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UNDERSTANDING THE STATE PLANE COORDINATE SYSTEMS

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ABSTRACT. In the more than 40 years since the development of the State plane coordinate systems, 35 States have passed legislation permitting their use in defining property boundaries and for other purposes; but many surveyors still do not utilize the systems. During this period several publications have been issued and numerous papers presented which describe the value of the State grids and provide, often in great detail, instructions on how to employ the systems including the computations involved. Although recently more surveyors have shown an increasing interest in State plane coordinates, their numbers are still small.

The major reasons offered for not using the systems relate primarily to several misconceptions about the State grids, and in particular to a lack of understanding of two corrections to distance measurements. These corrections involve the reduction to the sea level reference and for the scale distortions which result from confining a portion of the Earth's surface within the mathematically defined dimensions of the map projections used in the State systems. Although both corrections are arithmetical in nature and can be combined or even ignored in numerous cases, these arguments are often lost because of long held preconceived concepts.

In an attempt to overcome the reluctance of many in the profession to employ the State systems, the principal effort is directed to explaining these corrections. By using simple terminology and graphic demonstrations, the intent is to show that these reductions are far easier to understand than most believe. As a new generation of surveyors is about to assume their place in their chosen profession, this approach will hopefully lay aside many fears held previously.

INTRODUCTION

The State plane coordinate systems have been in existence for more than 40 years, and 35 States have enacted laws permitting their use to define positions on the surface of the Earth for describing property boundaries and other purposes. However, few surveyors use the systems on a day-to-day basis, although the number is slowly increasing. The geodetic community and others interested in a more general implementation of plane coordinates have conducted numerous inquiries directed toward the reluctance of the practicing surveyor to utilize the State grids. No one expects the average practitioner to expend the energy and cost to extend control over long distances to sites where none exists, even though modern equipment can often substantially reduce the effort and the cost. Most other reasons, and there are many, really are not acceptable when viewed in the clear light of day. Some cast the blame on the application of scale factors and reductions to the sea level reference which in fact may not be necessary for much of the work undertaken by many of these surveyors. Others take issue with the large numerical values assigned the coordinates and are difficult to convince that the solution is very simple -- only use those portions of the coordinate values sufficient to keep the quantities within the survey area positive. It is necessary, of course, to record on the documents, the constants which when added to the "abbreviated coordinates" will place them on the State grid. There are still others who believe that since the systems were developed by geodesists, they must be inherently difficult to understand and employ. Nothing is further from the truth. The State plane coordinate systems, for all intents and purposes, are nothing more than an adaptation of the practice of latitude and departures which have been used by surveyors in one form or another to define points on the Earth for perhaps 5,000 years. Nothing that will be presented here has not been discussed over the years. However, the major emphasis is placed on those segments of the plane coordinate concept which are least understood or where the greatest misconceptions exist. To fully understand how to employ the State grids may take a few hours, but once learned, the simplicity of the procedures involved are in marked contrast to the solution of many other surveying problems.

COORDINATE SYSTEMS

By definition, coordinates are linear or angular quantities, or both, which designate the position of a point in relation to a given reference frame. There are two general divisions of coordinates used in surveying: polar coordinates and rectangular coordinates. These may each be subdivided into three classes: plane coordinates, spherical coordinates, and space coordinates. In this paper the discussions are primarily directed to plane coordinates in a rectangular system, specifically the State grids. Brief mention is made of tangent plane and purely local grids (latitudes and departures) in order to demonstrate problems common to all plane coordinate systems.

Prior to the introduction of State grids, many cities and counties employed plane rectangular coordinates for general surveying and mapping purposes. These coordinates were calculated from the geographic positions determined from the rigorously adjusted results of well-scaled and oriented triangulation nets. As a rule, these basic frameworks were tied to the National network. To compute the coordinates, the geographic position of one of the stations or some point, preferably located near the center of the system, was selected as the origin. Using this position and the geographic positions obtained from the adjustments of the observations, computations quite similar to the geodetic inverse problem were carried out with the end result being rectangular coordinates expressed in the form of $X = -S \sin \alpha$ and $Y = -S \cos \alpha$, where "S" = geodetic distance and "α" = the geodetic azimuth (referenced from south) of the line from the origin to the point. In many instances, first-order and occasionally second-order traverses were fitted to the primary system to complete the network.

In some instances the results were published with the appropriate signs, in order to indicate the quadrant in which the points fell; but this was a source for mistakes. To eliminate the possibility of errors occurring from the signs being improperly applied in differing the values, the nominal coordinates of the origin $X = 0$ and $Y = 0$ were, in many cases, assigned sufficiently large constants to keep all coordinates positive. On occasion the coordinates were rotated for one reason or another, which often lead to confusion at later dates.

For simplicity sake, this system is identified as a tangent plane on the premise that only one point, the origin, was coincident with the spheroid (ellipsoid) of reference. In reality, it closely approximates the azimuthal equidistant map projection. By limiting the extent of the system from the origin, scale distortions and problems due to the convergence of the meridians are minimized.

In a purely local system of latitudes and departures or coordinates derived from these quantities, in which some arbitrary point in some meridian is selected as the origin, the problem related to the convergence of the meridians is the most bothersome, although scale distortions will eventually occur if the grid is considerably extended from the origin. It would indeed be rare when these systems are expanded to a point that scale problems arise, but this is not the case in the matter of convergence. Taking New York as an example, the convergence amounts to about 48" for each mile that a point is east or west of the origin. These systems are those favored by land surveyors and for engineering projects of very limited extent and for these purposes are entirely satisfactory.

Map projections are classified as conformal and non-conformal. For a projection to be conformal, the scale ratio (scale factor) at a point is identical for a finite distance in any azimuth; while for a non-conformal system, the scale at a point varies with the azimuth. The tangent plane system is non-conformal since the scale is exact on the lines between all points and the origin, but varies in other azimuths. In every case, except on lines to the origin, the grid distances are longer than measured distances. Since these systems seldom extend very far from the origin, this correction can be safely ignored. For example: at 20 miles from the origin, the maximum scale difference is only 1:70,000; but at 80 miles, the scale can differ by as much as 1:5,000. This brief discussion on projections has been presented for three reasons. One, to introduce the thought that scale factors exist in any coordinate system which extends an appreciable distance; two, to describe the major difference between conformal and non-conformal projections; and three, why the uniformity of scale at a point is essential, as a matter of convenience, in selecting a grid encompassing large areas. Furthermore, and perhaps most important, larger areas can be accommodated by using conformal projections.

The coordinate systems constructed by the National Ocean Survey (formerly the U.S. Coast and Geodetic Survey) are all derived from conformal map projections, except that developed for Guam which is on a tangent plane. For States whose primary extent is east-west, the Lambert conformal conic projection was selected because the scale varies with latitude; thus, a State the size of Tennessee could be covered with one zone. The transverse Mercator projection was best suited for those States whose areas lie mostly north-south, since the variation in scale depends on the difference in longitude. New Jersey is one of several States which has only a single zone on this system. The configurations of two States, Florida and New York, require that the Lambert projection be used for northern Florida and Long Island, while the remainder of the States are divided into zones served by the transverse Mercator grid. Michigan uses both systems. The original State grid was on the transverse Mercator projection. Later it was decided that the Lambert grid would be better for the State, since the population mass was concentrated in an east-west pattern. Subsequently, this projection, referenced to an elevation 800 feet above sea level, was adopted as the legal State grid by an act of the legislature in 1964. Alaska requires a skew projection for the southeast section, eight zones on the transverse Mercator system for the major land mass, and the Lambert grid for the Aleutian Islands.

In developing the State systems, there were three considerations. One, the zones were to be defined by County or equivalent political boundaries. This condition was met except for Alaska and Washington. In Washington it was necessary to delineate the zone boundary along a parallel passing through Grant County. Two, the scale reduction was not to be worse than 1:10,000. This limit was selected because for many surveyors at the time the State systems were prepared and even for much of the work undertaken today, the scale factor could and can be ignored without any serious consequences resulting. Also, by maintaining the scale ratio at 1:10,000, a zone could be 158 miles in width, extending around the world on the Lambert system and to within about 10° of the north pole for the transverse Mercator projection.

There are 130 zones which include both systems in Michigan to cover the 50 States. Including the systems prepared for Guam, Puerto Rico and the Virgin Islands, and American Samoa, there are 134 zones. Of this total, only 11 zones have maximum scale reductions in excess of 1:10,000, the worst being Alaska zone 10, covering the very sparsely settled Aleutian Islands, where the maximum reduction amounts to 1:6,600. About 58 percent of the scale reductions are better than 1:15,000. The third consideration was to select the zone boundaries such that the scale variations would be minimized in the vicinity of the major metropolitan regions. In many cases, this was accomplished; however, it was not possible to meet the condition in every situation without increasing the number of zones. To date, only one additional zone has been requested for that purpose. This is zone 7 in California, which covers Los Angeles County and has a maximum scale reduction of 1:87,300.

SEA LEVEL REDUCTION

State plane coordinates, with the exception of those computed on the Lambert projection for Michigan, are referenced to sea level. For those surveys where the most exact results are desired from the observations and the observations have been obtained using equipment and procedures commensurate with the desired exactness, then each length should be reduced to the sea level reference using the mean elevation of the two points involved. In many cases, however, a mean elevation for the entire area may be satisfactory; and in other instances where the survey procedures do not justify this computation or the elevations are near sea level, this reduction can be ignored.

Once a good understanding of the actual influence of sea level and scale reductions on measured lengths and the relationship between the corrections, which will be discussed later, is reached, a major source of apprehension in employing the State grids is eliminated. There is nothing mysterious about these reductions; they are simply arithmetical operations. What one has to learn is when the corrections should be applied and what errors are introduced when they are omitted. This matter will now be taken up. First, a graphic representation of the reduction to sea level, followed by the simplified formula for making this reduction, examples of the reduction of a length, computations of the sea level factor, and a table of factors with the corresponding proportional part changes to the lengths for selected heights above sea level seems in order. In this section, elevation and height are used interchangeably.

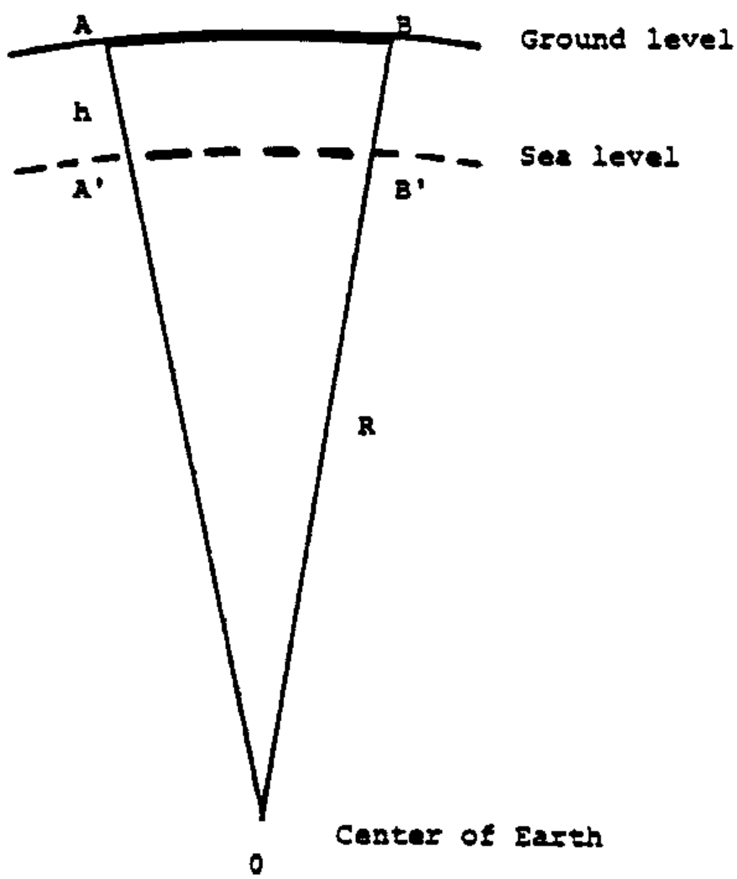


Figure 1. Reduction to sea level.

Figure 1 shows graphically the reduction of the horizontal ground length A-B to the sea level or geodetic length A'-B' Except for Death Valley and a few other locations where the land lies below sea level, this diagram is correct within the intent of this paper.

For computational purposes:

Horizontal ground distance A to B = S
Sea level distance A' to B' = S' = Geodetic distance
Mean elevation of A and B = h
Radius of Earth = R = 20,906,000 feet

NOTE 1: "S" may be in any linear unit (feet, meters, varas, etc.), but "h" must be in feet if "R" = 20,906,000 is used.

NOTE 2: "R" varies with latitude and azimuth; but for most purposes in the conterminous States, the value given is of sufficient accuracy.

$$S' = (S \times 20906000)/(20906000+h)$$

No proof for this formula is given here, as it may be found in many surveying texts and publications.

Where "h" is the average or adopted elevation of the area to be surveyed and the terrain is relatively level, a single factor can be determined by the formula $R/(R+h)$ and used as a multiplier to reduce all the horizontal ground distances to sea level. "Relatively level" is a function of the desired accuracy of a survey. For example: if the survey requirement is 1:20,000 or better, then differences in elevation of as much as 1000 ft. can probably be tolerated; where an accuracy of 1:5000 will satisfy the specifications, elevations differences of 2000 ft. are acceptable; and in some cases, the height variance could approach 4500 ft.

Examples follow:

a. Reduction to sea level:

$S = 23935.64$ (any linear unit) and $h = 1837$ ft.
$S' = (23935.64 \times 20906000)/(20906000 + 1837)$
$= (23935.64 \times 20906000)/(20907837)$
$S' = 23933.54$

b. Determination of sea level factor:

$h = 1837$ ft.
Sea level factor $= 20906000/(20906000 + 1837) = 0.9999121$

Table 1. Sea level factors and equivalent proportional part changes in lengths.

Elevation (ft.)	Factor	1:Part In	Elevation (ft.)	Factor	1:Part In
0	1.0000000	-	3000	0.9998565	6969
500	0.9999761	41841	3500	0.9998326	5974
1000	0.9999522	20920	4000	0.9998087	5227
1500	0.9999283	13947	4500	0.9997848	4647
2000	0.9999043	10449	5000	0.9997609	4182
2500	0.9998804	8361	5500	0.9997370	3802

If there was no other consideration in reducing the lengths, such as the scale factor, a surveyor knowing the mean elevation of the place to be surveyed or having decided that some height would best serve his interests could by a simple inspection of the proportional parts given in table 1 decide whether or not his measured lengths need be reduced. In the next section the combination of the sea level and scale factors into a single factor will be discussed.

Before leaving the subject of elevations, one further point must be made. For all intents and purposes, slope distances reduced to the horizontal are at the mean elevation of the two points. See figure 2 for a graphic display. The formulas for making the slope correction can be modified to place the horizontal distance at the elevation of one of the points; but this is seldom done. When a survey is made over terrain with a wide range of elevation and the computations are to be made on a local system (latitudes and departures), it is wise to determine the effect of these elevation differences and perhaps select some reference surface to which the distances would be reduced. This reference surface need not be sea level; but if the elevation chosen is greater or less than the actual minimum height, one has to keep in mind that those distances below the reference surface must be made longer and those above shorter. The factors given in table 1 can be used for this purpose by merely assuming the selected reference to have a zero elevation. Later, methods to use horizontal ground level lengths and modified State plane coordinates (Project Datum coordinates) will be discussed.

SCALE FACTORS

Scale factors are simply numerical values which when multiplied by the sea level distance between two points produces

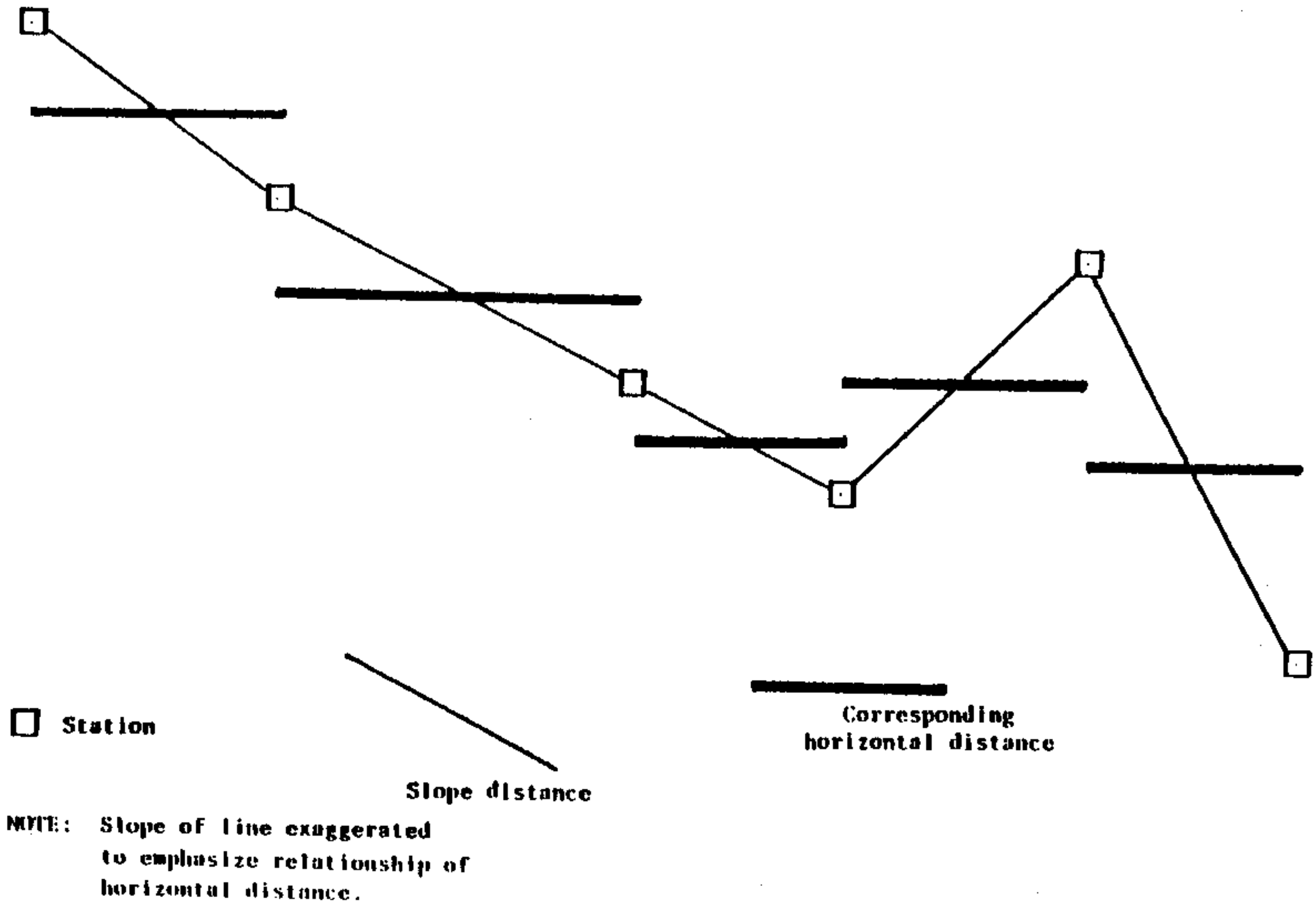


Figure 2.-Slope and corresponding horizontal distances

a grid distance. The question is often asked - why are scale factors necessary? The explanation can be demonstrated by considering a block of wood one inch square. Should the requirement arise to fit this block of wood exactly in a hole one half inch square, we have two choices. One, scale or reduce the block to fit the hole; or two, enlarge the hole. View the hole as being part of a structure which cannot be enlarged unless the entire structure is enlarged; and since the structure is completed, there is only one choice, the block of wood must be reduced in size. The structure is analogous to a map projection where a portion of the Earth, a curved surface, has been fitted to fixed dimensions, determined by mathematical processes; the hole represents a segment of the Earth within the portion and the block of wood the measured distances through that segment which must be reduced accordingly so that a homogenous structure is retained.

It is at this point that many surveyors simply cannot accept the State coordinate systems, since to change accurately measured distances often by very significant amounts is not consistent with time honored practices. The counter argument that these changes are merely computational processes is very often completely overshadowed by the initial reluctance to apply reductions which are not understood. There are many instances where the opposition remains, even when it is shown that the end results can easily be used to reproduce the measurements which differ only from the original values by adjustment corrections. By carefully viewing the examples which will be presented shortly, this reluctance should be largely overcome. Again, it must be emphasized that understanding is the key to what actually is a very simple problem.

Prior to presenting the examples, a few illustrations (figures 3, 4, and 5) may aid in obtaining a better grasp of scale factors and their relation to the sea level reductions. These illustrations are simple graphic representations and should not be construed as being exact portrayals of what are rather complex mathematical developments.

Figure 3 illustrates the regions where the scale factors are less than unity, exact and more than unity on the Lambert projection. For this projection, two-thirds of the zone fall between the standard parallels with one-sixth north and one-sixth south of standard parallels. Between the standard parallels, the scale factors are less than one, equal to one on the standard parallels, and greater than one to the north and south of these parallels. The maximum scale factor is about midway between the standard parallels. It now follows that since the grid distances are simply the sea level distances multiplied by the scale factor for the mean latitude of the line or, as is more often the case, the survey area, the following is true:

a. Grid distances between the standard parallels are shorter than the sea level distances.

b. Grid distances north and south of the standard parallels are longer than the sea level distances.

Figure 4 is the graphic illustration for the transverse Mercator; and rather than parallels, the scale factors are less than unity between the grid meridians (where the scale is exact), which are equally spaced from the central meridian and greater than one east and west of these grid meridians. There seems no need to repeat the explanations, as given for the Lambert projection since they are identical when grid meridians and east and west are substituted for standard parallels, and north and south respectively. There is one slight difference; the maximum scale reduction is along the central meridian, while for the Lambert projection the maximum reduction is not exactly midway between the standard parallels.

Figure 5 shows the relationship of ground measurements to sea level reference and the reduction to the grid plane for observations in the areas where the "scale is too small," as well as where the "scale is too large." This illustration applies to both the Lambert and transverse Mercator projections.

To further demonstrate the graphic presentations shown by figures 3, 4, and 5, the example noted previously is now presented in two parts. The first where the scale factor is less than one, and the second where it is greater than one. Either of the two projections employed by the great majority of the State systems could be used to illustrate the reductions. It is thought, however, since one scale factor depends on the mean latitude of the two points and the second on the mean distance (X') from the central meridian for the zone to provide a sample computation for each case.

Case 1. Lambert system - the distance between stations ODAIR USBR and DAVIS USBR in the State of Washington is measured with the intent to determine the accuracy of the distances in a specific network.

The horizontal ground distance, as reduced from the observations, is 25106.12 ft.; and the mean elevation (h) of the two stations is 2036 ft.

Washington north zone plane coordinates and the approximate latitudes follow:

Station	X Y	Latitude
ODAIR USBR	2388787.91 230531.04	47° 37' 16"
DAVIS USBR	2389121.53 255632.39	47 41 23

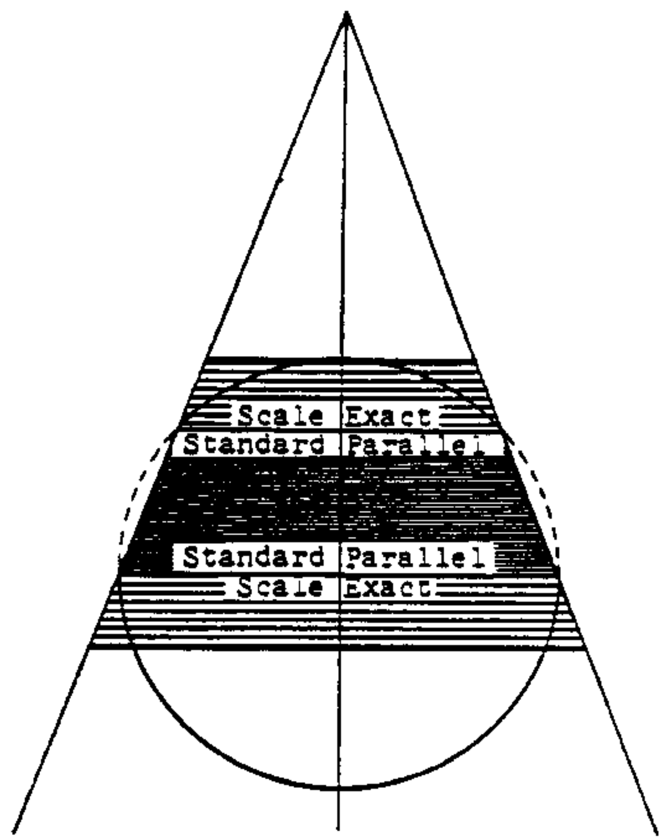


Figure 3.--Lambert projection - cone secant to sphere

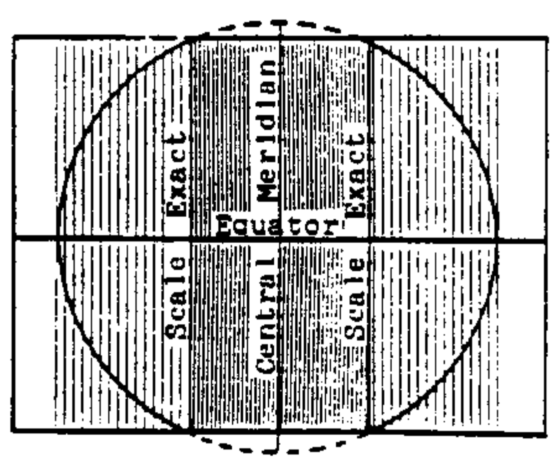
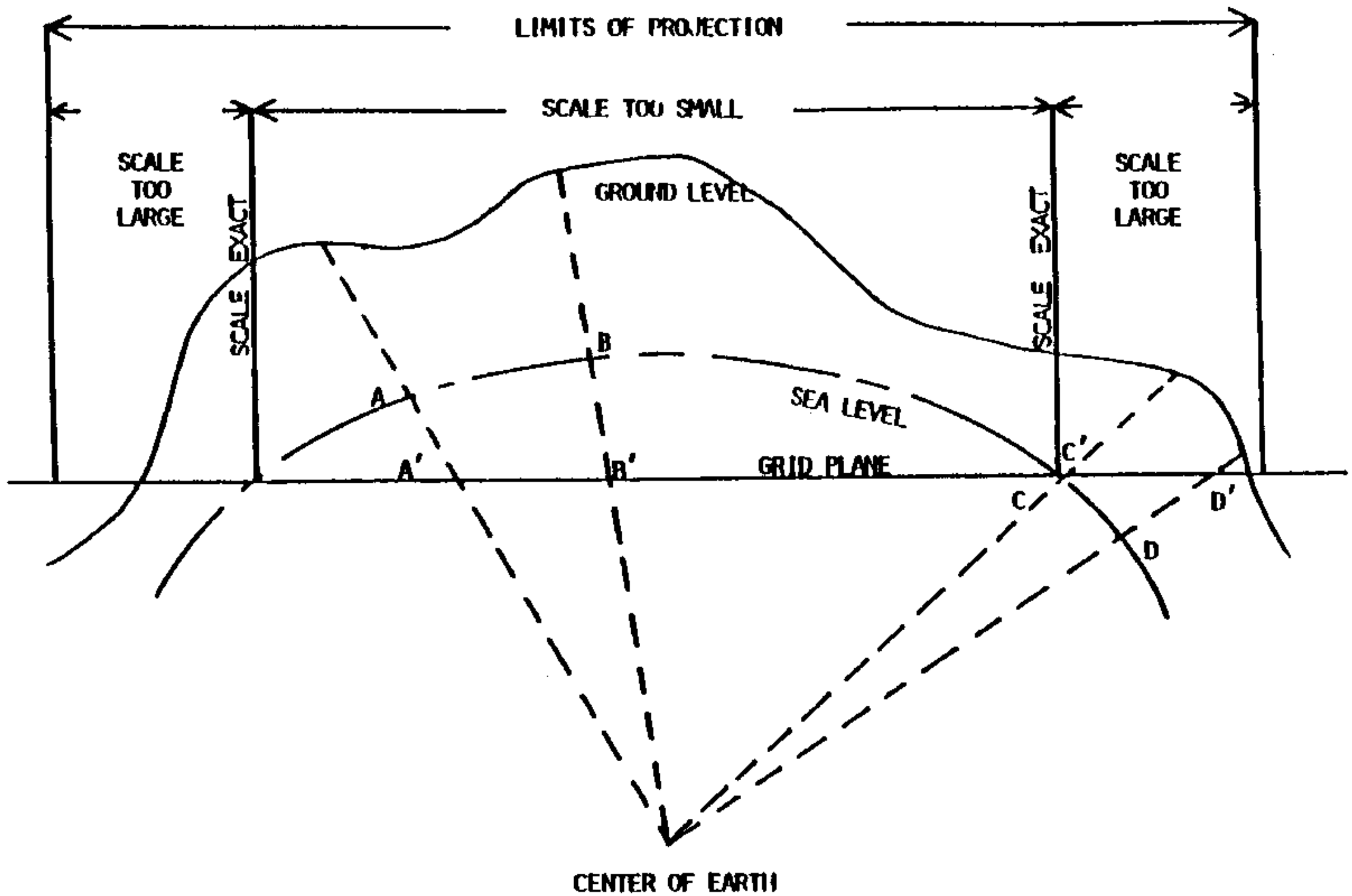


Figure 4.--Transverse Mercator projection - cylinder secant to sphere



GRID DISTANCE A' TO B' IS SMALLER THAN GEODETIC DISTANCE A TO B

GRID DISTANCE C' TO D' IS LARGER THAN GEODETIC DISTANCE C TO D

FIGURE 5.--RELATIONSHIP OF GROUND, SEA LEVEL, AND GRID DISTANCE.

Step 1 - Reduce the horizontal ground distance to sea level. Determine the sea level factor from table 1 or by formula:

0.9999043 - [(2036 - 2000)/500] x 0.0000239 = 0.9999026
Sea level distance = 25106.12 x 0.9999026 = 25103.675 ft.

Step 2 - Compute the scale factor for the mean latitude using table 2:

Mean latitude = 47° 39' 20" = 47° 39!3
Scale factor = 0.9999754 - (0.3 x 0.0000023) = 0.9999747

Step 3 - Determine the grid distance by multiplying the sea level distance (Step 1) by the scale factor from Step 2:

Grid distance = 25103.675 x 0.9999747 = 25103.04 ft. from observations.

Step 4 - Difference the known coordinates to obtain ΔX and ΔY and compute the grid distance by the formula $\sqrt{\Delta X^2 + \Delta Y^2}$:

ODAIR USBR to DAVIS USBR $\Delta X = -333.62$ $\Delta Y = -25101.35$
Grid distance = 25103.57 ft. from known coordinates.

Step 5 - Compare the results obtained in Steps 3 and 4. The comparison indicates the adjusted data with respect to the distances in the vicinity of the two stations is on the order of 1:47400 too long. For those surveys made in this area, which are primarily distance dependent, the position closures can be expected to approximate this ratio provided the same care and procedures are exercised in securing the measurements, as was taken on this observation.

By combining the sea level and scale factors, the reduction to the grid is accomplished by a single multiplication. To combine the factors, they may be multiplied together or added and 1.0000000 subtracted from the sum.

Combined factor = 0.9999026 x 0.9999747 = 0.9998773 or
0.9999026 + 0.9999747 - 1.0000000 = 0.9998773

Grid distance = 25106.12 x 0.9998773 = 25103.04 ft., which checks the value computed in Step 3.

Note: When the points lie in the area of a zone where the scale factor is less than unity, the combination of the scale and sea level factors have a compound effect. In the second example, the line selected falls in that part of the zone where the scale factor is greater than one; and in these instances, the combination of the factors has a lessening effect. This, of course, only holds true when the points are above sea level. In those few areas which are located below sea level, the opposite is true.

Figure 6 illustrates the reduction of the horizontal ground distance between stations ODAIR USBR and DAVIS USBR, as carried out in Steps 1-4.

Turning now to the case where the grid distance is known and there is a requirement to know the horizontal ground distance, this value is determined by dividing the grid value by the combined factor:

25103.04/0.9998773 = 25106.12 ft., which is identical to the given distance.

Table 2. Scale factors for use with coordinates on Lambert system.

LAMBERT PROJECTION FOR WASHINGTON (NORTH)

Lat.	R (feet)	y' y value on central meridian (feet)	Tabular difference for 1 sec. of lat. (feet)	Scale in units of 7th place of logs	Scale Factor: Scale expressed as a ratio
47° 36'	18,987,002.43	218,861.00	101.32333	- 74.5	0.9999828
37	18,980,923.03	224,940.40	101.32333	- 85.6	0.9999803
38	18,974,843.63	231,019.80	101.32350	- 96.4	0.9999778
39	18,968,764.22	237,099.21	101.32350	-106.8	0.9999754
40	18,962,684.81	243,178.62	101.32367	-116.9	0.9999731

Case 2 - Transverse Mercator system - the distance between stations REDWOOD and GRANITE in the State of New York was measured for the same reason as Case 1.

The horizontal ground distance, as reduced from the observation, is 39028.80 ft.; and the mean elevation (h) of the two stations is 610 ft.

New York east zone plane coordinates and the corresponding X' follow:

Station	X / Y	X'*
GRANITE	84086.55 1581012.62	415913
REDWOOD	122981.72 1584283.84	377018

* X' = X-500000.00 and for this purpose is always considered positive.

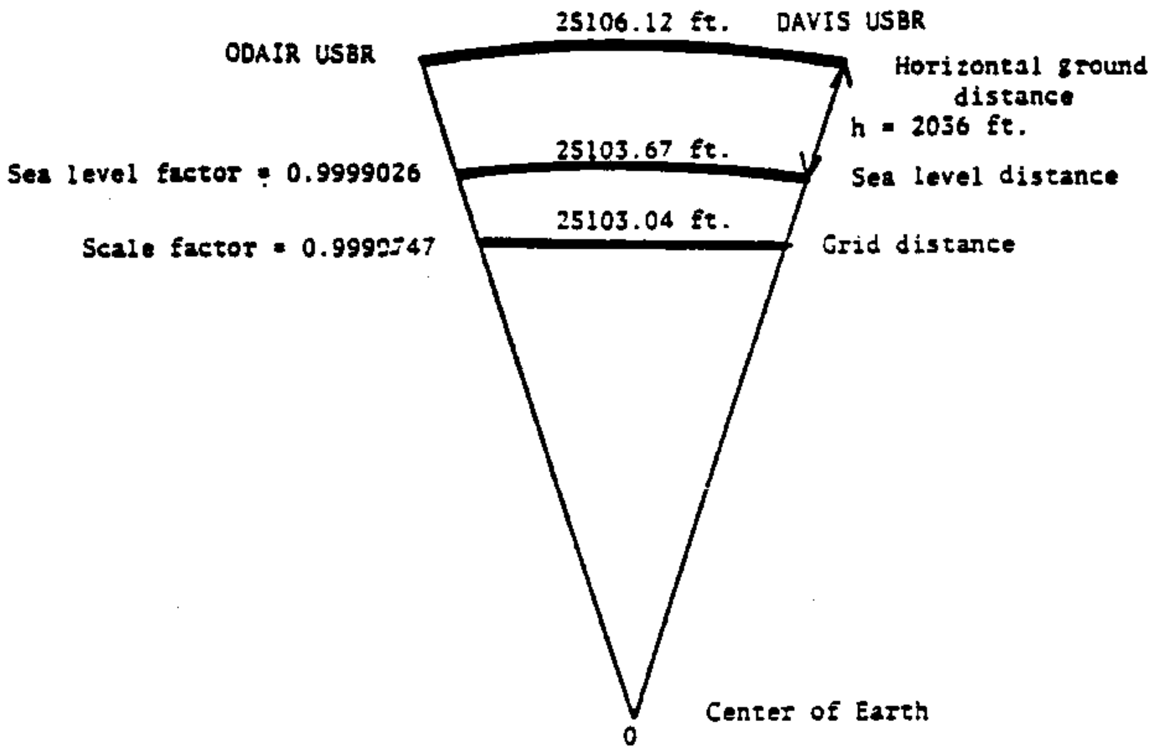


Figure 6.--Reduction of horizontal ground distance to sea level and grid- -ale factor less than unity.

Step 1 - Reduce the horizontal ground distance to sea level. Determine the sea level factor from table 1 or by formula:

0.9999761 - [(610-500)/500] x 0.0000239 = 0.9999708
Sea level distance = 39028.80 x 0.9999708 = 39027.660 ft.

Step 2 - Compute the scale factor for the mean X' using table 3:

Mean X' = 396466
Scale factor = 1.0001450 + (1466/5000) x 0.0000045 = 1.0001463

Step 3 - Determine the grid distance by multiplying the sea level distance (Step 1) by the scale factor from Step 2:

Grid distance = 39027.660 x 1.0001463 = 39033.37 ft. from observations.

Step 4 - Difference the known coordinates to obtain ΔX and ΔY and compute the grid distance from the formula $\sqrt{\Delta X^2 + \Delta Y^2}$:

GRANITE to REDWOOD $\Delta X = -38895.17$ $\Delta Y = -3271.22$
Grid distance = 39032.49 ft. from known coordinates.

Step 5 - Compare the results obtained in Steps 3 and 4. The comparison indicates the adjusted data, with respect to the distances, in the vicinity of the two stations in on the order of 1:44400 too short.

For those surveys made in this area, which are primarily distance dependent, the position closures can expect to approximate this ratio provided the same care and procedures are exercised in securing the measurements, as was taken on this observation.

As was done for Case 1, the sea level and scale factors are combined. Thus, the horizontal ground level distances can be reduced to the grid by a single multiplication.

Combined factor = 0.9999708 x 1.0001463 = 1.0001171
0.9999708 + 1.0001463 - 1.0000000 = 1.0001171

Grid distance = 39028.80 x 1.0001171 = 39033.37 ft., which checks the value computed in Step 3.

Where grid distance is known, the horizontal ground distance is computed in the same manner as for the Lambert coordinate system, i.e., 39033.37/1.0001171 = 39028.80 ft., which agrees with the given distance.

The practice of measuring the distances between known points, when this can be done, is highly recommended. By comparing the measured distances with the adjusted values one can determine the relative accuracy of the adjusted data. Many control surveyors utilize this procedure and in due course build up a file, with respect to anticipated closures between points in various areas.

Figure 7 illustrates the reduction of the horizontal ground distance between stations GRANITE and REDWOOD, as carried out in Steps 1-4.

The question often arises - how accurate need the various data (elevations, mean latitude, and mean X') from which the factors are computed be known? Rather than attempt to answer the question, with respect to the accuracy of the data, it is far easier to examine the influence on the lengths due to changes in the factors. One can then, by simple computations, determine from the maximum differences in the various data the accuracy required or whether one or several factors will be needed to assure the survey results are acceptable. Several examples will be given later.

Table 3. Scale factors for use with coordinates on transverse Mercator system.

TRANSVERSE MERCATOR PROJECTION FOR NEW YORK EAST

X' (feet)	Scale in units of 7th place of logs	Scale Factor: Scale expressed as a ratio
350,000	+463.0	1.0001067
355,000	+480.5	1.0001107
360,000	+498.2	1.0001148
365,000	+516.2	1.0001189
370,000	+534.4	1.0001231
375,000	+552.9	1.0001274
380,000	+571.6	1.0001317
385,000	+590.6	1.0001360
390,000	+609.8	1.0001405
395,000	+629.3	1.0001450
400,000	+649.0	1.0001495
405,000	+669.0	1.0001541
410,000	+689.2	1.0001587
415,000	+709.7	1.0001635
420,000	+730.4	1.0001682

Tabulated below are the effects on the lengths for a change of ±1 in the designated decimal places. These are linear effects, i.e., a change of ±3 at the same decimal place would effect the distance by three times the amount of a difference of ±1.

Decimal Place	1:Part In
Seventh	10,000,000
Sixth	1,000,000
Fifth	100,000
Fourth	10,000
Third	1,000

As can be easily seen, the average user need not be too concerned with variances in elevations, in mean latitude (for Lambert system) and mean X' for transverse Mercator coordinates unless the combined effect amounts to perhaps four in the fifth decimal place (1:25000); and in most cases a change of one in the fourth decimal (1:10000) would cause little worry This explanation is given to provide a further understanding of the use of these quantities in practical applications and not to encourage a careless approach to the use of the various factors.

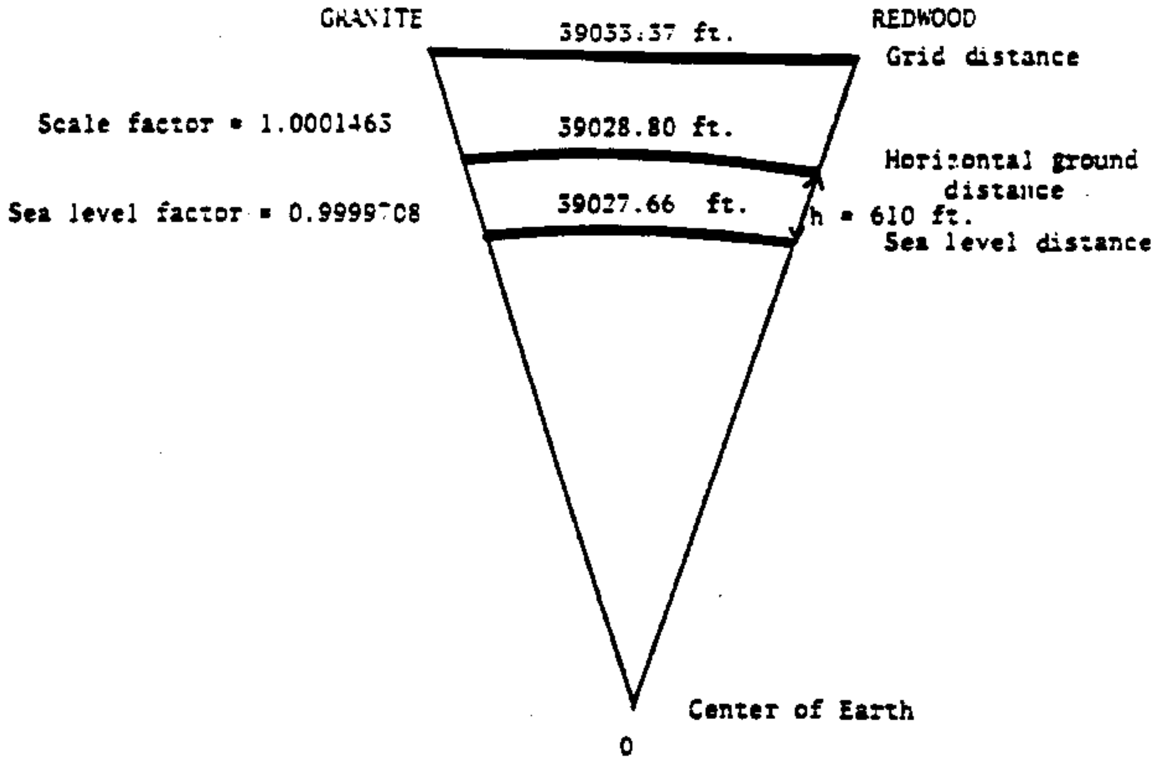


Figure 7. Reduction of horizontal ground distance to sea level and grid-scale factor greater than unity.

A few examples follow:

1. Assume that a survey is run in a north to south direction in the vicinity of the central meridians for New York east, central, or west zones at an elevation of 1000 ft., which is the average elevation for New York. The scale reduction is at the maximum less than unity on the central meridian. The scale factors on the central meridians are 0.9999667 for the east zone and 0.9999375 for the central and west zones. What is the maximum error introduced if the sea level and scale factor reductions are not applied?

(a) About 1:12500 for the east zone and 1:9100 for the central and west zones.

2. Assume the same survey is observed except the elevation is 2000 ft. What is the maximum error introduced if the sea level and scale factor reductions are not applied?

(a) About 1:7800 for the east zone and 1:6300 for the central and west zones.

3. These are the worst cases for the elevations given. A survey on the grid meridians, where the scale factor is unity and only the sea level correction is ignored, the distances would be in error by:

(a) At 1000 ft. elevation 1:20900 for all zones.

(b) At 2000 ft. elevation 1:10400 for all zones.

4. Assume the same survey as described in (1) and (2) is carried out, except that the distances are reduced from a mean elevation of 1500 ft. What are the maximum errors introduced at those points located at elevations of 1000 ft. and 2000 ft. by using a mean elevation?

(a) At points situated at elevations of 1000 ft. and 2000 ft. 1:41700 for all zones.

Although the preceding examples are all associated with differences in elevation, the same tests can be applied to variances in scale factors and the combination of the two factors.

As examples:

1. An east-west survey 70000 ft. in length on the transverse Mercator systems in New York is established between two points located 350000 ft. and 420000 ft. from the central meridian at an elevation of 1000 ft. The scale factors at 350000 ft. and 420000 ft. from the central meridian for the central and west zones are 1.0000775 and 1.0001390, respectively. What is the range of the errors introduced by omitting these reductions?

(a) At 1000 ft. elevation 1:8300 to 1:17000 for the east zone
1:11000 to 1:33700 for the central and west zones.

2. Same survey as (1), except the elevation is 2000 ft. What is the range of errors introduced by omitting these reductions?

(a) At 2000 ft. elevation 1:13800 to 1:91000 for the east zone
1:23100 to 1:55000 for the central and west zones.

3. Same survey as (1), except the distances are reduced by the combined factor derived from the mean scale factor and a mean elevation of 1500 ft. What is the errors in the lengths at the terminal points of the surveys?

(a) At 1000 ft. and 2000 ft. elevation and 350000 ft. from the central meridian 1:145000 and 1:18200 for all zones.*

(b) At 1000 ft. and 2000 ft. elevation and 420000 ft. from the central meridian 1:18300 and 1:149000 for all zones.*

* These are the proportional part errors at 1000 ft. and 2000 ft. elevation, respectively, assuming the elevations are at the points located 350000 ft. and 420000 ft. from the central meridian.

No computational details are provided for these examples. There is sufficient data given with the examples and elsewhere to verify the calculations. Tables 1 and 3 and the various computations concerned with the reduction of horizontal ground distances are the sources of information elsewhere.

PUBLISHED DATA, AZIMUTHS, BEARINGS, AND MAPPING ANGLES

Published data issued by the National Geodetic Survey include azimuths which are referenced clockwise from the south and X and Y values for the coordinates. The X and Y quantities are actually eastings and northings, respectively. Most surveyors prefer to use north as the azimuth reference and northings and eastings in that order. No harm whatsoever results if the user simply adds 180° 00' 00"0 to the published azimuths and interchanges the X's and Y's. The data are now in a familiar form.

Turning now to the use of bearings, bearings may be convenient in describing land boundaries but quite awkward to use in computations, since angles measured clockwise or counter-clockwise, as the case may be, are not added or subtracted in a uniform fashion. The argument that fewer errors result when bearings are employed is open to question. In fact, a counter argument can be made that there are probably more errors made in locating boundary corners from offset points, due to the use of bearings, than would occur if azimuths were employed. This is the author's contention; and while more and more surveyors are turning to azimuths, the great majority still prefer a system that emerged when compass surveys were the rule rather than the exception.

Mapping angles identified as "delta alpha ($\Delta\alpha$) angles" when associated with transverse Mercator coordinates and "theta (θ) angles" in the Lambert system are simply the small angles at points formed by the intersection of geodetic meridians passing through those points and corresponding grid meridians. These angles are equal to zero on the central meridian and increase in size the further the points are east and west of this meridian. East of the central meridian the angles are positive, and negative to the west of this meridian. Geodetic azimuths are equal to the grid azimuths plus the mapping angles taking into consideration the sign of this angle.

Although the computations can be made in different ways, the mapping angle in seconds at a point is actually equal to the sine of some latitude (ϕ) multiplied by the difference in seconds of longitude ($\Delta\lambda''$) between the point and the central meridian or the origin. This formula is expressed as follows:

Mapping angle = $\text{Sine}\,\phi \times \Delta\lambda''$

It may come as a surprise to some that mapping angles are inherent to all grid systems, including local ones. All surveyors are aware that the azimuths or bearings of lines are not in the same meridian as the survey is extended east or west of the origin; but few recognize the similarity between the differences on the local grid vis-a-vis the State grids.

Only the formulas are given here. Complete examples are given in the Preprint "Fundamentals of the State Plane Coordinate Systems," which is available free of charge from the National Geodetic Survey. The first formulas given in each instance are those that may be used if geographic positions, either computed or scaled from a good map, are available; and the second for use when only plane coordinates on a local or State grid are on hand.

Lambert system: (1) $\theta'' = \ell(\lambda_{cm}-\lambda_p)''$ difference $(\lambda_{cm}-\lambda_p)$ must be in seconds

(2) θ = arc tangent $(X-C)/(R_b-Y)$

Transverse Mercator system: (1) $\Delta\alpha'' = \text{Sine}\,\phi_p(\lambda_{cm}-\lambda_p)'' + g$ difference $(\lambda_{cm}-\lambda_p)$ must be in seconds

(2) $\Delta\alpha'' = M(X-C) - e$

Local system: (1) $\Delta\alpha'' = \text{Sine}\,\phi_m(\lambda_o-\lambda_p)''$ difference $(\lambda_o-\lambda_p)$ must be in seconds

(2) $\Delta\alpha'' = \text{Tan}\,\phi_p\,(\text{Departure}_{o-p})/102$ the sign of the Departure $_{o-p}$ must be taken into account and the quantity must be in feet.

NOTE: Departure $_{o-p}$ is the Departure (Easting) between the origin and the point.

ϕ = latitude λ = longitude λ_{cm} = central meridian

ϕ_p = latitude of point λ_p = longitude of point λ_o = longitude of origin

ϕ_{mn} = mean of latitude of the origin and latitude of point.

X and Y are State plane coordinates of point.

R_b, C, λ_{cm}, and ℓ are constants given in Table of Constants included with the projection tables for each State. See table 4 for example. A table containing values of "g" for latitudes 24° to 50° and Δλ" from 0" to 6000" is included with the projection tables for all States using the transverse Mercator system.

Values for "M" and "e" are given in a table included in the projection tables for those States using the transverse Mercator system except Alaska. See table 5 for example.

COMPUTATION OF A TRAVERSE

Computations for this example will be made as follows:

1. Plane azimuths from south, horizontal ground level lengths and X,Y State plane coordinates.

2. Plane bearings, grid distances and N(Y), E(X) State plane coordinates.

3. Plane azimuths from north, horizontal ground level lengths and x,y Project Datum coordinates.

Computations (2) and (3) will be shown to produce identical results when certain reductions are made. Computation (1) is for comparison purposes only. Balancing will be accomplished by a procedure identified in much of the world as the Bowditch Rule. In the United States, this method is known as the Compass Rule. It is indeed strange that in the country of his birth, Nathaniel Bowditch is not recognized for his development of this method for balancing traverses.

No detailed explanations for many of the computations are provided, since it is thought all surveyors are familiar with distributing angular closures and the Bowditch Rule (Compass Rule) for adjusting positional closures. Much of this paper has been directed to sea level and scale factors, individually and in combination; and there seems no need to repeat that information here. The Project Datum coordinate concept will be explained in sufficient depth to assure a good understanding of when the system may be beneficial and how the computations are made.

Table 4. Constants for New York

Constant	Zone			
	East	Central	West	Long Island
Central Meridian	74°20'00".000	76°35'00".000	78°35'00".000	74°00'00".000
log R	-144.8	-271.4	-271.4	
Scale reduction (Central Meridian)	1 : 30,000	1 : 16,000	1 : 16,000	
$\log\left(\frac{1}{6\rho_o^2}\right)_g$	4.580 7400 -20	4.580 7483 -20	4.580 7825 -20	
$\log\left(\frac{1}{6\rho_o^2 \sin 1''}\right)_g$	9.895 1651 -20	9.895 1734 -20	9.895 2076 -20	
$\left(\frac{1}{6\rho_o^2 \sin 1''}\right)_g$	0.7855×10^{-10}	0.7855×10^{-10}	0.7856×10^{-10}	
C				2,000,000.00 ft.
R_b				24,462,545.30 ft.
[illegible]$_o$				227,544 ft.
ℓ				0.654 08209
$\frac{1}{2\rho_o^2 \sin 1''}$				2.358×10^{-10}
$\log \frac{1}{2\rho_o^2 \sin 1''}$				0.372 4603 -10
$\log \ell$				9.815 63226 -10
log K				7.605 43607

"C" for all transverse Mercator systems except Alaska and New Jersey = 500,000.00 feet.

Table 5. M and e Values

TRANSVERSE MERCATOR PROJECTION

New York

$$\Delta\alpha'' = Mx' - e$$

y	East zone M	East zone ΔM	Central and west zones M	Central and west zones ΔM
0	0.008 2595	806	0.008 2598	805
100,000	0.008 3401	812	0.008 3403	813
200,000	0.008 4213	819	0.008 4216	819
300,000	0.008 5032	826	0.008 5035	825
400,000	0.008 5858	832	0.008 5860	832
500,000	0.008 6690	839	0.008 6692	840
600,000	0.008 7529	847	0.008 7532	847
700,000	0.008 8376	854	0.008 8379	854
800,000	0.008 9230	861	0.008 9233	861
900,000	0.009 0091	869	0.009 0094	869
1,000,000	0.009 0960	876	0.009 0963	876
1,100,000	0.009 1836	884	0.009 1839	884
1,200,000	0.009 2720	892	0.009 2723	892
1,300,000	0.009 3612	901	0.009 3615	901
1,400,000	0.009 4513	908	0.009 4516	908
1,500,000	0.009 5421	917	0.009 5424	917
1,600,000	0.009 6338	926	0.009 6341	926
1,700,000	0.009 7264	934	0.009 7267	934
1,800,000	0.009 8198	943	0.009 8201	943
1,900,000	0.009 9141		0.009 9144	

e

y \ x'	100,000	200,000	300,000	400,000
0	0.0	0.1	0.3	0.6
1,000,000	0.0	0.1	0.4	0.8
2,000,000	0.0	0.1	0.4	1.0

The coordinates are assumed to be New York east zone and the mean elevation of the survey is 1231 feet above sea level.

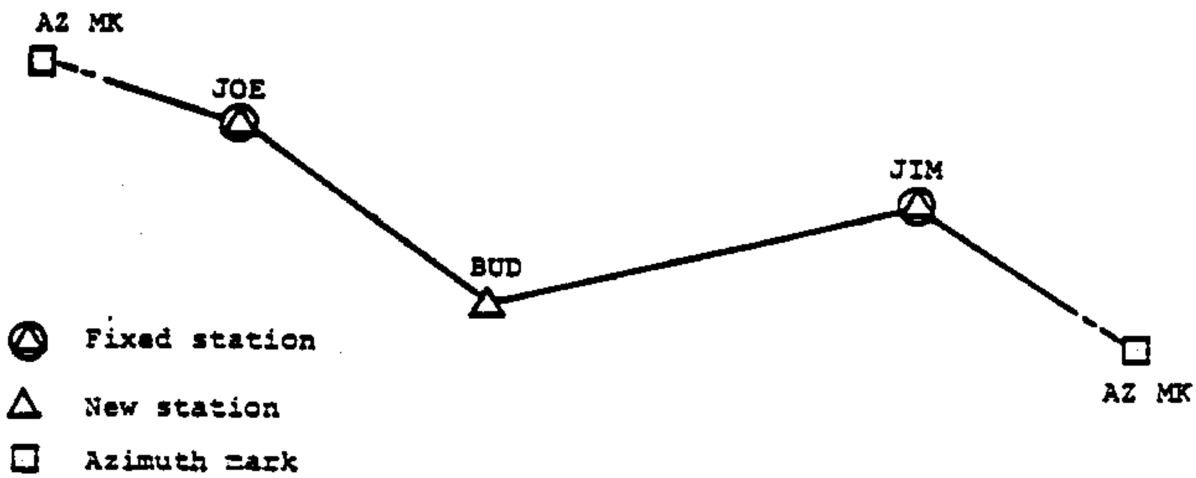


Figure 8. Sketch of sample traverse.

Observed Angles			
JOE			
AZ MK to BUD	201°	23'	45"
BUD			
JOE to JIM	129	42	35
JIM			
BUD to AZ MK	232	07	58

From	To	Distances in Feet Horizontal	Grid
JOE	BUD	5175.62	5175.91
BUD	JIM	7393.74	7394.15

Sea level factor = 0.9999411
Scale factor = 1.0001144
Combined factor = 1.0000555

Mn X' = 359463

Fixed Coordinates		Plane Az to Az Mk From South	Plane Az to Az Mk From North	Plane Bearing
JOE	X = 134895.36 Y = 476923.10	107° 16' 34"	287° 16' 34"	N 72° 43' 26" W
JIM	X = 146179.22 Y = 475177.82	311 30 40	131 30 40	S 48 29 20 E

Computation and Adjustment of Plane Azimuths

From	To	Azimuth	Corr.	Adj. Az	From North	Bearing
JOE	AZ MK	107° 16' 34"		107° 16' 34"	287° 16' 34"	N 72° 43' 26" W
∠		201 23 45				
JOE	BUD	308 40 19	- 4	308 40 15	128 40 15	S 51 19 45 E
BUD	JOE	128 40 19				
∠		129 42 35				
BUD	JIM	258 22 54	- 8	258 22 46	78 22 46	N 78 22 46 E
JIM	BUD	78 22 54				
∠		232 07 58				
JIM	AZ MK	311 30 52	-12	311 30 40	131 30 40	S 49 29 20 E
FIXED		311 30 40				
CLOSURE		- 12				

12/3 = -4" per angle

Computation and Adjustment of Coordinates

Computation No. 1

Station / Plane Azimuth*	Hor. Dist.	Depart.	Lat.	X	Y
JOE				134895.36	476923.10
		+4040.86			
308° 40' 15"	5175.62			138936.22	473689.14
			-3233.96	+0.33	-0.26
BUD				138936.55	473688.88
		+7242.19			
258° 22' 46"	7393.74			146178.41	475178.46
			+1489.32	+0.81	0.64
JIM				146179.22	475177.82
	Sum 12569.36				

* Azimuths from south

$$\text{Closure} = \sqrt{(0.81)^2 + (0.64)^2} = 1.03 \text{ ft.} = 1:12200$$

X factor = +0.81/12.56936 = +0.06444 per 1000 ft.
Y factor = -0.64/12.56936 = -0.05092 per 1000 ft.

Adjusted Departures, Latitudes, Horizontal Distances and Azimuths

From	To	Depart.	Lat.	Hor. Dist.	Azimuth
JOE	BUD	+4041.19	-3234.22	5176.04	308° 40' 15"
BUD	JIM	+7242.67	+1488.94	7394.13	258 22 59

Many may believe that the balancing process will take care of errors introduced by not applying the elevation-scale reduction to the horizontal ground level distances. As a general rule, this will not be true; although there are certainly occasions where there are only small differences between the adjusted coordinates. This seems to be the case here, since the adjusted coordinates determined in Computation No. 1 and No. 2 for station BUD differ by only 0.03 ft. in X and 0.14 ft. in Y. However, the horizontal ground level distances derived by dividing the adjusted grid distances from Computation No. 2 by the combined factor differ by considerably larger amounts from those determined in Computation No. 1. A comparison of results follow Computation No. 3.

<u>Computation and Adjustment of Coordinates</u>

Computation No. 2

Station Plane Bearing	Grid Dist.	Lat.	Depart.	N(Y)	E(X)
JOE				476923.10	134895.36
		-3234.14			
S 51° 19' 45" E	5175.91			473688.96	138936.44
			+4041.08	-0.22	+0.08
BUD				473688.74	138936.52
		+1489.40			
N 78° 22' 46" E	7394.15			475178.36	146179.03
			+7242.59	-0.54	+0.19
JIM				475177.82	146179.22
	Sum 12570.06				

Closure = $\sqrt{(0.54)^2 + (0.19)^2}$ = 0.57 ft. = 1:22100
N(Y) factors = -0.54/12.57006 = -0.04296 per 1000 ft.
E(X) factors = +0.19/12.57006 = +0.01512 per 1000 ft.

<u>Adjusted Latitudes, Departures, Grid Distances, and Bearings</u>

From	To	Lat.	Depart.	Grid Dist.	Bearing
JOE	BUD	-3234.36	+4041.16	5176.10	S 51° 19' 40" E
BUD	JIM	+1489.08	+7242.70	7394.19	N 78 22 55 E

<u>Project Datum Coordinates</u> - Among the reasons offered by some surveyors for not employing the State grids is the requirement that the final adjusted distances be horizontal ground level values. It is true that, except for the Michigan grid, all other State systems are referenced to sea level. Furthermore, every State system involves the use of scale factors. Earlier, it was shown that under some conditions the sea level and scale reductions could be omitted; but for higher accuracy surveys, this was not recommended. Also several examples were given of the reduction of grid distances to horizontal ground level lengths. All these examples involved State plane coordinates. However, for limited areas with moderate terrain relief, State plane coordinates can be converted to Project Datum plane

coordinates. Simply stated, the State coordinates are treated in such a fashion that the reductions to sea level and for scale are, within the limitations of the conversion process, effectively removed. The distances determined by differencing Project Datum coordinates are, for all intents and purposes, horizontal ground level quantities at the mean elevation of the survey site or whatever elevation reference was adopted. Since the combined elevation-scale factor enters into the conversion, tests should be made to assure that the results at the extremities of the survey fall within acceptable limits. Azimuths (bearings) computed from Project Datum coordinates are the same as those calculated from State plane coordinates.

Project Datum coordinates may be computed by two methods. One, compute and balance the survey using regular procedures such as Computation No. 2 and then divide each coordinate by the combined elevation-scale factor to obtain Project Datum values. Two, divide the coordinates for the fixed control points and use horizontal ground level lengths, as is done in Computation No. 3. In any case, the Project Datum coordinates should be made unique so that they would not be confused with State plane coordinates. In addition, each document showing these coordinates should carry an explanation including the combined factor and whatever constants are required to convert the values to the respective State grid. In the case of Computation No. 3, the fixed coordinates for JOE and JIM were divided by the combined factor 1.0000555 and 100,000.00 and 400,000.00 were then subtracted from the Project Datum coordinates, respectively, to obtain x and y. The azimuths or bearings are the same as used in other calculations. Horizontal ground level distances replace grid distances. The balancing of the traverse is carried out, as was done previously. A comparison of the results of the three computations is given later.

Computation and Adjustment of Project Datum Coordinates

Computation No. 3

Station Plane Azimuth*	Hor. Dist.	Depart.	Lat.	x	y
JOE				34887.87	76896.63
		+4040.86			
128° 40' 15"	5175.62			38928.73	73662.67
			-3233.96	+0.08	-0.22
BUD				38928.81	73662.45
		+7242.19			
78° 22' 46"	7393.74			46170.92	75151.99
			+1489.32	+0.19	-0.54
JIM				46171.11	75151.45
	Sum 12569.36				

* Azimuths from north Closure = $\sqrt{(0.19)^2 + (0.54)^2} = 0.57$ ft.
= 1:22,100

x factor = +0.19/12.56936 = +0.01512 per 1000 ft.
y factor = −0.54/12.56936 = −0.04296 per 1000 ft.

Adjusted Departures, Latitudes, Horizontal Distances and Azimuths

From	To	Depart.	Lat.	Hor. Dist.	Azimuth
JOE	BUD	+4040.94	−3234.18	5175.82	128° 40' 20"
BUD	JIM	+7242.30	+1489.00	7393.78	79 22 55

COMPARISON OF RESULTS

Computations

Coordinates	1-2	3-2*
BUD	ΔX = −0.14 ft.	ΔX = 0
	ΔY = +0.03 ft.	ΔY = 0
Distances		
JOE to BUD	+0.23 ft.	+0.01 ft.
BUD to JIM	+0.35 ft.	0
Azimuths (Bearings)		
JOE to BUD	+5"	0
BUD to JIM	+4"	0

Note: Computation No. 2 was used as the base for comparison purposes.

* The adjusted coordinates and distances from Computation No. 3 were multiplied by the combined factor 1.0000555 to obtain the comparison with similar quantities in Adjustment No. 2. Prior to converting the x,y coordinates from Adjustment No. 3, the constants 100,000.00 and 400,000.00 were added to these values, respectively.

SUMMARY

The main thrust of this paper was to describe graphically and in the simplest possible terms the reduction of horizontal ground level distances to sea level (elevation) and for scale when these distances are used in computations involving the State grids. Hopefully, this intent was successful. Discussions related to mapping angles and computations were presented merely to provide a broad overall picture of the concept of plane coordinates and their easy utilization. Information concerning small corrections to the observed angles, identified as second-term or (t-T) corrections was deliberately omitted since this is essentially a computational matter and as such outside the primary purpose for preparing this paper. It should be noted that these corrections are not difficult to understand or compute.

REFERENCES

Coast and Geodetic Survey, : Plane Coordinate Projection Tables, Special Publication series (available for all States, except Alaska. Computations for Alaska are made using the 2 1/2 minute intersection tables, USC&GS Publication 65-1, Part 49 for zone 1, Part 50 for zones 2-9, and Part 51 for zone 10). U. S. Department of Commerce, National Ocean Survey, NOAA, Rockville, Maryland 20852.

Dracup, J. F., 1973/1974: Fundamentals of the State Plane Coordinate Systems (preprint). Available from the National Geodetic Survey, NOS, NOAA, Rockville, Maryland 20852.

Dracup, J. F., and Kelley, C. F., 1973, reprinted 1975: Horizontal Control as Applied to Local Surveying Needs. American Congress on Surveying and Mapping, Falls Church, Virginia 22046.

Dracup, J. F., Kelley, C. F., Lesley, G. B., and Tomlinson, R. W., 1973: Surveying Instrumentation and Coordinate Computation Workshop Lecture Notes. American Congress on Surveying and Mapping, Falls Church, Virginia 22046.

Mitchell, H. C., and Simmons, L. G., 1945, revised 1974, reprinted 1975: The State Plane Coordinate Systems (A Manual for Surveyors), Special Publication No. 235. U. S. Department of Commerce, National Technical Information Service, Springfield, Va. 22151. (Order COM-72-50181). Price on request.

Reynolds, W. F., 1932, revised 1936: Relation Between Plane Rectangular Coordinates and Geographic Positions, Special Publication No. 71, U. S. Department of Commerce, Coast and Geodetic Survey. Out of print. Superseded, in a limited way, by a leaflet carrying the same title, published in 1954. U. S. Department of Commerce, National Ocean Survey, NOAA, Rockville, Maryland 20852.