Unified Modeling of Corporate Debt, Credit Derivatives, and Equity Derivatives*

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Introduction

- In the Black-Scholes-Merton model, the stock price follows geometric Brownian motion, a process with infinite lifetime (no possibility of bankruptcy).

- Real-world firms have positive probability of bankruptcy in finite time.

- While many popular stock options pricing models (local vol, stochastic vol, jump-diffusion, Levy,…) focus on modeling volatility skew and ignore the possibility of default of the firm underlying the option contract, modeling default is central to the area of credit risk, corporate debt and credit derivatives modeling.

Motivating Example

Data: General Motors price on **02.22.2006** was **$21.19**

Historical Volatility of GM stock price over the previous 12 months **~46%**

Question: Which exchange-traded option contract (strike and expiration) on GM had the largest Open Interest? Can you guess its Implied Volatility?

Motivating Example

Answer:

- Largest OI: **Jan 07 $10 Put @ $1.50 (IV ~100%)** with **OI 493,051 contracts**.
- 2nd largest OI: **Jan 07 $5 Put @ $0.50 (IV ~120%)** with **OI 205,703 contracts**.
- **Jan 07 $7.50 @ $1.00 (IV ~110%)** and **$2.50 @ $0.15 (IV ~133%) Puts** also have substantial OI.
- Total outstanding notional for Jan 07 Puts with strikes $2.50-$10 ~**100 million shares** (+ Jan 08 Puts with strikes $2.50-$10 ~**30 million shares**).

Implied Volatility Skew for GM 2007 Puts

Recall that 12-month Historical Volatility of GM **~46%!**

Stock Options as Credit Derivatives

- Put options on the stock provide default protection and can be used to manage default risk — close link between equity and credit derivatives.
- Deep out-of-the-money puts are essentially credit derivatives.
- A rough calculation for Jan 08 \$2.50 put. If GM does not go bankrupt, the put expires worthless. If GM goes bankrupt, GM stock will be worth pennies. Assume zero. Then the put pays \$2.50. Like CDS!
- Price = (discount factor) x (R.N. probability of GM bankruptcy) x \$2.50
- Price on 02.22.06 = \$0.50, so market implied

  Implied R.N. Probability of GM Bankruptcy by Jan 2008 = 22%

Further Observations

- On the flip side, pricing and risk management of equity derivatives should take into account the possibility of default.

- Possibility of default contributes to the implied volatility skew in stock options. Linkage between implied volatility skew in options markets and credit spreads in credit markets.

- Corporate debt, credit derivatives, and equity derivatives should be modeled within a unified framework.

Research Program

- We develop a parsimonious, analytically tractable unified modeling framework for corporate debt, credit derivatives, and equity derivatives.
- In the reduced-form, intensity-based framework, we model **Defaultable Stock** as the fundamental state variable.
- We view corporate debt, equity derivatives, and credit derivatives as contingent claims written on the defaultable stock.
- **<u>Our Research Program:</u>** introduce default into all the major equity derivatives models, preserving analytical tractability as much as possible.

A Catalogue of Models

I. Jump-to-Default Extended Diffusion (JDD)

I.1 Jump-to-Default Extended Black-Scholes-Merton (JDBSM)

I.2 JDBSM --- 2^{nd} Model Specification

I.3 Jump-to-Default Extended CEV (JDCEV)

II. Stochastic Volatility (SV) Models with Default

II.1 Affine SV Model with Default: Application to Convertible Bonds

II.2 Non-affine Local-Stochastic Volatility Models with Default

III. Models with Jumps

III.1 Introducing Jumps into JDD by Subordination

III.2 Models with Jumps, Stochastic Volatility, and Default

I. Jump-to-Default Extended Diffusion (JDD)

- Model the *pre-default* stock dynamics under an EMM $\mathbb{Q}$ as a diffusion:

$$dS_t = [r(t) - q(t) + \lambda(S_t, t)]S_t\, dt + \sigma(S_t, t)S_t\, dB_t, \quad S_0 = S > 0,$$

  r, q, σ and λ are the short rate, dividend yield, volatility, and default intensity.

- If the diffusion can hit zero, we kill it at the first hitting time of zero, T_0, and send it to a **cemetery (bankruptcy) state** Δ, where it remains forever.

- ***Jump-to-default*** arrives at the first jump time $\tilde{\zeta}$ of a doubly-stochastic Poisson process with intensity $\lambda(S_t, t)$. The **time of default** is $\zeta = \min\{T_0, \tilde{\zeta}\}$.

- Assume stock holders do not receive any recovery in the event of default. Addition of λ in the drift $r - q + \lambda$ compensates for default to insure that the discounted gain process to the stock holders is a martingale under the EMM.

Pricing Corporate Bonds

- The time-t price of a **defaultable zero-coupon bond** with face value of \$1 and no recovery in default:

$$B(S,t;T) = e^{-\int_t^T r(u)du} Q(S,t;T),$$

where the **(risk-neutral) survival probability** is:

$$Q(S,t;T) = \mathbb{E}[e^{-\int_t^T \lambda(S_u,u)du} \mathbf{1}_{\{T_0>T\}} | S_t = S].$$

Pricing Stock Options

- The time-t price of a **call option** with strike $K > 0$:

$$C(S,t;K,T) = e^{-\int_t^T r(u)du}\mathbb{E}\left[e^{-\int_t^T \lambda(S_u,u)du}(S_T - K)^+\mathbf{1}_{\{T_0>T\}}\middle| S_t = S\right].$$

- A **put option** with strike $K > 0$ can be decomposed into two parts:

$$(K - S_T)^+\mathbf{1}_{\{\zeta>T\}} + K\mathbf{1}_{\{\zeta\leq T\}},$$

  the put payoff given no default by T and a **recovery payment** at T equal to K in the event of default $\zeta \leq T$. The put price:

$$P(S,t;K,T) = e^{-\int_t^T r(u)du}\mathbb{E}\left[e^{-\int_t^T \lambda(S_u,u)du}(K - S_T)^+\mathbf{1}_{\{T_0>T\}}\middle| S_t = S\right]$$

$$+Ke^{-\int_t^T r(u)du}[1 - Q(S,t;T)].$$

- **Notice the default claim embedded in the put option!**

I.1 A Jump-to-Default Extended Black-Scholes-Merton

- ***Pre-default*** **stock price dynamics:**

$$dS_t = [r - q + \lambda(S_t)]S_t\,dt + \sigma S_t\,dB_t, \;\; S_0 = S > 0,$$

$$\lambda(S) = \frac{\alpha}{S^p}, \;\; \alpha > 0, \;\; p > 0.$$

- This process cannot diffuse to zero. **Time of default** ζ is the first jump time of a doubly stochastic Poisson process with intensity $\lambda(S)$.

- $\lambda(S) \to \infty$ as $S \to 0$, making default inevitable at low stock prices.

- $\lambda(S) \to 0$ as $S \to \infty$, making the stock asymptotically GBM at large values.

- **We obtain closed-form solutions in this model (V.L., "Pricing Equity Derivatives subject to Bankruptcy,"** ***Mathematical Finance*****, 2006, 16 (2), 255-282.**

Reduction of the Pricing Problem

- The pricing problem reduces to computing expectations of the form:

$$V_\Psi(S,T) = e^{-rT}\mathbb{E}\left[e^{-\int_0^T \lambda(S_t)dt}\Psi(S_T)\right].$$

- $e^{-\int_0^T \lambda(S_t)dt}$ can be removed by **changing measure via Girsanov**:

$$V_\Psi(S,T) = e^{-qT}S\,\widehat{\mathbb{E}}\left[S_T^{-1}\Psi(S_T)\right],$$

$\widehat{\mathbb{E}}$ is w.r.t. $\widehat{\mathbb{Q}}$ under which $\widehat{B}_t := B_t - \sigma t$ is a standard BM and

$$dS_t = (r - q + \sigma^2 + \alpha S_t^{-p})S_t\,dt + \sigma S_t\,d\widehat{B}_t,\;\; S_0 = S > 0.$$

- The pre-default stock process under $\hat{Q}$ can be represented as:

$$S_t = (\beta^{-1}X^{(\nu)}_{\tau(t)})^{\frac{1}{p}},$$

where X is a diffusion process

$$dX_t = [2(\nu+1)X_t + \mathbf{1}]dt + 2X_t dW_t,\;\; X_0 = x = \beta S^p,$$

$$\beta := p\sigma^2/(4\alpha),\;\; \nu := 2(r - q + \sigma^2/2)/(p\sigma^2),\;\; \tau(t) := p^2\sigma^2 t/4.$$

Reduction of the Pricing Problem

- The problem reduces to computing

$$V_\Psi(S,T) = e^{-qT} S E_x^{(\nu)}[(X_\tau/\beta)^{-\frac{1}{p}}\Psi((X_\tau/\beta)^{\frac{1}{p}})],$$

  where $E_x^{(\nu)}$ is w.r.t. the probability law of X starting at $x = \beta S^p$.

- The process X is closely related to the problem of pricing **Asian options** (Geman and Yor (1993), Donati-Martin and Yor (2001), Linetsky (2004)).

- The spectral expansion of the transition density of X is available in closed form, yielding closed-form pricing formulas for corporate bonds and stock options in the form of spectral expansions.

Closed Form Solution for Corporate Bonds

- **Zero-coupon bond** with unit face and no recovery:

$$B(S,0;T) = \mathbf{1}_{\{\nu>2/p\}} e^{-rT} \frac{\Gamma(\nu-1/p)}{\Gamma(\nu-2/p)} U\left(\frac{1}{p}, \frac{2}{p}-\nu+1, \frac{1}{2x}\right)$$

$$+\mathbf{1}_{\{\nu<0\}} e^{-qT} \frac{\Gamma(1/p-\nu)}{\Gamma(-\nu)} (2x)^{\frac{1}{p}}$$

$$+\mathbf{1}_{\{\nu<-2\}} e^{-qT} \sum_{n=1}^{[|\nu|/2]} e^{-2n(|\nu|-n)\tau} \frac{(|\nu|-2n)\Gamma(-1/p)\Gamma(1/p+|\nu|-n)}{\Gamma(1+|\nu|-n)\Gamma(1-1/p-n)} (2x)^{\frac{1}{p}+n} L_n^{(|\nu|-2n)}\left(\frac{1}{2x}\right)$$

$$+\frac{e^{-qT}}{4\pi^2\Gamma(1/p)} \int_0^\infty e^{-\frac{(\nu^2+\rho^2)\tau}{2}} (2x)^{\frac{1}{p}+\frac{1-\nu}{2}} e^{\frac{1}{4x}} W_{\frac{1-\nu}{2},\frac{i\rho}{2}}\left(\frac{1}{2x}\right) \left|\Gamma\left(\frac{\nu+i\rho}{2}\right)\Gamma\left(\frac{1}{p}-\frac{\nu+i\rho}{2}\right)\right|^2 \sinh(\pi\rho)\rho\, d\rho,$$

  $U(a,b,z)$ — **Tricomi confluent hypergeometric function**, $W_{\kappa,\mu}(z)$ — **2nd Whittaker function**, $L_n^{(\alpha)}(z)$ — **generalized Laguerre polynomials**, $\Gamma(z)$ — **Gamma function**, $[x]$ — integer part of x.

- **Continuous spectrum** above $\frac{\nu^2}{2}$ and **discrete eigenvalues** in $[0, \frac{\nu^2}{2})$.

- The formulas are **fast and easy to compute** in *Mathematica* and *Maple* where all special functions are available.

Closed Form Solution for Put Options

$$P(S,0;K,T)=e^{-rT}K-\mathbf{1}_{\{\nu>2/p\}}e^{-rT}K\frac{\Gamma(\nu-1/p)}{\Gamma(\nu-2/p)}U\left(\frac{1}{p},\frac{2}{p}-\nu+1,\frac{1}{2x}\right)$$

$$-\mathbf{1}_{\{\nu<0\}}\frac{e^{-qT}S}{\Gamma(|\nu|)}\left[\Gamma\left(|\nu|,\frac{1}{2k}\right)+(2k)^{\frac{1}{p}}\gamma\left(\frac{1}{p}+|\nu|,\frac{1}{2k}\right)\right]$$

$$-\mathbf{1}_{\{\nu<-2\}}e^{-qT}S\sum_{n=1}^{[|\nu|/2]}p_K(n)e^{-2n(|\nu|-n)\tau}\frac{2(|\nu|-2n)n!}{\Gamma(1+|\nu|-n)}(2x)^nL_n^{(|\nu|-2n)}\left(\frac{1}{2x}\right)$$

$$-e^{-qT}S\frac{1}{2\pi^2}\int_0^\infty P_K(\rho)e^{-\frac{(\nu^2+\rho^2)\tau}{2}}x^{\frac{1-\nu}{2}}e^{\frac{1}{4x}}W_{\frac{1-\nu}{2},\frac{i\rho}{2}}\left(\frac{1}{2x}\right)\left|\Gamma\left(\frac{\nu+i\rho}{2}\right)\right|^2\sinh(\pi\rho)\rho d\rho,$$

where

$$P_K(\rho)=k^{\frac{1+\nu}{2}}e^{-\frac{1}{4k}}W_{-\frac{1+\nu}{2},\frac{i\rho}{2}}\left(\frac{1}{2k}\right)$$

$$+2\Re\left\{\frac{(2k)^{\frac{\nu-i\rho}{2}}(2/p-\nu-i\rho)\Gamma(-i\rho)}{\Gamma(\nu/2-i\rho/2)(\rho^2+(\nu-2/p)^2)}\ {}_2F_2\left[\frac{1}{p}-\frac{\nu}{2}+\frac{i\rho}{2},1-\frac{\nu}{2}+\frac{i\rho}{2};1+i\rho,\frac{1}{p}-\frac{\nu}{2}+\frac{i\rho}{2}+1;-\frac{1}{2k}\right]\right\},$$

$$p_K(n)=-\frac{1}{2n}(2k)^{n-|\nu|}e^{-\frac{1}{2k}}L_{n-1}^{(|\nu|-2n)}\left(\frac{1}{2k}\right)$$

$$+\frac{\Gamma(|\nu|-n+1)(2k)^{n-|\nu|}}{2(|\nu|-n+1/p)n!\Gamma(|\nu|-2n+1)}\,{}_2F_2\left[|\nu|-n+1,|\nu|-n+\frac{1}{p};|\nu|-2n+1,|\nu|-n+1+\frac{1}{p};-\frac{1}{2k}\right],$$

${}_2F_2[a_1,a_2;b_1,b_2;z]$ — **Gauss hypergeometric function**, $\gamma(a,z)$ — **incomplete Gamma function**, $\Gamma(a,z)$ — **complementary incomplete Gamma function**, $\Re(z)$ — real part of a complex number z.

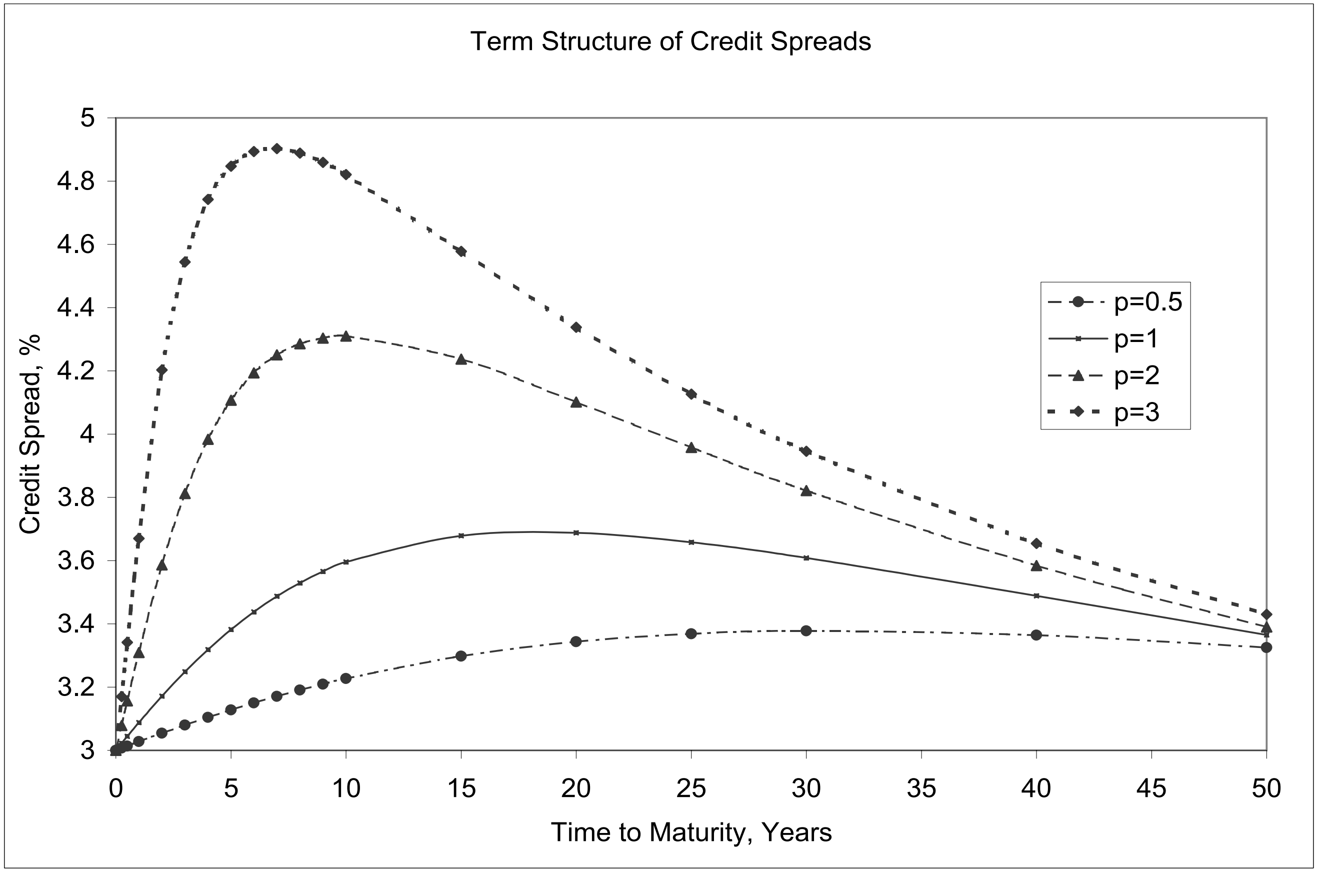

Figure 1: **Term Structure of Credit Spreads.** Parameter values: $S = S^* = 50$, $\sigma = 0.3$, $r = q = 0.03$, $h^* = 0.03$, $p = 0.5, 1, 2, 3$. $\lambda(S) = h^* \left(\frac{S^*}{S}\right)^p$, where $S^* > 0$ is some reference price level and $h^* = \lambda(S^*) > 0$ (h^* is the scale parameter).

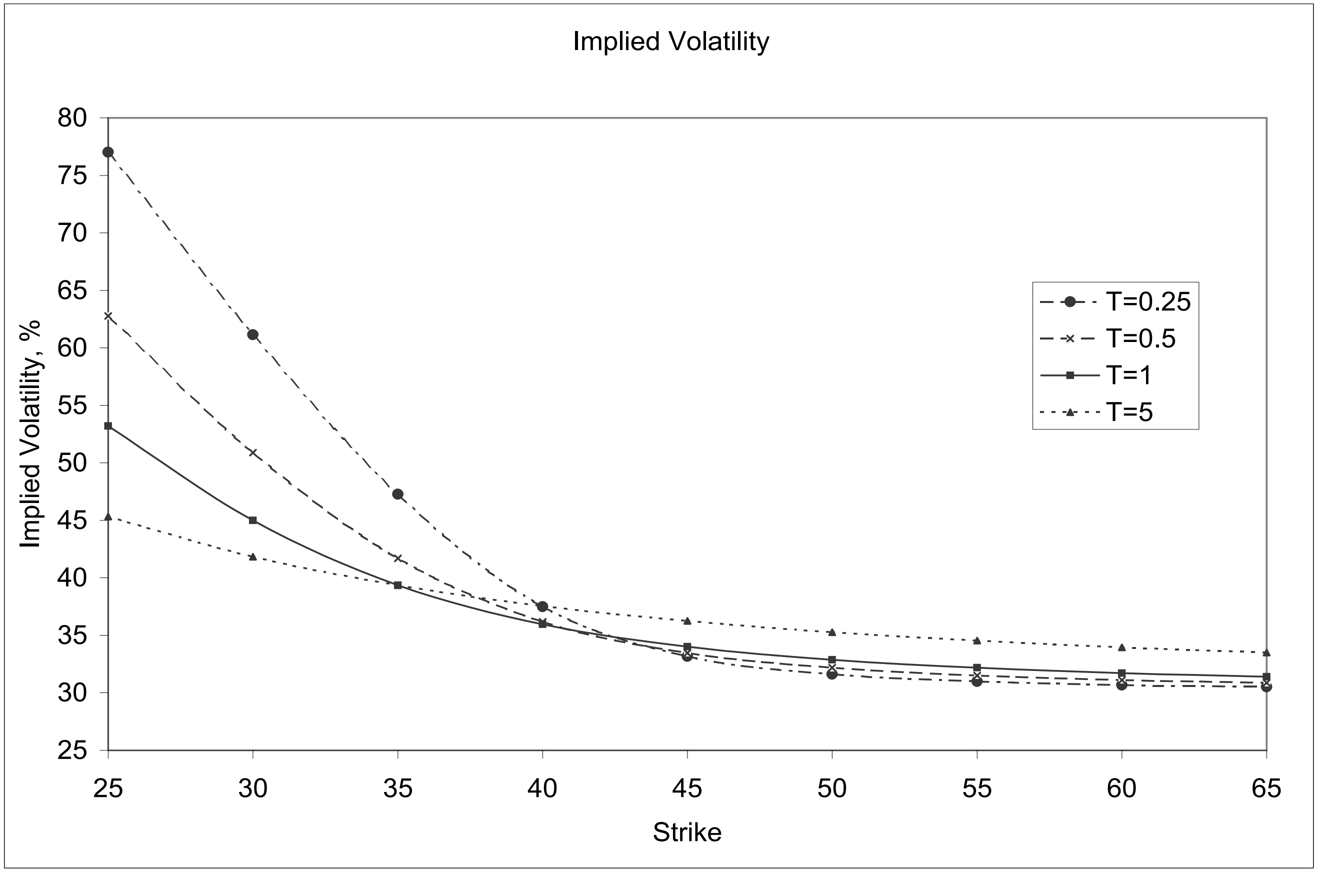

Figure 2: **Implied Volatilities for Times to Expiration $T = 0.25, 0.5, 1, 5$.** Parameter values: $S = S^* = 50$, $\sigma = 0.3$, $r = q = 0.03$, $h^* = 0.03$, $p = 2$ (the same parameters used to compute credit spreads).

I.2 Alternative Intensity Specification

- Alternative intensity specification:

$$\lambda(S) = \frac{c}{\ln(S/B)}, \quad c > 0, \quad B > 0, \quad S > B.$$

  This specification is similar to the one used in Madan and Unal (1998).

- $\lambda(S) \to \infty$ as $S \to B$, making default inevitable as the stock falls towards B. $\lambda(S) \to 0$ as $S \to \infty$.

- The pricing problem reduces to computing expectations of the form:

$$V_\Psi(S,T) = e^{-rT}\mathbb{E}\left[e^{-\int_0^T \lambda(S_t)dt}\Psi(S_T)\right] = e^{-qT}S\,\widehat{\mathbb{E}}\left[S_T^{-1}\Psi(S_T)\right],$$

  $\widehat{\mathbb{E}}$ is w.r.t. $\widehat{\mathbb{Q}}$ under which $\widehat{B}_t := B_t - \sigma t$ is a standard BM and

$$dS_t = (r - q + \sigma^2 + c/\ln(S_t/B))S_t\,dt + \sigma S_t\,d\widehat{B}_t, \quad S_0 = S > B.$$

- **We obtain closed-form solutions in this model**.

Reduction to Bessel Process with Drift

- Let X be a Bessel process with drift:

$$dX_t = \left(\frac{\nu + 1/2}{X_t} + \mu\right) dt + dW_t, \quad X_0 = x > 0.$$

- The pre-default stock process under $\hat{\mathbb{Q}}$ can be represented as:

$$S_t = Be^{\sigma X_t} \text{ where } \nu = c/\sigma^2 - 1/2, \ \mu = (r-q)/\sigma + \sigma/2, \ x = \ln(S/B)/\sigma.$$

- The problem reduces to computing

$$V_\Psi(S,T) = e^{-qT}(S/B)E_x^{(\nu,\mu)}[e^{-\sigma X_T}\Psi(Be^{\sigma X_T})],$$

  where $E_x^{(\nu,\mu)}$ is w.r.t. the probability law of X starting at $x = \ln(S/B)/\sigma$.

- Laplace transform of transition density of X was obtained by Yor (1984). It was inverted by V. Linetsky, "The Spectral Representation of Bessel Processes with Drift," *J. Appl. Probability*, 41 (2004) 327-344. This yields an analytical solution to our model (in preparation).

I.3 A Jump-to-Default Extended CEV Model (JDCEV)

- Pre-default stock dynamics:

$$dS_t = [r(t) - q(t) + \lambda(S_t, t)]S_t\,dt + \sigma(S_t, t)S_t\,dB_t, \quad S_0 = S > 0.$$

- To be consistent with the **leverage effect**, **constant elasticity of variance (CEV)** volatility specification:

$$\sigma(S, t) = a(t)S^{\beta},$$

  $\beta < 0$ is the **volatility elasticity** and $a(t) > 0$ is the (time-dependent) **volatility scale parameter**.

- To be consistent with the evidence linking credit spreads to stock price volatility, **default intensity** — affine function of the instantaneous variance of the stock:

$$\lambda(S, t) = b(t) + c\,\sigma^2(S, t) = b(t) + c\,a^2(t)S^{2\beta}, \; b(t) \geq 0, \;\; c > 0.$$

- Peter Carr and V.L., "A Jump-to-Default Extended CEV Model: An Application of Bessel Processes," *Finance and Stochastics*, 10 (3), 303-330.

Reduction of JDCEV to CEV

Linetsky & Mendoza recently (two weeks ago) proved that calculations in JDCEV can be reduced to the standard CEV without jump-to-default by changes of variables and changes of measure:

$$V_\Psi(S,t;T) = e^{-\int_t^T r(u)du} E_{t,S}\left[e^{-\int_t^T \lambda(S_u,u)du}\Psi(S_T)\mathbf{1}_{\{T_0>T\}}\right]$$

$$= e^{-\int_t^T \tilde{r}(u)du} x^{-\frac{2c}{2c+1}} \hat{E}_{t,x}\left[X_T^{\frac{2c}{2c+1}}\Psi(X_T^{\frac{1}{2c+1}})\mathbf{1}_{\{T_0>T\}}\right],$$

where $x = S^{1+2c}$ and, under $\hat{\mathbb{Q}}$, X follows a standard CEV process:

$$dX_t = [\tilde{r}(t) - q(t)]X_t dt + \tilde{a}(t) X_t^{\tilde{\beta}+1} d\hat{B}_t$$

with parameters:

$$\tilde{r}(t) = (2c+1)(r(t)+b(t)) - 2cq(t), \;\; \tilde{a}(t) = (2c+1)a(t), \;\; \tilde{\beta} = (2c+1)\beta.$$

Non-central Chi-square Distribution

- CEV transition density is expressed in terms of non-central chi-square.
- $\chi^2(\delta, \alpha)$ with δ degrees of freedom and non-centrality parameter $\alpha > 0$:

$$f_{\chi^2}(x; \delta, \alpha) = \frac{1}{2} e^{-\frac{\alpha + x}{2}} \left(\frac{x}{\alpha} \right)^{\frac{\nu}{2}} I_\nu(\sqrt{x\alpha}) \mathbf{1}_{\{x>0\}},$$

where $\nu = \delta/2 - 1$ and I_ν is the Bessel function.

- For $p > -(\nu + 1)$ and $k > 0$, $p-$th moment and truncated $p-$th moments:

$$\mathcal{M}(p; \delta, \alpha) = E^{\chi^2(\delta,\alpha)}[X^p] = 2^p e^{-\frac{\alpha}{2}} \frac{\Gamma(p + \nu + 1)}{\Gamma(\nu + 1)} \, {}_1F_1(p + \nu + 1, \nu + 1, \alpha/2),$$

$$\Phi^+(p, k; \delta, \alpha) = E^{\chi^2(\delta,\alpha)}[X^p \mathbf{1}_{\{X>k\}}] = 2^p \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}} \left(\frac{\alpha}{2} \right)^n \frac{\Gamma(\nu + p + n + 1, k/2)}{n! \Gamma(\nu + n + 1)},$$

$\Gamma(a)$ — Gamma function, $\gamma(a, x)$ — incomplete Gamma function, $\Gamma(a, x)$ — complementary incomplete Gamma function, ${}_1F_1(a, b, x)$ — confluent hypergeometric function.

Results for Survival Probability and Options

- Define $x := \frac{1}{|\beta|}S^{|\beta|} > 0$, $\nu_+ := \frac{c+1/2}{|\beta|} > 0$, $\delta_+ := 2(\nu_+ + 1) > 0$,

$$\tau = \tau(t,T) := \int_t^T a^2(u) e^{-2|\beta| \int_t^u \alpha(s)ds} du.$$

- Assume no default by time $t \geq 0$. The **risk-neutral survival probability** is:

$$Q(S,t;T) = e^{-\int_t^T b(u)du} \left(\frac{x^2}{\tau}\right)^{\frac{1}{2|\beta|}} \mathcal{M}\left(-\frac{1}{2|\beta|}; \delta_+, \frac{x^2}{\tau}\right).$$

- The **call option price** is:

$$C(S,t;K,T) = e^{-\int_t^T q(u)du} S\Phi^+\left(0, \frac{k^2}{\tau}; \delta_+, \frac{x^2}{\tau}\right)$$

$$-e^{-\int_t^T [r(u)+b(u)]du} K \left(\frac{x^2}{\tau}\right)^{\frac{1}{2|\beta|}} \Phi^+\left(-\frac{1}{2|\beta|}, \frac{k^2}{\tau}; \delta_+, \frac{x^2}{\tau}\right),$$

$$k = k(t,T) = \frac{1}{|\beta|} K^{|\beta|} e^{-|\beta| \int_t^T \alpha(u)du}.$$

Numerical Examples

- Consider a **time-homogeneous model**.
- Parameterize **volatility** as follows:

$$\sigma(S) = \sigma_* \left(\frac{S}{S^*} \right)^{\beta},$$

  $S^* > 0$ is some reference stock price level and $\sigma_* > 0$ is the volatility at that reference level, $\sigma(S^*) = \sigma_*$ (e.g., $S^* = S_0$, initial stock price).

- Assume $S_0 = 50$, $\sigma_* = 0.2$, $r = 0.05$, $q = 0$, and the elasticity parameter $\beta = -1$.
- **Default intensity**:

$$\lambda(S) = b + c\,\sigma_*^2 \left(\frac{S}{S^*} \right)^{2\beta}.$$

  Consider cases $b = 0$ and $b = 0.02$ and $c = 1/2$ and $c = 1$.

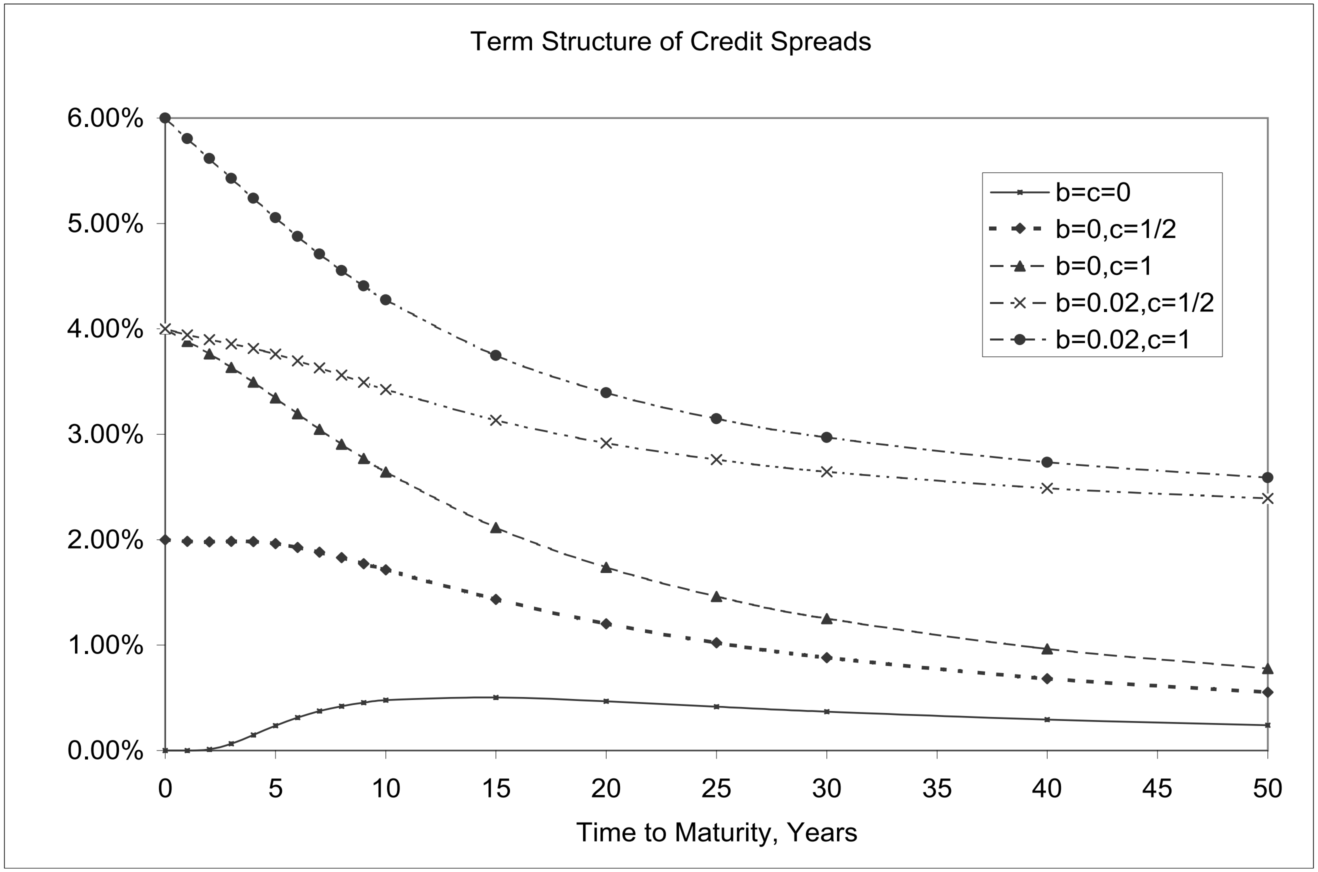

Figure 3: **Term structures of credit spreads.** Parameter values: $S_0 = 50$, $\sigma_* = 0.2$, $\beta = -1$, $r = 0.05$, $q = 0$, $b = 0, 0.02$, $c = 0, 1/2, 1$.

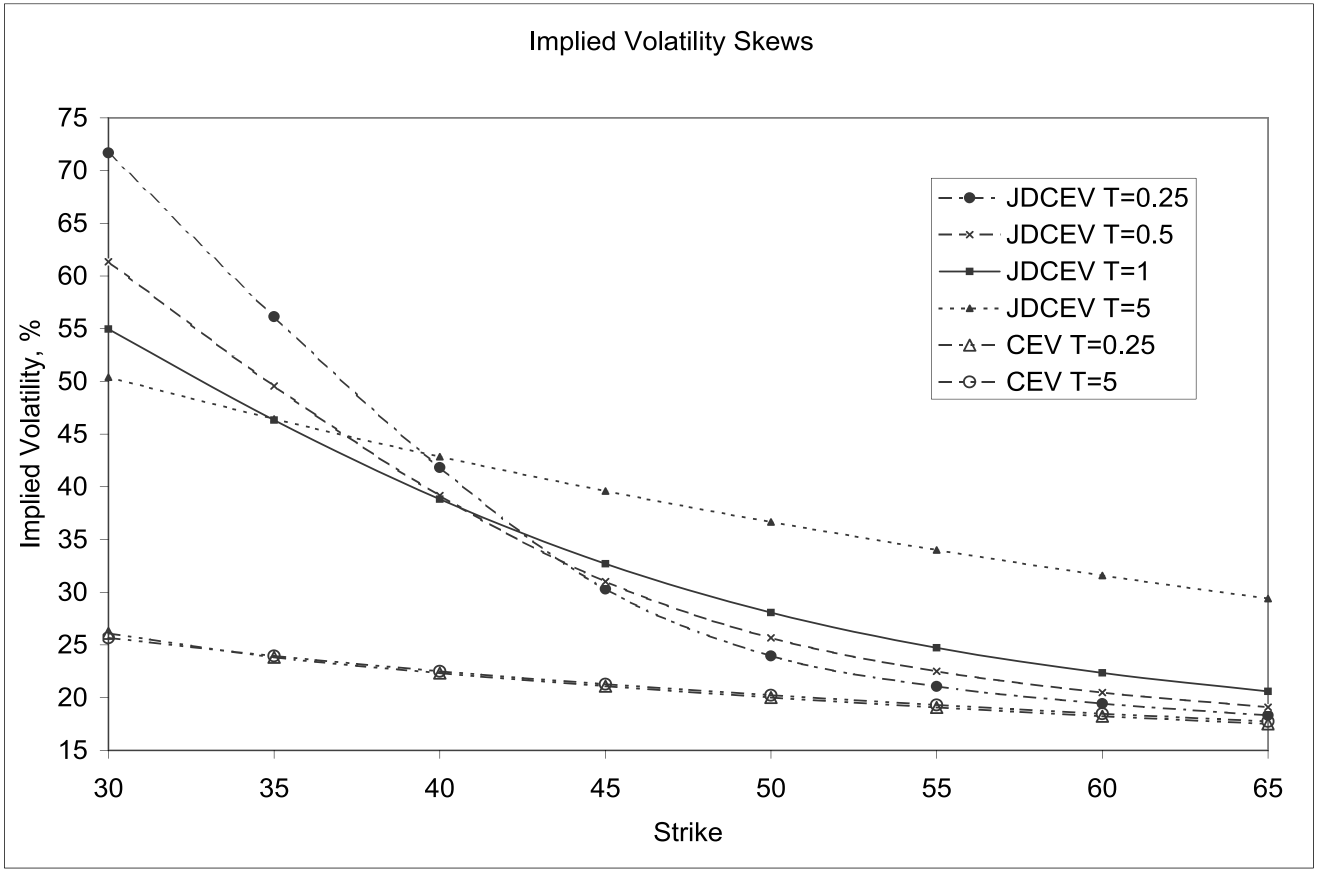

Figure 4: **Implied volatility skews.** Parameter values: $S = S^* = 50$, $\sigma_* = 0.2$, $\beta = -1$, $r = 0.05$, $q = 0$. For CEV model: $b = c = 0$. For JDCEV model: $b = 0.02$, $c = 1$. JDCEV times to expiration are $T = 0.25, 0.5, 1, 5$ years. Implied volatilities are plotted against strike.

II.1 Affine Stochastic Volatility Model with Default

- Affine SV model with default and stochastic rates (extenstion of Carr & Wu (2005) with stochastic rates):

$$dS_t = (r_t - q + \lambda_t)S_t dt + \sqrt{V_t} S_t dW_t^S,$$

$$dr_t = \kappa_r(\theta_r - r_t)dt + \sigma_r \sqrt{r_t}\, dW_t^r,$$

$$dV_t = \kappa_V(\theta_V - V_t)dt + \sigma_V \sqrt{V_t}\, dW_t^V,$$

$$dz_t = \kappa_z(\theta_z + \gamma V_t - z_t)dt + \sigma_z \sqrt{z_t}\, dW_t^z,$$

$$\lambda_t = z_t + \alpha V_t + \beta r_t,$$

$$dW_t^S dW_t^V = \rho_{SV} dt, \;\; \rho_{SV} < 0,$$

other correlations equal to zero.

- The model is affine and analytically tractable for European-style securities, incl. defaultable bonds and stock options, up to Fourier inversion.

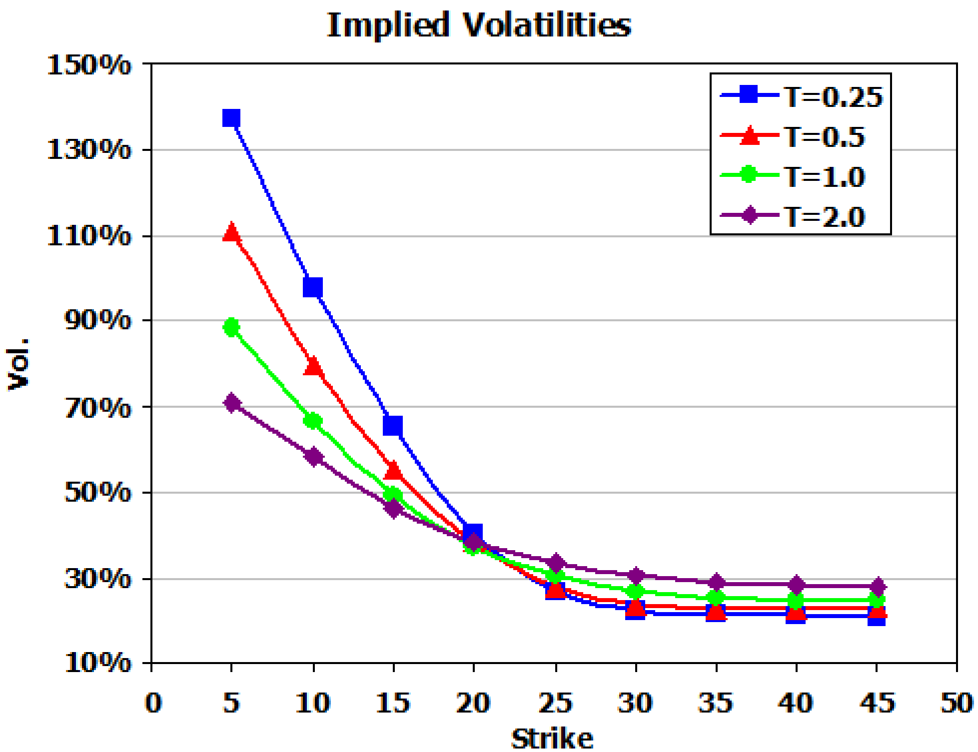

Figure 5: **Implied Vol Skews for $T = 0.25, 0.5, 1,$ and 2 years. Current stock price $S_0 = 25$.**

Volatility Skew

- Similar to JDCEV, default intensity linearly depends on local variance.
- The model features realistic volatility skews linked with credit spreads.

Application to Convertible Bonds

- Convertible bonds are American-style. The problem is to find an **optimal conversion strategy** for the bondholder and an **optimal call strategy** for the firm, and value the **convertible bond** assuming both players behave optimally: a **differential game problem**.
- Solving it in the **4-factor model** is computationally **very challenging**!
- We convert the differential game to a **non-linear penalized PDE** and solve it numerically by the **finite element method-of-lines** with the adaptive time-stepping package SUNDIALS from the Lawrence Livermore National Laboratory. Kovalov and Linetsky, "Valuing Convertible Bonds with Stock Price, Volatility, Interest Rate, and Default Risk", working paper.

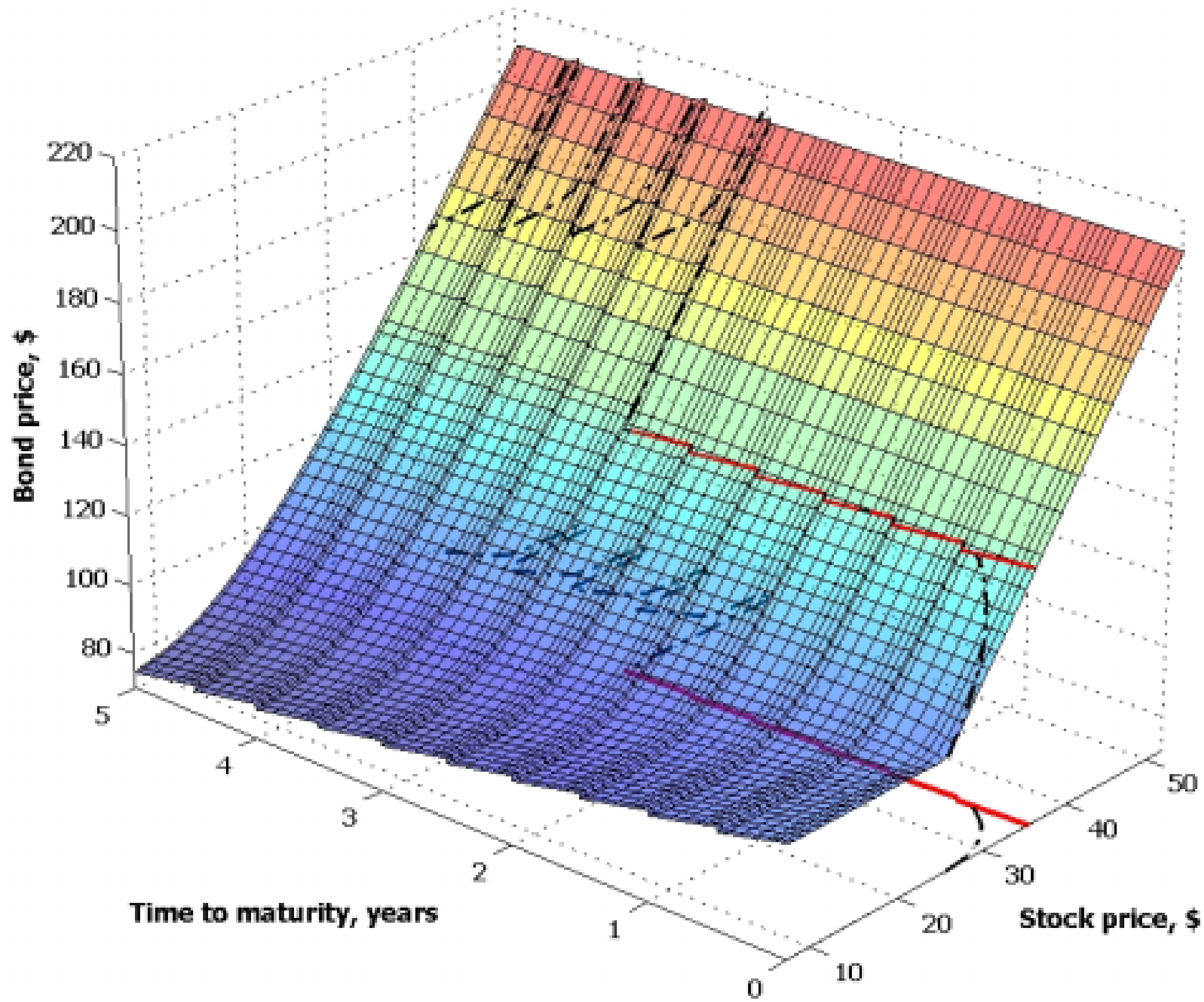

Figure 6: **Solution for the 5-year Convertible Bond (semiannual 3% coupon, 2 years call protection period, and clean call price $1400). Solid lines – call boundary, dashed lines – conversion boundary.**

II.2 Non-affine Local-Stochastic Vol. Models with Default

- Start with a model with local volatility $\sigma(x)$:

$$dX_t = \lambda(X_t)X_t dt + \sigma(X_t)X_t dB_t$$

where default arrives with intensity $\lambda(X_t)$, sending X to zero.

- Suppose V_t follows a CIR process:

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t.$$

Consider an integral: $T_t = \int_0^t V_u du$.

- Do a time change $S_t = X_{T_t}$. The SDE is:

$$dS_t = \lambda(S_t, V_t)S_t dt + \sigma(S_t)\sqrt{V_t} S_t dB_t$$

and default intensity is $\lambda(S, V) = \lambda(S)V$.

- If X has an analytically tractable spectral expansion, then S is also tractable since we know the Laplace transform $E[e^{-sT_t}]$. We introduce SV into JDD models, and default intensity is linear in stochastic volatility! For the JDCEVSV, $\lambda(S, V) = bV + cVa^2 S^{2\beta}$. **Work in progress**.

Time Changing Diffusions with Known Spectral Expansions

- Transition densities of 1D diffusions admit spectral expansions. If the spectrum of the infinitesimal generator of the diffusion X is discrete with eigenvalues λ_n and eigenfunctions $\varphi_n(x)$, then the transition density has the spectral expansion:

$$p(t;x,y) = m(y) \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y),$$

  where $m(y)$ is the speed density of X. If the spectrum is continuous, the sum is replaced with the integral.

- Suppose T_t is a non-decreasing process with the known Laplace transform:

$$\mathcal{L}(t,\lambda) = E[e^{-\lambda T_t}].$$

- Then the time-changed process X_{T_t} has the transition density:

$$p(t;x,y) = m(y) \sum_{n=1}^{\infty} \mathcal{L}(t,\lambda_n) \varphi_n(x) \varphi_n(y).$$

- We can construct new tractable processes from a diffusion X with the known spectral expansion and a time change T with the known Laplace transform.

III.1 Introducing Jumps by Subordination

- Start with a model with local volatility $\sigma(x)$ and default intensity $\lambda(x)$:

$$dX_t = \lambda(X_t)X_t dt + \sigma(X_t)X_t dB_t.$$

- Suppose $\{T_t, t \geq 0\}$ is a *Lévy subordinator*, i.e., a non-decreasing Lévy process (only positive jumps). Its Laplace transform is:

$$E[e^{-sT_t}] = e^{-t\phi(s)}$$

  with *Laplace exponent* $\phi(s)$ given by the Lévy-Khintcine Theorem.

- Do a time change $S_t = X_{T_t}$. It is a jump-diffusion process with the same diffusion volatility as X, with jumps with Lévy density and default intensity:

$$\pi^{\phi}(S, S') = \int_0^{\infty} p(t; S, S')\nu(dt), \;\; \lambda^{\phi}(S) = \lambda(S) + \int_0^{\infty} P_d(S,t)\nu(dt),$$

  $p(t; x, y)$ — transition density of X_t, $P_d(x,t)$ — probability of default of X by time t starting from state x at time zero, ν — Lévy measure of T.

- S has an analytically tractable spectral expansion if X is, and we have a way of introducing jumps into our JDD framework. Work in progress with Peter Carr.

III.2 Models with Jumps, SV, and Default

- If we compose the two types of time changes, then we can build very rich models with stochastic volatility, jumps, and default.

- Start with X as before, first do the time change with the Lévy subordinator T_t^1, and then do the time change with $T_t^2 = \int_0^t V_u du$. The result is a jump-diffusion process with jumps, SV, and default with diffusion volatility $\sigma(S,V) = \sigma(S)\sqrt{V}$, Lévy measure

$$\pi^{\phi}(V,S,S') = V \int_0^{\infty} p(t;S,S)\nu(dt),$$

and default intensity

$$\lambda^{\phi}(S,V) = V \left[\lambda(S) + \int_0^{\infty} P_d(S,t)\nu(dt)\right].$$

- S is analytically tractable if X is, and we have a way of introducing jumps and SV into our JDD framework. Work in progress with Peter Carr.

Conclusion

- We develop a framework for unified modeling of corporate debt, credit derivatives, and equity derivatives.

- Within this framework, we are able to go surprisingly far in obtaining analytical solutions to credit-equity models with diffusion, jumps, SV, and default.

- Our research program is to introduce default into all the major equity models, incl. SV, Lévy, etc.

- These models feature linkages between corporate credit spreads in the credit markets and implied volatility skews in the options markets.

- Equity options may be used as indicators of market's assessment of credit risk of the underlying firm along with credit spreads, and are emerging as an important source of market data to potentially improve credit risk measurement.

This talk is based on:

- Linetsky, "Pricing Equity Derivatives subject to Bankruptcy," *Mathematical Finance*, 2006, 16 (2), 255-282.
- Carr and Linetsky, "A Jump-to-Default Extended CEV Model: An Application of Bessel Processes," *Finance and Stochastics*, 10 (3), 303-330.
- Linetsky, "Spectral Methods in Derivatives Pricing," to appear in *Handbook of Financial Engineering*, Eds. Birge and Linetsky, Elsevier.
- Kovalov and Linetsky, "Valuing Convertible Bonds with Stock Price, Volatility, Interest Rate, and Default Risk," working paper.
- Linetsky and Mendoza, "A Note on the Jump-to-Default Extended CEV Model," in preparation.
- Linetsky, "Unified Valuation of Corporate Debt, Credit Derivatives, and Equity Derivatives," in progress.
- Carr and Linetsky, "Time Changed Markov Processes in Asset Pricing," in progress.