ON THE HOT-SPOTS CONJECTURE FOR TRIANGLES

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1. Introduction

Hot-spots conjecture, posed by J. Rauch, states that the hottest point on an insulated plate with arbitrary initial heat distribution shifts toward the boundary. More precisely, consider Neumann eigenvalue problem

$$\Delta u = -\mu u, \text{ on } D$$
$$\frac{\partial u}{\partial n} = 0, \text{ on } \partial D.$$

It is well known that for nice domains D (in particular convex) there exists increasing sequence of eigenvalues satisfying

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \to \infty.$$

It is also known that μ_2 has multiplicity at most 2. We are interested in the hot-spots conjecture in its strongest form

Conjecture 1.1. *Every eigenfunction for μ_2 attains its maximum and minimum on the boundary of the domain.*

For discussion on various other formulations see [2]. The conjecture is false for arbitrary sets but it is most likely true for convex domains. For the overview of the known results see Section 1.2.

In this paper we will be concerned with triangular domains. In this case, the conjecture is settled only for obtuse, right and isosceles triangles. It is however open for any nonsymmetric acute triangle. Recently, acute triangles became the subject of the Polymath7 project [14]. The goal of the project is to give a proof in this special case, as well as establish new numerical and analytical tools for studying this and related problems.

In this paper we refine the method of boundary critical points used by Miyamoto [12] to give an analytic proof of the hot-spots conjecture for isosceles triangles. As a result we resolve the conjecture for triangles with one small angle via a reduction to a problem on the boundary of the domain.

1.1. Main results.

First we establish an auxiliary result, simplicity for μ_2. In this case all versions of the hot-spots conjecture are equivalent (any eigenfunction = all eigenfunctions). Simplicity is also important for numerical stability and theoretical estimates for eigenfunctions (see e.g. [14]). We also use it in symmetry based arguments for triangles and kites.

Theorem 1.2. *The second Neumann eigenvalue μ_2 is simple for all non-equilateral triangles.*

This result was already known for obtuse and right triangles [1], and for isosceles triangles [12]. The proof is given in the last section using ideas contributed to the Polymath7 project [14] by the author of this paper.

Next we reduce the hot spots problem to a similar problem on the boundary of a triangle.

Theorem 1.3. *If the second Neumann eigenfunction has no critical points on two sides of an acute triangle, than the hot-spots conjecture holds.*

As a consequence we get a partial proof of the conjecture.

Theorem 1.4. *Hot spots conjecture holds for acute triangles with an angle smaller or equal to $\pi/6$.*

In fact in Section 3.1 we prove the conjecture for these, and few more triangles. This result relies on two key lemmas: symmetry of the second Neumann eigenfunction of kites, and a critical point elimination lemma.

Let T be a triangle with vertices $(0,0)$, $(1,0)$ and (a,b) with $0 \le a \le 1/2$ and $b > 0$. Let K be the kite obtained by mirroring the triangle along x-axis (fourth vertex $(a,-b)$). We have

Lemma 1.5. *If $3b^2 \le 1 - a + a^2$ then the second Neumann eigenfunction of K is simple and symmetric with respect to x-axis, except for the square ($a = b = 1/2$) and equilateral triangle ($a = 0$ and $b = 1/\sqrt{3}$). In these cases K has double eigenvalue.*

The above lemma is used in conjunction with the following lemma to prove that there are no critical points on two sides of a triangle.

Lemma 1.6 (Generalization of [12, Lemma 3.2])**.** *Suppose that $\mu_2(T) \le \frac{\pi^2}{b^2}$ and φ is the eigenfunction for μ_2.*

- *φ has at most one critical point on the side on x-axis. This critical point (if exists) is a positive minimum or a negative maximum.*
- *If K has symmetric second Neumann eigenfunction, then φ has no critical point on x-axis and it is changing sign there.*

However to apply this lemma we need a bound for $\mu_2(T)$.

Lemma 1.7. *If $b^2 \le a^2 + (1-a)^2$ then*

$$\mu_2 \le \frac{\pi^2}{b^2}.$$

The condition holds in particular for any triangle with the longest or middle side on x-axis and the angle up to $\pi/4$ at vertex $(1,0)$.

For the discussion on how all these results can be applied to the hot-spots problem, see Section 3.1, in which we prove Theorem 1.4.

1.2. **History of the problem.** Hot spots conjecture was posed by J. Rauch in 1975 [**?**] for arbitrary open sets. The first positive result was obtained by Kawohl [8] for products of an arbitrary domain and an interval. In the same manuscript author also restates the conjecture just for convex sets. Subsequent counterexamples by Burdzy and Werner [3] (two holes) and Burdzy [4] (one hole) show that the restriction to convex domains might be necessary.

The hot-spots conjecture for convex domains remains open, however many special cases were solved. Bañuelos and Burdzy were able to handle domains with a line of symmetry and a few more technical assumptions [2]. A year later Jerison and Nadirashvili proved that the conjecture holds for domains with two lines of symmetry. In a different direction, Burdzy and Atar [1] assumed that the domain is bounded by graphs of two Lipschitz functions with constant 1.

All known results assume some degree of symmetry or special shape of the boundary. Surprisingly, domains as simple as acute triangles are not covered by any known result (note that obtuse

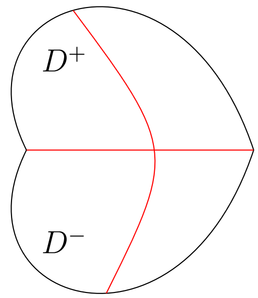


FIGURE 1. Second antisymmetric mode. Red curves denote nodal lines.

and right triangles were solved [2, 1]). The conjecture for isosceles triangles can be obtained by combining [1] and [10], or directly using new method due to Miyamoto [12]. Refinement of this new method lead to the results of this paper. There is also an active Polymath7 project [14] proposed by Chris Evans and moderated by Terrence Tao. The current focus of the project is on developing robust validated numerical methods that would lead to the proof of the hot spots conjecture for acute triangles and possibly other domains.

2. Preliminary results

2.1. **Symmetric modes.** Note that on a domain with a line of symmetry, the second Neumann eigenfunction is either symmetric or antisymmetric. In case the eigenvalue is double, we can decompose any eigenfunction into symmetric and antisymmetric parts.

Any symmetric mode on a symmetric domain satisfies Neumann condition on the line of symmetry. Hence it is also a mode for the half of the domain. Therefore the lowest symmetric mode on the symmetric domain must be the same as the eigenfunction for μ_2 for each half. Note however, that this symmetric mode does not need to belong to μ_2 on the whole domain. We need the following stronger results for symmetric modes.

Lemma 2.1. *Suppose D is a domain with a line of symmetry. Then there cannot be two orthogonal antisymmetric eigenfunctions in the span of the eigenspaces of μ_2 and μ_3 (note that μ_2 might equal μ_3).*

This means that either μ_2 or μ_3 must have a symmetric eigenfunction. It is also possible that all eigenfunctions for these eigenvalues are symmetric, as is the case for narrow subequilateral triangles, or narrow sectors.

Proof. Suppose that the line of symmetry divides D into D^+ and D^-, see Figure 1. Suppose also there are two orthogonal antisymmetric eigenfunctions in the span of the eigenspaces of μ_2 and μ_3. One of them must change sign in D^+ (and by antisymmetry in D^-), otherwise these would not be orthogonal. This eigenfunction will have at least 4 nodal domains, contradicting Courant's nodal domain theorem. □

Let $\lambda_1(D)$ be the smallest Dirichlet eigenvalue of D. To prove the next result we need the following eigenvalue comparison result

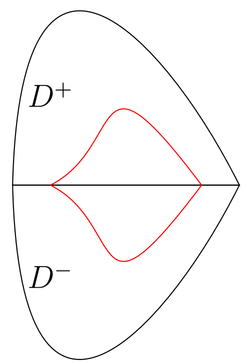


FIGURE 2. Nodal line cannot start and end on the straight part of the boundary (as shown).

Theorem 2.2 (Friedlander '95)**.** *For convex domains*

$$\lambda_1(D) \geq \mu_3(D).$$

Lemma 2.3 (Polymath7 [14])**.** *Suppose that we have a convex domain D^+ with a straight part of the boundary, such that the domain D obtained by mirroring about the straight part is also convex, see Figure 2 The second Neumann eigenvalue of D^+ cannot have an eigenfunction with nodal line starting and ending on this straight piece of the boundary. This includes the endpoints of the straight piece.*

Proof. Suppose the second eigenvalue $\mu_2(D^+)$ has an eigenfunction φ with nodal line starting and ending on the same straight piece of the boundary. We can unfold the domain D^+ and φ to get a symmetric domain $S = D^+ \cup D^-$ and its symmetric eigenfunction with closed nodal domain N inside. The Dirichlet eigenvalue of this nodal domain ($\lambda_1(N) = \mu_2(D^+)$) is strictly larger than the first Dirichlet eigenvalue $\lambda_1(S)$. This one is however larger than or equal to $\mu_3(S)$ (Theorem 2.2). Hence

$$\mu_2(D^+) = \lambda_1(N) > \lambda_1(S) \geq \mu_3(S). \tag{1}$$

Eigenfunction φ is the lowest symmetric mode of S, since it belongs to the smallest positive eigenvalue on D^+. By Lemma 2.1 it must belong to either $\mu_2(S)$ or $\mu_3(S)$. In either case we get a contradiction with Equation 1. □

2.2. **Nodal line approach.** In this section we collect the results needed for the approach due Miyamoto [12]. We generalize most of the key lemmas to avoid the symmetry assumptions for triangles.

First we need the following consequence of real analyticity for eigenfunctions

Lemma 2.4 ([12, Corollary 2.2])**.** *Suppose u satisfies $\Delta u = -\mu u$ on D (without any boundary condition). If $u(x,y) = u_x(x,y) = u_y(x,y) = 0$ (degenerate zero) then either $u \equiv 0$ or $\{u = 0\}$*

has at least 4 branches from (x, y) and $\{u > 0\}$ (and $\{u < 0\}$) has at least 2 connected components near (x, y) (but these might be globally connected).

We generalize [12, Lemma 2.3] (for convex domains) using Theorem 2.2.

Lemma 2.5. *Let D be a convex domain, and u be any function satisfying $\Delta u = -\mu u$ on D (without boundary conditions). If $\mu \leq \mu_3(D)$, then $\{u = 0\}$ has no loop in $\overline{D}$ (no nodal domain with boundary contained in $\{u = 0\}$).*

In particular nodal lines of partial derivatives of the first two eigenfunctions cannot have loops.

Proof. Suppose there is a loop and let F be the set enclosed by the loop. Then

$$\mu = \lambda_1(F) > \lambda_1(D) \geq \mu_3(D) \geq \mu.$$

Giving contradiction. □

We can also strengthen the first part of [12, Lemma 2.4].

Lemma 2.6. *If u is an eigenfunction for convex D belonging to μ_2 or μ_3 then u does not have a degenerate zero in D.*

Proof. Degenerate zero implies at least 4 branches for $\{u = 0\}$. Therefore locally we have two nodal domains where eigenfunction is positive, and between them there are two domains with negative sign. If the two positive nodal domains are globally connected, then there is curve that connects a point near the critical point from one of them to a point in the other. Hence the negative nodal domain between the two positive subdomains is closed inside the original domain. Hence the negative nodal domain forms a closed loop as part of the nodal set. But Lemma 2.5 states that there is no loop. Hence the positive nodal domains near the critical point are not connected, similarly the negative nodal domains. This contradicts Courant's nodal domain theorem, since we have at least 4 nodal domains. □

Define

$$\mathcal{H}_\mu[u] = \int_D (|\nabla u|^2 - \mu u^2) dA$$

Then by variational formula for eigenvalues

Lemma 2.7. *If $\int_D u = 0$, then $\mathcal{H}_{\mu_2}[u] \geq 0$.*

If $\int_D u = 0$ and u is symmetric, then $\mathcal{H}_{\mu_s}[u] \geq 0$. Here μ_s is the lowest symmetric mode for a symmetric D.

In general whenever u is a valid test function for λ, then $\mathcal{H}_\lambda[u] \geq 0$. Here λ can be any type of eigenvalue.

This observation was used by Miyamoto [12] in contradiction arguments. One needs to construct u such that $\mathcal{H}[u] < 0$. Suppose that $\Delta u = -\mu u$ on D (no boundary conditions). Then

$$\mathcal{H}_\mu[u] = \int_{\partial D} u \partial_\nu u d\sigma.$$

In particular for any mixed Dirichlet-Neumann boundary conditions $\mathcal{H}_\mu[u] = 0$. However, one can get a contradiction in the above lemma by controlling the sign of the product $u\partial_\nu u$.

Note that in Lemma 2.3 we can drop the assumption that we have an eigenfunction, cf. Lemma 2.5.

Lemma 2.8. *Let D be a convex domain with a straight piece of boundary, $\Delta u = -\mu_2 u$ and $\partial_\nu u \leq 0$ on the straight piece whenever $u > 0$. Then $\partial\{u > 0\}$ cannot have a connected component bounded by a curve starting and ending on the straight piece of the boundary.*

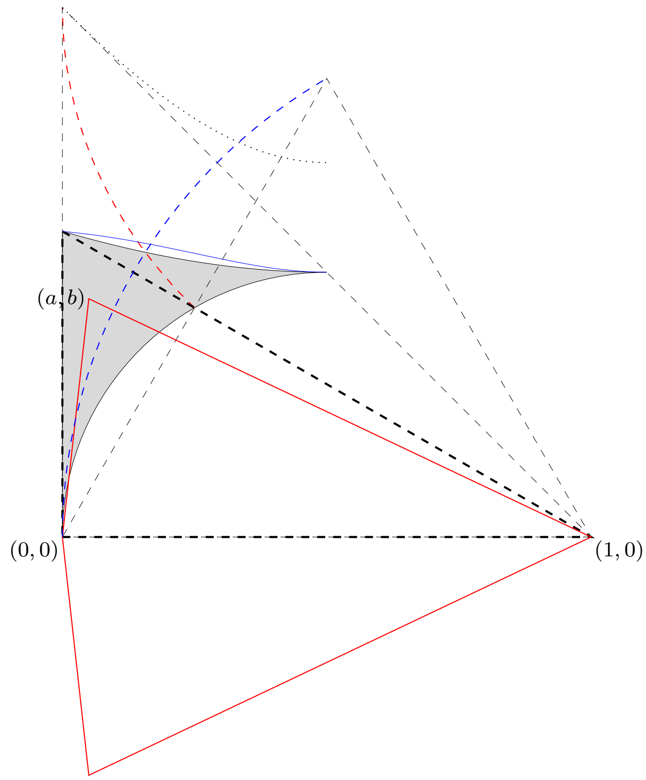


FIGURE 3. Triangle $T(a, b)$, its kite and various conditions on a and b.

Proof. If this was the case then $H_{\mu_2}[u] \leq 0$ on this connected component and μ_2 would be larger than or equal to the mixed Dirichlet-Neumann eigenvalue for the component. Now we can apply the argument from the proof of Lemma 2.3. □

3. PROOFS OF THE MAIN RESULTS

3.1. **Hot-spots conjecture.** Here we prove a result stronger than Theorem 1.4.

Note that the condition in Lemma 1.5

$$3b^2 < 1 - a + a^2 \tag{2}$$

implies the condition in Lemma 1.7

$$b^2 < a^2 + (1-a)^2. \tag{3}$$

Therefore as soon as we can apply Lemma 1.5, we also have eigenvalue bound from Lemma 1.7. Therefore Lemma 1.6 implies no critical point on the side on x-axis. Therefore the hot-spots conjecture is true for any triangle for which the former is satisfied for two ways to put the triangle in the coordinate system (the longest or the middle side on x-axis), by Theorem 1.3.

The dashed lines on Figure 3 form 3 triangles: equilateral, right isosceles and a half-of-equilateral right triangle (angle $\pi/6$ near $(1, 0)$). The dashed blue line gives (a, b) pairs for isosceles triangles. Below this line the longest side is on the x-axis, above it the middle (or the shortest) side. Hence all triangles can be uniquely described by a pair (a, b) with $(1-a)^2 + b^2 \leq 1$, while acute triangles satisfy $a^2 + b^2 > a$ (lower boundary of the gray area). The gray area contains all (a, b) pairs for which kite K has symmetric second eigenfunction according to (2), while a solid blue line just

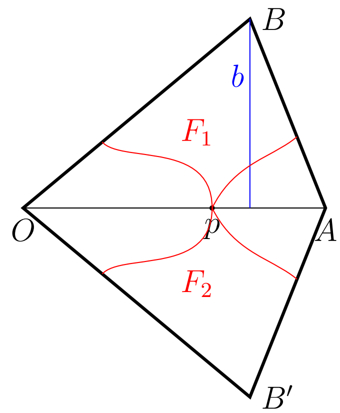


FIGURE 4. For branches from a critical point p inside a kite.

above the area is a numerical curve on which the symmetric mode equals antisymmetric mode. The black dotted line depicts the boundary of the condition (3) from Lemma 1.7.

Finally the red dotted line is the inversion of the upper part of the boundary of the gray area with respect to "isosceles circle" $(a-1)^2 + b^2 = 1$. It happens that the inversion of (a, b) gives a new placement for the same triangle (the middle side interchanges with the longest). Indeed, if (a, b) is inside the "isosceles circle", then the longest side of the triangle is on x-axis and has length 1. The middle side has vertices (a, b) and $(1, 0)$. If we invert the point (a, b) with respect to the "isosceles circle", then the length ratio between the longest and the middle side does not change, however the middle side is now on x-axis. Therefore rescaling to get the longest side of length 1 leads to the same triangle as the original triangle (same two sides and the angle between).

Therefore the hot spots conjecture is true for any triangle in the gray area and below the dashed red line. This clearly contains all triangles with smallest angle up to $\pi/6$ (below the thick dashed line near $(1, 0)$), implying Theorem 1.4.

3.2. **Proof of Lemma 1.6.** Let T be the triangle OAB and K be the kite $OBAB'$ (see Figure 4). We can assume that $|OA| = 1$. Let φ be the eigenfunction for $\mu = \mu_2(T)$. It is also the lowest symmetric mode for K, and it belongs to either $\mu_2(K)$ or $\mu_3(K)$. Suppose φ has a critical point p on OA. It cannot be 0 there, by Lemma 2.6. Without loss of generality we can assume that $\varphi(p) > 0$. If there is more than one critical point take the one with maximal value of $\varphi(p)$. Let

$$\psi(x, y) = \cos(\sqrt{\mu} y).$$

By assumption $\mu y^2 \leq \mu b^2 \leq \pi$, hence $\psi_y(x, y) < 0$ when $y > 0$. Therefore outward normal derivative $\partial_\nu \psi < 0$ on the boundary of K.

Take $u(x, y) = \varphi(p)\psi(x, y) - \varphi(x, y)$. Then $u(p) = u_x(p) = u_y(p) = 0$ (degenerate zero). Hence there are four branches of $\{u = 0\}$ around p (by Lemma 2.4), unless u is the eigenfunction, but it does not satisfy Neumann boundary condition. Furthermore $\{u > 0\}$ has at least 2 connected components near p and these cannot be globally connected since $\{u = 0\}$ has no loops. Therefore there are at least two disjoint subsets F_1 and F_2 of K such that $u \geq 0$ on $\overline{F_i}$. Finally $\partial_\nu u = \partial_\nu \psi < 0$ on ∂K.

Suppose $u \leq 0$ on OA, then $\varphi(x, y) \geq \varphi(p) > 0$ on OA, and the eigenfunction is strictly positive on OA. Furthermore all points such that $\varphi(x, y) = \varphi(p)$ are also critical points for the side and u is zero there. We will eliminate the possibility of 2 degenerate zeros later. Note also

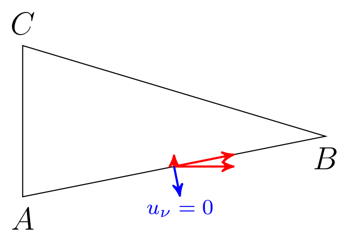


FIGURE 5. Acute triangle with one vertical side and no critical points on sloped sides.

that if we had two critical points with different value of φ, then we have taken the larger one as p, hence u will be positive somewhere and this case does not apply.

Suppose $u > 0$ somewhere on OA. Then at least one of F_i must contain a part of OA and it must be symmetric with respect to OA. Suppose F_1 has this property. Take $G = \{u > 0\} \setminus F_1$, then G is also symmetric, since $F_1 \cup G$ is symmetric.

Define a symmetric test function v

$$v = u1_{F_1} - cu1_G,$$

where c is chosen so that $\int_K v = 0$.

This is a valid test function for μ (regardless if it equals $\mu_2(K)$ or $\mu_3(K)$). Note that

$$v\partial_\nu v = (u\partial_\nu u)1_{F_1} + c^2(u\partial_\nu u)1_{F_2} \leq 0, \text{ on } \partial(F_1 \cup G).$$

Indeed, either $u = 0$ or $u > 0$ and outward normal is negative. Therefore $\mathcal{H}_\mu[v] \leq 0$. But it cannot be equal 0 since v equals 0 on an open set, and it cannot be the eigenfunction. This contradicts Lemma 2.7.

We already showed that if $u \leq 0$ on OA then we have a global minimum, possibly at two or more points. However this means two or more degenerate zeros for u. In this case degenerate zeros generate disjoint sets $\{u > 0\}$, since there are no loops. Take these sets as F_1 and G to get a contradiction. Hence there is only one global minimum.

Finally if $\mu_2(K)$ has symmetric eigenfunction we do not need symmetry of v and we can take any F_1 and F_2 in its definition. This proves that even if $u \leq 0$ on OA, we still cannot have a critical point. Moreover, since the $\mu_2(K)$ has symmetric eigenfunction, this eigenfunction must be 0 somewhere on the line of symmetry, otherwise we would have at least 3 nodal domains.

3.3. **Proof of Theorem 1.3.** Position the triangle as on Figure 5. Since this triangle is acute, the bottom side is sloped up, and the upper side is sloped down. Let u be the second Neumann eigenfunction of ABC. We know that there are no critical points on two sloped sides, and we have Neumann boundary conditions there. Therefore u_x and u_y cannot change sign on these sides and they are never 0 there. Note also that $u_x = 0$ on AC, u_x and u_y must have the same signs on AB, and opposite signs on BC.

If $u_x > 0$ (or $u_x < 0$) on both AB and CB, then $u_x \geq 0$ on the boundary of the triangle. If there was a point p inside ABC such that $u_x(p) = 0$, then real analyticity implies that $u_x < 0$ at some point near p. Therefore $u_x < 0$ would have to form an open nonempty subset inside (possibly with a piece of the boundary on AC). Hence $u_x = 0$ would have a loop, contradicting Lemma 2.5. Therefore $u_x > 0$ inside ABC. Therefore the global maximum and the global minimum of u must be on the boundary. One of them must be at B, the other on AC. This is the case for subequilateral triangles.

If $u_x > 0$ on AB and $u_x < 0$ on CB, then $u_y > 0$ on AB and on CB (similar argument for opposite signs). Furthermore u_y satisfies Neumann boundary condition on AC, since $(u_y)_x = (u_x)_y = 0$ on AC. As before $\{u_y < 0\}$ would need to form a nonempty subset of T, possibly with a part of the boundary on AC. But this contradicts Lemma 2.8. Hence $u_y > 0$ on T. Hence the maximum is at C and the minimum at A. This is the case for superequilateral triangles.

As a consequence we obtain

Corollary 3.1 (Atar, Burdzy [1]). *Hot-spots conjecture holds for the lowest symmetric mode of any acute isosceles triangle (note that for superequilateral triangles this is the third eigenfunction). This in turn implies that the conjecture holds for all right triangles.*

Note that for subequilateral triangles and the corresponding right triangles our proof follows closely Miyamoto's proof of the same result.

Proof. For right triangles with the longest side of length 1 and the shortest altitude b, Theorem 3.1 from [9] gives

$$\mu_2 b^2 \leq \frac{4\pi^2 b^2}{3\sqrt{3}A} \leq \frac{4\pi^2 b^2}{3\sqrt{3}b^2} = \frac{4\pi^2}{3\sqrt{3}} \leq \pi^2.$$

Hence we can apply Lemma 1.6 to the half of the acute isosceles triangle mirrored along the longest side. We prove below that corresponding kite has symmetric eigenfunction hence there are no critical points on the equal sides of the isosceles triangle. Now we apply Theorem 1.3. □

3.4. **Proof of Lemma 1.7.** Start with the second eigenfunctions for two right isosceles triangles with $(a, b) = (0, 1)$ and $(a, b) = (1/2, 1/2)$. These are

$$\varphi_1(x, y) = \cos(\pi y) - \cos(\pi x),$$
$$\varphi_2(x, y) = \cos(\pi x)\cos(\pi y).$$

The only property of these functions we actually need is that they integrate to 0 over their respective right isosceles triangles (orthogonal to constants). Now we apply linear transformations to obtain functions on $T(a, b)$ with arbitrary (a, b). We still have orthogonality to constants. Take

$$f(x, y) = (1/2 - a)\varphi_1(x - ay/b, y/b) - a\varphi_2(x + (1 - 2a)y/2b, y/2b).$$

This is a valid test function for $\mu_2(T)$ (integrates to 0 over $T(a, b)$). Note that when $a = 0$ or $a = 1/2$ we recover the exact eigenfunctions for the right isosceles triangles we considered. Let $c = a(a - 1)$. We get

$$\mu_2(T) \leq \frac{\pi^2(2c(2 + b^2 + c) + b^2 + 1) - 16c(b^2 + c)}{2(3c + 1)b^2} \overset{?}{\leq} \frac{\pi^2}{b^2},$$

where we need to prove the last inequality. Hence we need

$$\pi^2(2c(2 + b^2 + c) + b^2 + 1) - 16c(b^2 + c) - 2\pi^2(3c + 1) \leq 0.$$

Put $d = b^2 + c$. Now we need

$$0 \geq \pi^2(2cd + d + 1) - 16cd - \pi^2(3c + 2) = d(\pi^2(1 + 2c) - 16c) - \pi^2(3c + 1)$$

Note that $0 \geq c \geq -1/4$, hence the coefficient in front of d is positive. But

$$d = b^2 + a^2 - a \leq a^2 + (1 - a)^2 + a^2 - a = 3a^2 - 3a + 1 = 3c + 1.$$

Hence the desired inequality is true if

$$0 \geq (3c + 1)(\pi^2(1 + 2c) - 16c) - \pi^2(3c + 1) = (3c + 1)c(2\pi^2 - 16)$$

But $3c+1>0$, $c\le 0$ and $2\pi^2-16>0$. Hence the inequality is true.

4. Kites and Lemma 1.5

Recall that for a triangle $T(a,b)$ we define a kite K by mirroring the triangle with respect to the x-axis. We consider the lowest antisymmetric modes and their eigenvalues $\mu_a(K)$. We prove that if $3b^2\le 1-a+a^2$, then the eigenvalues μ_a are above the second Neumann eigenvalue. This ensures that all eigenfunctions for $\mu_2(K)$ are symmetric. But then they are also eigenfunctions for $T(a,b)$ (with simple eigenvalue). Hence these kites have simple second eigenvalue. This proves Lemma 1.5.

Let μ_a be the lowest antisymmetric mode on K. Then μ_a is the lowest eigenvalue of the mixed Dirichlet-Neumann problem on $T(a,b)$ with Dirichlet condition on x-axis. We find lower bound for this eigenvalue using unknown trial function method developed in [10] and [11].

Then we find an upper bound for $\mu_2(K)=\mu_2(T)$ that is smaller than the lower bound from the first step.

4.1. Lower bound for μ_a.

Let $\lambda(a,b)=\mu_a(a,b)$ be the lowest eigenvalue of the mixed problem with Dirichlet condition on x-axis and Neumann on the other two sides. We will use the following unknown trial function lemma

Lemma 4.1 ([11, Lemma 4.1])**.** *The inequality*

$$\lambda(a,b)\ge C_{a,b,c,d}\lambda(c,d)$$

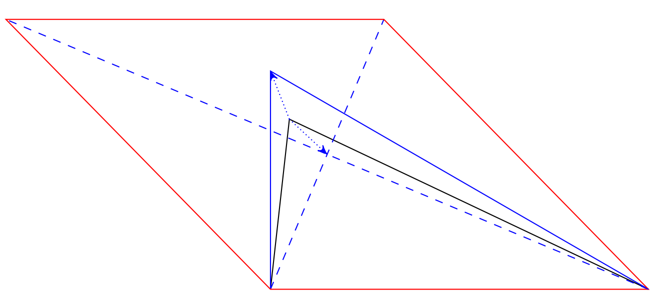
is true if

$$((a-c)^2+d^2)(1-\gamma)+2b(a-c)\delta+b^2\gamma\le d^2/C_{a,b,c,d},$$

where δ and γ are some numbers (unfortunately unknown) depending only on a and b and satisfying $|\delta|\le 1/2$ and $0\le\gamma\le 1$.

Remark. *This lemma relies on linear transformation between triangles. However the result holds for any family of domains that can be obtained using the same linear transformation. In particular, the same is true for triangles $T(a,b)$ (with vertices $(0,0)$, $(1,0)$ and (a,b)), the triangle notation in this paper. Furthermore, this lemma applies to any mixed boundary conditions (see also* [10, Corollary 5.5]*).*

Remark. *In order to use this inequality we would need to prove the "if" part for any γ and δ. Instead, we can choose a few sets of values of c and d so the eigenvalues on the right are explicit, effectively obtaining 2 inequalities involving a,b,γ,δ. For fixed a and b we need to show that at least one of those inequalities is true for any admissible pair (γ,δ).*



We need to consider two pairs (c,d).

(1) $(c,d) = (0, 1/\sqrt{3})$ (blue half-equilateral triangle above). On this triangle $\lambda(0, 1/\sqrt{3}) = 4\pi^2/3$ (the lowest Neumann eigenvalue of the equilateral triangle with side length 4/3, see e.g. McCartin ??).
(2) $(c,d) = (c, \sqrt{c-c^2} =: h)$, where $a \le c \le 1/2$ will be chosen later. Here we get $\lambda = \lambda_1(R)$ (the first Dirichlet eigenvalue of the red rhombus, see picture). We can use Hooker-Protter bound [6] to get

$$\lambda(c,h) \ge \frac{\pi^2(1+2h)}{4h^2}.$$

See also [5] for comparisons of known bounds for rhombi, showing that Hooker-Protter bound is the best for relatively square rhombi.

Suppose we want to prove

$$\mu_a((a,b)) = \lambda(a,b) \ge \frac{4\pi^2}{3F},$$

for some, not yet known $F = F(a,b)$.

(1) First consider $c = 0$ and $d = 1/\sqrt{3}$. To get the bound we want we need

$$(3a^2+1)(1-\gamma) + 6ab\delta + 3b^2\gamma \overset{?}{\le} F, \tag{4}$$

(2) Using $d = h = \sqrt{c-c^2}$ we need

$$((a-c)^2 + h^2)(1-\gamma) + 2b(a-c)\delta + b^2\gamma \overset{?}{\le} \frac{3(1+2h)F}{16} \tag{5}$$

We need to show that at least one of the inequalities (4,5) is true. We can achieve that by proving that one positive linear combination of those inequalities is true. We can choose this linear combination so that δ cancel.

Therefore we combine $c - a$ times (4) and $3a$ times (5) to get

$$(3ca(1-a) + c - a)(1-\gamma) + 3b^2c\gamma \overset{?}{\le} \left(c - a + \frac{9a(1+2h)}{16}\right) F. \tag{6}$$

This inequality must be true for any $0 \le \gamma \le 1$. To simplify the task we may choose c so that the expressions in front of $1-\gamma$ and γ are equal, effectively eliminating γ. That is

$$c = \frac{a}{1 - 3(b^2 + a^2 - a)},$$

with $\delta = a^2 + b^2 - a \ge 0$. Note that $c = a$ for all right triangles ($\delta = 0$), hence $T(c,h) = T(a,b)$ for right triangles and we are using Hooker-Protter bound for all right triangles (no contribution from the first case, no deformation in the second). This gives

$$\frac{3b^2a}{1-3\delta} \le a\left(\frac{3\delta}{1-3\delta} + \frac{9(1+2h)}{16}\right) F$$

We can treat the equality case of this inequality as the definition of F. Hence we can take

$$\frac{1}{F} = \frac{3 + 7\delta + 6\sqrt{a(1-3\delta-a))}}{16b^2}$$

Therefore we proved that

$$\mu_a(a,b) \geq \frac{\pi^2\left(3+7\delta+6\sqrt{a(1-3\delta-a)}\right)}{12b^2}$$

The assumption $3b^2 < 1-a+a^2$ in Lemma 1.5 is equivalent to $1-3\delta-a < -4a^2+3a$. Under this assumption the bound simplifies to

$$\mu_a(a,b) \geq \frac{\pi^2\left(3+7\delta+6a\sqrt{3-4a}\right)}{12b^2}$$

Note that the c we choose above, for $3b^2 = 1-a+a^2$, satisfies $c = \frac{1}{4(1-a)} > \frac{1}{4}$. Hence even for $a \approx 0$ (near equilateral) we are using rhombi that are not far from square, hence Hooker-Protter bound is the most accurate known according to Figure 12 in [5] (in the notation of this paper we are dealing with rhombi with $a \geq \sqrt{3}/3$). Obviously for smaller values of b we are using much smaller c's, but these cases are far from critical numerical curve.

4.2. **General variational upper bound approach.** Variational upper bounds involving linear combinations of any number of transplanted exact eigenfunctions always have the following form

$$\mu_2(a,b) \leq \frac{A(a)+B(a)b^2}{C(a)b^2}, \tag{7}$$

where $A(a)$, $B(a)$ and $C(a)$ are polynomials. One way to show that $\mu_a > \mu_2$ is to first show that $C(a) > 0$ for $0 \leq a \leq 1/2$. Then prove that

$$12A(a) + 12B(a)b^2 \leq C(a)\pi^2(3+7a^2+7b^2-7a+6a\sqrt{3-4a}).$$

This is however equivalent to

$$12A(a) + (12B(a) - 7\pi^2C(a))b^2 \leq C(a)\pi^2(3+7a^2-7a+6a\sqrt{3-4a}). \tag{8}$$

Next we show that the polynomial in front of b^2 is positive and we get that the left hand side is increasing with b, while the right hand side is decreasing. Therefore we can put any upper bound for b involving a and we get an inequality for a. This inequality will have one square root, but it can be transformed into a high order polynomial inequality, and we need to prove it for all $0 \leq a \leq 1/2$.

Note that we do not need a sharp inequality in the above inequalities, as long as we find upper bound that is not sharp for any $T(a,b)$ (except for known cases with double eigenvalue, square and equilateral). Then obviously $\mu_2(K)$ has both symmetric and antisymetric eigenfunction.

4.3. **Upper bound and the proof of Lemma 1.5.** We will take a linear combination of 3 eigenfunctions, 1 from a half-equilateral triangle and 2 from a right isosceles triangle. To be more precise we need the second eigenfunction on $T(0,1/\sqrt{3})$ and the first two nonconstant eigenfunctions from $T(1/2,1/2)$. Take

$$\begin{aligned}\varphi(x,y) &= (2a-1)\cos\left(\frac{2\pi y}{3b}\right)\left(1-2\cos\left(\frac{\pi(bx-ay)}{b}\right)\right)+\\ &+ 4a\cos\left(\frac{\pi y}{2b}\right)\cos\left(\frac{\pi(2bx+(1-2a)y)}{2b}\right)+\\ &+ 2a(2a-1)\cos\left(\frac{\pi(bx+(1-a)y)}{b}\right)\cos\left(\frac{\pi(bx-ay)}{b}\right).\end{aligned}$$

Note that only the first term is present for $a=0$, in particular for $T(0,1/\sqrt{3})$. On the other hand only the middle term is nonzero for $T(1/2,1/2)$. Therefore we recover exact eigenfunctions for these special cases. We will not need this fact, nor that we used eigenfunction in our test function. We only need to know that this function integrates to 0 over $T(a,b)$, hence it is a good test function for μ_2. This gives an upper bound in the form (7) with

$$\begin{aligned}C(a) &= 67200\pi^2a^4 + 12(74976-5600\pi^2)a^3 + 12(15400\pi^2-182346)a^2+\\&\quad + 12(72429-8400\pi^2)a + 25200\pi^2\\&\ge 12(15400\pi^2-182346)a^2 + 12(72429-8400\pi^2)a + 25200\pi^2,\end{aligned}$$

Coefficients for a^2 and a are negative, hence we can put $a=1/2$ and we get $C(a)>0$ for $0\le a\le 1/2$.

We also have

$$\begin{aligned}B(a) &= \pi^2\Big(134400\pi^2a^4 + (650880-134400\pi^2)a^3 + (168000\pi^2-1867740)a^2+\\&\quad + (784800-67200\pi^2)a + (127575+16800\pi^2)\Big),\\A(a) &= \pi^2\Big(134400\pi^2a^6 + (650880-268800\pi^2)a^5 + (369600\pi^2-2518620)a^4+\\&\quad + (2105752-235200\pi^2)a^3 + (89600\pi^2-924757)a^2+\\&\quad + (227938-22400\pi^2)a + (5600\pi^2-42525)\Big).\end{aligned}$$

We need to show that the following polynomial is positive

$$\begin{aligned}\frac{12B(a)-7\pi^2C(a)}{12\pi^2} &= 95200\pi^2a^4 + (126048-95200\pi^2)a^3 + (60200\pi^2-591318)a^2+\\&\quad + (277797-8400\pi^2)a + (2100\pi^2+127575)\end{aligned}$$

Note that only the coefficient for a^3 is negative, and replacing a^3 with $1/8$ still gives positive constant coefficient. Hence we proved that $12B-7\pi^2C>0$. Therefore in (8) we can replace b with its maximal value $\sqrt{(1-a+a^2)/3}$ and we need to prove that

$$\begin{aligned}&P(a)a \le 18Q(a)a\sqrt{3-4a}, \qquad (9)\\&Q(a) = 5600\pi^2a^4 + (74976-5600\pi^2)a^3 + 2(7700\pi^2-91173)a^2+\\&\qquad + (72429-8400\pi^2)a + 2100\pi^2,\\&P(a) = 380800\pi^2a^5 + (504192-761600\pi^2)a^4 + (868000\pi^2-2869464)a^3-\\&\qquad - 480(665\pi^2-2682)a^2 - 8(44211+2450\pi^2)a + 525(347+80\pi^2)\end{aligned}$$

Again we note that coefficients of a^4 and a^3 in Q are positive, hence we can disregard these terms. While coefficients of a^2 and a in Q are negative, hence we can put $a=1/2$ and we get $Q(a)>0$. Therefore it is enough to show

$$P^2(a) - 18^2Q^2(a)(3-4a) \le 0. \qquad (10)$$

Note that for $T(0,1/\sqrt{3})$ and $T(1/2,1/2)$ we have exact eigenvalues. Hence we can expect that $a=0$ and $a=1/2$ are roots of the above polynomial. In fact we already eliminated $a=0$ in (9). However $a=1/2$ is a double root of (10) and we can reduce the degree by 2. Numerical results suggest that the inequality is roughly true for $a\in[-0.49,0.52]$ ($(0,1/2)$ is needed). To avoid closeness of the positive root substitute $a \to 1/2 - a$. We still need to prove the new inequality for $a \in [0, 1/2]$ and it is numerically true up to $a \approx 1$. We are left with 8-degree polynomial inequality

$$\begin{aligned} 0 \geq & 36252160000\pi^4 a^8 + (-95998156800\pi^2 - 46412800000\pi^4)a^7 + \\ & + (63552393216 - 34277644800\pi^2 + 83354880000\pi^4)a^6 + \\ & + (-1352162962944 + 582294182400\pi^2 - 149461760000\pi^4)a^5 + \\ & + (-3554482258800 - 300435206400\pi^2 + 121433760000\pi^4)a^4 + \\ & + (-1682712947520 + 1404232972800\pi^2 - 139740160000\pi^4)a^3 + \\ & + (4864275678312 - 1107844970400\pi^2 + 70309120000\pi^4)a^2 + \\ & + (-1418249685780 + 311172170400\pi^2 - 20603520000\pi^4)a + \\ & + (3669120000\pi^4 - 36985183200\pi^2) \end{aligned}$$

We will reduce this polynomial to a negative constant by increasing it and simplifying at the same time. We apply the following steps

(1) Coefficient for a^8 is positive, hence we can replace a^8 with $a^7/2$. Similarly for a^6.
(2) Coefficient for a^4 is positive, hence we can replace a^4 with $a^3\frac{\frac{2}{7}+\frac{7}{2}a^2}{2} \geq a^4$. Here simple linear estimate is too rough yielding false inequality.
(3) Similarly replace a^2 with $a\frac{\frac{1}{2}+2a^2}{2} \geq a^2$.
(4) New coefficients for a^7 and a^5 are still negative, hence we can replace a^7 and a^5 with 0.
(5) New coefficient for a^3 is positive, hence we can replace a^3 with $a/4$.

After all these steps we get a linear function with negative coefficients, proving the inequality is true and it is strict.

Therefore we get strict inequality $\mu_a > \mu_2$ for all cases except $T(0, 1/\sqrt{3})$ and $T(1/2, 1/2)$. These triangles have double eigenvalues.

5. Simplicity for μ_2

Simplicity argument goes here (already posted on Polymath7).

References

[1] **R. Atar and K. Burdzy**, *On Neumann eigenfunctions in lip domains*, J. Amer. Math. Soc. **17** (2004), *no. 2*, 243–265 (electronic). MR2051611
[2] **R. Bañuelos and K. Burdzy**, *On the "hot spots" conjecture of J. Rauch*, J. Funct. Anal. **164** (1999), *no. 1*, 1–33. MR1694534
[3] **K. Burdzy and W. Werner**, *A counterexample to the "hot spots" conjecture*, Ann. of Math. (2) **149** (1999), *no. 1*, 309–317. MR1680567
[4] **K. Burdzy**, *The hot spots problem in planar domains with on hole.*, Duke Math. J. **129** (2005), *no. 3*, 481–502. doi:10.1215/S0012-7094-05-12932-5
[5] **P. Freitas and B. Siudeja**, *Bounds for the first Dirichlet eigenvalue of triangles and quadrilaterals*, ESAIM Control Optim. Calc. Var. **16** (2010), *no. 3*, 648–676. MR2674631
[6] **W. Hooker and M. H. Protter**, *Bounds for the first eigenvalue of a rhombic membrane*, J. Math. and Phys. **39** (1960/1961), 18–34. MR0127610
[7] **D. Jerison and N. Nadirashvili**, *The "hot spots" conjecture for domains with two axes of symmetry*, J. Amer. Math. Soc. **13** (2000), *no. 4*, 741–772. MR1775736
[8] **B. Kawohl**, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Mathematics, vol. 1150, Springer-Verlag, Berlin, 1985. MR810619

[9] **R. S. Laugesen and B. A. Siudeja**, *Maximizing Neumann fundamental tones of triangles*, J. Math. Phys. **50** (2009), *no. 11*, 112903, 18. MR2567204

[10] **R. S. Laugesen and B. A. Siudeja**, *Minimizing Neumann fundamental tones of triangles: an optimal Poincaré inequality*, J. Differential Equations **249** (2010), *no. 1*, 118–135. MR2644129

[11] **R. S. Laugesen and B. A. Siudeja**, *Dirichlet eigenvalue sums on triangles are minimal for equilaterals*, Comm. Anal. Geom. **19** (2011), *no. 5*, 855–885. MR2886710

[12] **Y. Miyamoto** *A planar convex domain with many isolated "hot spots" on the boundary*, Japan Journal of Industrial and Applied Mathematics **30** (2013), *no. 1*, 145-164. doi:10.1007/s13160-012-0091-z

[13] **J. Rauch**, *Five problems: an introduction to the qualitative theory of partial differential equations*, Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974), Springer, Berlin, 1975, pp. 355–369. Lecture Notes in Math., Vol. 446. MR0509045

[14] Polymath7 project: The hot spots conjecture, `http://michaelnielsen.org/polymath1/index.php?title=The_hot_spots_conjecture`