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Perturbation theory of massive Yang-Mills fields

by

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Abstract : Perturbation theory of massive Yang-Mills fields is investigated with the help of the Bell-Treiman transformation. Diagrams containing one closed loop are shown to be convergent if there are more than four external vector boson lines. The investigation presented does not exclude the possibility that the theory is renormalizable.

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1. Introduction

The structure of weak interactions discovered so far suggests very strongly the existence of charged, and possibly also neutral vector mesons. First of all, there is the current × current form of leptonic and semi-leptonic weak interactions, and possibly also the non-leptonic weak interactions; secondly there is the structure of the hadron currents, very similar to the structure of the electromagnetic currents in the sense that they may be thought of as to be constructed with the help of some gauge principle. More precisely, Gell-Mann[1] suggested current commutation rules for vector and axial-vector currents that can be understood as a simple extension of the commutation rules known for e.m. currents; as is well known these commutation rules have led to a large number of successes, in particular the so-called low-energy theorems.

These very same low energy theorems have been derived also by means of some gauge principle[2]. One is led then to divergence equations for the currents of weak interactions of the form

$$\partial_\mu \vec{J}_\mu = -g \, \vec{W}_\mu \times \vec{J}_\mu \, , \tag{1}$$

a very natural extension of the equation

$$\partial_\mu \vec{J}_\mu = -e \, \vec{A}_\mu \times \vec{J}_\mu \tag{2}$$

obtained if one introduces e.m. corrections in a "minimal" fashion to the equation $\partial_\mu \vec{J}_\mu = 0$ [3]. Again, in a natural way a vector boson enters into the theory, but again merely as a matter of technique, since the results do not depend on the mass of the boson in question.

One is very much tempted to ask: why does nature choose currents in such a way that equation (1) holds ? What is so special about these currents ? The outstanding fact about these currents is that they are related to some gauge invariance in the strong interaction Lagrangian, and in fact to lowest order in the weak and e.m. coupling constants they are the currents that by virtue of that invariance are conserved. And the structure of the right hand side of (1) merely reflects the particular symmetry involved (note that $\vec{W}_\mu \times \vec{J}_\mu \equiv \epsilon_{abc} W^b_\mu J^c_\mu$, where the ϵ_{abc} are the structure constants of SU(2)).

It turns out that, by introducing a trilinear and quadratilinear interaction between the vector-bosons involving those same structure constants one can arrange things in such a way that the source current of the W-field is exactly divergence free. If the W-mass is zero, one obtains just the Yang-Mills theory, which was constructed on the basis of considerations of gauge invariance[4)].

One is thus led to the study of massive Yang-Mills fields, which is the subject of the present paper. Here we will not deal with complications due to symmetry breaking resulting in the occurrence of extra terms in the right hand side of (1); in our opinion the situation without symmetry breaking has to be understood before one can attack the more general problem. Furthermore we will direct our attention to the properties of the perturbation expansion, in particular the question of renormalization.

In this direction very beautiful work has been done, for the mass-less Yang-Mills theory, by Feynman, De Witt, Fadeev and Popov, and Mandelstam[5)]. These authors have shown that for this case a unitary, renormalizable perturbation expansion of the S-matrix exists with rather peculiar Feynman rules. Here one must add that the zero mass theory contains horrible infrared divergencies, which in fact prohibits the study of an S-matrix with in- and outgoing particles of zero four-momentum squared. Nevertheless, the basis of the following discussion is the belief that this zero-mass theory can be obtained in the limit of zero mass from the non-zero mass S-matrix, with in- and outgoing particles on the mass-shell.

The motivation for our investigation is essentially the following remark. The propagator for a massive vector-boson is of the form

$$\frac{\delta_{\mu\nu} + k_\mu k_\nu / M^2}{k^2 + M^2 - i\epsilon} .$$

If the limit $M \to 0$ exists, then somehow the factors M^{-2} must be cancelled by other factors M^2 arising from the integration over closed loops and application of the relation $p^2 = -M^2$ for in- and outgoing bosons. But because of dimensional reasons any factor M decreases by one the possible degree of divergence of a particular diagram, or sets of diagrams.

In the following we will with the help of the Bell-Treiman transformation* bring the theory into a form which permits the direct comparison of the massive and mass-less case. In particular, we will be able to split off a set of diagrams whose limit for $M \to 0$ (M = boson mass) can easily be seen to correspond to the set of Feynman diagrams as given by Feynman et al., having also the same infrared divergencies. Moreover, our theory is unitary and causal[6] for any nonzero M and for the lowest order terms in the coupling constant the limit $M \to 0$ exists and corresponds to the zero mass case. Since the requirements of unitarity and causality essentially determine the higher order S-matrix elements, and since the zero-mass S-matrix is unitary and causal it appears very plausible that also the higher order non zero mass S-matrix elements go over into the higher order zero mass S-matrix elements. It must be emphasized that this is plausible, but not a must; in the sense of Bogoliubov[7] the so-called counter terms are just such that they can be added to the S-matrix without spoiling unitarity and causality. Or, stated differently, the non-zero mass Lagrangian may be changed by introducing counter terms proportional to the W-mass without affecting the limit $M \to 0$. However, it is very difficult to construct

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*The author is indebted to Profs. Bell and Treiman for discussions on this point.

such counter terms in a perturbation expansion simply because of dimensional reasons. For instance, consider a counter term introduced in order to make the process $3W \to 3W$ finite in 6^{th} order. Such a term must have the form:

$$g^6 \, C \, (\vec{W}_\mu \vec{W}_\mu)^3$$

where g is the dimensionless coupling constant, and C must have the dimensions of $(\text{mass})^{-2}$. Suppose we have a cut-off mass Λ in the theory, everything being finite for finite Λ. There are two parameters on which C can depend, namely M, the boson mass, and Λ. We are not interested in constants C that go to zero as $\Lambda \to \infty$; the only serious terms are those proportional to some power of Λ, i.e.

$$C = c \frac{\Lambda^m}{M^{m+2}} , \qquad m > 0 .$$

However, if we know that in the limit $M \to 0$ no such counter term exists then obviously c must be zero. Here we want to stress that we do not consider this argument a proof, but rather a plausibility argument. Mainly we may learn from these arguments that one should investigate the dependence on M rather than what happens for large momenta; moreover by formulating the theory in a manner that exhibits clearly the desired properties and the known infrared divergencies in the limit $M \to 0$ this investigation might be facilitated.

Our technique is roughly as follows. The W-field propagator contains a term $k_\mu k_\nu / M^2$. As is well known this term is modified if one performs a gauge transformation of the second kind. It is also well known that the mass-term breaks the local gauge invariance of the Lagrangian; nevertheless we perform a transformation whereby then the Lagrangian changes (essentially a power series in g/M is added) while the W-propagator takes the form

$$\frac{\delta_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 + M^2 - i\epsilon} \, .$$

The Feynman rules stemming from the new Lagrangian are then used to investigate the perturbation expansion of the S-matrix. We emphasize that the on-mass shell S-matrix is not affected by these manipulations, described below.

2. Equations of motion[8]

In the following we will limit ourselves to a world without strange particles, and with exact conservation of isospin. Suppose there exists a triplet of vector boson fields, $W_\mu^a(x)$, $a = 1,2,3$, coupled to a triplet of hadrons $J_\mu^a(x)$ such that

$$\partial_\mu J_\mu^a = -g \, \epsilon_{abc} \, W_\nu^b \, J_\nu^c \, . \tag{3}$$

The equations of motion for the W-field will contain the hadron currents. By introducing interactions of the W_μ^a with themselves the W-source current can be made to be divergence free, as we will show by writing down equations of motion with the desired properties:

$$\partial_\nu(\partial_\nu W_\mu^a - \partial_\mu W_\nu^a) - M^2 W_\mu^a = j_\mu^a \tag{4}$$

$$j_\mu^a = -g \, \epsilon_{abc} \, \{\partial_\nu \, (W_\nu^b \, W_\mu^c) + W_\nu^b \, (\partial_\nu W_\mu^c - \partial_\mu W_\nu^c) - $$

$$-g \, \epsilon_{bef} \, W_\mu^e \, W_\nu^f \, W_\nu^c \} - g \, J_\mu^a \, . \tag{5}$$

Everywhere we suppressed the dependence on space-time, writing for instance W_μ^a instead of $W_\mu^a(x)$.

Taking now the divergence of (5), using (3), (4) and (5) one easily establishes

$$\partial_\mu j_\mu^a = 0 \tag{6}$$

Since for our purposes the occurrence of hadrons and hadron currents J^a_μ is a trivial complication we will drop them from now on.

The equations of motion (4),(5) may be derived from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - \frac{1}{2} M^2 W^a_\mu W^a_\mu \tag{7}$$

with

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\, \epsilon_{ade} W^d_\mu W^e_\nu . \tag{8}$$

In detail the Lagrangian is:

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu) - \frac{1}{2} M^2 W^a_\mu W^a_\mu - \frac{1}{2} g\, \epsilon_{abc} (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu) W^b_\mu W^c_\nu$$
$$- \frac{1}{4} g^2 \epsilon_{abc} \epsilon_{ade} W^b_\mu W^c_\nu W^d_\mu W^e_\nu . \tag{9}$$

Note that g is a dimensionless coupling constant. For $M = 0$ this is just the theory introduced by Yang and Mills. We write:

$$\mathcal{L} = \mathcal{L}_{YM}(W^a_\mu) - \frac{1}{2} M^2 W^a_\mu W^a_\mu . \tag{10}$$

The Feynman rules corresponding to this Lagrangian involve a vector meson propagator of the form

$$\frac{\delta_{\mu\nu} + k_\mu k_\nu / M^2}{h^2 + M^2 - i\epsilon} . \tag{11}$$

Furthermore there is a vertex with three bosons and a vertex with four bosons. Simple power counting of the diagrams indicates that an infinite number of subtraction terms has to be added to $\mathcal{L}$ in order to make the S-matrix finite. This would not be so if the $k_\mu k_\nu / M^2$ term in the W-propagator were not present. The fact now that the divergence of W-source current is zero implies that probably a good many of the $k_\mu k_\nu$ terms may be dropped, or at least behave effectively much less than quadratic in the limit of large momenta. In order to investigate this point we perform the Bell-Treiman transformation which leads to a new set of Feynman rules where the $k_\mu k_\nu / M^2$ term in the W-propagator has been replaced by $-k_\mu k_\nu / k^2$ at the possible expense of having to introduce new vertices. The Bell-Treiman transformation may be described in various ways ; we will give two identical prescriptions.

i Consider the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi^a)(\partial_\mu \varphi^a) - \frac{1}{2} M'^2 \varphi^a \varphi^a + \mathcal{L}_{YM}(W^a_\mu + \frac{\lambda}{M} \partial_\mu \varphi^a) - \frac{1}{2} M^2 W^a_\mu W^a_\mu$$
$$+ R(\lambda\varphi, W) \ . \tag{12}$$

It differs from (10) by the addition of a scalar triplet of fields φ^a. The replacement W^a_μ by $W^a_\mu + \frac{\lambda}{M} \partial_\mu \varphi^a$ implicates that φ interacts with the W and the φ-fields themselves. The unspecified extra term $R(\lambda\varphi, W)$ must be chosen in such a way that φ^a satisfies the free field equation of motion

$$(\Box - M'^2)\, \varphi^a = 0 \ . \tag{13}$$

Of course, there is always the trivial and uninteresting solution where R is such that all terms containing φ in the rest of the interaction Lagrangian are explicitly cancelled; but if the Lagrangian obeys some symmetry or partial symmetry one can often find, by using the equation of motion of the W (and eventual other fields if present), another non-trivial R. In this case after some work one convinces oneself that there exists an R of the form

$$R(\lambda\varphi, W) = M\lambda\, (W^a_\mu + \frac{\lambda}{M} \partial_\mu \varphi^a)\, \partial_\mu \varphi^b\, f_1^{ab} \left(\frac{g\lambda}{M}\varphi\right) + f_2 \left(\frac{g\lambda}{M}\varphi, \frac{g\lambda}{M} \partial_\mu \varphi\right) \tag{14}$$

where f_1 and f_2 are power series in $g\lambda / M$. Of interest is the first term of f_1 :

$$f_1^{ab} = \frac{g\lambda}{2M} \epsilon_{abc}\, \varphi^c + O\left(\frac{g^2\lambda^2}{M^2}\right) . \tag{15}$$

It may be verified that with the inclusion of this term alone the φ-field satisfies (13) up to terms of order g^2. Here one must use the W-equation of motion, in order to obtain some expression for $\partial_\mu W^a_\mu$.

By this method it is very difficult to obtain R completely; the equations determining R are in this case rather complicated. Any-way, if now φ^a satisfies (13) we can be sure that the Lagrangian (12) gives rise to the same S-matrix for vector-boson processes as the original Lagrangian (10).

ii . Alternatively to determine R in closed form one may proceed as follows. The YM part of the Lagrangian (10) is invariant under the infinitesimal transformation

$$W^a_\mu(x) \rightarrow W^a_\mu(x) + \epsilon_{abc} W^b_\mu(x) \epsilon^c(x) - \frac{1}{g} \partial_\mu \epsilon^a(x)$$

where $\epsilon^a(x)$ is an arbitrary triplet of infinitesimal functions of x. One derives to first order in $\epsilon(x)$:

$$G^a_{\mu\nu}(x) \rightarrow G^a_{\mu\nu}(x) + \epsilon_{abc} G^b_{\mu\nu}(x) \epsilon^c(x) .$$

This is an infinitesimal rotation in I-spin space; because of the anti-symmetry of ϵ_{abc} the product $G^a_{\mu\nu} G^a_{\mu\nu}$ is invariant up to first order in $\epsilon^c(x)$.

Here we are not interested in infinitesimal gauge transformations, but in finite transformations obtained by integration of the above one. Yang and Mills[4] give as result for a finite transformation:

$$W^a_\mu(x) \rightarrow f_{ab}(x) W^b_\mu(x) - \frac{1}{2g} \epsilon_{abc} (\partial_\mu f(x))_{cd} (f^{-1}(x))_{db} \tag{16}$$

where $f_{ab}(x)$ is a rotation in I-spin space depending on x. The general form of the 3×3 matrix f is:

$$f(x) = e^{\rho^a \psi^a(x)} \tag{17}$$

where ρ^a is a triplet of 3×3 matrices:

$$(\rho^a)_{bc} = \epsilon_{abc} \tag{18}$$

and $\psi^a(x)$ is a triplet of arbitrary functions of x. Indeed, inserting (16) into the formula (8) for $G^a_{\mu\nu}$, one finds, using $f^{-1}_{ab} = f_{ba}$ and $\partial_\mu f^{-1} = -f^{-1}(\partial_\mu f)f$:

$$G^a_{\mu\nu} \to f_{ab}\, G^b_{\mu\nu}$$

and $G^a_{\mu\nu}\, G^a_{\mu\nu}$ is invariant. However,

$$\begin{aligned}
-\frac{1}{2} M^2 W^a_\mu W^a_\mu \to & -\frac{1}{2} M^2 W^a_\mu W^a_\mu + \frac{M^2}{2g} \epsilon_{abc}\, f_{ad}\, W^d_\mu\, (\partial_\mu f)_{ce}\, f^{-1}_{eb} \\
& - \frac{M^2}{8g^2} \epsilon_{abc}\, \epsilon_{ade}\, (\partial_\mu f)_{cf}\, f^{-1}_{fb} (\partial_\mu f)_{eg}\, f^{-1}_{gd} = \\
= & -\frac{1}{2} M^2 W^a_\mu W^a_\mu + \frac{M^2}{2g} \epsilon_{def}\, W^d_\mu\, f^{-1}_{fc}\, (\partial_\mu f)_{ce} \\
& - \frac{M^2}{8g^2} (\epsilon_{afh}\, f^{-1}_{hc}\, \partial_\mu f_{cf})(\epsilon_{agi}\, f^{-1}_{ie}\, \partial_\mu f_{eg}) .
\end{aligned} \tag{19}$$

Suppose now we find an f such that

$$\epsilon_{deh}\, f^{-1}_{hc}\, \partial_\mu\, f_{ce} = \frac{2\lambda g}{M}\, \partial_\mu \varphi^d + \frac{\lambda^2 g^2}{M^2}\, R^d_\mu . \tag{20}$$

R^d shall be some power series in g. There is a large class of f such that this holds, but we have not been able to exploit this freedom to our advantage. With such an f we have:

$$\begin{aligned}
-\frac{1}{2} M^2 W^a_\mu W^a_\mu \to & -\frac{1}{2} M^2 \left(W^a_\mu - \frac{\lambda}{M} \partial_\mu \varphi^a\right)\left(W^a_\mu - \frac{\lambda}{M} \partial_\mu \varphi^a\right) \\
& + \frac{\lambda^2 g}{2} W^d_\mu R^d_\mu - \frac{\lambda^3 g}{2M} \partial_\mu \varphi^d R^d_\mu - \frac{\lambda^4 g^2}{8 M^2} R^d_\mu R^d_\mu .
\end{aligned} \tag{21}$$

Note that this is different to zeroth order in g ! To restore at least the zeroth order part of R to its original form we replace everywhere W^a_μ by $W^a_\mu + \frac{\lambda}{M}\partial_\mu \varphi^a$. With this we obtain finally:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi^a)(\partial_\mu \varphi^a) - \frac{1}{2} M'^2 \varphi^a \varphi^a + \mathcal{L}_{YM}(W^a_\mu + \frac{\lambda}{M}\partial_\mu \varphi^a) - \frac{1}{2} M^2 W^a_\mu W^a_\mu$$
$$+ \frac{\lambda^2 g}{2}(W^d_\mu + \frac{\lambda}{M}\partial_\mu \varphi^d) R^d_\mu - \frac{\lambda^3 g}{2M}\partial_\mu \varphi^d R^d_\mu - \frac{\lambda^4 g^2}{8M^2} R^d_\mu R^d_\mu \ . \tag{22}$$

One verifies that to zeroth order in g the W-part of this $\mathcal{L}$ is the same as in (10), implying the propagator (11) for the W-field. Choosing $\lambda = 1$, an indefinite metric and mass $M'^2 = 0$ for the φ-field one has however the propagator

$$\frac{\delta_{\mu\nu} + k_\mu k_\nu / M^2}{k^2 + M^2 - i\epsilon} - \frac{k_\mu k_\nu}{M^2(k^2 - i\epsilon)} = \frac{\delta_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 + M^2 - i\epsilon} \tag{23}$$

for the combination $\Omega_\mu = W_\mu + \frac{1}{M}\partial_\mu \varphi$ that enters in the interaction.

It is worthwhile to note again that φ obeys the free field equation (13). This we know for sure, because with the replacement $W_\mu \to W_\mu - \frac{1}{M}\partial_\mu \varphi$ followed by a gauge transformation we can eliminate the φ-field from the interaction.

Let us now evaluate R^d to lowest order. In all generality we take

$$f(x) = e^{\rho^a \psi^a} \tag{24}$$

where ψ is some as yet unspecified function of the dimensionless quantity φ/M. One has:

$$f_{ab} = \delta_{ab} + \frac{\sin \psi}{\psi}\epsilon_{abc}\psi^c + \frac{1 - \cos \psi}{\psi^2}(\psi^a \psi^b - \psi^2 \delta_{ab}) \tag{25}$$

with $\psi = \sqrt{\psi^d \psi^d}$. Further

$$(\partial_\mu f^{-1})_{hc}\, f_{ce} = \epsilon_{ahe} \left\{ \frac{1-\cos\psi}{\psi^2}\, \epsilon_{abc}\, \psi^c - \frac{\psi - \sin\psi}{\psi^3} (\psi^a \psi^b - \psi^2\, \delta_{ab}) - \delta_{ab} \right\} \partial_\mu \psi^b \tag{26}$$

and

$$\epsilon_{deh}\, f^{-1}_{hc}\, \partial_\mu f_{ce} = 2 \left\{ \frac{1-\cos\psi}{\psi^2}\, \epsilon_{dbc}\, \psi^c - \frac{\psi - \sin\psi}{\psi^3} (\psi^d \psi^b - \psi^2\, \delta_{db}) - \delta_{db} \right\} \partial_\mu \psi^b . \tag{27}$$

Series expansion gives

$$\begin{aligned} \epsilon_{deh}\, f^{-1}_{hc}\, \partial_\mu f_{ce} = {} & 2\, \partial_\mu \psi^b \Big\{ \left(\tfrac{1}{2} - \tfrac{1}{4!}\psi^2 + \tfrac{1}{6!}\psi^4 \ldots\right) \epsilon_{dbc}\, \psi^c \\ & - \left(\tfrac{1}{3!} - \tfrac{1}{5!}\psi^2 + \tfrac{1}{7!}\psi^4 \ldots\right)(\psi^d \psi^b - \psi^2\, \delta_{db}) - \delta_{db} \Big\} . \end{aligned} \tag{28}$$

Choosing $\psi^b = -\frac{\lambda g}{M}\, \varphi^b$ one obtains the desired result (20), with

$$\frac{\lambda g}{M} R^d_\mu = -2\, \partial_\mu \varphi^b \left\{ \frac{1-\cos\psi}{\psi^2}\, \epsilon_{dbc}\, \psi^c - \frac{\psi - \sin\psi}{\psi^3} (\psi^d \psi^b - \psi^2\, \delta_{db}) \right\} . \tag{29}$$

To lowest order:

$$R^d_\mu = \epsilon_{dbc}\, \partial_\mu \varphi^b \varphi^c \tag{30}$$

and the lowest order extra term in $\mathcal{L}$ is:

$$\frac{1}{2} \lambda^2\, g\, \epsilon_{dbc}\, \Omega^d_\mu\, \partial_\mu \varphi^b \varphi^c \tag{31}$$

as given before.

3. The Feynman rules

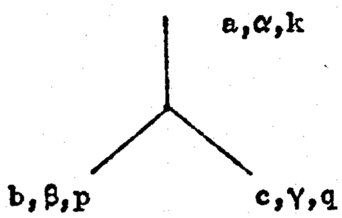
Our starting point is the Lagrangian (22). From this $\mathcal{L}$ we infer the following Feynman rules:

Propagators:

$$\Omega_\mu^a \equiv W_\mu^a + \frac{1}{M}\partial_\mu \varphi^a \qquad \text{———} \qquad \delta_{ab}\,\frac{\delta_{\mu\nu} - k_\mu k_\nu/k^2}{k^2 + M^2 - i\epsilon} \tag{32}$$

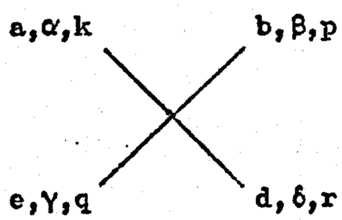
$$\varphi^a \qquad \text{- - - - - - -} \qquad -\delta_{ab}\,\frac{1}{k^2 - i\epsilon} \tag{33}$$

Vertices:



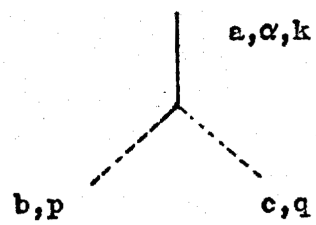
$$-g\,\epsilon_{abc}\,\{\delta_{\alpha\gamma}(k-q)_\beta + \delta_{\beta\gamma}(q-p)_\alpha + \delta_{\alpha\beta}(p-k)_\gamma\} \tag{34}$$

$p + k + q = 0$



$$-g^2\,[\epsilon_{gdc}\,\epsilon_{gba}\,\{2\,\delta_{\alpha\gamma}\,\delta_{\beta\delta} - \delta_{\alpha\delta}\,\delta_{\beta\gamma} - \delta_{\alpha\beta}\,\delta_{\gamma\delta}\} + \epsilon_{gdb}\,\epsilon_{gca}\,\{2\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} - \delta_{\alpha\delta}\,\delta_{\gamma\beta} - \delta_{\alpha\gamma}\,\delta_{\beta\delta}\}] \tag{35}$$

$p + k + q + r = 0$

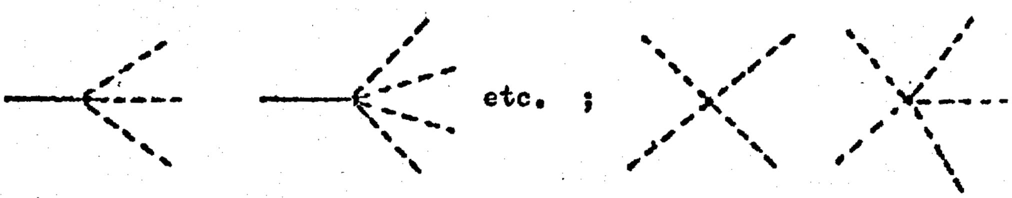


$$\frac{1}{2}\,g\,\epsilon_{abc}\,(p-q)_\alpha \tag{36}$$

$p + k + q = 0$

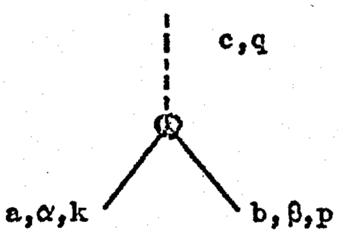
Further vertices involving one Ω and 3 or more φ- lines, and vertices involving four or more φ- lines. These vertices have factors $g\left(\frac{g}{M}\right)^n$

where $n + 2$ is the number of φ-lines:



$$(37)$$

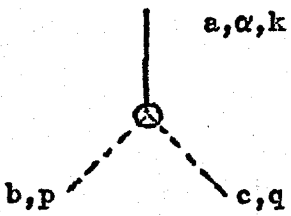
Finally there are vertices resulting from a contraction of a φ-line with the $\partial_\mu \varphi$ part in Ω_μ in the vertices (34) and (35). This type of vertices will be denoted by a circle, and they are obtained from the vertices (34), (35) by multiplication with the appropriate four momentum



$$-\frac{g}{M}\epsilon_{abc}\left[(p^2 - k^2)\,\delta_{\alpha\beta} - p_\alpha p_\beta + k_\alpha k_\beta\right] . \qquad (38)$$

$k + p + q = 0$

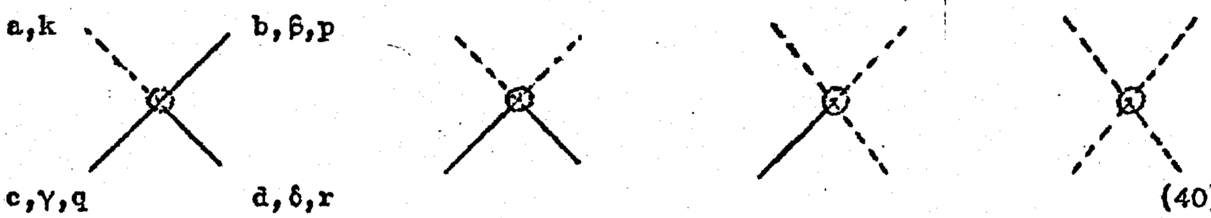
In view of the form of the Ω propagator (32) one may in fact drop the terms containing k_α or p_α.



$$-\frac{g}{M^2}\epsilon_{abc}\left[-\frac{1}{2}(qk - pk)\,k_\alpha - \frac{1}{2}k^2(p - q)_\alpha\right] . \qquad (39)$$

$k + p + q = 0$

Here the k_α term may be dropped. Further there are vertices



$$(40)$$

The first one is obtained from (35) by multiplication with k_α. Similarly for the others. Care must be taken that symmetry or antisymmetry properties under exchange of two lines are as indicated in the interaction Lagrangian.

Four important remarks must be made:

i No φ- line connects two circled vertices;

ii The rules given by Feynman et al. are: (32)-(36) with $M \to 0$ and a factor -1 for every closed loop of an even nr, of φ- propagators;

iii If one counts every occurrence of a factor M^{-1} as a (four-momentum)$^{-1}$ one may convince oneself that the theory is renormalizable. Every primitive diagram containing five or more external Ω- lines is convergent. Conversely, if we can show that for a certain set of diagrams the limit $M \to 0$ exists then those diagrams behave for large momentum as in a renormalizable theory;

iv In the zero-mass Yang-Mills theory one has a number of identities connecting W-wave function renormalization, 3-vertex and 4-vertex renormalization. Essentially one counter term, of the form

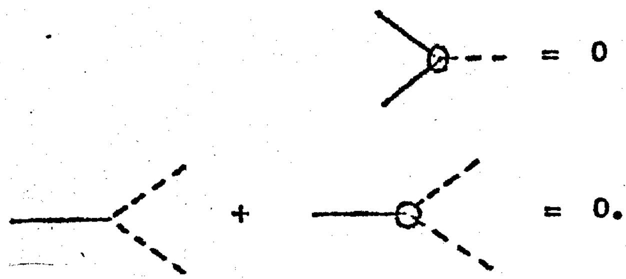
$$Z\, G^a_{\mu\nu}\, G^a_{\mu\nu}$$

should make the S-matrix finite (apart from infrared troubles).

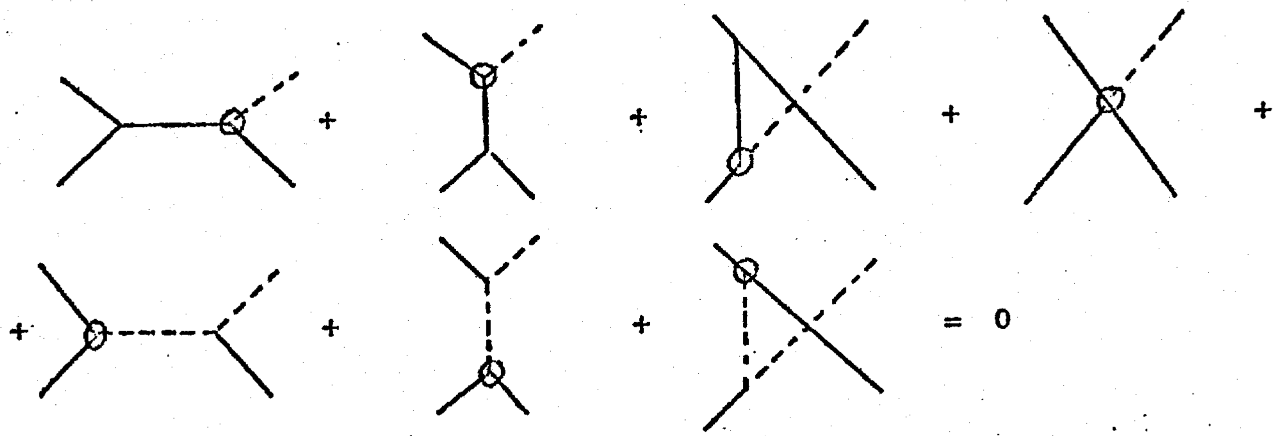
4. Some special cases

The φ- particle is a free particle, and any S-matrix element containing one or more outgoing φ- particles (on or off the mass shell) is zero. It must be stressed that this holds only provided the in- and outgoing W are on the mass shell (and their polarization vectors e_μ satisfy $k_\mu e_\mu = 0$). This fact may be used to establish a large amount of relations between diagrams.

In order g :

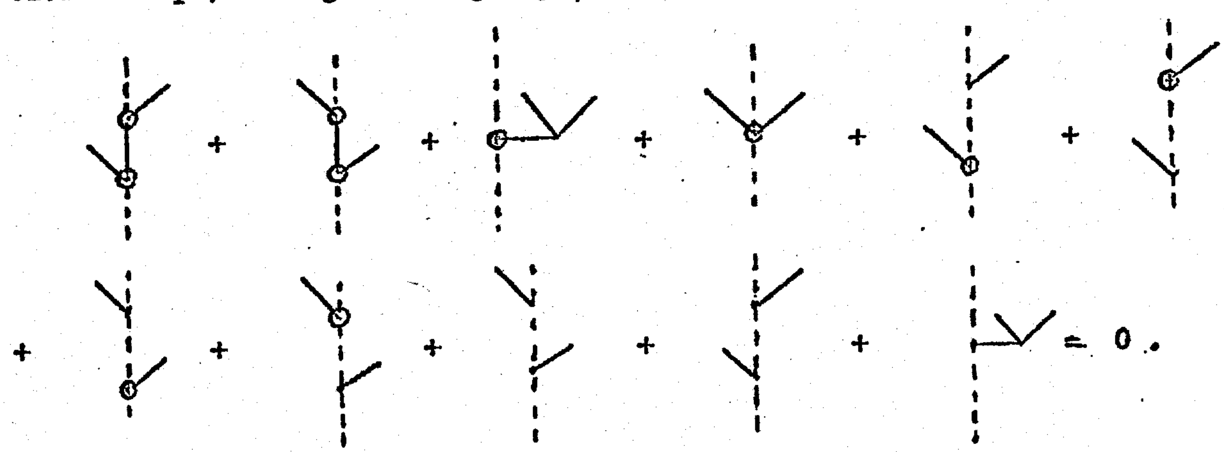


In order g^2 :



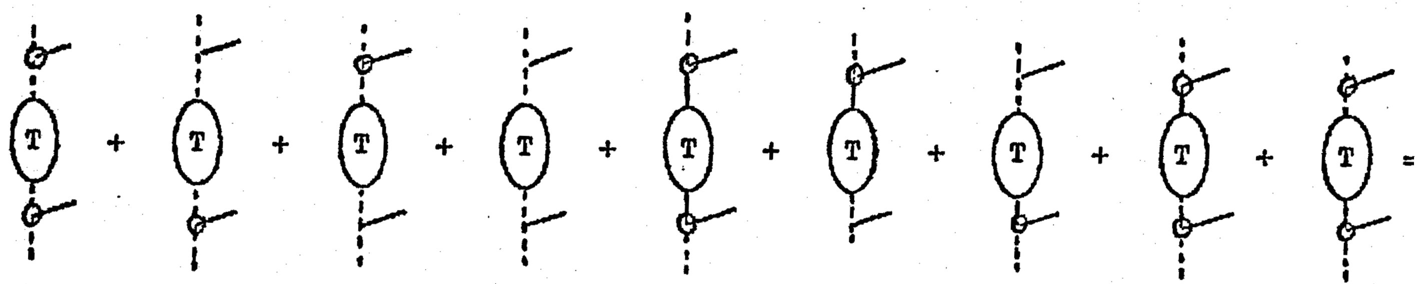
Note that there exist already relations in certain subclasses of these diagrams by virtue of the results in order g .

Consider now in second order the so-called tree diagrams (no closed loops) having two outgoing φ-particles:



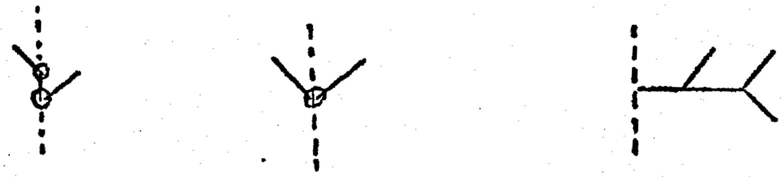
We note a very important fact: the collection of diagrams containing an internal φ-line and a circled vertex are equal to a multiple of the collection of diagrams with an internal φ-line but no circled vertex, on account of the results in order g. Therefore also the collection of diagrams containing internal boson lines and circled vertices equals a multiple of the diagrams without internal boson lines and without circled vertices.

This property remains true also for trees of arbitrary length with two outgoing φ-lines (all W being on the mass shell). This may be proved by induction. Let a circle with a T denote a general tree, having in addition to the lines drawn an arbitrary number of external W-lines. One has:



T_1 T_2 T_3 T_4 T_5 T_6 T_7 T_8 T_9

We did not explicitly indicate vertices



etc.

On account of results in lower orders one has

$$T_9 \propto T_7, \qquad T_8 \propto T_6, \qquad T_1, T_2, T_3 \propto T_4, \qquad T_6 + T_3 + T_4 = 0,$$

therefore T_6 proportional to T_4.

Similarly $T_7 \propto T_4$.
Therefore $T_5 \propto T_4$.

In this way we see, by induction, that any collection of tree diagrams having two φ-lines and an arbitrary number of on-mass-shell bosons equals a collection of tree diagrams having no circled vertices.

From this we infer the following result: consider in a given order of perturbation theory all diagrams having at most one closed loop, and no external φ-lines. That excludes occurrence of vertices (37), having factors $1/M$. Then the collection of these diagrams behaves as diagrams containing no circled vertices (but containing internal Ω and φ-lines !). Since such diagrams contain no factors $1/M$ we conclude that diagrams with one closed loop are finite if there are more than four external boson lines[9].

5. Conclusions

From the foregoing it is clear that many diagrams of the massive Yang-Mills theory are convergent in the sense of a renormalizable field theory. We have not been able to treat diagrams that involve vertices with more than 2 φ-particles and factors $1/M$. These result from the perturbation expansion of expressions like

$$\sin\left(\frac{g\varphi}{M}\right)$$

and one may suspect that these vertices are summable in some sense, because the limit $M \to 0$ seems to exist. However, we have not found any way to understand the details of the theory involved.

Finally we wish to note that the above methods should, if they work in this case, also be applicable to the case where one takes the limit of the mass of the W_3 field to be zero and considers that field as the photon field[10]. In this way perhaps also symmetry breaking may be introduced.

6. Acknowledgements

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Appendix

It appears to us that in the further study of the massive Yang-Mills fields expressions of the form

$$F_n(x,y) = < 0 \mid (\varphi^a(x)\, \varphi^a(x))^n\, (\varphi^b(y)\, \varphi^b(y))^n \mid 0 >$$

will play an important role. Evaluating this expression amounts to counting combinations. One finds, with

$$\delta_{ab}\, \Delta^+(x-y) = < 0 \mid \varphi^a(x)\, \varphi^b(y) \mid 0 > ,$$

the result

$$F_n(x,y) = (\Delta^+(x-y))^{2n}\, (n!)^2 \Big\{ 3\, \frac{2^{2n-1}}{n} + 3^2\, \frac{2^{2n-2}}{2!} \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1(n-\ell_1)} +$$

$$+ \dots + 3^n\, \frac{2^{2n-n}}{n!} \sum_{\ell_1=1}^{1} \sum_{\ell_2=1}^{1} \dots \sum_{\ell_{n-1}=1}^{1} \frac{1}{\ell_1 \ell_2 \dots \ell_{n-1}(n-\ell_1-\ell_2 \dots -\ell_{n-1})}$$

$$= (\Delta^+(x-y))^{2n}\, (n!)^2 \Big\{ \sum_{m=1}^{n} \frac{3^m\, 2^{2n-m}}{m!} \sum_{\ell_1=1}^{n-m+1} \sum_{\ell_2=1}^{n-\ell_1-m+2} \dots$$

$$\dots \sum_{\ell_{m-1}=1}^{n-\ell_1-\ell_2 \dots -1} \frac{1}{\ell_1 \ell_2 \dots \ell_{m-1}(n-\ell_1-\ell_2 \dots -\ell_{m-1})} \Big\} .$$

Of interest are series of the form

$$\sum_k \frac{1}{((2k+3)!)^2}\, F_k(x,y) \left(\frac{g}{M}\right)^{4k}$$

and ultraviolet properties are studied by considering the behavior for x in the neighbourhood of y , where the Δ-functions become singular. Again, the supposedly decent behaviour for $M \to 0$ inspires confidence concerning the ultraviolet problem.

References

1) M. Gell-Mann, Physics 1, 63 (1964).
2) J.S. Bell, Varenna Lectures 1967;
J.S. Bell, Il Nuovo Cim. 50, 129 (1967).
M. Veltman, Phys. Rev. Letters 17, 553 (1966).
3) S.L. Adler, Phys. Rev. 139, B1638 (1965).
4) C.N. Yang and R.L. Mills, Phys. Rev. 96, 191 (1954).
5) R.P. Feynman, Acta Phys. Polonica 24, 697 (1963).
B.S. De Witt, Phys. Rev. 162, 1195, 1239 (1967).
L.D. Fadeev and V.N. Popov, Phys. Letters 25B, 29 (1967).
S. Mandelstam, Berkeley preprint, July 1968.
6) Unitarity, causality, relation between Feynman diagrams and equations of motion etc. are properties that can be understood very easily in terms of diagrams of a perturbation expansion if one remains in configuration space rather than in momentum space. See M. Veltman, Physica 29, 186 (1963). See also Feynman, ref. 5.
7) N. Bogoliubov, D. Shirkov, Introduction to the theory of quantized fields. Interscience Publishers (1959).
8) Part of the work in this section has been discussed in great detail in the lecture notes for the Copenhagen Summer Institute, July 1968.
9) Feynman, ref. 5, seems to have obtained his results by considering explicitly diagrams and taking the limit $M \to 0$. The reader will understand why he could only give the proof for diagrams with one closed loop: in these diagrams no "difficult" vertices stemming from the series expansion of $\sin(g\varphi/M)$ etc. appear.
10) This case has been investigated by T.D. Lee and C.N. Yang, Phys. Rev. 128, 885 (1962), and T.D. Lee, Phys. Rev. 128, 899, (1962). However these authors allowed for a general magnetic moment for the W, and did not include a vertex involving 4 charged bosons.