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An Algorithm to Compute the Eigenvectors of a Symmetric Matrix

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(Continued at end of publication)

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U.S. DEPARTMENT OF COMMERCE
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AN ALGORITHM TO COMPUTE THE EIGENVECTORS OF A SYMMETRIC MATRIX

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ABSTRACT. A method is outlined and illustrated by an example to compute iteratively both the eigenvalues and the corresponding matrix of eigenvectors simultaneously. The program is applicable to symmetric matrices with real eigenvalues and, in particular, to the positive definite matrices of least-squares theory. A high degree of precision, limited only by the capacity of the computer, is attainable with relatively few iterations that approach the result exponentially.

In a previous paper (Schmid 1971) a method was proposed for computing, by iteration, the orthogonal matrix of unit eigenvectors, the so-called modal matrix, of a symmetric matrix with real coefficients (entries) together with the corresponding set of eigenvalues. The present paper reports the effort to implement the theory with computational results, as justification for claims made for the efficiency and accuracy of the method. During this investigation a number of effective refinements in the theory were found for the case of symmetric matrices. These I have not been able to generalize sufficiently to apply effectively to nonsymmetric matrices. The following will therefore be restricted to symmetric matrices, which are of paramount importance in the theory of least squares.

This restriction makes it feasible to simplify somewhat the notation used previously. For this reason and for making this presentation self-sufficient, relevant sections of the previous paper will be repeated.

Essentially, the theory is based on a theorem, the validity of which depends on the convergence of the Left-Right Transformations (LRT) of Rutishauser (1958) and their identity with the convergents developed here.

Given is a symmetric matrix N $(=N^T)$ with real coefficients that, for the present, we will assume positive definite such as the matrix of the normal equations in least-squares theory. A product of the type TNT^{-1} is known as a similarity transformation of N and has the same eigenvalues, or characteristic roots, as N. It has been proved that for such N there exists an orthogonal matrix X, i.e., one for which $X^T = X^{-1}$, such that

$$XNX^{-1} = XNX^T = D,$$

where D is a diagonal matrix whose elements are the eigenvalues of the matrix N. The orthogonal matrix X (or its transpose X^T) is designated the modal matrix and its rows (columns) are the unit eigenvectors associated with the corresponding eigenvalues in D. The theorem, cited above, on which the proposed algorithm is based now reads:

Theorem A: The matrix $C_n^{-1} N^n$ where $N^{2n} = C_n C_n^T$, converges to the modal matrix X as n increases.

The product of $C_n C_n^T$ is the familiar Cholesky factorization of the symmetric matrix N^{2n} into the lower triangular matrix (LTM) C_n times its transpose C_n^T, the corresponding upper triangular matrix (UTM). The inversion of the triangular matrix C_n needed to form the product $C_n^{-1} N^n = X_n$ is a relatively simple and straightforward process that is at the root of all matrix inversions based on the Gauss algorithm. The corresponding convergent N_n to the diagonal matrix D is then

$$N_n = X_n N X_n^T, \qquad (1)$$

and the magnitude of the off-diagonal terms of this convergent (1) is the criterion for the sufficiency of the iteration.

To show the orthogonality of $C_n^{-1} N^n$, we form the product

$$X_n X_n^T = (C_n^{-1} N^n)(C_n^{-1} N^n)^T = C_n^{-1} N^{2n} (C_n^{-1})^T = C_n^{-1} C_n C_n^T (C_n^T)^{-1} = I,$$

substituting $C_n C_n^T$ for N^{2n} and making use of the symmetry of N^n. The transpose X_n^T is therefore X_n^{-1} of X_n, and hence $X_n = C_n^{-1} N^n$ is orthogonal.

A superior method of computing X_n follows from the following considerations. Interchanging the factors of

$$N^{2n} = C_n C_n^T \qquad (2)$$

forms the product $C_n^T C_n$, the LRT of $C_n C_n^T$. Factoring this product by the Cholesky algorithm we obtain

$$C_n^T C_n = K_n K_n^T,$$

so that

$$N^{4n} = N^{2n} N^{2n} = C_n C_n^T C_n C_n^T = C_n K_n K_n^T C_n^T$$

or

$$N^{4n} = (C_n K_n)(C_n K_n)^T. \qquad (4)$$

Since C_n and K_n are by definition LTM's, their product is an LTM and (4) is consequently the Cholesky factorization of N^{4n}, which is unique. From the definition (2), $N^{4n} = C_{2n} C^T_{2n}$ and therefore,

$$C_{2n} = C_n K_n. \quad (5)$$

The corresponding convergent to the modal matrix is

$$X_{2n} = C^{-1}_{2n} N^{2n} = C^{-1}_{2n} C_n C^T_n = K^{-1}_n C^{-1}_n C_n C^T_n \text{ or}$$

$$X_{2n} = K^{-1}_n C^T_n. \quad (6)$$

A simple iterative procedure for computing C_n, and hence K_n and X_{2n}, that is remarkably free of error accumulation is suggested by (5) and (6).

The Cholesky factorization of the given matrix N in our notation is $N = C_{1/2} C^T_{1/2}$. An LRT and a second factorization gives $C^T_{1/2} C_{1/2} = K_{1/2} K^T_{1/2}$. The multiplication $C_{1/2} K_{1/2} = C_1$ completes the first iterative cycle. Factoring the product $C^T_1 C_1$ into $K_1 K^T_1$ yields $C_1 K_1 = C_2$ etc., the index n of C_n being doubled at each step. Thus after $i+1$ iterations we obtain C_n and K_n with $n = 2^i$.

This procedure is superior to raising N to the $2n^{th}$ power by successive exponentiation as implied in the original paper. A well-known property of the LRT is that it tends to rearrange the rows and columns of the transformed matrix such that the diagonal terms, and eventually the eigenvalues, are in progressive decreasing order. One consequence of this is that a considerably higher indexed C_n can be obtained by the iterative approach of the previous paragraph than by straightforward Cholesky factorization of N^{2n}. In fact, with a properly scaled matrix, the only limitation to this iterative process appears to be the limitation of the electronic computer in floating point mode with respect to the size of the exponents of 10. The exponents of the elements of the C_n and K_n matrices increase numerically with increasing n and eventually bring the computation to a halt.

Further improvement in precision results from the use of X_{2n} from (6) rather than from $X_n = C^{-1}_n N^n$ specified in theorem A. Not only is the index of the convergent resulting from the use of the matrix (6) twice as large but, by the nature of its formation, this formula produces an orthogonal matrix almost precisely, as the numerical example below will indicate. For a similarity transformation with an orthogonal matrix this fact is of basic importance.

An electronic computer operating in floating point mode has an upper and lower limit to the magnitude of exponents it can handle. The small desk computer used in the computation of the numerical example, for instance, has a range of numbers between 10^{-100} and 10^{100} in floating point with 14 significant digits. In order to keep the elements of the matrix N^{2n} within these limits for maximum n, it is expedient to scale N so that its determinant det N = 1. For an m×m matrix N, this can be accomplished by dividing each element of the given N by the mth root of the determinantal value, which is determined in the first operation of the program, the Cholesky factorization of N into $C_{1/2} C^T_{1/2}$. The determinant of each of the factors $C_{1/2}$ and $C^T_{1/2}$ is the product of the diagonal terms of either of these two triangular matrices, and the determinant of N is the square of this product. Let this divisor of the elements of N be k and the resulting matrix be N*, so that det N* = 1. Then

$$1/k\, XNX^T = XN^*X^T = 1/k\, D,$$

the matrix on the right being the diagonal matrix D with each element divided by k. The eigenvalues of N* are thus 1/k times those of N and the eigenvectors are unaffected.

Another useful device in the application of this algorithm is to increase or decrease each eigenvalue of N by a scalar k simply by adding or subtracting k to each diagonal term of N. This transformation also does not affect the eigenvectors, as is seen from the identity

$$X(N+kI)X^T = XNX^T + XkIX^T = D + kI,$$

since $XX^T = I$ and the scalar k is permutable in matrix multiplication. If the given N is not positive definite, i.e., if it has one or more negative or zero eigenvalues, the ordinary Cholesky factorization with real numbers will fail because some reduced diagonal term becomes negative or zero. By adding a sufficiently large constant to each diagonal term of the given matrix N, a solution for the modal matrix and eigenvalues is obtained which differs from the solution for the given matrix only in that all eigenvalues are positive and too large by this constant. This transformation is also effective in the case of a matrix in which the two smallest eigenvalues are very nearly alike. The convergence for such matrices is notoriously slow, a fact which is known from the theory of LRT's, although exactly equal eigenvalues create no problem whatever in the algorithm. By subtracting from all the diagonal terms of the given matrix a quantity equal to the leading common digits of these nearly equal eigenvalues, a matrix is produced that converges normally. The digits in question are obtained, of course, from the preliminary computation that has failed to converge sufficiently.

Another application of this device is for a matrix, all of whose eigenvalues are known to exceed a large number, say 1,000. In such cases it is advisable to subtract 1,000 from all diagonal terms of the given matrix and thus gain 3 additional significant digits in the result.

However, the number subtracted must not be too large, i.e., too close to the smallest eigenvalue, because in that case the resulting matrix approaches singularity and the algorithm becomes unstable. In fact, if the matrix as given is of this nature, it may be necessary to add a constant to the diagonal terms to obtain a sufficiently precise result. One indication of this situation, i.e., that the ratio between the largest and smallest eigenvalue is excessive for the computer in question, is that the number of iterations the computer can carry out without overflowing becomes small.

Before going into further theoretical questions, we show some results of a numerical computation which was carried out on a programmable electronic desk calculator. In order to have a running check on the results, the symmetric matrix N to be tested was constructed from the formula $N = X^T D X$, where X is the arbitrary orthogonal 4×4 matrix

$$\begin{pmatrix} 4.39515906683864\text{–}01 & 5.96805461178517\text{–}01 & -6.56490076001253\text{–}01 & 1.40245821146132\text{–}01 \\ -8.24429075665667\text{–}01 & 5.54017559026016\text{–}01 & -6.82528566377838\text{–}02 & -9.33958820785540\text{–}02 \\ 3.18702046279928\text{–}01 & 3.98245306050589\text{–}01 & 4.14387835519257\text{–}01 & -7.53732315845578\text{–}01 \\ 1.59910826807145\text{–}01 & 4.22242182903664\text{–}01 & 6.26613239226594\text{–}01 & 6.35213282930980\text{–}01 \end{pmatrix}$$

and D is the diagonal matrix D(4, 9, 16, 25). The resulting symmetric input matrix N is then

$$\begin{pmatrix} 9.15426935587807\ 00 & 6.57261234300050\text{–}01 & 3.97039027466662\ 00 & -3.64473866463880\text{–}01 \\ 6.57261234300050\text{–}01 & 1.11819268468828\ 01 & 7.34750309941285\ 00 & 1.77173163233321\ 00 \\ 3.97039027466662\ 00 & 7.34750309941285\ 00 & 1.43294267817752\ 01 & 4.64251675641154\ 00 \\ -3.64473866463880\text{–}01 & 1.77173163233321\ 00 & 4.64251675641154\ 00 & 1.93343770126634\ 01 \end{pmatrix}$$

The first step in the program is the Cholesky factorization of N into $C_{\frac{1}{2}} C^T_{\frac{1}{2}}$. The product of the diagonal terms of these two triangular matrices gives the values of det N. From the construction we know in this particular instance that det N is $4 \times 9 \times 16 \times 25 = 14{,}400$, i.e., the product of the diagonal entries of D, the imposed eigenvalues of N. Dividing each element of N by $14400^{1/4}$, we get the matrix N^* whose determinant = 1, and which now replaces N in our computations. This divisor is stored and eventually used as a multiplier for the eigenvalues of N^* to give the corresponding values of N.

The computer repeats this cycle eight times:

1) multiplying C^T_n by C_n and factoring the product $C^T_n C_n$ into $K_n K^T_n$

2) multiplying C_n by K_n to obtain with (5) the quantity C_{2n} for step (1) of the next iteration, until the entries of the matrix $C^T_n C_n$ get too large for the computer to handle.

Having found C_n and K_n for maximum n, all that remains to be done is to invert the LTM K_n and form $K^{-1}_n C^T_N = X_{2n}$ in accordance with (6) and the convergent $N_{2n} = X_{2n} N X^T_{2n}$. The number of iterations (eight) appears to be typical for this particular program and computer and permits the computation of $N_{128} = X_{128} N X^T_{128}$, the 128th convergent to the diagonal matrix of eigenvalues D.

The results to 12 decimals for X_{128} are

$$\begin{pmatrix} 0.159910826807 & 0.422242182904 & 0.626613239266 & 0.635213282940 \\ -0.318702046279 & -0.398245306051 & -0.414387835192 & 0.753732315846 \\ 0.824429075667 & -0.554017559262 & 0.068252856637 & 0.093395882078 \\ 0.439515906836 & 0.596805461790 & -0.656490760016 & 0.140245821461 \end{pmatrix}$$

and for N_{128}

$$\begin{pmatrix} \underline{24.999999999990} & 0.000000000005 & -0.00000000024 & -0.000000000013 \\ & \underline{15.999999999996} & 0.000000000002 & -0.000000000002 \\ & & \underline{9.000000000030} & -0.000000000051 \\ & & & \underline{4.000000000028} \end{pmatrix}$$

To produce this output requires only one inversion of a triangular matrix and only one modal matrix and one convergent, i.e., the final ones, X_{2n} and N_{2n}, need to be computed.

Comparison of the output X_{128} with the X used initially to form N shows that the rows are in reverse order, a consequence of the LRT's characteristic rearrangement of eigenvalues in order of descending magnitudes. With the use of permutation matrices it can be shown that a permutation of the rows of X and a similar permutation of the rows and columns of the D matrix produce an orthogonal matrix $\overline{X}$ and diagonal $\overline{D}$ respectively, such that the resulting $\bar{N} = \bar{X}^T \bar{D} \bar{X}$ is identical with $N = X^T D X$. It should also be noted that some of the eigenvectors have changed sign. This, too, is a consequence of the rearrangement of the eigenvalues.

The orthogonality of the X_{2n} matrix computed with (6) can be verified from the product $X X^T = I$. Unlike the algorithm suggested in Schmid (1971), it needs no correction in general, the product I being diagonal to within practically the last decimal on the computer. This is also apparent from the terms of the convergent N_{128}, demonstrating not only the orthogonality of X_{128} but also its validity as a convergent to the modal matrix.

A significant improvement in the convergence and precision of the result above can be obtained by permuting the rows and columns of the input symmetric N so as to put its diagonal terms in descending order of magnitude from top to bottom. It will then be necessary to permute the columns of the output matrix X correspondingly in order to associate the individual eigenvectors with their proper eigenvalues.

Returning now to the theoretical aspects, we continue the series of LRT's beginning with $C_n C_n^T$, $C_n^T C_n$, $K_n K_n^T$ . . . , as follows

$$\begin{aligned} N^{2n} &= & &= C_n C_n^T \\ N_n^{2n} &= C_n^T C_n & &= K_n K_n^T \\ N_{2n}^{2n} &= K_n^T K_n & &= P_n P_n^T \\ N_{3n}^{2n} &= P_n^T P_n & &= Q_n Q_n^T \\ N_{4n}^{2n} &= Q_n^T Q_n & &= R_n R_n^T \\ N_{5n}^{2n} &= R_n^T R_n & &= S_n S_n^T \\ N_{6n}^{2n} &= S_n^T S_n \text{ etc.} \end{aligned} \tag{7}$$

Since it has been postulated that $C_n^T C_n$ is factorable into $K_n K_n^T$, the LRT $K_n^T K_n$ will also be factorable, producing in sequence $P_n^T P_n$, $Q_n^T Q_n$ etc. All of these products will be factorable because the LRT does not increase the order of magnitude of the coefficients or terms of this sequence of matrices and because, in accordance with the Rutishauser theory, the sequence becomes increasingly well-conditioned. It remains to be shown that, as indicated by the notation in the first column of tabulation in (7), this sequence of LRT's is actually identical with the sequence of convergents of the 2nth powers N^{2n} of the given matrix N evaluated at intervals n, 2n, 3n . . . The proof is by induction.

According to the theorem A, N^{2n} factored into $C_n C_n^T$ yields the orthogonal matrix $X_n = C_n^{-1} N^n$ and its transpose $X_n^T = X_n^{-1} = N^{-n} C_n$ such that

$$\begin{aligned} N_n &= X_n N X_n^T \\ &= C_n^{-1} N^n N N^{-n} C_n \\ &= C_n^{-1} N C_n \end{aligned} \tag{8}$$

and

$$N_n^{2n} = C_n^{-1} N^{2n} C_n \tag{8a}$$

by actual multiplication of (8) by itself 2n times. From (8a) follows

$$N_n^{2n} = C_n^{-1} N^{2n} C_n = C_n^{-1} C_n C_n^T C_n = C_n^T C_n,$$

as indicated in the second line of (7). Similarly for the next line in (7), after factoring $C_n^T C_n$ into $K_n K_n^T$ application of the theorem A yields an orthogonal matrix $K_n^{-1} N_n^n$, which advances the convergent N_n by means of the transformation

$$\begin{aligned} K_n^{-1} N_n^n N_n N_n^{-n} K_n &= K_n^{-1} N_n K_n \\ &= K_n^{-1} C_n^{-1} N C_n K_n \text{ using (8)} \\ &= C_{2n}^{-1} N C_{2n} \qquad \text{from (5).} \end{aligned}$$

This convergent is, therefore, equal to N_{2n}, according to (8) and hence

$$\begin{aligned} (N_{2n})^{2n} = N_{2n}^{2n} &= C_{2n}^{-1} N^{2n} C_{2n} \\ &= K_n^{-1} C_n^{-1} C_n C_n^T C_n K_n = K_n^{-1} K_n K_n^T K_n \\ &= K_n^T K_n, \end{aligned}$$

as indicated in the third line of (7). Induction in the subsequent lines requires the relations

$$\begin{aligned} C_{3n} &= C_n K_n P_n \\ C_{4n} &= C_n K_n P_n Q_n \\ &\text{etc.,} \end{aligned}$$

which can be readily established from N^{6n}, N^{8n} . . . in analogy to the derivation of (4) and (5) from N^{4n}.

If, therefore, $N^{2n} = C_n C_n^T$ is computable, N_n, N_{2n}, N_{3n} . . . are computable, as are X_n, X_{2n}, X_{3n} . . . , using the LRT's N_n^{2n}, N_{2n}^{2n} N^{2n} . . . Theoretically there is no limit to the degree of approximation attainable, and in practice any degree of accuracy desired can be obtained by testing the orthogonality of the last X computed and correcting it if necessary to make it rigorously orthogonal. A method of correction is shown in detail in Schmid (1973).

Setting $n = 1/2$ in (7) produces the sequence of convergents N, $N_{1/2}$, N_1 . . . , which in our notation is the original series of Rutishauser LR transforms for the case of a symmetric N and Cholesky factorization. This identity proves theorem A and, by comparison, illustrates some advantages of the proposed method.

1. Not only the eigenvalues but the corresponding eigenvectors are produced.

2. Corresponding to i cycles of this method, 2^i LRT's would be required to obtain the same convergent to the diagonal matrix of eigenvalues.

3. Suprisingly enough, although the Rutishauser LRT is relatively free of error accumulation, comparison with the numerical results from the above method, even though

the latter implicitly involves matrices with the eigenvalues raised to the $2n^{th}$ power, shows even less error accumulation. Furthermore, any error accumulation present can be eliminated by making the matrix X_n rigorously orthogonal, a device which is not available in the Rutishauser approach.

The basic iteration cycle used in the numerical example comes to an end with X_{2n} computed from (6) when $C^T_{2n} C_{2n}$ can no longer be factored. Inspection of the off-diagonal terms of the matrix $X_{2n} N X^T_{2n} = N_{2n}$ will show whether N has been sufficiently diagonalized.

If further convergents are needed, the most obvious procedure that presents itself is to treat this convergent N_{2n} in a manner analogous to the initially given N and find the $2n^{th}$ convergent of N_{2n}. It is however, unnecessary to repeat the entire cycle, since the tabulation (7) shows that N^{2n}_{2n}, the first intermediate result to be computed, is equal to $K^T_n K_n$, both of which factors are available from the last cycle. From this product compute

$$K^T_n K_n = P_n P^T_n$$

and

$$P^T_n P_n = Q_n Q^T_n .$$

According to theorem A, the orthogonal matrix $Y_n = P^{-1}_n N^n_{2n}$ used as a similarity transformation on N_{2n} produces the higher convergent

$$Y_n N_{2n} Y^T_n = P^{-1}_n N^n_{2n} N_{2n} N^{-n}_{2n} P_n = P^{-1}_n N_{2n} P_n$$

and, therefore,

$$\begin{aligned} Y_n N^{2n}_{2n} Y^T_n = P^{-1}_n N^{2n}_{2n} P_n &= P^{-1}_n K^T_n K_n P_n, \\ &= P^{-1}_n P_n P^T_n P_n \\ &= P^T_n P_n \\ &= N^{2n}_{3n} \end{aligned}$$

from (7).

It is, however, more efficient at this point to compute by analogy with (6) the orthogonal matrix

$$Y_{2n} = Q^{-1}_n P^T_n , \qquad (9)$$

which will advance N_{2n} to N_{4n} by means of the relation

$$N_{4n} = Y_{2n} N_{2n} Y^T_{2n}$$

or, since $N_{2n} = X_{2n} N X^T_{2n}$,

$$N_{4n} = Y_{2n} X_{2n} N X^T_{2n} Y^T_{2n} .$$

Hence

$$N^{2n}_{4n} = (Y_{2n} X_{2n})\ N^{2n} (Y_{2n} X_{2n})^T . \qquad (10)$$

Also

$$N^{2n}_{4n} = X_{4n} N^{2n} X^T_{4n} \text{ by definition} \qquad (11)$$

and

$$N^{2n}_{4n} = Q^T_n Q_n \text{ from (7).} \qquad (12)$$

It is apparent from (10) and (11) that the new convergent to the X matrix is

$$X_{4n} = Y_{2n} X_{2n}, \qquad (13)$$

with Y_{2n} computed from (9). This completes the first iteration of the optional new cycle and additional iterations of this type can be repeated as often as desired. The foregoing considerations show that this type of iteration cycle is based solely on the application of a few basic principles used previously and repetitions are based on a continuation of the tabulation (7).

The next iteration replaces (9) with

$$Y \text{ new} = S^{-1}_n R^T_n$$

where $\left.\begin{aligned} Q^T_n Q_n &= R_n R^T_n \\ R^T_n R_n &= S_n S^T_n \end{aligned}\right\}$ are taken from tabulation (7).

The new (13) is then

$$X \text{ new} = Y \text{ new } X \text{ old}$$

etc.

Each iteration of this type increases the subscript 2_{mn} of the previous convergent N_{2mn} by 2_n, which is an increase no longer exponential as in the first series of iterations but, in view of the magnitude of n, still a considerable linear rate. In this supplemental cycle, the Y matrix of (9) and (13) approaches the unit matrix I in the limit, which can serve as a test for the convergence.

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(Continued from inside front cover)

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NOS NGS-6 Determination of North American Datum 1983 coordinates of map corners. T. Vincenty, October 1976, 8 p. (PB262442). Predictions of changes in coordinates of map corners are detailed.

NOS NGS-7 Recent elevation change in Southern California. S.R. Holdahl, February 1977, 19 p. (PB265940). Velocities of elevation change have been determined from Southern Calif. leveling data for 1906-62 and 1959-76 epochs.

NOS NGS-8 Establishment of calibration base lines. Joseph F. Dracup, Charles J. Fronczek, and Raymond W. Tomlinson, August 1977, 22 p. (PB277130). Specifications are given for establishing calibration base lines.

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NOS NGS-9 National Geodetic Survey Publications on surveying and geodesy 1976. September 1977, 17 p. (PB275181). This compilation lists publications authored by NGS staff in 1976, sources of availability of out-of-print Coast and Geodetic Survey publications, and information on subscriptions to the Geodetic Control Data Automatic Mailing List.

NOS NGS-10 Use of calibration base lines. Charles J. Fronczek, December 1977, 38 p. (PB279574). A detailed explanation is given for evaluating electronic distance measuring instruments.

NOS NGS-11 Applicability of Array Algebra. Richard A. Snay, February 1978, 22 p. (PB281196). Conditions required for the transformation from matrix equations into computationally more efficient array equations are considered.

NOS NGS-12 The TRAV-10 horizontal network adjustment program. Charles R. Schwarz, April 1978, 52 p. The design, objectives, and specifications of the horizontal control adjustment program are presented.

NOS NGS-13 Application of three-dimensional geodesy to adjustments of horizontal networks. T. Vincenty and B. R. Bowring, June, 1978, 7 p. A method is given for adjusting measurements in three-dimensional space without reducing them to any computational surface.

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NOS 65 NGS 1 The statistics of residuals and the detection of outliers. Allen J. Pope, May 1976, 133 p. (PB258428). A criterion for rejection of bad geodetic data is derived on the basis of residuals from a simultaneous least-squares adjustment; subroutine TAURE is included.

NOS 66 NGS 2 Effect of Geoceiver observations upon the classical triangulation network. R. E. Moose, and S. W. Henriksen, June 1976, 65 p. (PB260921). The use of Geoceiver observations is investigated as a means of improving triangulation network adjustment results.

NOS 67 NGS 3 Algorithms for computing the geopotential using a simple-layer density model. Foster Morrison, March 1977, 41 p. (PB266967). Several algorithms are developed for computing the gravitational attraction with high accuracy of a simple-density layer at arbitrary altitudes. Computer program is included.

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NOS 68 NGS 4 Test results of first-order class III leveling. Charles T. Whalen and Emery Balazs, November 1976, 30 p. (PB265-421). Specifications for releveling the National vertical control net were tested and the results published.

NOS 70 NGS 5 Selenocentric geodetic reference system. Frederick J. Doyle, Atef A. Elassal, and James R. Lucas, February 1977, 53 p. (PB266046). Reference system was established by simultaneous adjustment of 1,244 metric-camera photographs of the lunar surface from which 2,662 terrain points were positioned.

NOS 71 NGS 6 Application of digital filtering to satellite geodesy. C. C. Goad, May 1977, 73 p. (PB270192). Variations in the orbit of GEOS-3 were analyzed for M_2 tidal harmonic coefficient values which perturb the orbits of artificial satellites and the Moon.

NOS 72 NGS 7 Systems for the determination of polar motion. Soren W. Henriksen, May 1977, 55 p. Methods for determining polar motion are described and their advantages and disadvantages compared.

NOS 73 NGS 8 Control leveling. Charles T. Whalen, May 1978, 23 p. This publication describes the history of the National network of geodetic control from its origin in 1878 until today and presents the latest observational and computational procedures.

NOS 74 NGS 9 Survey of the McDonald Observatory radial line scheme by relative lateration techniques. William E. Carter and T. Vincenty, June 1978, 33 p. This report contains the results of experimental application of the "ratio method" of electromagnetic distance measurements for high resolution crustal deformation studies in the vicinity of the McDonald Lunar Laser Ranging and Harvard Radio Astronomy Stations.

NOAA--S/T 78-277