made, using the above method for evaluating the three-body energy by Sprung, Bhargava and Dahlblom<sup>24</sup>, and by Kirson<sup>12</sup>. Kirson uses the full standard hard-core potential, including the short and long range parts, to obtain an energy of -5.15 MeV for the three-body clusters. His curves for the functions  $F_1(r_{23})$ ,  $F_a(r_{23})$  and  $f_b(r_{23})$  defined earlier are depicted in fig. 14. As anticipated, for large  $\frac{r_{23}}{c}$ , all three functions approach the same value, 1. At small distances,  $F_1$  is about a third of  $F_a$ , as expected from eq. (4.26). The curves have been drawn to a scale where  $F_0(r_{23})$ , the corresponding function for just the third order bubble diagram is taken to be unity.

The diminution from -38.35 MeV to -5.15 MeV in going from the two-body to the three-body contribution indicates that our hopes of



convergence in powers of density appear justified. It will be seen in section 7 that the long-range part  $v_l$  contributes most of the -5 MeV in the three-body energy, and that the energy for just the short range part is a small positive amount. We therefore can expect a four-body energy of much less than 1 MeV for  $v_l$ . It will be shown in section 7 that most of the higher order effects of  $v_l$  can, on the other hand, be absorbed in the single-particle potential energy  $U(b)$ .

A more recent calculation by Dahlblom of the functions  $F_1$ ,  $F_a$  and  $f_b$  is given in fig. 15. This calculation uses the recent Reid potential<sup>7</sup> for triplet states, which has a hard core of about 0.52 F and consequently a stronger attraction outside. The resulting three-body energy is consequently more negative (about -7 MeV).

## 5. Effect of the Momentum Dependence of the g Matrix

We now proceed to a systematic evaluation of the three-body energy. In particular, we shall take into account the momentum dependence of  $g$  which was mentioned below eq. (4.14) but was then ignored in the remainder of Sec. 4. To do this, we must go back to the fundamental eq. (4.12), line before the last. From this, or from Fig. 16 (which is just Fig. 6 with different features emphasized) we find, in slightly different notation,

$$W_3(K_0) = \int dK_1 dK_2 \langle K_0 | \hat{\eta}_{12} | K_1 \rangle \langle K_1 | \hat{g}_{23} | K_2 \rangle \langle K_2 | \hat{Z}^{(1)} | K_0 \rangle \quad (5.1)$$

Here each  $K_i$  stands for the momenta of all three particles; specifically  $K_0$  refers to the initial and the others to the two intermediate states. The final state is of course identical with the initial. The  $\eta$ ,  $g$ , and  $Z$  are operators and have therefore been denoted by a  $\hat{\phantom{x}}$ . The first operator represents

$$\hat{\eta}_{12} = \hat{g}_{12} \frac{Q}{e} \quad (5.2),$$

i.e. the last interaction and the preceding propagator. The last interaction is separated from the rest in this manner because the  $e$  here corresponds to the excitation of 2 particles while "earlier" propagators correspond to excitation of 3 particles, as explained in (4.5), (4.6). (5.1) is somewhat more general than (4.12) in that the momenta in initial and final state are not yet put to zero. Following Kirson and the discussion in Sec. 4, we shall assume that the three-body wave function operator  $Z^{(1)}$ , operating on the unperturbed wave function  $|K_0\rangle = \Phi$ , may be written as

$$\hat{z}^{(1)} | K_0 \rangle = z^{(1)}(r_{12}, r_{23}, r_{31}) \bar{\Phi}(K_0) \quad (5.3)$$

where  $\bar{\Phi}(K_0)$  is the unperturbed wave function (produce of three plane waves) and the function  $z^{(1)}$  depends only on the distances between the three particles, not on the directions of the vectors  $\vec{r}_{12}$ , etc.

The problem then is the matrix element of  $g_{23}$ . This quantity depends appreciably on the relative momentum of particles 2 and 3 in state  $K_1$ . To see this, we write

$$\hat{g}_{23} | K_1 \rangle = e \hat{\zeta}_{23} | K_1 \rangle \quad (5.4)$$

where  $\hat{\zeta}_{23}$  is the two-body defect function as defined in Sec. 3, and the operator  $e$  may be written in the reference spectrum approximation

$$e = \gamma^2 - \nabla_{23}^2 \quad (5.5)$$

As is shown in BBP,  $\gamma$  increases with increasing excitation of the state  $K_1$ . Therefore, just because  $\gamma$  is very insensitive to  $K_1$  (ref. 6, p. 809),  $g$  is very sensitive: it increases rapidly with increasing energy. This is particularly true for the contribution from inside the core; we have

$$\hat{\zeta}_{23} | K_1 \rangle = | K_1 \rangle, \quad r_{23} < c \quad (5.6)$$

$$\hat{g}_{23} | K_1 \rangle = (k_{23}^2 + \gamma^2) | K_1 \rangle, \quad \text{if } r_{23} < c \quad (5.7)$$

where  $k_{23}$  is the relative momentum of particles 2 and 3 in state  $K_1$ . The contribution from the core surface (ref. 4, Eq.(5.28)) is less sensitive to the energy of state  $K_1$ , and that from the long-range, attractive forces is insensitive, viz.

$$\hat{g}_{23} | K_1 \rangle \approx v(r_{23}) | K_1 \rangle, \quad r_{23} \gg c \quad (5.8)$$

A simple approximation was made by Kirson: He replaced  $g$  by its value for the average momentum  $k_{23}$  of the interacting particles. He justified this approximation by his finding that the distribution of the momentum  $k_{23}$  in state  $K_1$  is sharply peaked around its average value of about  $0.6/c$  (cf. this paper above (4.19)). Hence Kirson puts

$$\hat{g}_{23} | K_1 \rangle = g_{23}(k_{23} \text{ Av}, r_{23}) \bar{\Phi}(K_1) \quad (5.9)$$

where  $\bar{\Phi}(K_1)$  is the product of three plane waves corresponding to the momenta in state  $K_1$ , and  $g_{23}$  is simply a function of  $r_{23}$ . If this is assumed, the integration over  $K_1$  and  $K_2$  in (5.1) can be done immediately and gives

$$W_3(K_0) = \langle K_0 | \eta^*(r_{12}) g_{23}^*(r_{23}) z^{(1)}(r_{12}, r_{23}, r_{31}) | K_0 \rangle \quad (5.10),$$

substantially equivalent to (4.12).

Because of the considerable sensitivity of  $g_{23}$  to  $k_{23}$ , the choice of  $k_{23} \text{ Av}$  in the Kirson approximation is rather critical, and it is not clear what criteria to use. Bethe has given a more general solution. He writes

$$\hat{g}_{23} | K_1 \rangle = g_{Pk}(\vec{r}_{23}) \bar{\Phi}(K_1) \quad (5.11)$$

where  $P$  and  $k$  are, respectively, the average and the relative momentum of particles 2 and 3 in the state  $K_1$ . The function  $g$  depends on both these parameters, and it is therefore not possible to effect "closure" with respect to  $K_1$ . However, closure is possible with respect to  $K_2$ , and leads to the result

$$W_3(K_0) = \pi^{-3} \int d^3k \langle K_0 | \eta_{12} | K_1 \rangle \langle K_1 | g^{*z(1)} | K_0 \rangle \quad (5.12)$$

$$\langle K_1 | g^{*z(1)} | K_0 \rangle = \int d^3r_{12} \int d^3r_{23} g_{Pk}^*(r_{23}) z^{(1)}(r_{12}, r_{23}, r_{31}) \exp i(\vec{p}_3 - \vec{p}_2 + 2\vec{k}) \cdot \vec{r}_{12} \quad (5.13)$$

$$\langle K_0 | \eta | K_1 \rangle = \int d^3r_{12} \eta^*(r_{12}) \exp i(\vec{p}_3 - \vec{p}_2 + 2\vec{k}) \cdot \vec{r}_{21} \quad (5.14)$$

Here  $\vec{p}_1, \vec{p}_2, \vec{p}_3$  are the initial momenta of the three particles in state  $K_0$ , and  $2\vec{k}$ , as in (5.11), is the difference between the momenta of particles 2 and 3 in the intermediate state  $K_1$ .

Two alternatives are now open. One is to calculate and use the exact expression for  $g_{Pk}(r_{23})$ ; then no further simplification of (5.13) seems possible. The other is to find a manageable approximation to the dependence of  $g$  on  $P$  and  $k$ ; this is probably sufficient because the entire three-body correlation contributes less than 20% to the total potential energy of nuclear matter. Such an approximation is suggested by (5.7) and (5.8), together with (4.5) and the discussion of BBP Sec. 7. The state  $K_1$  is characterized by the excitation of only two particles, 1 and 2. In this case, if  $k_b$  denotes the momentum of particle 2, (4.5) shows that in (5.7)

$$k_{23} \equiv k \simeq \frac{1}{2}k_b$$

and,

$$k_{23}^2 + \gamma^2 \simeq k_b^2 \quad (5.15)$$

neglecting  $k_F^2$  compared with  $k_b^2$ . Then, considering the two components (5.7) and (5.8) of  $g$  and assuming that the contribution from the core surface has an intermediate behavior, it is reasonable to set

$$g_{Pk}(r_{23}) = g_1(r_{23}) + k^2 g_2(r_{23}) \quad (5.16)$$

Here we have used both equations (5.15).

Before (5.16) can be used, it is still necessary to make assumptions about the nuclear force, as follows:

1. An ordinary (non-exchange, central) force between the nucleons, which may include a repulsive core, can be represented quite well by (5.16). In evaluating  $g_1$  and  $g_2$ , the definition of  $g$  should be remembered:

$$g_{pk}(\vec{r}) \phi(\vec{r}) = v(r) \psi_{pk}(\vec{r}) \quad (5.17)$$

where  $\phi = e^{i\vec{k} \cdot \vec{r}}$  and  $\psi = \phi - \zeta$  is the two-particle wave function with interaction. In practice it is probably best to calculate  $g_{pk}$  by explicit integration of the Schrodinger equation for two values of  $k$  on both sides of the most probable value of  $k$  determined by Kirson ( $k=0.6/c$ ), and then to deduce  $g_1$  and  $g_2$  from these. Such a program is presently being carried out by Dahlblom.<sup>26</sup>

If (5.16) is accepted, integration of (5.13) over  $K_1$  is straightforward for  $g_1$ <sup>28</sup>. The factor  $k^2$  multiplying  $g_2$ , on the other hand, can be combined with the  $\eta$  matrix element, if we assume

$$|\vec{p}_3 - \vec{p}_2| \ll 2k \quad (5.18)$$

which is generally a good approximation since  $p_3$  and  $p_2$  are  $< k_F$ .

The result is then<sup>29</sup>

$$W_3(K_0) = \int d^3r_{21} d^3r_{23} \left[ \eta^*(r_{12}) g_1^*(r_{23}) - \frac{1}{4} \nabla^2 \eta^*(r_{12}) g_2^*(r_{23}) \right] Z^{(1)}(r_{12}, r_{23}, r_{31}) \quad (5.19)$$

Since both  $\eta$  and  $Z^{(1)}$  are essentially independent of the momenta  $\vec{p}_1, \vec{p}_2, \vec{p}_3$ , this result may simply be multiplied by  $p^2$  to give the energy per particle. The effect of non-vanishing momenta  $\vec{p}_1, \vec{p}_2, \vec{p}_3$  can be treated approximately, using the method of Kirson<sup>12</sup>.

Eq. (5.19) can be further simplified. Since  $g$  depends only on  $r_{23}$ , we can integrate over the position of particle 1, i.e. over  $\vec{r}_{12}$ , keeping  $r_{23}$  fixed. This yields

$$F_1(r_{23}) = \int d^3r_{12} \eta^*(r_{12}) z^{(1)}(r_{12}, r_{23}, r_{31}) \quad (5.20)$$

$$F_2(r_{23}) = -\frac{1}{4} \int d^3r_{12} (\nabla^2 \eta^*(r_{12})) z^{(1)}(r_{12}, r_{23}, r_{31}) \quad (5.21)$$

$$W_3(K_0) = \int d^3r_{23} \left[ g_1^*(r_{23}) F_1(r_{23}) + g_2^*(r_{23}) F_2(r_{23}) \right] \quad (5.22)$$

The function  $F_1$  is identical with the  $F$  introduced in ref.6, Eq. (5.1) which was there shown to be small if  $r_{23}$  is inside the core and to increase rapidly (by about a factor 3) outside. The other correlation function,  $F_2$ , was introduced in ref. 29.

2. A more realistic nuclear force may be considered, in first approximation, as a superposition of two parts, (a) a Serber force

$$\hat{v}_e \psi(\vec{r}) = \frac{1}{2} v_e(r) \left[ \psi(\vec{r}) + \psi(-\vec{r}) \right] \quad (5.23)$$

and (b) an additional attractive force acting in the S state only

$$v_s(\vec{r}) = v_s(r) \psi_0(r) \quad (5.24)$$

with  $\psi_0$  denoting the  $L = 0$  component of the wave function (subscripts e for even L, S for S-state). The Serber force acts in all two-body states of even L, with  $L = 0$  and 2 being the only important ones, while the need for the force (b) arises from the observed fact<sup>30,31</sup> that the nucleon-nucleon potential is more attractive in the  $^1S$  than in the  $^1D$  state.

a. If it is assumed that the Serber force  $v_e$  acts only for large  $r$ , we can proceed as in (5.8) and can replace  $\psi(\vec{r})$  by the unperturbed wave function  $\phi(\vec{r}) = e^{i\vec{k} \cdot \vec{r}}$ . In this case, (5.22)

is replaced by <sup>29</sup>

$$W_{3e}(K_0) = \frac{1}{2} \int d^3r_{12} d^3r_{23} \left[ \eta^*(r_{12}) + \eta^*(r_{13}) \right] v_e(r_{23}) \cdot Z^{(1)}(r_{12}, r_{13}, r_{23}) \quad (5.25)$$

Here the first term  $\eta(r_{12})$ , arises from the "direct" term in the Serber force,  $\psi(\vec{r})$  in (5.23), while  $\eta(r_{13})$  arises from the exchange term,  $\psi(-\vec{r})$ . But  $Z^{(1)}$  is clearly symmetric in  $r_{12}$  and  $r_{13}$ , therefore the two terms in the bracket of (5.23) give equal contributions, and the Serber force gives exactly the same result as an ordinary force. This is in accord with the discussion at the end of Sec. 4.

It is then possible to drop the assumption that  $v_s$  acts only at long distances  $r$ , and to assume instead

$$\hat{v}_e \psi(\underline{r}) = \frac{1}{2} \left[ g_{e1}(r) + k^2 g_{e2}(r) \right] \left[ \phi(\vec{r}) + \phi(-\vec{r}) \right] \quad (5.26)$$

in analogy with (5.16). The result is exactly (5.22).

b. The S-state force (5.24) is somewhat more complicated.

Assuming, again in analogy with (5.16),

$$v_s(r) \psi_0(r) = \left[ g_{s1}(r) + k^2 g_{s2}(r) \right] \phi_0(r) \\ \phi_0(r) = j_0(kr), \quad (5.27)$$

Bethe obtains

$$W_{3s}(K_0) = \int d^3r_{23} \left[ g_{s1}(r_{23}) F_{s1}(r_{23}) + g_{s2}(r_{23}) F_{s2}(r_{23}) \right], \quad (5.28)$$

$$F_{s1}(r_{23}) = r_{23}^{-1} \int d^3r_{12} R^{-1} \left[ \chi(R - \frac{1}{2}r_{23}) - \chi(R + \frac{1}{2}r_{23}) \right] Z^{(1)}, \quad (5.29)$$

$$F_{s2}(r_{23}) = (4r_{23})^{-1} \int d^3r_{12} R^{-1} \left[ \omega(R - \frac{1}{2}r_{23}) - \omega(R + \frac{1}{2}r_{23}) \right] Z^{(1)}, \quad (5.30)$$

$$R = \frac{1}{2} \left| \vec{r}_{12} + \vec{r}_{13} \right|, \quad R^2 = \frac{1}{2} (r_{12}^2 + r_{13}^2) - \frac{1}{4} r_{23}^2$$

$$\chi(x) = \int_0^\infty \eta(y) y dy$$

$$\omega(x) = \int_0^\infty \eta(x) + x \left( \frac{d\eta}{dx} \right) dx. \quad (5.31)$$



3. The tensor force between nucleons 2 and 3 is conveniently replaced by the central force which results from it<sup>32</sup> in second order perturbation theory. This force depends slightly on  $k$ , so that it should be a good approximation to write

$$\hat{g}_T \phi(\vec{r}) = (g_{T1}(r) + k^2 g_{T2}(r)) \phi(\vec{r}) \quad (5.32)$$

with  $g_{T1}$  attractive and  $g_{T2}$  repulsive. Thereby the tensor force is reduced to the same form as the other parts.

Summary The total nuclear force may be well represented as a sum of an ordinary, a Serber, an S-state and a tensor force. The ordinary force may be chosen to include the effect of the repulsive force in the  $^1P$  state and the repulsive core. All terms except the s-state force reduce to a result of the type (5.20)-(5.22), the s-state force gives (5.28)-(5.32).

## 6. Tensor Forces

The theories of Bethe<sup>6</sup>, Day<sup>21</sup> and Kirson<sup>12</sup> assume that the initial and final interactions in the three-particle ladder involve central forces, including a repulsive core. Dahlblom<sup>26</sup> has treated the case of a tensor force in the initial and final interaction, with central forces in all intermediate interactions. He assumes that the initial and final particle momenta are zero. Then the force in the  $^3S$ -state modifies the two-particle function  $\phi = 1$  into

$$\psi = \phi - \eta^c(r) - \eta^T S_{12} \quad (6.1)$$

where  $\eta^c$  is the s-state defect function, previously called  $\eta$  in Sec.4 and 5, while the last term is the D-state which is introduced by the tensor force.  $S_{12}$  is the usual tensor operator

$$S_{12} = 3\hat{r}_1 \cdot \vec{\sigma}_1 \hat{r}_2 \cdot \vec{\sigma}_2 - \vec{\sigma}_1 \cdot \vec{\sigma}_2 \quad (6.2)$$

with  $\hat{r}$  a unit vector in the direction  $\vec{r}$ . The contribution of the tensor force to the three-body energy is written (apart from a numerical factor)

$$W_T = \int \eta^T_{12} g_{23} z^{(1)T} \quad (6.3)$$

Because of the spin operators in  $S_{12}$ ,  $W_T$  is not symmetric in particles 2 and 3, in contrast to (4.33). Using otherwise Day's approximation, Dahlblom finds

$$\begin{aligned} z^{(1)T} = & \eta^T_{12} \left[ 4 - \zeta_{13} + \frac{1}{2} \zeta_{13} \zeta_{23} \right] \\ & - \frac{1}{2} \eta^T_{13} \left[ 4 - \zeta_{12} + \zeta_{12} \zeta_{23} \right] \\ & + \frac{1}{2} \eta^T_{23} \left( \zeta_{12} + \zeta_{13} - \zeta_{12} \zeta_{13} \right) \end{aligned} \quad (6.4)$$

In this expression,  $\xi$  is the wave-function defect due to central forces in an intermediate state, the same as in (4.33). (6.4) contains no counterpart to the last term in (4.33) which arose there from the Z inside the repulsive core: The tensor force inside the core is zero, hence this term is absent in (6.4). Since  $V_T = 0$  for  $r < c$ , the function  $\eta^T$  rises only slowly outside the core, (see the curve  $u_{21}^{(0)}$  in ref. 4, Fig.13). On the other hand  $\xi$  drops rapidly for  $r > c$  (see curve  $\chi_0$  in the same figure). Hence products like  $\eta_{12}^T \xi_{13}$  which occur in (6.4) are rather unimportant: If  $r_{23}$  is small, then  $r_{12}$  and  $r_{13}$  are not very different, and the product  $\eta^T(r_{12}) \xi(r_{13})$  is small because we cannot make  $r_{12}$  large while  $r_{13}$  remains of order  $c$ . If  $r_{23} \gg c$ , we may integrate over  $r_{13}$  essentially independently of  $r_{12}$ ; then again the term  $\xi(r_{13})$  has little influence because it is appreciable only over such a small volume. Moreover, the terms  $\eta \xi$  have small coefficients. Thus (6.4) reduces essentially to the simple expression

$$z^{(1)T} \approx 4\eta_{12}^T - 2\eta_{13}^T \quad (6.5)$$

Dahlblom has confirmed this qualitative argument by calculating

$$F_T(r_{23}) = \int \eta^T(r_{12}) z^{(1)T}(r_{12}, r_{23}, r_{13}) d^3r_1 \quad (6.6)$$

which is analogous to (5.20) and may be used to calculate  $W_T$  in (6.3). He finds that for all values of  $r_{23}$ ,  $F_T$  is within about 5 percent equal to

$$F_{To}(r_{23}) = \int \eta^T(r_{12}) [4\eta^T(r_{12}) - 2\eta^T(r_{13})] d^3r_1 \quad (6.7)$$

which corresponds to the approximation (6.5). Now this

approximation does not contain  $\xi$ , and hence corresponds exactly to the third order of the Goldstone expansion. Thus, if the initial and final interaction are both tensor, the old-fashioned third order calculation of the three-body energy is adequate.

If (6.7) is inserted into (6.3), the first term,  $[\eta^T(r_{12})]^2$ , corresponds to the "bubble" diagram 3a, the second term to the "ring" diagram 3b. In the bubble diagram, the middle interaction  $g_{23}$  leaves the states of the two interacting particles unchanged, viz. b and r, respectively; in 3b, the two particles are effectively exchanged by the middle interaction, i.e. particle 2 goes from the excited state b to the state m in the Fermi sea, while 3 goes from its normal state m to a highly excited state whose momentum is close to b. Hence, taking into account that (6.7) contains  $2\eta_{12} - \eta_{13}$ , the effective middle interaction is

$$\begin{aligned}
 & 2g(\text{ordinary}) - g(\text{exchange}) = \\
 & = 2 [g(\text{even}) + g(\text{odd})] - [g(\text{even}) - g(\text{odd})] \\
 & = g(\text{even}) + 3g(\text{odd}) \qquad (6.8)
 \end{aligned}$$

This is in marked contrast to the case when initial and final interaction are central: in that case, as discussed in the last part of Sec. 4, the interaction  $g_{23}$  is purely in even states.

It is well known that the interaction is more attractive in even than in odd states: The long-range force is attractive only in even states, repulsive in  $1P$  and essentially zero in other odd states. The repulsive core is assumed to exist in all states. Hence the three-body energy is less attractive if the initial and final interactions are tensor than if they are central.

### Tensor Forces at other levels

We have only discussed the case when tensor forces act at the beginning and the end of the Goldstone diagram. At any intermediate level, we can have a tensor force act twice in succession on the same pair of particles: this gives an effective central force which can be treated like a central force, cf. item 3 near the end of Sec. 5.

Two tensor interactions between two different pairs of particles, at any two levels, give only a small contribution, because the average over spins give nearly zero. An exception is the case of an initial and final tensor interaction between different pairs, 12 and 13, which contributes the term  $2 \eta(r_{13})$  in (6.5): as Dahlblom has shown, this is due to the fact that the momenta of the various particles are strongly correlated in this simple case which is not true in general.

Three tensor interactions, without any central ones, have been treated by Dahlblom<sup>33</sup> using the OPEP interaction without modification. This overestimates the interaction because we know that the  $1/r^3$  singularity of OPEP must be compensated by other contributions. Even so, Dahlblom found only about 1/2 Mev contribution.

## 7. Single-Particle Potential Energies

We will now discuss the choice of the single-particle potential  $U$  (See A), which affects the value of the  $g$  matrix, and consequently every diagram in the expansion. In principle, of course, any convenient choice of  $U$  which gives a finite  $g$  matrix is permitted, but an educated choice that enhances the convergence of the Brueckner-Goldstone expansion is clearly preferable.

In simple physical terms, the potential  $U$ , which is added and subtracted from the total Hamiltonian is supposed to reduce the size of the perturbation  $H_1$ , by absorbing some of the inter-particle potential energy  $\sum_{i \neq j} v_{ij}$  into the unperturbed  $H_0$ . This would clearly enhance convergence. In diagrammatic language, the introduction of  $U$  results in some additional Goldstone diagrams besides the pure  $g$  matrix diagrams, and the choice of  $U$  is designed to cancel some of the latter by the former.

Thus, one choice of  $U(b)$  for "particle"-states<sup>3</sup> would be such as to cancel the third order "particle-bubble" diagram (fig.17b) with the corresponding diagram involving  $U(b)$ . In order to effect such a cancellation between Fig.17a and 17b for a given value of momenta  $l$ ,  $a$  and  $m$ , we clearly require

$$U(b) = \sum_{n < k_F} \langle bn | g(W) | bn \rangle \quad (7.1)$$

where  $W$  is the starting energy as defined in BBP<sup>4</sup> and in A.

Now,  $W$ , and hence  $\langle bn | g(W) | bn \rangle$  depend on the states  $l$ ,  $m$  and  $a$ , so that the choice  $U(b)$  eq. (7.1) is not a function of the state  $b$  alone. Thus, the cancellation of Fig.17a by Fig.17b for all  $l$ ,  $m$  and  $a$ , can be achieved only in an average sense, by choosing

U(b) as in eq. (7.1) with some typical values of l, m and a. This is precisely what BBP do in their choice of U(b) for particle states. They not only try to cancel Fig.17b on the average, but also the diagram with the middle g matrix exchanged. Thus, they choose

$$U(b) = \sum_{n < k_F} \left\{ \langle bn | g(\Delta E) | bn \rangle - \langle bn | g(\Delta E) | nb \rangle \right\} \quad (7.2)$$

with  $\Delta E$  corresponding  $\langle l^2 \rangle = \langle m^2 \rangle = 0.6k_F^2$  and  $\langle a \rangle \sim 4F^{-1}$

which are typical values. This choice is somewhat modified by including other third order diagrams of comparable size as shown by Rajaraman<sup>14</sup>.

It should be noted that there is a self-consistency requirement implied in eq. (7.2) since the g(W) used to define U(b) itself depends on U(b). BBP define a self-consistent U(b) according to the above prescription and show that this potential can be approximated by a quadratic form  $A + Bb^2$ , which agrees with the exact U(b) in the important range of  $b = 2$  to  $5F^{-1}$ . This is BBP's Reference Spectrum for particle states. Extensive work goes into the calculation of U(b), ensuring self consistency, and the reader is referred to BBP and the subsequent work of Sprung<sup>34</sup> and Razavy<sup>10,34</sup> for details.

The BBP choice for the hole-state potential energy U(m) is similar to (7.2), except for the important difference that the g matrices are on the energy shell.

$$U(m) = \sum_{n < k_F} \left\{ \langle mn | g(W_0) | mn \rangle - \langle mn | g(W_0) | nm \rangle \right\} \quad (7.3)$$

where  $W_0 = E_m + E_n$ . This choice is simpler because it does not depend on the other particles in the diagram, unlike U(b) in eq. (7.1) where such a dependence arises because of the off-energy-shell nature. BBP also show, using an elegant identity

generalized from an idea of Brueckner and Goldman<sup>35</sup> that this choice of  $U(m)$  cancels not only the "hole-bubble" diagram, but a whole sequence of diagrams shown in Fig. 18. There is no corresponding identity for the particle energy  $U(b)$  and the off-energy shell dependence of (7.2) has to be retained.

We will now show that according to our present understanding of the subject, the choice of the particle potential energy  $U(b)$  above should be modified. As mentioned in Sec.3, at the time of the BBP work, it was hoped that the Brueckner-Goldstone expansion converges order by order. If this were so, then the first order diagrams, using the BBP choice of  $U(b)$  and  $U(m)$  should give a good approximation to the binding energy since second order diagrams don't exist and third order diagrams are cancelled by the above potential energies. We now know from the subsequent research described in sections 3 to 6 that there is no convergence order by order in the  $g$  matrix, and that the third order diagrams cancelled by  $U(b)$  are the lowest order terms in an alternating and non-convergent series of three-body ladders. As equations (4.23) and (4.26) show, the third order terms are nearly three times the full three-body energy, and therefore by cancelling them off the BBP choice of  $U(b)$  may do more harm than good.

On the other hand, the choice (7.3) of the hole energy  $U(m)$  is still good because the diagrams in Fig. 18 that it absorbs do not belong to the three-body ladder sequence of sec. 4. This can be readily seen by attempting to draw the diagrams in Fig.18 in the ladder notation. Thus, we will retain the BBP choice of



$U(m)$  which absorbs Goldstone diagrams not summed in the three-body energy, but look for alternate prescriptions for the particle potential  $U(b)$ .

Several choices have been suggested in the literature. We have mentioned in Sec.3, that the Coon and Dabrowsky choice<sup>17</sup> amounts to the BBP prescription. More recently Brandow<sup>1</sup> has suggested that  $U(b) = 0$  be used. This is appealing not only because of its simplicity, but because it gives a sizeable energy gap between particle and hole spectra, thus facilitating the use of the reference method. (The average value of  $U(m)$  is around -90 MeV.) On the other hand, the Brandow choice does not absorb any specific higher order correction terms. Thus the problem of evaluating four-body corrections, etc., is still left open.

Bethe<sup>6</sup> has suggested that  $U(b)$  be defined so as to absorb the full three-body energy evaluated in Sec.4. In other words,  $U(b)$  should be chosen such that the three-body energy may be written as

$$W_3 = \int |y(\vec{b})|^2 U(\vec{b}) d^3b \quad (7.4)$$

where

$$y(\vec{b}) = \int \eta(\vec{r}) e^{i\vec{b} \cdot \vec{r}} d^3r \quad (7.5)$$

and  $|y(\vec{b})|^2$  is the probability that a pair of nucleons is excited from a state in the Fermi sea to a state of momenta  $+\vec{b}$  and  $-\vec{b}$  (the momenta of the initial hole states have been neglected, and  $\eta$  has been assumed independent of these momenta). Obviously, (7.4) is only one condition on the function  $U(b)$  which still permits

wide latitude in its choice. Bethe<sup>6</sup> proposed to set

$$U(b) = |y(b)|^{-2} \int d^3r_{23} g_1(b, r_{23}) F_1(r_{23}) \quad (7.6)$$

with  $F_1$  given by (5.20). At least approximately, this should satisfy (7.4). Using this prescription, Kirson<sup>12</sup> and Sprung, Bhargava and Dahlblom<sup>24</sup> have calculated  $U(b)$ , and the latter workers also include to some extent the effect of tensor forces.

(7.6) may be criticized on five counts: 1) It is not established by any fundamental theory, 2) it is not even proved to satisfy (7.4), 3) it has a singularity if  $y(b) = 0$  for some  $b$ , 4) Sprung et al<sup>24</sup> found that for  $b$  slightly greater than  $k_F$ ,  $U(b)$  tended to become smaller than  $U(m)$  for states  $m$  slightly below  $k_F$ , thus giving a "negative gap" in the energy spectrum. They arbitrarily removed this unacceptable negative gap. 5) In three-body ladder diagrams, all three nucleons are interacting with each other, and it is therefore somewhat artificial to treat this effect as a single particle potential on one of them.

A more systematic definition of  $U(b)$  has recently been given by Bethe.<sup>29</sup> He starts from our eq. (5.13) and finds

$$U(b) = [y(b)]^{-1} \int d^3r_{23} g_{PK}^*(r_{23}) Y_b(r_{23}) \quad (7.7)$$

$$Y_b(r_{23}) = \int d^3r_{12} j_0(br_{12}) z^{(1)}(r_{12}, r_{23}, r_{31}) \quad (7.8)$$

For the values of  $b$  which are most important according to Kirson, viz.  $b \approx 0.6/c$ ,  $r_{12} j_0(br_{12})$  has its peak at about the same value of  $r_{12}$  as  $\eta(r_{12})$ ; then  $Y_b(r_{23})$  is proportional to  $F_1$ , eq. (5.20), and the new definition (7.7) of  $U(b)$  reduces to the old one, (7.6).

On the other hand, for small  $b$ ,  $j_0$  in (7.8) is  $\approx 1$  so that  $Y_b$  is nearly independent of  $r_{23}$  (while for larger  $b$ ,  $Y_b$  increases with  $r_{23}$ ); therefore, for small  $b$ , the repulsive contribution from  $r_{23} \leq c$  in (7.7) is not suppressed by the factor  $Y_b$ : This automatically eliminates the negative energy gap of Sprung et al., discussed as point 4 above. Points 1 and 2 are clearly satisfied by the new definition (7.7); point 5 of course remains.

The problem about the denominator  $y(b)$  is removed by remembering that the initial interaction may be alternatively tensor or central. Then it can easily be shown that a suitable definition is

$$U(b) = \frac{y_c^2(b) U_c(b) + y_T^2(b) U_T(b)}{y_c^2(b) + y_T^2(b)} \quad (7.9)$$

where  $y_c(b)$  and  $y_T(b)$  are the expressions (7.5), calculated respectively with the defect functions  $\eta_c$  and  $\eta_T$  for central and tensor forces, and  $U_c, U_T$  are defined correspondingly. The denominator of (7.9) does not vanish for any  $b$ , so that criticism 3 above is now also taken care of.

#### Multi-particle clusters

Rajaraman<sup>13</sup> has emphasized the usefulness of defining a  $U(b)$  in order to absorb the principal effect of many-body clusters. His argument runs briefly as follows: Consider all three-body ladders to be divided into the three groups as suggested in Sec.3: (i) those involving  $g_{\text{short}}$  only, (ii) those with  $v_\ell$  only, and (iii) those mixed in  $g_s$  and  $v_\ell$ . For class (i), we expect the

four-body energy to be much smaller than the three-body energy and so on, because of the short-rangedness of the force ( $\rho c^3 < 1$ ). The class (ii) diagrams are similar to higher Born terms in  $v_e$ , and should be very small since even the second Born term  $v_e Q v_e$  is only 2% of the first Born term.<sup>11,12</sup> This then leaves diagrams of class (iii) as the leading contributors to the four-body energy and larger clusters. In fact, even for the three-body energy of about -5 MeV, we will show that the dominant contribution comes from a diagram of this class.

Now, for any typical potential, such as the standard hard-core potential, the matrix element  $\langle \vec{k} | v_e | \vec{k}_0 \rangle$  is strongly dependent on the momentum transfer  $\vec{q} = \vec{k} - \vec{k}_0$ , although the diagonal element  $\langle \vec{k} | v_e | \vec{k} \rangle$  is relatively independent of  $k$ . A graph of  $\langle \vec{k} | v_e | \vec{k}_0 \rangle$  for  $k_0 = \sqrt{0.3} k_F$  is given in Fig. 19, and shows that a typical off-diagonal matrix element is about 1/7 or less, of the diagonal one. This, for instance, is the reason why the second order term in  $v_e$ , namely  $v_e Q v_e$ , is only 2% of the first order  $v_e$ . The former contains two off-diagonal matrix elements compared to the diagonal first order term. On the other hand, the dependence of  $g_s$  on the momentum transfer is not strong.

From this, we would expect that the most important diagrams of class (iii) should involve diagonal elements of  $v_e$ . A diagonal matrix element in a Goldstone diagram corresponds to "bubble interactions." Thus, we would expect a diagram of the type in Fig. 20a, to be larger than the one in 20b although both belong to class (iii) and are four-body terms. In addition, a diagram

with a bubble-interaction contains one less excited particle, and hence smaller energy denominators, which also enhances its importance. It should be noted here that if one uses a Serber force for  $v_b$  instead of a Wigner force, then the exchange of the bubble interaction is also important. The best way to take into account diagrams of the type in Fig. 20a with exchanges, is to absorb them into the single-particle energy with a potential

$$U_{\text{long}}(b) = \sum_{n < k_F} \langle bn | v_b | bn \rangle - \frac{1}{4} \langle bn | v_b | nb \rangle \quad (7.10)$$

The factor of  $-\frac{1}{4}$  in front of the exchange term arises for spin-independent forces because of the statistical weight arguments given earlier<sup>14</sup>. For Serber forces, of course,

$$\langle bn | v_b | nb \rangle = \langle bn | v_b | bn \rangle \quad (7.10a)$$

if states b and n agree in spin and isospin.

It is worth noting that three-body ladders can have at most one bubble interaction, so that the only diagram of class (iii) with a diagonal  $v_b$ , is the third order diagram in Fig. 21. An estimate of this diagram, along with its exchange using a typical value of b, gives about -5 MeV. While this is only an estimate, it reveals the dominance of such diagonal  $v_b$  terms in the three-body energy. Accordingly, Rajaraman<sup>13</sup> proposed to use only the long-range bubble terms in the definition of  $U(b)$ . However, if this were taken literally, this would enhance the trouble found by Sprung et al. and listed as criticism, point 4 above. The new Bethe prescription (7.7) or (7.9) also emphasizes the long-range force acting on highly excited states b, and is therefore acceptable from the point of view of Rajaraman.

Since Rajaraman has shown that bubble interactions involving  $v_0$  give the main contribution to the many-body energy, we may expect that the use of (7.9) for  $U(b)$  will absorb a major part of these many-body clusters. The prescription for nuclear matter calculations is then the same as has been used in the past by Brueckner's group and others, viz:

Calculate the 2-body  $g$  matrix from the integral equation

$$g | mn \rangle = v | mn \rangle - (2\pi)^{-3} \int d^3b v | ab \rangle \frac{Q(a,b)}{T(a) + T(b) + U(a) + U(b) - E(m) - E(n)} \langle ab | g | mn \rangle \quad (7.11)$$

where  $T(a)$  is the kinetic and  $U(a)$  the potential energy of the particle state  $a$ , and  $E(m)$  the total energy of a hole state  $m$ . Similar prescriptions hold if the initial state includes one or two particles rather than two holes. Calculate  $U(b)$  from (7.9).

Summing the diagonal elements of  $g$ , we get the total, two-body nuclear energy. If now the denominator of (7.11) is expanded in powers of  $U(a) + U(b)$ , the term independent of  $U$  will give the two-body energy according to Brandow's prescription.<sup>1</sup> The term linear in  $U(a) + U(b)$  will give the three-body energy, because  $U$  satisfies (7.4). The remainder represents the part of the multi-body energy which we can take into account by our simple scheme. This remainder then is the only term by which our prescription differs from Brandow's if both are consistently carried out. Since the potential  $U(b)$  is likely to be negative for all important intermediate states, our prescription will

provide more attraction than his. This will be especially important for the low particle states, with  $b$  slightly above  $k_F$ , which are reached mostly by the tensor force. We guess that the effect of putting  $U(a) + U(b)$  into the denominator of (7.11) may be about 1-3 MeV. This attraction is, however, already contained in such calculations as that of Sprung et al.<sup>24</sup>, so that it cannot be invoked to give a larger binding energy than previously found.

### Acknowledgements

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## Figure Captions

- Fig. 1. S-wave part of the two-body defend function for low relative momentum.
- Fig. 2. First order diagrams.
- Fig. 3. Third order direct diagrams.
- Fig. 4. Two fourth order diagrams. Diagram (a) contains one extra independent hole line as compared to fig. 3b, while diagram (b) has an extra particle line.
- Fig. 5. A fourth order term represented in (a) the Goldstone convention, and (b) the Rajaraman convention.
- Fig. 6. A typical three-body cluster term belonging to  $\langle lmn | T | lmn \rangle$ .
- Fig. 7. Two "three-body ladder" diagrams which have no Goldstone analogue.
- Fig. 8. Two Goldstone diagrams in third order, which are not contained in  $\langle lmn | T | lmn \rangle$ .
- Fig. 9. Triangle formed by the three particles in coordinate space.
- Fig. 10. Two graphs due to Kirson<sup>12</sup> comparing Day's analytical solution (solid line) and its first iterate (dashed line). Graph (a) is for all  $r_{ij} = r$  while graph (b) has  $r_{13} = |r_{23} - r_{12}|$  and  $r_{12} = 1.5c$ .
- Fig. 11. A comparison of the Day and Bethe solutions, exhibiting the strong discontinuities of the latter (solid line) as compared to the former (dashed line). The graph is drawn for  $r_{12} = 1.5c$ ,  $r_{13} = |r_{23} - r_{12}|$ .
- Fig. 12. An exchange diagram represented in two equivalent ways.
- Fig. 13. Comparison of a direct diagram belonging to  $\langle lmn | T | lmn \rangle$  and an exchange diagram belonging to  $\langle lnm | T | lmn \rangle$ .

- Fig. 14. A graph of the functions  $F_a$ ,  $F_1$  and  $f_b$  defined in text, for a standard hard core potential <sup>12</sup><sub>b</sub>.
- Fig. 15. The functions  $F_a$ ,  $F_1$  and  $f_b$  using the Reid potential<sup>7</sup> for triplet states.
- Fig. 16. A three-body diagram.  $K_0$ ,  $K_1$  and  $K_2$  each stand for the momenta of all three particles at the respective stages.
- Fig. 17. The third order "bubble diagram" and the third order U-diagram. Goldstone<sup>3</sup> suggested a choice of  $U(b)$  designed to cancel these terms with one another.
- Fig. 18. Diagrams showing the absorption of a whole sequence of terms by the choice of the hole potential energies  $U(m)$  on the energy shell.
- Fig. 19. Dependence of the long-range force matrix elements on the momentum transfer.
- Fig. 20. Two four-body terms. Diagram (a) contains one diagonal  $v_\ell$  matrix element while (b) has two off-diagonal ones.
- Fig. 21. The long range "bubble" diagram which dominates the three-body energy.

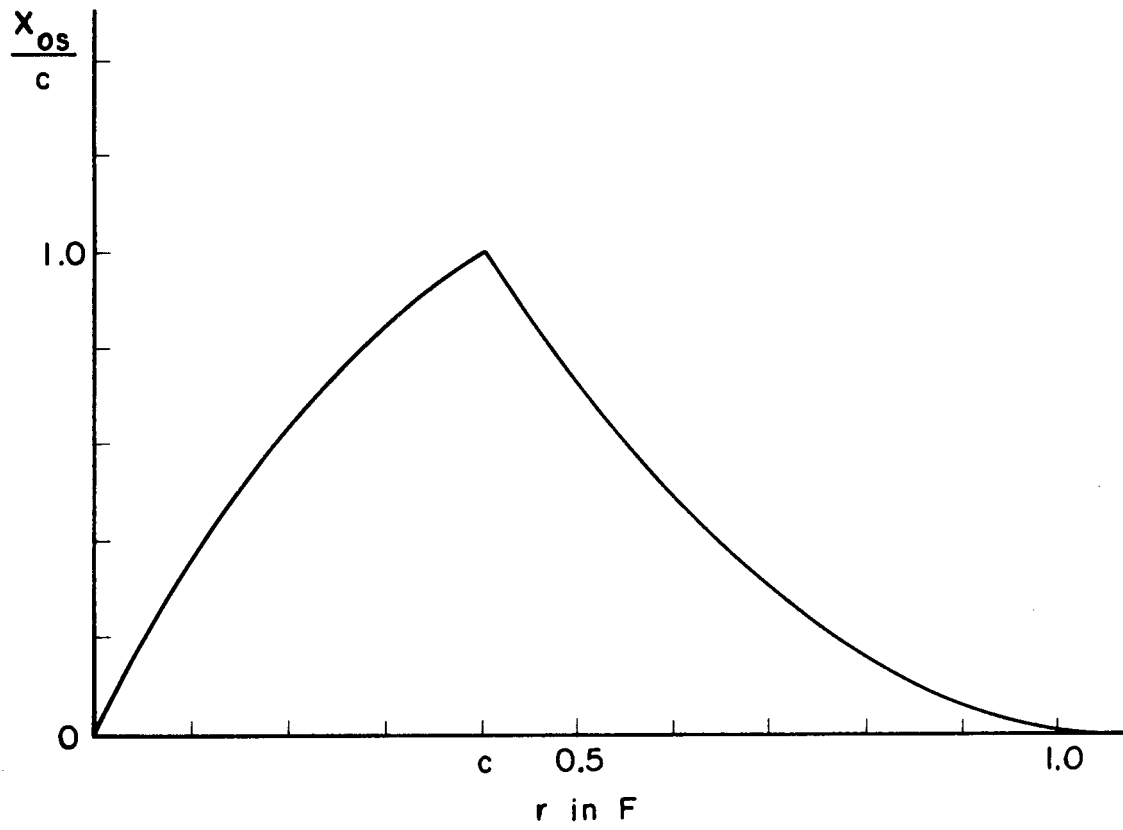
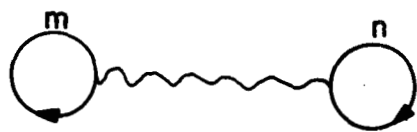
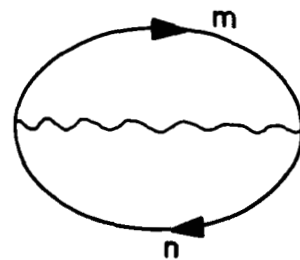


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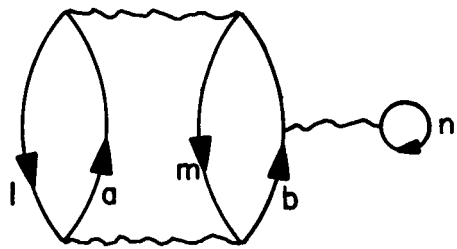


(a)

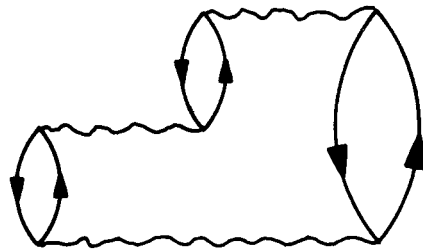


(b)

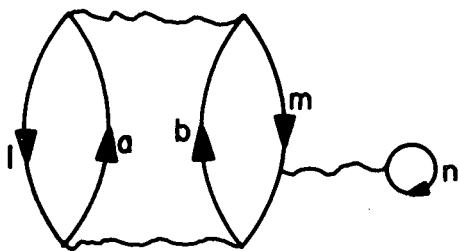
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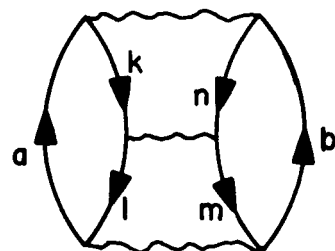
(a)



(b)



(c)



(d)

Figure 3



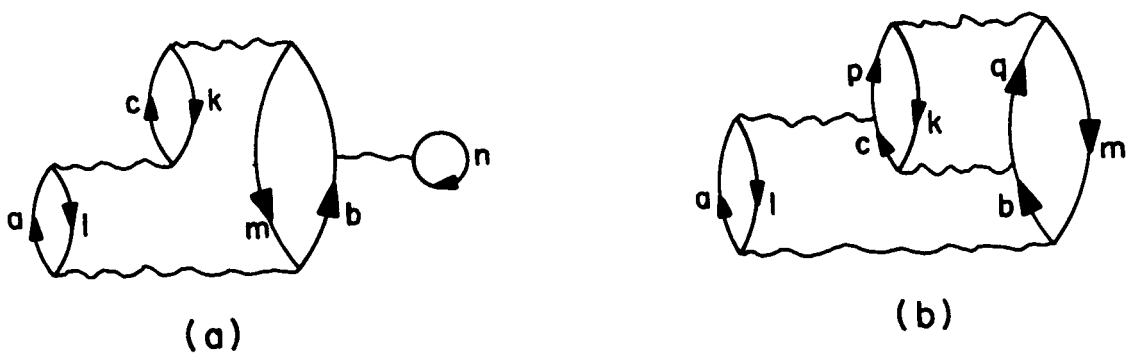
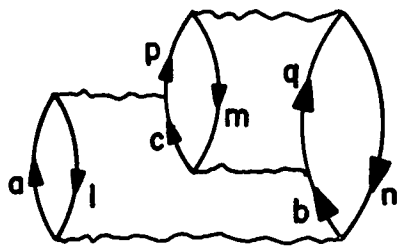
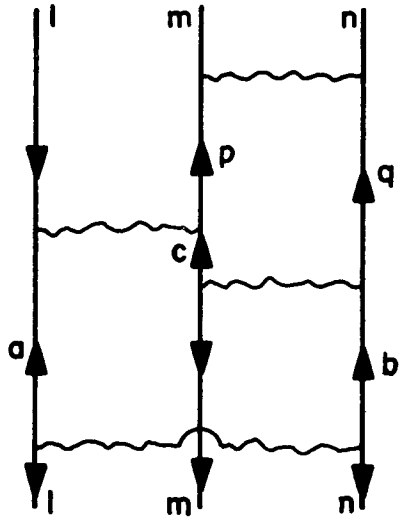


Figure 4



(a)



(b)

Figure 5

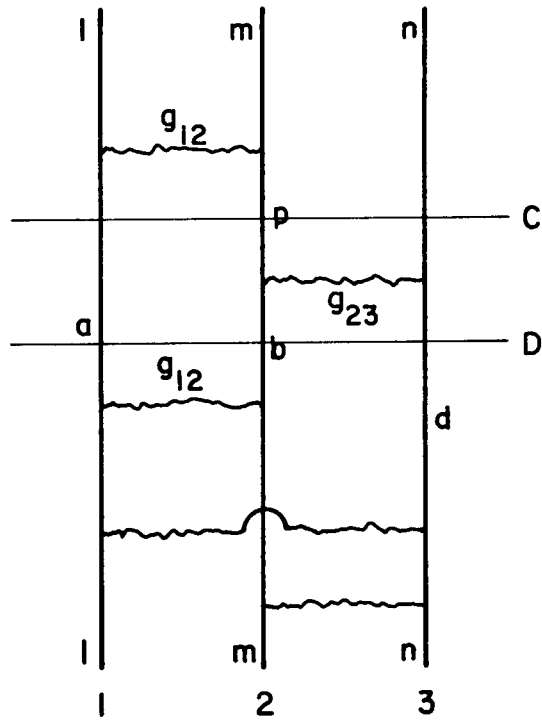


Figure 6

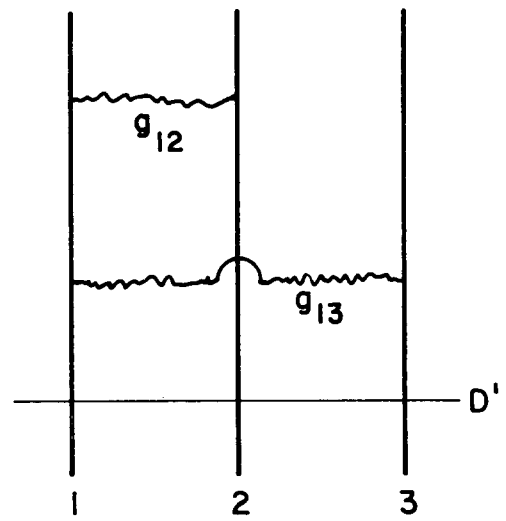
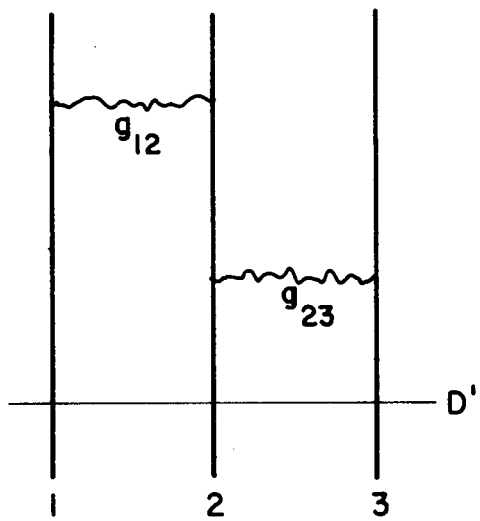
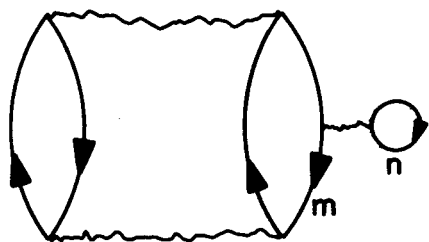
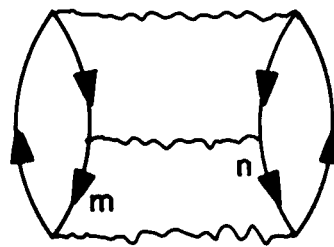


Figure 7



(a)



(b)

Figure 8

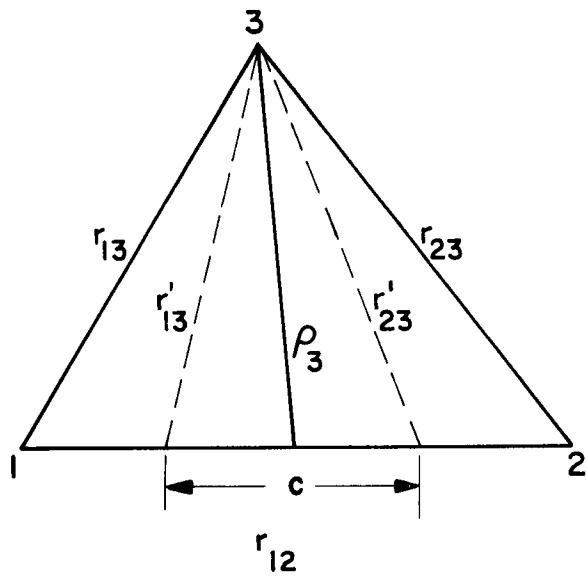


Figure 9

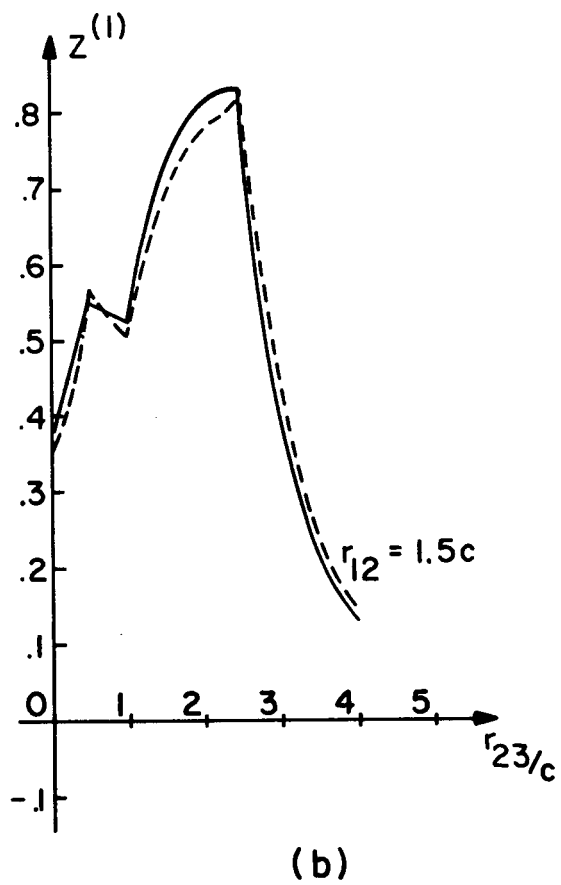
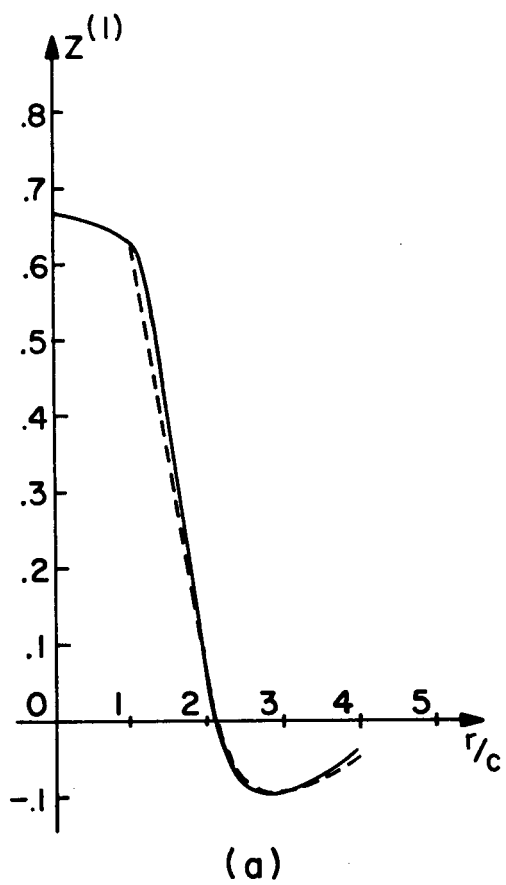


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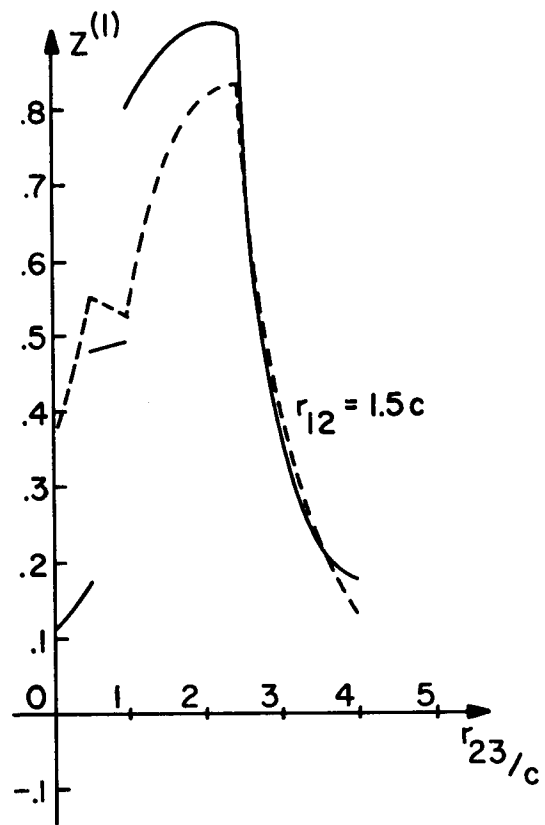
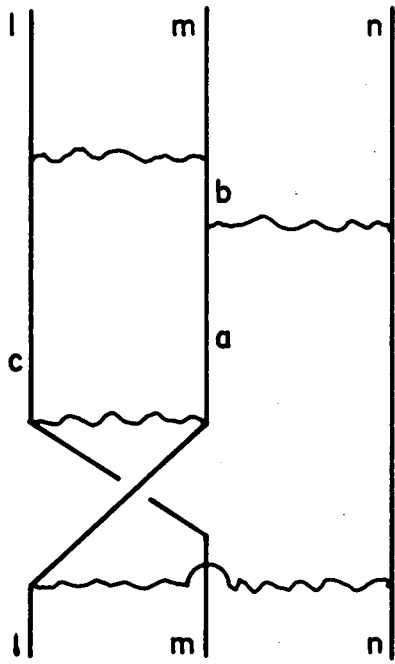
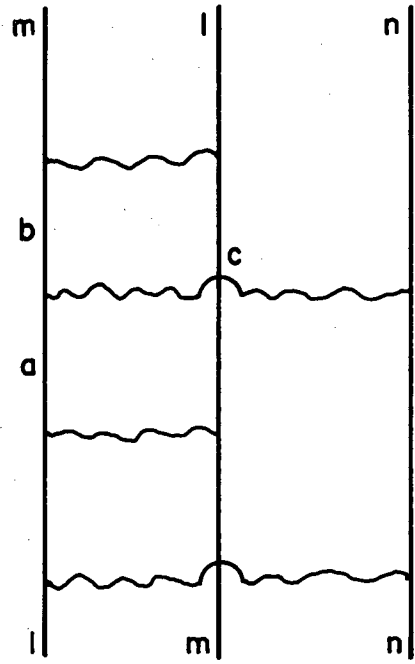


Figure 11



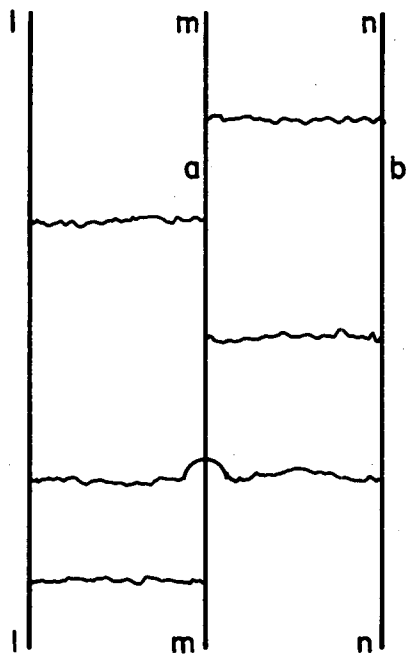


(a)

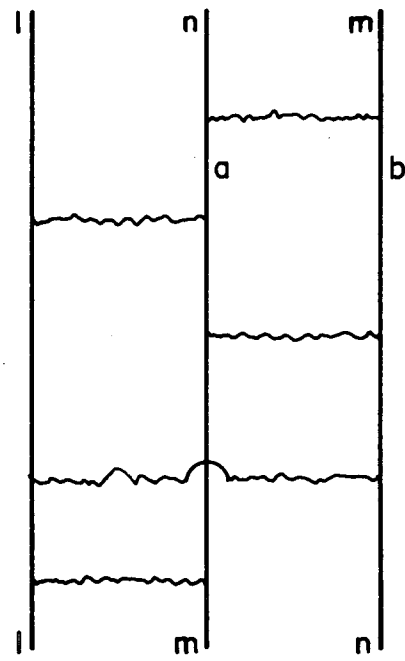


(b)

Figure 12



(a)



(b)

Figure 13

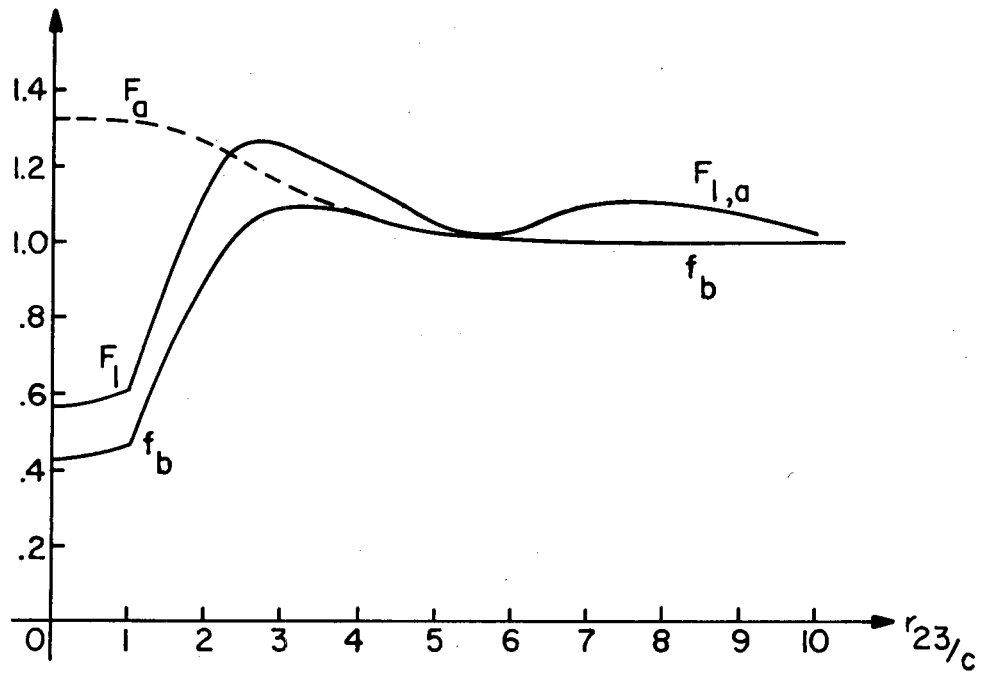


Figure 14

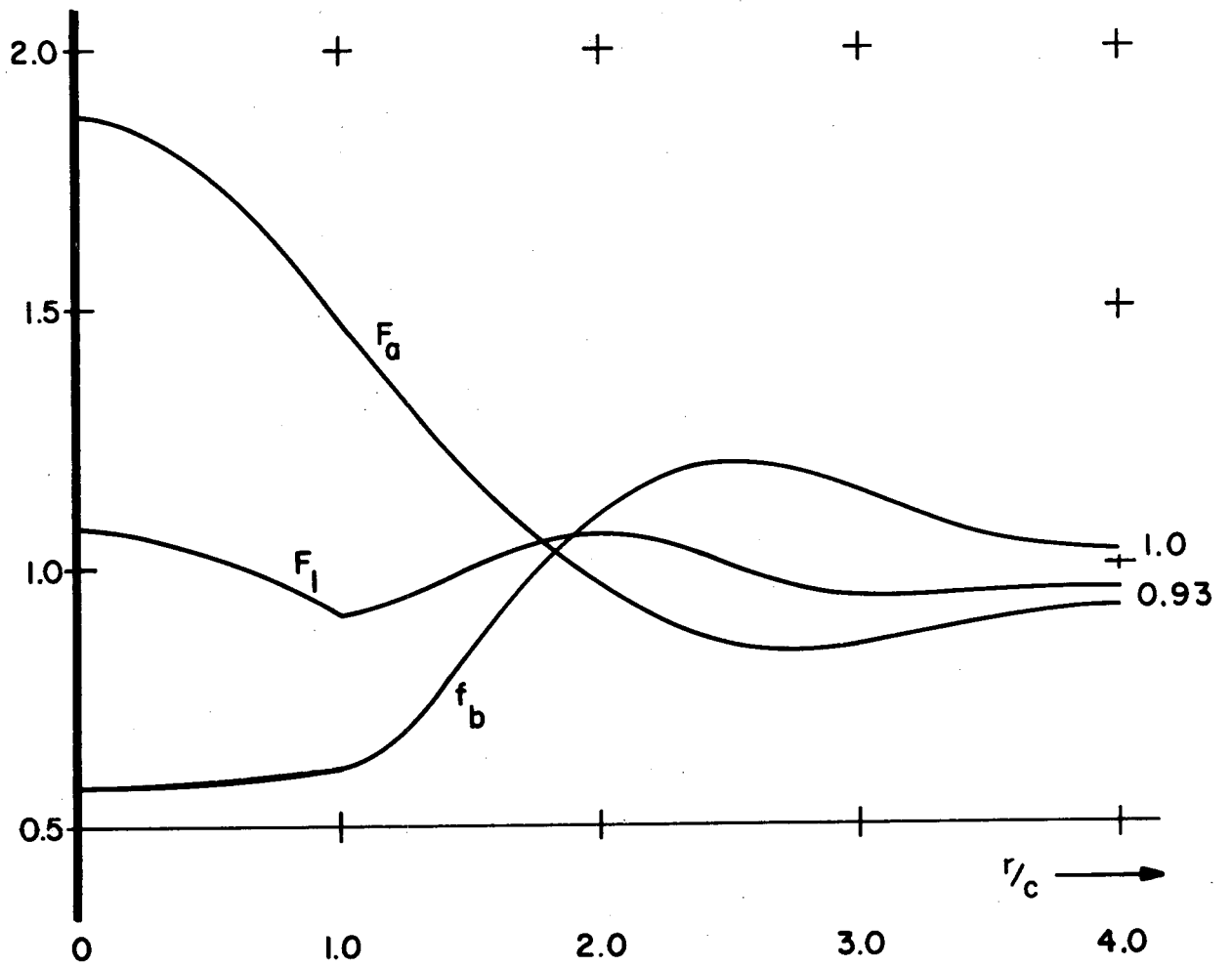


Figure 15

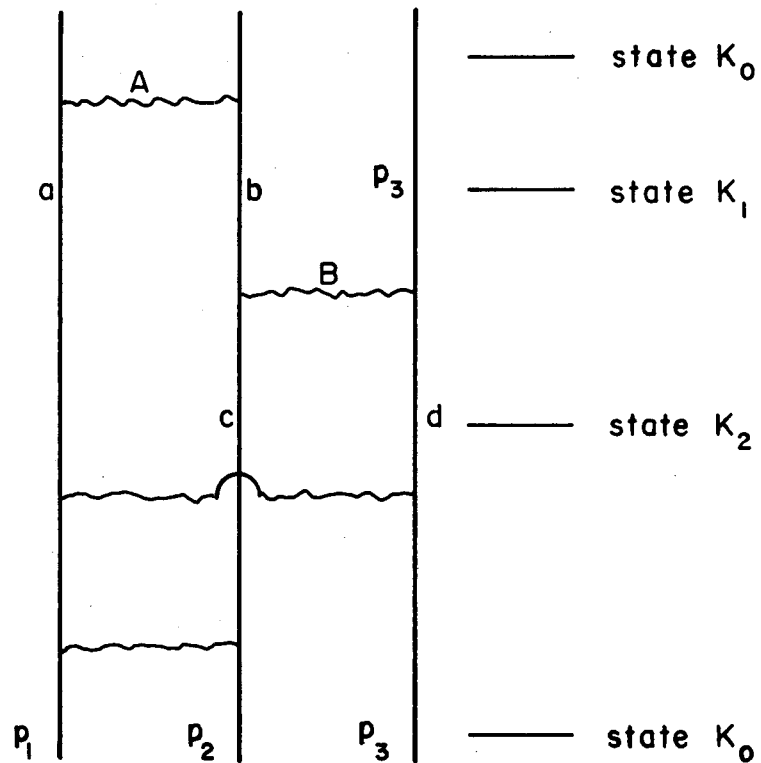


Figure 16

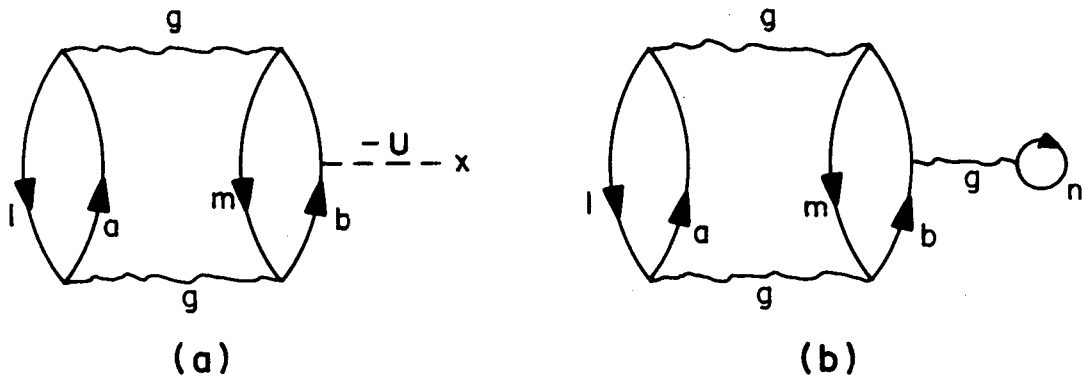


Figure 17

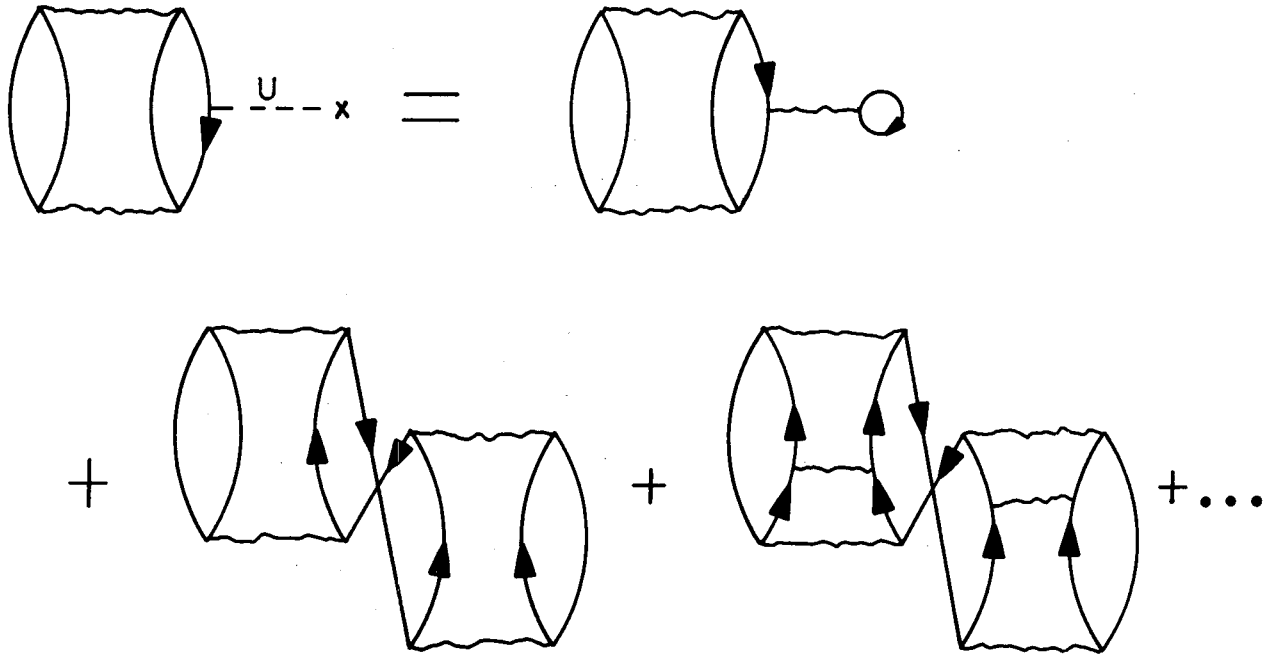


Figure 18

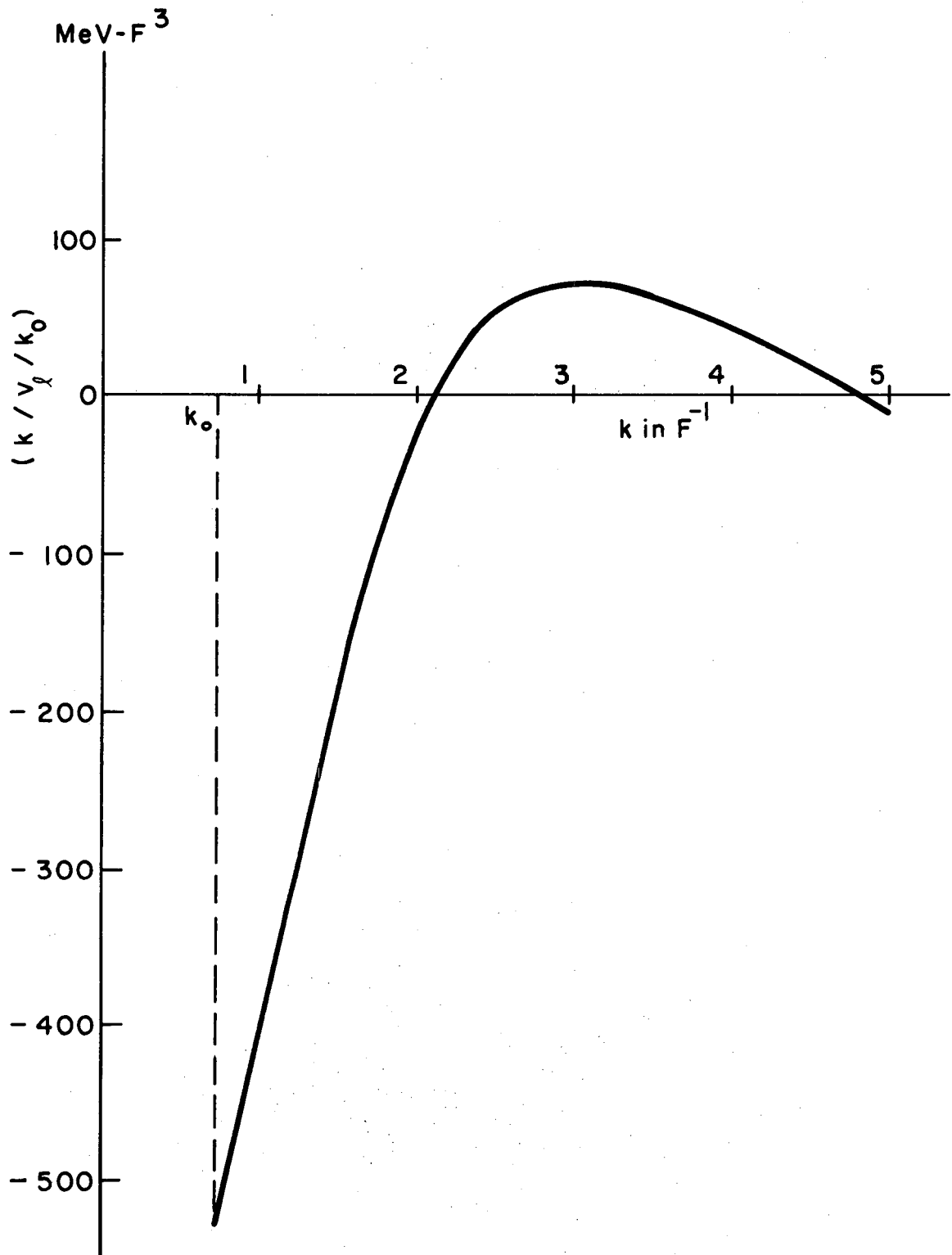


Figure 19



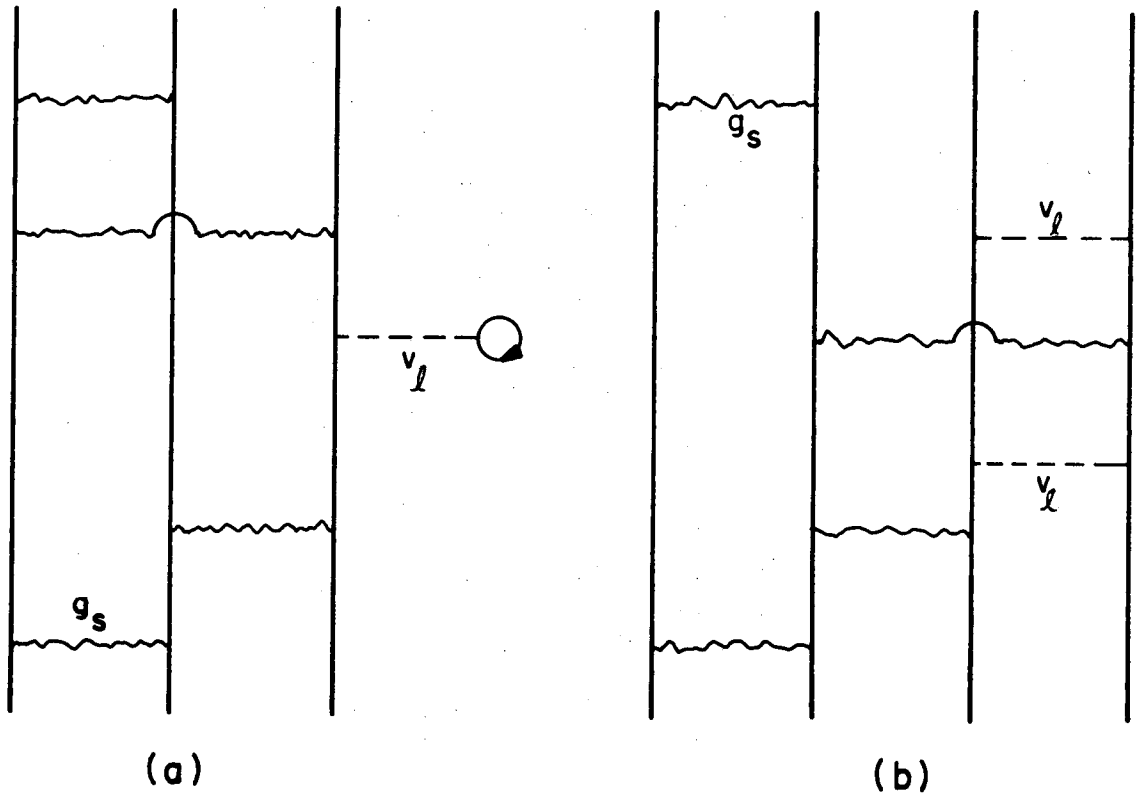


Figure 20

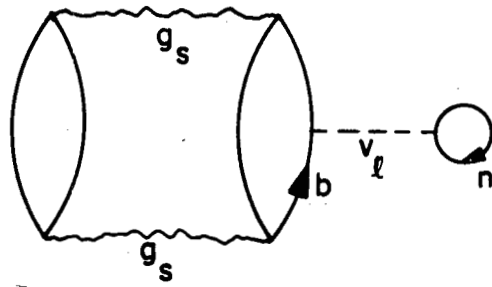


Figure 21