The Partition of Unity Finite Element Method: Basic Theory and Applications

J.M. Melenk and I. Babuška[1]

Research Report No. 96-01
January 1996

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

[1]TICAM, The University of Texas at Austin, Austin, TX 78712, USA

The Partition of Unity Finite Element Method: Basic Theory and Applications

J.M. Melenk and I. Babuška[1]

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Research Report No. 96-01 January 1996

Abstract

The paper presents the basic ideas and the mathematical foundation of the partition of unity finite element method (PUFEM). We will show how the PUFEM can be used to employ the structure of the differential equation under consideration to construct effective and robust methods. Although the method and its theory are valid in n dimensions, a detailed and illustrative analysis will be given for a one dimensional model problem. We identify some classes of non-standard problems which can profit highly from the advantages of the PUFEM and conclude this paper with some open questions concerning implementational aspects of the PUFEM.

Keywords: Finite element method, meshless finite element method, robust finite element methods, finite element methods for highly oscillatory solutions

[1]TICAM, The University of Texas at Austin, Austin, TX 78712, USA

1 Introduction

The aim of this paper is to present a new method for solving differential equations, the "partition of unity finite element method" (PUFEM). We explain the mathematical foundation of the PUFEM and discuss some of its features. The most prominent among them are

1. the ability to include a priori knowledge about the local behavior of the solution in the finite element space;

2. the ability to construct finite element spaces of any desired regularity (as may be important for the solution of higher order equations);

3. the fact that the PUFEM falls into the category of "meshless" methods; a mesh in the classical sense does not have to be created and thus the complicated meshing process is avoided;

4. the fact that the PUFEM can be understood as a generalization of the classical h, p, and hp versions of the finite element method.

In this paper, we will mostly concentrate on the first of these four features. In particular, the one dimensional example of section 4 illustrates the fact that the PUFEM enables us to construct finite element spaces which perform very well in cases where the classical finite element methods fail or are prohibitively expensive. The success of the PUFEM in this example is precisely due to the fact that the PUFEM offers an easy way to include analytical information about the problem being solved in the finite element space. A similar example was analyzed in [15] for a problem with a boundary layer. Again, the PUFEM permitted the construction of finite element spaces which account for the boundary layer behavior and thus led to a robust method in the sense that the performance of the method is independent of the actual strength of the boundary layer. An application of the PUFEM to the Timoshenko beam with hard elastic support can be found in [16]. The paper is organized as follows. The rest of section 1 establishes that the two main ingredients of finite element spaces are local approximation properties and some interelement continuity. The PUFEM constructs a global *conforming* finite element space out of a set of given local approximation spaces – the precise construction is described in section 2. Therefore, the PUFEM separates the issues of interelement continuity and local approximability and allows us to concentrate on finding good local approximation spaces for a given problem. In section 3, we give a few examples of spaces with good local approximation properties for several differential equations. A detailed example of the PUFEM is presented in section 4 for a one dimensional model problem with rough coefficients. In section 4 we construct local approximation spaces which reflect the rough behavior of the solution and show that they are robust. The numerical example of 4.3 illustrates the robust performance of the PUFEM. We conclude the paper in section 5 with an application of the PUFEM to the two dimensional Helmholtz equation and identify some open questions concerning implementational aspects.

1.1 The Finite Element Method

The finite element method (FEM) for the solution of linear problems can be understood as follows. The problem is formulated in a *weak* form

$$\text{find } u \in \mathcal{X} : \; B(u,v) = F(v) \qquad \forall v \in \mathcal{Y} \tag{1}$$

where $\mathcal{X}$, $\mathcal{Y}$ are Hilbert spaces with norms $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$. $B : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ is continuous and bilinear, and $F : \mathcal{Y} \mapsto \mathbb{R}$ is continuous and linear. Of course, in all problems of interest, the spaces $\mathcal{X}$, $\mathcal{Y}$ are infinite dimensional. In the FEM finite dimensional subspaces $\mathcal{X}_n \subset \mathcal{X}$ (called the trial spaces), $\mathcal{Y}_n \subset \mathcal{Y}$ (called the test spaces) of dimension n are chosen and the finite element approximation u_{FE} is defined as the solution of

$$\text{find } u_{FE} \in \mathcal{X}_n : \; B(u_{FE},v) = F(v) \qquad \forall v \in \mathcal{Y}_n. \tag{2}$$

In order for the approximations u_{FE} to converge to the exact solution u, the following two conditions are necessary:

- *Approximability*: u can be approximated well by the subspaces $\mathcal{X}_n$; at least, we need $\inf\{\|u - v\|_{\mathcal{X}_n} \,|\, v \in \mathcal{X}_n\} \to 0$ as $n \to \infty$;
- *Stability*: The bilinear form B (together with the subspaces $\mathcal{X}_n$, $\mathcal{Y}_n$) satisfies an inf-sup condition (also known as the BB condition, see [1]).

In particular, if the stability condition holds, then problem (2) has a unique solution u_{FE} which satisfies

$$\|u - u_{FE}\|_{\mathcal{X}} \le C \inf_{v \in \mathcal{X}_n} \|u - v\|_{\mathcal{X}} \tag{3}$$

with a constant $C > 0$ independent of u and n. Thus the error of the finite element approximation is – up to the constant C – as small as the error of the best approximant in the space $\mathcal{X}_n$. Therefore, given stability, the performance of the finite element method is determined by the approximation properties of the spaces $\mathcal{X}_n$ for the approximation of the solution u. These observations lead to the problem of constructing spaces $\mathcal{X}_n$ which are conforming (i.e., $\mathcal{X}_n \subset \mathcal{X}$) and which have good approximation properties for the approximation of the exact solution u.

1.2 Local Approximability and Interelement Continuity

Let us now consider some of the classical choices of the trial spaces $\mathcal{X}_n$ and see how the condition to be conforming and the approximation properties are realized. In many applications (e.g., the heat equation, the elasticity equations in displacement formulation) the space $\mathcal{X}$ is a subspace of the Sobolev space H^1. We will therefore concentrate on the classical piecewise polynomial subspaces of H^1. In the classical FEM the spaces $\mathcal{X}_n$ are chosen such that they have good local approximation properties and are conforming; more precisely, they are chosen to consist of piecewise polynomials (or mapped polynomials) and are continuous across element boundaries. In the h-version of the FEM, the polynomial degree is fixed (typically, $p \le 2$) and the approximation is realized by decreasing the mesh size h. If the function u to be approximated is sufficiently smooth

(in H^k, say), an appropriate interpolant Iu (for example, for $p = 1$ and piecewise linear functions on triangles, one can choose the nodal interpolant) satisfies an estimate of the form

$$\|u - Iu\|_{H^1} \leq C_{k,p} h^{\min(k-1,p)} |u|_{H^k} \tag{4}$$

where $C_{k,p}$ is independent of u and h. We see that the approximation properties of these classical h-version type finite element spaces are good whenever the exact solution is not "rough". By "rough" we mean here and in the rest of this paper that either higher derivatives of u are not square integrable (i.e., the case that k is close to 1) or that they exist but are very large (i.e., $|u|_{H^k}$ is big). In both cases, the approximation with piecewise polynomial functions performs very poorly and the mesh size h has to be chosen very small before a reasonable accuracy is achieved (cf. section 4 and lemma 4.1).
In the p version of the FEM, the mesh is fixed and the local approximation is realized by polynomials (or mapped polynomials) of increasing degree. Again, continuity across the interelement boundaries is enforced in order to ensure conformity of the finite element spaces. The error estimates typically have the form

$$\|u - Iu\|_{H^1} \leq C_k p^{-(k-1)} |u|_{H^k}, \tag{5}$$

and thus the p version can be expected to work well whenever the exact solution is reasonably smooth; however, the p version exhibits the same deficiencies as the h version whenever the exact solution is rough.
In conclusion, the approximation properties of both the h and the p version of the finite element method are based on the fact that

1. (local approximability) a smooth function can be approximated locally by polynomials, and

2. (conformity of the finite element spaces/interelement continuity) polynomial spaces are big enough to absorb extra constraints of continuity across interelement boundaries without loosing the approximation properties.

Conversely, any system of functions which have good local approximation properties and can be constrained to satisfy some interelement continuity leads to a good finite element method.
Let us first elaborate the problem of local approximability. There are many systems of functions which have good local approximation properties. For certain types of equations, one can exploit the structure of the differential equation to construct spaces of functions which can approximate the solution even better than the spaces of polynomials. In section 3 we give a few examples of spaces which have very good approximation properties for the solution of Laplace's equation, the homogeneous Helmholtz equation, and the elasticity equations in two dimensions. For example, for the approximation of harmonic functions, it is enough to approximate locally with *harmonic* polynomials–it is not necessary to use the full space of polynomials. In the example of Helmholtz's equation, we see below that local approximation can be done with systems of plane waves or with spaces based on radial Bessel functions. Finally, in section 4, we consider a one dimensional model problem with rough coefficients and construct spaces of functions (which take into

account the rough behavior of the coefficients of the differential equation) which have good local approximation properties for the approximation of the (also rough) solution. In this example, the PUFEM based on these special functions leads to a robust method, i.e., a method which performs as well as the classical FEM performs for a problem with smooth coefficients. This is due to the fact that the special ansatz functions incorporate the rough behavior of the solution.
Let us now turn to the problem of conformity of the finite element space/interelement continuity. We have just seen that it is possible to construct many spaces of functions (typically non-polynomial) which have good local approximation properties for the approximation of a solution u of a differential equation. In general, it is not possible to enforce conformity, i.e., interelement continuity, for these non-polynomial local approximation spaces. The PUFEM, however, offers a means to construct a conforming space out of any given system of local approximation spaces without sacrificing the approximation properties. This is done as follows. Let $\{\Omega_i\}$ be a system of overlapping patches which cover the domain Ω of interest. Let $\{\varphi_i\}$ be a partition of unity subordinate to the cover. On each patch Ω_i, let $V_i \subset H^1(\Omega_i)$ be a space of functions by which $u|_{\Omega_i}$ can be approximated well. The global finite element space V is then defined by $\sum_i \varphi_i V_i$. Theorem 2.1 below states that the global space V inherits the approximation properties of the local spaces V_i, i.e., the function u can be approximated on Ω by functions of V as well as the functions $u|_{\Omega_i}$ can be approximated in the local spaces V_i. Moreover, the space V inherits the smoothness of the partition of unity φ_i. In particular, the smoothness of the partition of unity enforces the conformity of the global space V.

1.3 Potential Applications of the PUFEM

We already mentioned above that one potential field of application of the PUFEM are problems where the classical polynomial based FEM fail. In this category fall problems where the solution is rough (or highly oscillatory) and the usual piecewise polynomial spaces cannot resolve the essential features of the solution unless the mesh size h is very small or the polynomial degree p is very large. In both cases the computational costs are high or even too high for today's computers. Examples of problems with rough or highly oscillatory solutions are the elasticity equations for laminated materials, materials with stiffeners, or the Helmholtz equation for large wave numbers to mention but a few. In section 4 the PUFEM is applied to a problem with rough coefficients.
Problems of singularly perturbed type or problems where the solution exhibits a boundary layer can also be dealt with very successfully in the framework of the PUFEM. If the singular behavior of the solution is known, the PUFEM allows us to incorporate this knowledge directly into the finite element space. In contrast to this, the classical FEM has to use very small mesh sizes in order to resolve the singular behavior of the solution ([15]).
We mentioned above that the PUFEM falls in the general category of "meshless" methods. This feature might be exploited for certain problems where the usual methods involve frequent remeshing. For example, in the problem of the optimal placement of a fastener, the engineer has to try several locations of the fastener. For each run, he has to remesh parts of the domain in order to account for the changed position of the fastener.

One could construct a local approximation space which models the fastener and then changing the position of the fastener simply means changing the local approximation spaces.

2 Mathematical Foundation of the PUFEM

In this section, we present a method of constructing conforming subspaces of $H^1(\Omega)$. We construct finite element spaces which are subspaces of $H^1(\Omega)$ as an example because of their importance in applications. We would like to stress at this point that the method leads to the construction of smoother spaces (subspaces of H^k, $k > 1$) or subspaces of Sobolev spaces $W^{k,p}$ in a straight forward manner. The main technical notion in the construction of the PUFEM spaces is the (M, C_∞, C_G) partition of unity.

Definition 2.1 *Let $\Omega \subset \mathbb{R}^n$ be an open set, $\{\Omega_i\}$ be an open cover of Ω satisfying a pointwise overlap condition*

$$\exists M \in \mathbb{N} \qquad \forall x \in \Omega \qquad \text{card}\{i \,|\, x \in \Omega_i\} \leq M.$$

Let $\{\varphi_i\}$ be a Lipschitz partition of unity subordinate to the cover $\{\Omega_i\}$ satisfying

$$\begin{aligned} \text{supp}\,\varphi_i &\subset \text{closure}(\Omega_i) \qquad \forall i, && (6)\\ \sum_i \varphi_i &\equiv 1 \text{ on } \Omega, && (7)\\ \|\varphi_i\|_{L^\infty(\mathbb{R}^n)} &\leq C_\infty, && (8)\\ \|\nabla\varphi_i\|_{L^\infty(\mathbb{R}^n)} &\leq \frac{C_G}{\text{diam}\Omega_i}, && (9) \end{aligned}$$

where C_∞, C_G are two constants. Then $\{\varphi_i\}$ is called a (M, C_∞, C_G) partition of unity subordinate to the cover $\{\Omega_i\}$. The partition of unity $\{\varphi_i\}$ is said to be of degree $m \in \mathbb{N}_0$ *if $\{\varphi_i\} \subset C^m(\mathbb{R}^n)$. The covering sets $\{\Omega_i\}$ are called* patches.

Definition 2.2 *Let $\{\Omega_i\}$ be an open cover of $\Omega \subset \mathbb{R}^n$ and let $\{\varphi_i\}$ be a (M, C_∞, C_G) partition of unity subordinate to $\{\Omega_i\}$. Let $V_i \subset H^1(\Omega_i \cap \Omega)$ be given. Then the space*

$$V := \sum_i \varphi_i V_i = \{\sum_i \varphi_i v_i \,|\, v_i \in V_i\} \subset H^1(\Omega)$$

is called the PUFEM space. *The PUFEM space V is said to be of degree $m \in \mathbb{N}$ if $V \subset C^m(\Omega)$. The spaces V_i are referred to as the* local approximation spaces.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be given. Let $\{\Omega_i\}$, $\{\varphi_i\}$, and $\{V_i\}$ be as in definitions 2.1, 2.2. Let $u \in H^1(\Omega)$ be the function to be approximated. Assume that the local approximation spaces V_i have the following approximation properties: On each patch $\Omega_i \cap \Omega$, u can be approximated by a function $v_i \in V_i$ such that*

$$\begin{aligned} \|u - v_i\|_{L^2(\Omega_i \cap \Omega)} &\leq \epsilon_1(i),\\ \|\nabla(u - v_i)\|_{L^2(\Omega_i \cap \Omega)} &\leq \epsilon_2(i). \end{aligned}$$

Then the function

$$u_{ap} = \sum_i \varphi_i v_i \in V \subset H^1(\Omega)$$

satisfies

$$\|u - u_{ap}\|_{L^2(\Omega)} \leq \sqrt{M} C_\infty \left(\sum_i \epsilon_1^2(i) \right)^{1/2}, \tag{10}$$

$$\|\nabla(u - u_{ap})\|_{L^2(\Omega)} \leq \sqrt{2M} \left(\sum_i \left(\frac{C_G}{\mathrm{diam}\Omega_i} \right)^2 \epsilon_1^2(i) + C_\infty^2 \epsilon_2^2(i) \right)^{1/2}. \tag{11}$$

Proof: We will only show estimate (11) because (10) is proved similarly. Let u_{ap} be defined as in the statement of the theorem. Since the functions φ_i form a partition of unity, we have $1 \cdot u = (\sum_i \varphi_i)u = \sum_i \varphi_i u$ and thus

$$\begin{aligned} \|\nabla(u - u_{ap})\|^2_{L^2(\Omega)} &= \|\nabla \sum_i \varphi_i (u - v_i)\|^2_{L^2(\Omega)} \\ &\leq 2\|\sum_i \nabla\varphi_i (u - v_i)\|^2_{L^2(\Omega)} + 2\|\sum_i \varphi_i \nabla(u - v_i)\|^2_{L^2(\Omega)}. \end{aligned}$$

Now, since not more than M patches overlap in any given point $x \in \Omega$, the sums $\sum_i \nabla\varphi_i(u - v_i)$ and $\sum_i \varphi_i \nabla(u - v_i)$ also contain at most M terms for any fixed $x \in \Omega$. Thus, $|\sum_i \nabla\varphi_i(u - v_i)|^2 \leq M \sum_i |\nabla\varphi_i(u - v_i)|^2$ and $|\sum_i \varphi_i \nabla(u - v_i)|^2 \leq M \sum_i |\varphi_i \nabla(u - v_i)|^2$ for any $x \in \Omega$. Hence, if we observe that $\mathrm{supp}\varphi_i \subset \Omega_i$

$$\begin{aligned} &2\|\sum_i \nabla\varphi_i (u - v_i)\|^2_{L^2(\Omega)} + 2\|\sum_i \varphi_i \nabla(u - v_i)\|^2_{L^2(\Omega)} \leq \\ &2M \sum_i \|\nabla\varphi_i (u - v_i)\|^2_{L^2(\Omega)} + 2M \sum_i \|\varphi_i \nabla(u - v_i)\|^2_{L^2(\Omega)} \leq \\ &2M \sum_i \|\nabla\varphi_i (u - v_i)\|^2_{L^2(\Omega_i \cap \Omega)} + 2M \sum_i \|\varphi_i \nabla(u - v_i)\|^2_{L^2(\Omega_i \cap \Omega)} \leq \\ &2M \sum_i \left(\frac{C_G^2}{(\mathrm{diam}\Omega_i)^2} \epsilon_1^2(i) + C_\infty^2 \epsilon_2^2(i) \right) \end{aligned}$$

which finishes the proof. □

Remark 2.1: The constant M controls the overlap of the patches. In particular, not more than M patches overlap in any given point $x \in \Omega$ of the domain. The patches have to overlap because the functions φ_i are supposed to form a sufficiently regular (here: Lipschitz) partition of unity. Condition (9) expresses the fact that we need to control the gradient of the partition of unity functions φ_i if we are interested in H^1 estimates. Note that typically $\epsilon_1(i) \leq C(\mathrm{diam}\,\Omega_i)\epsilon_2(i)$ so that the terms in the sum of (11) are in a sense balanced.
The usual piecewise linear hat functions on a regular (triangular) mesh in two dimensions satisfy the above conditions of a (M, C_∞, C_G) partition of unity; actually, $M = 3$, $C_\infty = 1$, and condition (9) is satisfied because of the regularity of the mesh, i.e., the minimum angle condition satisfied by the triangulation. Similarly, the classical bilinear finite element

functions on quadrilateral meshes form a (M, C_∞, C_G) partition of unity ($M = 4$, $C_\infty = 1$).
The PUFEM has approximation properties very similar to the usual h and p version if the local approximation spaces V_i are chosen to be spaces of polynomials. In fact, if the local approximation spaces consist of polynomials of fixed degree p and the approximation in V_i is achieved through the smallness of the patch Ω_i, the method behaves like the h version. If the patches are kept fixed and the local approximation is achieved by increasing the degree p of the polynomials, which comprise the local spaces V_i, the method behaves like the p version. In this sense, the PUFEM is a generalization of the h and p version.

3 Examples of Local Approximation Spaces

Let us consider a few examples of systems of functions which have good approximation properties for the solutions to a given differential equation and additionally solve the differential equation themselves. A minimal condition on such a system is that it be dense in the set of all solution. We will see that these systems are not unique and that there are many dense system for a given differential equation. The choice of a particular system thus depends on practical aspects (cost of constructing the functions, ease of evaluation of the functions, i.e., cost of construction of the stiffness matrix; conditioning number of the resulting stiffness matrix) and theoretical aspects (optimality of the system; see remark 3.5 below).

3.1 Laplace's Equation

Let us begin with Laplace's equation

$$-\Delta u = 0 \tag{12}$$

on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$. The classical approximation theory in L^∞ with *harmonic polynomials* leads to results of the following form.

Theorem 3.1 (Szegö) *Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded Lipschitz domain. Let $\tilde{\Omega} \supset\supset \Omega$ and assume that $u \in L^2(\tilde{\Omega})$ is harmonic on $\tilde{\Omega}$. Then there is a sequence $(u_p)_{p=0}^\infty$ of harmonic polynomials of degree p such that*

$$\begin{aligned} \|u - u_p\|_{L^\infty(\Omega)} &\leq Ce^{-\gamma p}\|u\|_{L^2(\tilde{\Omega})}, \\ \|\nabla(u - u_p)\|_{L^\infty(\Omega)} &\leq Ce^{-\gamma p}\|u\|_{L^2(\tilde{\Omega})} \end{aligned}$$

where γ, $C > 0$ depend only on Ω, $\tilde{\Omega}$.

Proof: See [17], [20]. □

Theorem 3.2 *Let Ω be a bounded Lipschitz domain, star-shaped with respect to a ball. Let the exterior angle of Ω be bounded from below by $\lambda\pi$, $0 < \lambda < 2$. Assume that $u \in H^k(\Omega)$, $k > 1$, is harmonic. Then there is a sequence $(u_p)_{p=2}^{\infty}$ of harmonic polynomials of degree p such that*

$$\|u - u_p\|_{H^j(\Omega)} \leq C(\mathrm{diam}\Omega)^{k-j} \left(\frac{\ln p}{p}\right)^{\lambda(k-j)} \|u\|_{H^k(\Omega)}, \qquad j = 0, \ldots, [k]$$

where $C > 0$ depends only on the shape of Ω and k.

See [8] for a proof of theorem 3.2. Note that typically $\lambda \leq 1$ and that for domains with re-entrant corners, λ can be significantly less than 1.

Remark 3.2: The restriction in theorem 3.2 that Ω be star-shaped with respect to a ball is not a big constraint for our purposes because we are interested in local estimates on patches and the patches are typically chosen to be star-shaped.

Remark 3.3: We note that the error estimates of theorem 3.1 are (up to the constants involved) similar to the estimates one obtains for the approximation with full spaces of polynomials in that the dependence on p is essentially the same. However, since the number of harmonic polynomials of degree p is $2p + 1$ and the number of polynomials of degree p is $\frac{p(p+1)}{2}$, the approximation with harmonic polynomials is (asymptotically) better in terms of error versus degrees of freedom.

Remark 3.4: We formulated theorem 3.2 in an H^1 framework. Similar results in an L^∞ setting can be found in [9], for example. Those estimates also exhibit the loss in the rate of the approximability when the domain Ω has re-entrant corners.

Remark 3.5: Harmonic polynomials are not the only system of functions which are dense in the class of solutions to Laplace's equation. For example, the systems $\{\mathrm{Re}\, e^{nz}, \mathrm{Im}\, e^{nz} \,|\, n \in \mathbb{N}_0\}$, or $\{\mathrm{Re}\, z^{-n}, \mathrm{Im}\, z^{-n} \,|\, n \in \mathbb{N}_0\}$ (if $0 \notin \Omega$), or the system of rational functions are dense in the set of solutions of Laplace's equation. The system of harmonic polynomials is optimal in the sense of n-width for the approximation of rotationally invariant spaces of harmonic functions on discs (see [15]).

3.2 Elasticity Equations

The solutions of the equations of linear elasticity (in the absence of body forces) in two dimensions can be expressed in terms of two holomorphic functions (see [10]). Let us consider the case of plain strain on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and let λ, μ be the Lamé constants of the material (for the case of plain stress, replace in what follows λ by $\lambda^* = 2\lambda\mu/(\lambda + 2\mu)$). The displacement field (u, v) can be expressed by two holomorphic functions φ, ψ:

$$2\mu\,(u(x,y) + iv(x,y)) = \kappa\varphi(z) - z\overline{\varphi'}(z) - \overline{\psi}(z) \tag{13}$$

where $\kappa = (\lambda + 3\mu)/(\lambda + \mu)$ and we set $z = x + iy$. For a given displacement state, the holomorphic functions φ, ψ are unique up to the normalization of $\varphi(z_0) = 0$ in a

point $z_0 \in \Omega$. Thus, we may approximate the displacement field (u, v) by "generalized harmonic polynomials"

$$2\mu(u + iv) = \kappa\varphi_p(z) - z\overline{\varphi_p'}(z) - \overline{\psi_p}(z) \tag{14}$$

where the functions φ_p, ψ_p are complex polynomials of degree p

$$\varphi_p(z) = \sum_{n=1}^{p} a_n(z - z_0)^n$$
$$\psi_p(z) = \sum_{n=0}^{p} b_n(z - z_0)^n$$

with complex coefficients a_n, b_n. In a real formulation, the displacements u and v are obtained by taking the real and imaginary parts of the elements of the space V (as a vector space over $\mathbb{R}$ of dimension $2 + 4p$)

$$\begin{aligned} V = \text{ span } \{&1, i, \overline{(z - z_0)^n}, i\overline{(z - z_0)^n}, \\ &\kappa(z - z_0)^n - n(z - z_0)\overline{(z - z_0)^{n-1}}, \\ &i\kappa(z - z_0)^n - in(z - z_0)\overline{(z - z_0)^{n-1}} \mid n = 1, \ldots, p\}. \end{aligned}$$

The approximation properties of these "generalized harmonic polynomials" are very similar to the approximation properties of the harmonic polynomials for the approximation of solutions of Laplace's equation. Obviously, in the case that the displacement field satisfies the elasticity equations on a domain $\tilde{\Omega} \supset\supset \Omega$, the estimates of theorem 3.1 produce similar estimates for the error in the displacement field and stress field for the approximation with "generalized harmonic polynomials". The analogous theorem to theorem 3.2 takes the form

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, star-shaped with respect to a ball. Let the exterior angle of Ω be bounded from below by $\tilde{\lambda}\pi$. Assume that the displacement field $(u, v) \in H^k(\Omega)$, $k \geq 1$. Then (u, v) can be approximated by "generalized harmonic polynomials" of degree p such that*

$$\|2\mu(u + iv) - (\kappa\varphi_p - (z - z_0)\overline{\varphi_p'} - \overline{\psi_p})\|_{H^j(\Omega)} \leq C(\mathrm{diam}\Omega)^{k-j} \left(\frac{\ln p}{p}\right)^{\tilde{\lambda}(k-j)} \|(u, v)\|_{H^k(\Omega)}$$

for $j = 0, 1$. The constant C depends only on the shape of Ω and k.

Proof: The proof can be found [8]. A density assertion for these "generalized harmonic polynomials" in the space of solutions of the elasticity equations can also be found in [6] (under stronger assumptions, however). □

Remark 3.6: As in the example with Laplace's equation, we are not restricted to using "harmonic polynomials". Analogous systems based on functions of the form e^{nz}, or polynomials in $1/z$ are also dense in the set of solutions of the elasticity equations.

Remark 3.7: The theory can be extended to problems with certain loads. In many practical applications the load is simple (constant, polynomial) and an explicit particular

solution of the elasticity equations is known. Thus, augmenting the space V by this particular solution allows us to deal with these problems successfully in the framework of approximating the sought solution by functions which solve the differential equation.

3.3 Helmholtz's Equation

In this section let us consider the Helmholtz equation on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$:

$$-\Delta u + k^2 u = 0 \qquad \text{on } \Omega \subset \mathbb{R}^2 \tag{15}$$

where $k > 0$ is the wave number. For this problem we discuss two sets of functions which have good approximation properties for the general solution of (15). Define "generalized harmonic polynomials" of degree p by

$$V^V(p) = \text{span}\,\{e^{\pm in\theta} J_n(kr) \,|\, n = 0, \ldots, p\} \tag{16}$$

where we used polar coordinates (r, θ) and the functions J_n are the usual Bessel functions of the first kind (see, e.g., [5]). The nomenclature "generalized harmonic polynomials" comes from the fact that these functions are the direct analogues of harmonic polynomials via the theory of Bergman and Vekua ([3], [19]). In fact, the approximation results for the approximation of harmonic functions with harmonic polynomials carry over to the case of the approximation of the solutions of (15) with "generalized harmonic polynomials":

Theorem 3.4 *Let Ω be a bounded Lipschitz domain, star-shaped with respect to a ball. Let the exterior angle of Ω be bounded from below by $\lambda\pi$ and assume that $u \in H^s(\Omega)$, $s \geq 1$, solves (15). Then there are functions $u_p \in V^V(p)$ such that*

$$\|u - u_p\|_{H^j(\Omega)} \leq C(\Omega, s, k) \left(\frac{\ln p}{p}\right)^{\lambda(s-j)} \|u\|_{H^s(\Omega)} \qquad j = 0, 1$$

where $C(\Omega, s, k) > 0$ depends only on Ω, k, and s.

Proof: see [8]. □

As in the case of the approximation of solutions to Laplace's equation, there are many other alternatives to the choice of "generalized harmonic polynomials". For example, one can approximate the solutions of (15) with systems of *plane waves*:

$$W(p) = \text{span}\,\{\exp\,[ik(x\cos\theta_j + y\sin\theta_j)] \,|\, \theta_j = \frac{2\pi}{p} j, j = 0, \ldots, p-1\}. \tag{17}$$

One can show that these systems of plane wave have approximation properties which are very similar to the approximation with "generalized harmonic polynomials":

Theorem 3.5 *Under the same assumptions as in theorem 3.4 there are functions $u_p \in W(p)$ such that*

$$\|u - u_p\|_{H^j(\Omega)} \leq C(\Omega, s, k) \left(\frac{\ln^2 p}{p}\right)^{\lambda(s-j)} \|u\|_{H^s(\Omega)} \qquad j = 0, 1$$

where $C(\Omega, s, k) > 0$ depends only on Ω, k, and s.

What are the differences between the generalized harmonic polynomials and the systems of plane waves? Just as the harmonic polynomials were optimal in the sense of n-width for the approximation of rotationally invariant spaces of harmonic functions on discs, the generalized harmonic polynomials are optimal in the sense of n-width for rotationally invariant spaces of solutions of (15) for the special case of Ω being a disc.
An advantage of systems of plane waves is that they might be easier to use in practical applications. Plane waves can be written as products of functions of x and of y only; thus, if the patches consists of rectangles aligned with the coordinate axes, then the integrals appearing in the stiffness matrix can written as products of one dimensional integrals and evaluated cheaply. This observation has been exploited in section 5.1. Let us finish this section by mentioning that these "generalized harmonic polynomials" and the systems of plane waves lead to exponential rates of convergence if the function u is analytic up to boundary:

Theorem 3.6 *Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded Lipschitz domain. Let $\tilde{\Omega} \supset\supset \Omega$ and assume that $u \in L^2(\tilde{\Omega})$ solves the homogeneous Helmholtz equation on $\tilde{\Omega}$. Then*

$$\begin{aligned} \inf_{u_p \in V^V(p)} \|u - u_p\|_{H^1(\Omega)} &\leq C e^{-\gamma p} \|u\|_{L^2(\tilde{\Omega})} \\ \inf_{w_p \in W(p)} \|u - w_p\|_{H^1(\Omega)} &\leq \tilde{C} e^{-\tilde{\gamma} p / \ln p} \|u\|_{L^2(\tilde{\Omega})} \end{aligned}$$

where C, $\tilde{C}$, γ, and $\tilde{\gamma}$ depend only on Ω, $\tilde{\Omega}$, and k.

3.4 Change of Variables Techniques: Rough Coefficients and Elasticity Equations with Corners

The idea of the PUFEM is to enable the user to employ functions with good local approximation properties. These functions do not necessarily have to solve the differential equation. In fact, it can sometimes be too costly to create "optimal" functions. One method to create functions which have good local approximation properties is obtained by an appropriate change of variables. Let us assume that the change of variables $x \mapsto \tilde{x}$ transforms the sought solution u into a function $\tilde{u}$ which is smoother than u. Then, this transformed function $\tilde{u}$ can be approximated well by polynomials $\tilde{P}(\tilde{x})$. This suggests that a good choice for the approximation of u are the mapped "polynomials" $P(x) = \tilde{P}(\tilde{x})$ where the functions $\tilde{P}$ are polynomials.
This idea has been analyzed for a model problem with unilaterally rough coefficients in [13] (the next section considers in detail the one dimensional analogue of the problem considered in [13]).
The idea of exploiting the improved approximation properties of mapped "polynomials" has been applied very successfully to the problem of the elasticity equations with singularities ([11], [12]). The natural change of variables (in a two dimensional setting) is a conformal map which makes corner singularities or singularities arising at interfaces less pronounced. The mapped function can be approximated well by polynomials. Mapping the polynomials back under this conformal map leads to the ansatz functions used.

3.5 The Choice of the Partition of Unity Functions

In the preceding subsections, we described various choices of local approximation spaces which have better approximation properties than the spaces of polynomials of degree p. Let us now turn to the problem of the choice of the partition of unity which puts a given set of local approximation spaces together to produce a conforming global space. The conditions on the partition of unity are very weak: a Lipschitz partition of unity suffices to construct a subspace of H^1 according to theorem 2.1.

Let us consider a domain $\Omega \subset \mathbb{R}^2$. One possible choice of a partition of unity is a collection of finite element functions. Let $\tilde{\Omega} \supset \Omega$ be any domain on which a mesh (consisting of triangles or rectangles, say) has been defined. The usual piecewise linear or bilinear hat functions associated with the nodes of this mesh form a partition of unity for $\tilde{\Omega}$ and therefore for Ω as well. The supports of these hat functions can then be taken as the patches Ω_i. If the mesh satisfies a minimum angle condition, this partition of unity satisfies all the requirements of theorem 2.1. This particular choice has been made for the numerical example of section 5.1.

A more general choice of a partition of unity is given by the following procedure. Let $\{\Omega_i\}$ be a collection of overlapping patches which cover Ω and let $\{\psi_i\}$ be a collection of functions which are supported by the patches Ω_i. Then the normalization

$$\varphi_i = \frac{\psi_i}{\sum_j \psi_j} \tag{18}$$

yields a partition of unity subordinate to the cover $\{\Omega_i\}$. Note that for given i the sum in (18) actually only extends over those j which satisfy $\Omega_i \cap \Omega_j \neq \emptyset$. The functions φ_i inherit the smoothness of the functions ψ_i and thus this normalization technique gives one possible construction of finite element spaces with higher regularity, for example, subspaces of H^2.

We have seen in the introduction that a finite element method is completely determined by the bilinear form and the finite dimensional trial and test spaces. In order to solve (2) in practice, we have to find bases for the test and trial spaces. Since the finite element spaces V constructed by the PUFEM are of the form $V = \sum \varphi_i V_i$ where the φ_i are a partition of unity and the V_i are the local approximation spaces, it is natural to seek a basis of V based on bases of the spaces V_i. If $\{v_{i,p} \,|\, p = 0, \ldots\}$ are basis functions of the local spaces V_i, one can hope that the functions

$$\mathcal{B} = \{\varphi_i v_{i,p}\} \tag{19}$$

form a basis of V. However, there are a few cases, where the set $\mathcal{B}$ is not linearly independent. In order to see this, let us consider a one dimensional example. Define $\Omega = (0,1)$, $h = 1/n$, $x_i = ih$, $i = 0, \ldots, n$, $\Omega_i = (x_i - h, x_i + h)$, and let φ_i be the usual piecewise linear hat function associated with the node x_i. Now choose for the local approximation spaces $V_i = \text{span}\,\{1, x, \ldots, x^p\}$, $p \in \mathbb{N}$. The PUFEM space V is then precisely the space of continuous functions which are piecewise polynomials of degree $p+1$, i.e., $\dim V = n(p+1)+1$. On the other hand, the set $\mathcal{B}$ contains $n(p+1)+p+1$ elements. Thus, $\mathcal{B}$ cannot form a basis of V. Of course, this particular example is somewhat contrived and in general the set $\mathcal{B}$ will form a basis of V. However, this

example shows that we may have to expect that the elements of $\mathcal{B}$ could be nearly linearly dependent which would lead to badly conditioned stiffness matrices.
One way to ensure that the sets $\mathcal{B}$ of the form (19) are linearly independent is to constrain the partition of unity in such a way that each function φ_i is identically 1 on a subset of Ω_i and all other functions φ_j vanish on this subset.
The linear dependencies in the one dimensional example above can be removed by a slight change of the partition of unity functions. It is enough to change those partition of unity functions which are close to boundary. Since we will use this particular partition of unity for the numerical example in section 4.3, we describe it in more detail:

$$\begin{aligned}
x_i &= ih && i = 1, \dots, n-1 \\
\Omega_1 &= (0, 2h) \\
\Omega_i &= (x_i - h, x_i + h) && i = 2, \dots, n-2 \\
\Omega_{n-1} &= (1-2h, 1) \\
\varphi_1 &= \begin{cases} 1 & \text{if } x \in (0, h) \\ 1 - \frac{x-h}{h} & \text{if } x \in (h, 2h) \end{cases} \\
\varphi_i &= \begin{cases} 1 + \frac{x-x_i}{h} & \text{if } x \in (x_i - h, x_i) \\ 1 - \frac{x-x_i}{h} & \text{if } x \in (x_i, x_i + h) \end{cases} && i = 2, \dots, n-2 \\
\varphi_{n-1} &= \begin{cases} 1 + \frac{x-(1-h)}{h} & \text{if } x \in (1-2h, 1-h) \\ 1 & \text{if } x \in (1-h, 1) \end{cases}
\end{aligned} \tag{20}$$

4 A Robust Method for an Equation with Rough Coefficients

4.1 Construction of Robust Local Approximation Spaces

In this section, we want to construct a robust method for the approximation of the solution of an equation with rough coefficients. As a model problem let us consider the elliptic boundary value problem

$$\begin{aligned}
Lu = -(a(x)u')' + b(x)u &= f \qquad \text{on } \Omega = (0,1) \\
u(0) = u(1) &= 0
\end{aligned} \tag{21}$$

where the coefficients a, $b \in L^\infty(\Omega)$ satisfy

$$0 < a_0 \le a(x) \le \|a\|_{L^\infty} < \infty, \qquad 0 \le b(x) \le \|b\|_{L^\infty} < \infty \qquad \text{on } \Omega.$$

We assume that the function $f \in L^\infty$. Observe that the solution u of (21) and the function au' are Lipschitz continuous, i.e.,

$$u, au' \in W^{1,\infty}(\Omega).$$

However, if a is merely in L^∞, we cannot expect the solution u to be in some $H^{1+\epsilon}(\Omega)$, $\epsilon > 0$. Thus, the classical piecewise polynomial finite element spaces may perform very poorly. In fact, the following result holds:

Lemma 4.1 *Let $b = 0$, $f = 1$ in problem (21) and let $\Phi(n)$ be any sequence of numbers which decreases monotonically to 0. Then one can find a function $a \in L^\infty$ with $1 \leq a(x) \leq 2$ and a constant $C > 0$ such that for* any *n dimensional space V_n of continuous, piecewise linear functions*

$$\inf_{u_n \in V_n} \|u - u_n\|_{H^1(\Omega)} \geq C\Phi(n). \tag{22}$$

Proof: [2] □

The lemma shows that the usual FEM may converge arbitrarily slowly (as the number of degrees of freedom n is increased) if the coefficient a is sufficiently rough. Note that (22) holds for all spaces of continuous, piecewise linear functions, and thus we cannot improve the rate of convergence by choosing the meshes judiciously. In practice this means that the classical FEM breaks down for these rough coefficients because "convergence" is only achieved for extremely small mesh sizes h.

Remark 4.1: The case that the coefficients a, b are smooth but highly oscillatory (i.e., large derivatives) is also covered by the ensuing theory. When the coefficients are smooth but highly oscillatory, the exact solution u may be smooth (in H^2, say), but $\|u\|_{H^2(\Omega)}$ is so large that the asymptotic behavior of the FEM is visible for very small mesh sizes only. The special ansatz functions constructed below circumvent this phenomenon and lead to robust finite element methods which behave like the usual FEM for smooth coefficients a, b (with reasonable bounds on the derivatives).

The goal of this subsection is to construct (local) approximation spaces for the approximation of u which are robust. We construct spaces with any desired order of approximability (for sufficiently smooth right hand side f – the coefficients a, b, however, are still assumed to be merely in L^∞). In proposition 4.2 we exhibit such spaces. However, since the functions of proposition 4.2 are the solutions of auxiliary problems, which are not necessarily easier to solve than the original problem, we present approximations of these functions in theorem 4.1 which have approximation properties as good as those of proposition 4.2.

Define

$$B = \frac{\|b\|_{L^\infty}}{a_0}$$

and let us consider the approximation of u on an interval $I \subset \Omega$ of length h by two functions u_0, u_1 which form a fundamental system for L, i.e., any solution v of the equation $Lv = 0$ can be expressed as a linear combination of u_0, u_1.

Proposition 4.1 (Approximation with fundamental systems) *Let u be the solution of (21), $I \subset \Omega$ be an interval of length h, and let u_0, u_1 be a fundamental system for L. Under the assumption that $Bh \leq \gamma < 1$, there is $u_h \in V = \operatorname{span}\{u_0, u_1\}$ such that*

$$\begin{aligned} \|u - u_h\|_{L^\infty(I)} &\leq \frac{1}{a_0(1-\gamma)} h^2 \|f\|_{L^\infty(I)} \\ \|(u - u_h)'\|_{L^\infty(I)} &\leq \frac{1}{a_0(1-\gamma)} h \|f\|_{L^\infty(I)}. \end{aligned}$$

Proof: Fix $x_0 \in I$. Choose $u_h \in V$ such that the function $e = u - u_h$ satisfies

$$Le = f \qquad e(x_0) = 0 \qquad (ae')(x_0) = 0.$$

Then we have an explicit formula for the error e

$$(ae')(x) = -\int_{x_0}^{x} f - be\, dt. \tag{23}$$

Since $e(x_0) = 0$, we have $\|e\|_{L^\infty(I)} \leq h\|e'\|_{L^\infty(I)}$ and hence (23) allows us to bound

$$a_0\|e'\|_{L^\infty(I)} \leq \|ae'\|_{L^\infty(I)} \leq h\|f\|_{L^\infty(I)} + \|b\|_{L^\infty(I)}h\|e'\|_{L^\infty(I)}.$$

With the assumption that $1 - Bh \geq 1 - \gamma$ we conclude

$$\|e'\|_{L^\infty(I)} \leq \frac{1}{a_0(1-\gamma)}\|f\|_{L^\infty(I)}.$$

□

Remark 4.2: One choice of the fundamental system is the following one. Let $x_0 \in I$ be a reference point and let u_0, u_1 solve the initial value problems

$$\begin{array}{lll} Lu_0 = 0 & u_0(x_0) = 1 & (au_0')(x_0) = 0 \\ Lu_1 = 0 & u_1(x_0) = 0 & (au_1')(x_0) = 1. \end{array}$$

Then the function

$$u_h = u(x_0)u_0 + (au')(x_0)u_1 \in V$$

satisfies the estimates of proposition 4.1.

We note that the estimates of proposition 4.1 are robust in the following sense. The exact solution, in spite of being merely in $W^{1,\infty}$, can be approximated with accuracy $O(h)$ independently of the roughness of the coefficients a and b: Only the bounds a_0 and $\|b\|_{L^\infty}$ enter in the estimates.
Proposition 4.1 gives local approximation spaces which are first order accurate. Let us now construct local approximation spaces which have higher order of accuracy (assuming that the right hand side f is sufficiently smooth). To that end we will augment the space V of proposition 4.1 by particular solutions to certain right hand sides.

Proposition 4.2 (Approximation with augmented fundamental systems) *Let u be the solution of (21), $I \subset \Omega$ be an interval of length h, $x_0 \in I$ be a reference point, and let u_0, u_1 be a fundamental system for L. Let v_i, $i \in \mathbb{N}_0$, be functions such that $Lv_i = (x - x_0)^i$. For $p \in \mathbb{N}_0 \cup \{-1\}$ define the space*

$$V_p = \begin{cases} \text{span}\,\{u_0, u_1, v_0, \ldots, v_p\} & \text{if } p \in \mathbb{N}_0 \\ \text{span}\,\{u_0, u_1\} & \text{if } p = -1. \end{cases}$$

Under the assumption that $Bh \leq \gamma < 1$ and $f \in C^{p+1}(\overline{\Omega})$, there is $u_h \in V_p$ such that

$$\begin{aligned} \|u - u_h\|_{L^\infty(I)} &\leq \frac{1}{a_0(1-\gamma)(p+1)!}h^{p+3}\|f^{(p+1)}\|_{L^\infty(I)} \\ \|(u - u_h)'\|_{L^\infty(I)} &\leq \frac{1}{a_0(1-\gamma)(p+1)!}h^{p+2}\|f^{(p+1)}\|_{L^\infty(I)}. \end{aligned}$$

Proof: The case $p = -1$ has been handled in proposition 4.1. Let therefore $p \in \mathbb{N}_0$. Taylor's theorem allows us to write $f = \sum_{n=0}^{p} f_n (x - x_0)^n + R(x)$ where $\|R\|_{L^\infty(I)} \leq \frac{h^{p+1}}{(p+1)!} \|f^{(p+1)}\|_{L^\infty(I)}$. Then the function $e = u - \sum_{n=0}^{p} f_n v_n$ satisfies $Le = R$ on I. Using proposition 4.1 we can approximate e with the functions u_0, u_1 and arrive at the desired estimates. □

Proposition 4.2 permits us to construct robust methods of any desired order (assuming that the right hand side f is sufficiently smooth) if we can find the local approximation functions u_0, u_1, $v_0, \ldots$. In the special case $b \equiv 0$, these functions are explicitly available:

$$u_0 = 1 \qquad u_1(x) = \int_{x_0}^{x} \frac{1}{a(t)} dt \qquad v_i(x) = -\frac{1}{i+1} \int_{x_0}^{x} \frac{(t - x_0)^{i+1}}{a(t)} dt.$$

In the general case, $b \not\equiv 0$, finding u_0, u_1, and the v_i amounts to solving appropriate auxiliary problems on I. In practice, we have to find approximations to the functions u_0, u_1, v_i. In the rest of this section, we will describe one method to approximate these functions and analyze how accurate these approximations have to be. For the approximation of these functions, we will use the fact that they can be written as the solutions of appropriate Volterra integral equations which can be solved by an iterative method. We will see that only few iterations are necessary to yield satisfactory approximations of the functions u_0, u_1, v_i.
For the remainder of the section, let $I \subset \Omega$ be an interval of length h and let $x_0 \in I$ be a reference point in I. Let us consider the initial value problem

$$Lw = g \in L^\infty(I) \qquad w(x_0) = w_0 \qquad (aw')(x_0) = w_1. \tag{24}$$

The function w is the solution of the following Volterra integral equation

$$w = Kw + \tilde{w} \tag{25}$$

where the operator K and the function $\tilde{w}$ are defined by

$$(Kw)(x) = \int_{x_0}^{x} \frac{1}{a(t)} \int_{x_0}^{t} b(\tau) w(\tau) d\tau dt \tag{26}$$

$$\tilde{w}(x) = w_0 + w_1 \int_{x_0}^{x} \frac{1}{a(t)} dt - \int_{x_0}^{x} \frac{1}{a(t)} \int_{x_0}^{t} g(\tau) d\tau dt. \tag{27}$$

The theory of Volterra integral equations (see, e.g., [7]) allows us to expand the solution operator $(I - K)^{-1}$ in a Neumann series, and we can write

$$w = \sum_{n=0}^{\infty} K^n \tilde{w}.$$

We introduce now approximations to the exact solution w by partial sums of this series:

$$w^N = \begin{cases} \sum_{n=0}^{N} K^n \tilde{w} & \text{if } N \in \mathbb{N}_0 \\ 0 & \text{if } N = -1. \end{cases} \tag{28}$$

We need to estimate $w - w^N$. The next two lemmas clarify the approximation properties of the approximants w^N.

Lemma 4.2 *Let the operator K be defined as in (26). Then for any $w \in L^\infty(I)$ and any $n \in \mathbb{N}_0$ we have*

$$\begin{aligned} |(K^n w)(x)| &\leq \frac{B^n |x - x_0|^{2n}}{(2n)!} \|w\|_{L^\infty(I)} \\ |(K^{n+1} w)'(x)| &\leq \frac{B^{n+1} |x - x_0|^{2n+1}}{(2n+1)!} \|w\|_{L^\infty(I)} \end{aligned}$$

where again $B = a_0^{-1} \|b\|_{L^\infty}$.

Proof: The first estimate is the classical estimate for Volterra integral equations (in a C^0 setting) and may be proved by induction. The second estimate follows from the first one with the observation

$$|(K K^n w)'(x)| = \left| \frac{1}{a(x)} \int_{x_0}^x b(t)(K^n w)(t) dt \right| \leq B \left| \int_{x_0}^x |(K^n w)(t)| \, dt \right|.$$

□

Remark 4.3: Lemma 4.2 shows that the fixed point equation (25) can be solved by a Neumann series expansion in an C^0 or an $W^{1,\infty}$ setting. The Neumann series converges for any $h > 0$.

Lemma 4.3 *Let w be the solution of the fixed point problem (25) and let w^N be the approximation given by (28) for $N \in \mathbb{N}_0 \cup \{-1\}$. Then*

$$\begin{aligned} \|w - w^N\|_{L^\infty(I)} &\leq h^{2N+2} C_1(N, h, B) \|\tilde{w}\|_{L^\infty(I)} \\ \|(w - w^N)'\|_{L^\infty(I)} &\leq \begin{cases} h^{2N+1} C_2(N, h, B) \|\tilde{w}\|_{L^\infty(I)} & \text{if } N \in \mathbb{N}_0 \\ \|\tilde{w}'\|_{L^\infty(I)} + h C_2(0, h, B) \|\tilde{w}\|_{L^\infty(I)} & \text{if } N = -1 \end{cases} \end{aligned}$$

where C_1, C_2 are defined by

$$\begin{aligned} C_1(N, h, B) &= \frac{B^{N+1}}{(2N+2)!} \sum_{n=0}^\infty \frac{(2N+2)!}{(2N+2n+2)!} (Bh^2)^n \\ C_2(N, h, B) &= \frac{B^{N+1}}{(2N+1)!} \sum_{n=0}^\infty \frac{(2N+1)!}{(2N+2n+1)!} (Bh^2)^n. \end{aligned}$$

Proof: We can write $w - w^N = \sum_{n=N+1}^\infty K^n \tilde{w}$ and use the bounds on the operators K^n obtained in lemma 4.2. □

Remark 4.4: Under the assumption $\tilde{w} \in W^{1,\infty}(I)$, $\tilde{w}(x_0) = 0$, the estimate on $(w - w^N)'$ can be formulated in the following, more compact form:

$$\|(w - w^N)'\|_{L^\infty(I)} \leq h^{2N+2} C_3(N, h, B) \|\tilde{w}'\|_{L^\infty(I)}$$

where C_3 is given by

$$C_3(N, h, B) = \begin{cases} C_2(N, h, B) & \text{if } N \in \mathbb{N}_0 \\ 1 + h^2 C_2(0, h, B) & \text{if } N = -1. \end{cases}$$

Remark 4.5: Under the assumptions $Bh \le \gamma < 1$, $h < 1$, we can easily bound C_1, C_2 by

$$\begin{aligned} C_1(N,h,B) &\le \frac{B^{N+1}}{(2N+2)!}\frac{1}{1-\gamma h} \qquad \text{for } N \in \mathbb{N}_0 \cup \{-1\} \\ C_2(N,h,B) &\le \frac{B^{N+1}}{(2N+1)!}\frac{1}{1-\gamma h} \qquad \text{for } N \in \mathbb{N}_0. \end{aligned}$$

This analysis of the fixed point problem (25) is now the tool for the approximation of a fundamental system u_0, u_1 and for the approximation of particular solutions v_i. Let u_0, u_1, v_i be given by

$$\begin{array}{rrr} Lu_0 = 0 & u_0(x_0) = 1 & (au_0')(x_0) = 0 \\ Lu_1 = 0 & u_1(x_0) = 0 & (au_1')(x_0) = 1 \\ Lv_i = (x-x_0)^i & v_i(x_0) = 0 & (av_i')(x_0) = 0 \end{array}$$

which are solutions of problem (25) for appropriately chosen w_0, w_1, and g. Let u_0^N, u_1^N, and v_i^N, $N \in \mathbb{N}_0 \cup \{-1\}$, be the approximations to the exact solutions as defined by (28). Then, the following lemma holds:

Lemma 4.4

$$\begin{aligned} \|u_0 - u_0^N\|_{L^\infty(I)} &\le h^{2N+2} C_1(N,h,B) \\ \|(u_0 - u_0^N)'\|_{L^\infty(I)} &\le h^{2N+1} C_2(N,h,B) \qquad \textit{for } N \in \mathbb{N}_0 \\ \|u_1 - u_1^N\|_{L^\infty(I)} &\le h^{2N+3} C_1(N,h,B)\frac{1}{a_0} \\ \|(u_1 - u_1^N)'\|_{L^\infty(I)} &\le h^{2N+2} C_3(N,h,B)\frac{1}{a_0} \\ \|v_i - v_i^N\|_{L^\infty(I)} &\le h^{2N+4+i} C_1(N,h,B)\frac{1}{a_0(i+1)(i+2)} \\ \|(v_i - v_i^N)'\|_{L^\infty(I)} &\le h^{2N+3+i} C_3(N,h,B)\frac{1}{a_0(i+1)}. \end{aligned}$$

Proof: The proof follows directly from lemma 4.3 and remark 4.4. □

We would like to construct an approximation of the space V_p of proposition 4.2. Lemma 4.4 enables us now to calculate how many terms of the Neumann expansion suffice. Recall that the error estimate of proposition 4.2 for the approximation in V_p is $O(h^{p+2})$ (for the error in the derivative). The approximations u_0^N, v_1^N, and v_i^N have to be calculated with the same accuracy. This gives for the number of terms:

$$\begin{aligned} N_0 &\ge \frac{p+1}{2} \qquad \text{for the approximation of } u_0 \\ N_1 &\ge \frac{p}{2} \qquad \text{for the approximation of } u_1 \\ \tilde{N}_i &\ge \frac{p-i-1}{2} \qquad \text{for the approximation of } v_i \end{aligned}$$

where N_0, N_1, $\tilde{N}_i \in \mathbb{N}_0 \cup \{-1\}$. Choosing the smallest N_0, N_1, and $\tilde{N}_i$ such that these three inequalities are satisfied, we can define

$$\tilde{V}_p = \begin{cases} \text{span}\,\{u_0^{N_0}, u_1^{N_1}, v_i^{\tilde{N}_i} \mid i = 0, \ldots, p\} & \text{for } p \in \mathbb{N}_0 \\ \text{span}\,\{u_0^0, u_1^0\} & \text{for } p = -1. \end{cases} \tag{29}$$

For example, we have

$$\begin{aligned} \tilde{V}_{-1} &= \text{span}\,\{u_0^0, u_1^0\} = \text{span}\,\{1, \int_{x_0}^x \frac{1}{a(t)} dt\} \\ \tilde{V}_0 &= \text{span}\,\{u_0^1, u_1^0, v_0^0\} = \text{span}\,\{1 + \int_{x_0}^x \frac{1}{a(t)} \int_{x_0}^t b(\tau) d\tau dt, \int_{x_0}^x \frac{1}{a(t)} dt, \int_{x_0}^x \frac{t - x_0}{a(t)} dt\} \\ \tilde{V}_1 &= \text{span}\,\{u_0^1, u_1^1, v_0^0, v_1^0\} \\ \tilde{V}_2 &= \text{span}\,\{u_0^2, u_1^1, v_0^1, v_1^0, v_2^0\}. \end{aligned}$$

We now show that the space $\tilde{V}_p$ has indeed the desired approximation properties, i.e., the approximation properties of $\tilde{V}_p$ are essentially the same as those of the spaces V_p.

Theorem 4.1 (approximate augmented fundamental system) *Let $I \subset \Omega$ be an interval of length h, $x_0 \in I$ be any reference point in I, and let u be the solution of (21). Let $p \in \mathbb{N}_0 \cup \{-1\}$, $f \in C^{p+1}(\overline{\Omega})$, $\tilde{V}_p$ be defined as in (29) and assume that $Bh \leq \gamma < 1$. Then there is $u_h \in \tilde{V}_p$ such that*

$$\begin{aligned} \|u - u_h\|_{L^\infty(I)} &\leq h^{p+3} C(p, B, a_0, \gamma) \|f\|_{C^{p+1}(\overline{\Omega})} \\ \|(u - u_h)'\|_{L^\infty(I)} &\leq h^{p+2} C(p, B, a_0, \gamma) \|f\|_{C^{p+1}(\overline{\Omega})} \end{aligned}$$

where $C(p, B, a_0, \gamma)$ depends only on p, B, a_0, γ, and Ω.

Proof: The proof follows very closely the proof of proposition 4.2. Let us write

$$f(x) = \sum_{n=0}^{p} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + R(x)$$

where the remainder $R(x)$ satisfies $\|R\|_{L^\infty(I)} \leq \frac{h^{p+1}}{(p+1)!} \|f^{(p+1)}\|_{L^\infty(I)}$. If we agree to assign the empty sum the value 0, the estimate for R also holds for $p = -1$. The approximant of proposition 4.2 could be chosen to be (cf. also remark 4.2)

$$u_{ap} = u(x_0) u_0 + (au')(x_0) u_1 + \sum_{n=0}^{p} \frac{f^{(n)}(x_0)}{n!} v_n.$$

Because the functions v_n satisfy $v_n(x_0) = (a v_n')(x_0) = 0$, the error $r = u - u_{ap}$ satisfies

$$Lr = R \qquad r(x_0) = 0 \qquad (ar')(x_0) = 0.$$

Let us approximate u in $\tilde{V}_p$ by

$$u_h = u(x_0) u_0^{N_0} + (au')(x_0) u_1^{N_1} + \sum_{n=0}^{p} \frac{f^{(n)}(x_0)}{n!} v_i^{\tilde{N}_i},$$

and we get the following representation for the error:

$$u - u_h = u(x_0)(u_0 - u_0^{N_0}) + (au')(x_0)(u_1 - u_1^{N_1}) + \sum_{n=0}^{p} \frac{f^{(n)}(x_0)}{n!}(v_i - v_i^{\tilde{N}_i}) + r.$$

From lemma 4.3 with $N = -1$, we can bound r

$$\begin{aligned}
\|r\|_{L^\infty(I)} &\leq C_1(-1,h,B)\frac{h^2}{a_0}\|R\|_{L^\infty(I)} \leq C_1(-1,h,B)\frac{h^{p+3}}{a_0(p+1)!}\|f^{(p+1)}\|_{L^\infty(I)} \\
\|r'\|_{L^\infty(I)} &\leq C_3(-1,h,B)\frac{h}{a_0}\|R\|_{L^\infty(I)} \leq C_3(-1,h,B)\frac{h^{p+2}}{a_0(p+1)!}\|f^{(p+1)}\|_{L^\infty(I)}.
\end{aligned}$$

Applying the estimates of lemma 4.4 to the remaining terms of the error representation finishes the proof, if we observe that $\|u\|_{L^\infty(I)}$, $\|au'\|_{L^\infty(I)}$ can be bounded by $C(a_0, B, \Omega)\|f\|_{L^\infty(\Omega)}$; $C(a_0, B, \Omega)$ depends only on a_0, B, and Ω according to standard regularity theory. □

Remark 4.6: The approximation properties of the space $\tilde{V}_{-1}$ can be understood with the ideas of section 3.4 as well. If one introduces the change of variables $\tilde{x} = \int_0^x \frac{1}{a(t)}dt$, then problem (21) is transformed to a problem of the form

$$-\tilde{u}'' + \tilde{b}\tilde{u} = \tilde{f}$$

where $\tilde{b}$, $\tilde{f}$ are still in L^∞ and hence $\tilde{u} \in W^{2,\infty}$. The elements of $\tilde{V}_{-1}$ transform to linear functions. Therefore, the approximation of u in $\tilde{V}_{-1}$ can be expected to behave like the approximation of a $W^{2,\infty}$ function by linear functions.

4.2 Construction of the Global Finite Element Space

We will now construct a global conforming finite element space from the spaces $\tilde{V}_p$ (cf. (29)), which have good local approximation properties for the approximation of the solution of (21). We proceed as outlined in section 2. Let $(\Omega_i)_{i=1}^N$ be a covering of $\Omega = (0,1)$ satisfying the overlap condition. Let $(\varphi_i)_{i=1}^N$ be a (M, C_∞, C_G) partition of unity associated with this covering (Ω_i). The local approximation spaces V_i are given by theorem 4.1 as follows. In each patch, we choose a reference point $z_i \in \Omega_i$ (which plays the role of the point x_0 of theorem 4.1). For $p \in \mathbb{N}_0 \cup \{-1\}$, the local approximation spaces $V_i = V_i(p)$ are then taken as the spaces $\tilde{V}_p$ of (29) with reference point z_i instead of x_0. Theorem 4.1 immediately gives for the local approximation properties (expressed in the notation of theorem 2.1)

$$\begin{aligned}
\epsilon_1(i) &\leq C(p, B, a_0, \gamma, \Omega)(\mathrm{diam}\Omega_i)^{p+3+1/2}\|f\|_{C^{p+1}(\overline{\Omega})} \\
\epsilon_2(i) &\leq C(p, B, a_0, \gamma, \Omega)(\mathrm{diam}\Omega_i)^{p+2+1/2}\|f\|_{C^{p+1}(\overline{\Omega})}.
\end{aligned}$$

We define the global approximation space $V = V(p) = \sum_{i=1}^N \varphi_i V_i(p)$. Hence, for u solving (21), there is $u_h \in V(p)$ such that

$$\|u - u_h\|_{L^2(\Omega)} \leq \sqrt{M}C_\infty C(p, B, a_0, \gamma, \Omega)\|f\|_{C^{p+1}(\overline{\Omega})} \left(\sum_{i=1}^N (\mathrm{diam}\Omega_i)^{2(p+3)+1}\right)^{1/2}$$

$$\|(u-u_h)'\|_{L^2(\Omega)} \leq \sqrt{2M(C_G^2+C_\infty^2)}C(p,B,a_0,\gamma,\Omega)\|f\|_{C^{p+1}(\overline{\Omega})}\left(\sum_{i=1}^{N}(\mathrm{diam}\Omega_i)^{2(p+2)+1}\right)^{1/2}$$

So far we have not dealt with the essential boundary conditions at $x=0$ and $x=1$. However, they are easily enforced by a judicious choice of the reference point for the patches Ω_i close to the boundary, i.e., $\Omega_i \cap \partial\Omega \neq \emptyset$. For these patches, we choose the reference point z_i to be the boundary point and then simply leave out the approximations $u_0^{N_0}$ to u_0 because all the other elements of $\tilde{V}_p$ vanish at the reference point. The finite element space $V(p)$ is thus a subspace of $H^1(\Omega)$ and satisfies the boundary conditions, i.e., it is a conforming finite element space.

Let us give a more concrete example of the abstract procedure given above for the construction of the global space $V(p)$. Let $n \in \mathbb{N}$, $h=1/n$ and define the patches Ω_i and the partition of unity φ_i as in equations (20). The local approximation spaces $V_i(p)$ associated with the patches Ω_i are given by (29) where the reference point in each patch Ω_i is chosen to be the node x_i for $i=2,\ldots,n-2$. For $i=1$ the reference point is chosen to the left boundary point $x=0$ and for $i=n-1$ the reference point is chosen to be right boundary point $x=1$. The approximation space $V_1(p)$ and $V_{n-1}(p)$ associated with the first and last patch are constrained to satisfy the essential boundary conditions by omitting the approximations to u_0. For example, the two simplest spaces are

$$\begin{aligned} V(-1) &= \mathrm{span}\,\{\varphi_1(x)\int_0^x \frac{1}{a(t)}dt, \varphi_{n-1}(x)\int_1^x \frac{1}{a(t)}dt, \\ &\quad \varphi_i(x), \varphi_i(x)\int_{x_i}^x \frac{1}{a(t)}dt \,|\, i=2,\ldots,n-2\} \qquad (30) \\ V(0) &= \mathrm{span}\,\{\varphi_1(x)\int_0^x \frac{1}{a(t)}dt, \varphi_1(x)\int_0^x \frac{t}{a(t)}dt, \\ &\quad \varphi_{n-1}(x)\int_1^x \frac{1}{a(t)}dt, \varphi_{n-1}(x)\int_1^x \frac{t-1}{a(t)}dt, \\ &\quad \varphi_i(x)\left(1+\int_{x_i}^x \frac{1}{a(t)}\int_{x_i}^t b(\tau)d\tau dt\right), \\ &\quad \varphi_i(x)\int_{x_i}^x \frac{1}{a(t)}dt, \varphi_i(x)\int_{x_i}^x \frac{t-x_i}{a(t)}dt \,|\, i=2,\ldots,n-2\}. \qquad (31) \end{aligned}$$

And the above theory gives that the spaces $V(-1)$, $V(0)$ approximate the solution u of (21) such that

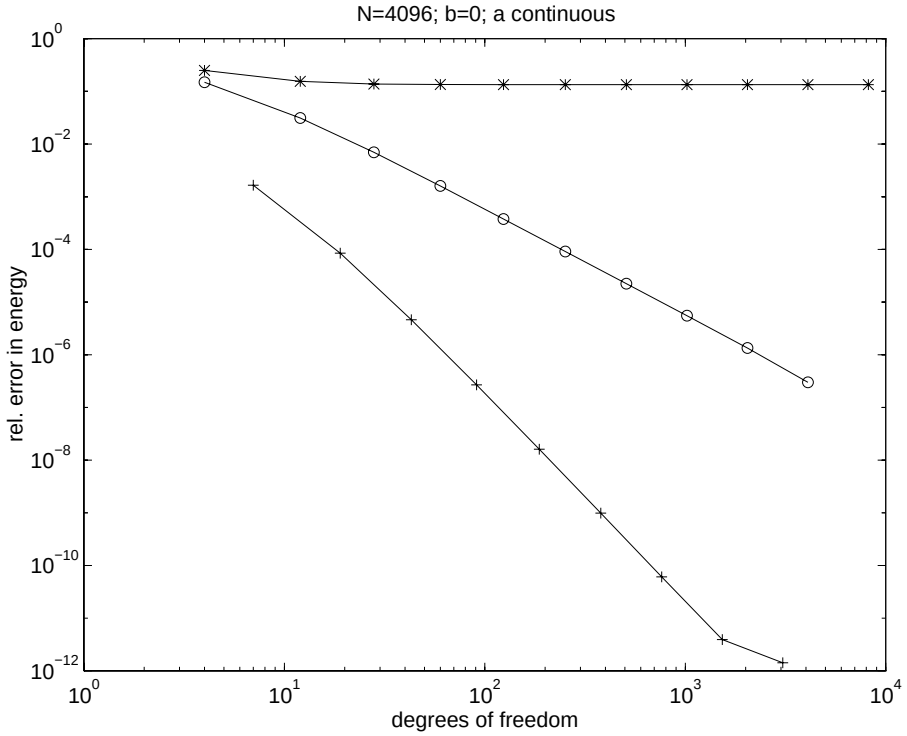
$$\inf_{u_h\in V(-1)} \|u-u_h\|_{L^2(\Omega)} + h\|(u-u_h)'\|_{L^2(\Omega)} \leq C(B,a_0,\gamma)\|f\|_{L^\infty(\Omega)}h^2 \qquad (32)$$

$$\inf_{u_h\in V(0)} \|u-u_h\|_{L^2(\Omega)} + h\|(u-u_h)'\|_{L^2(\Omega)} \leq C(B,a_0,\gamma)\|f\|_{C^1(\overline{\Omega})}h^3 \qquad (33)$$

where the constant $C(B,a_0,\gamma)$ depends only on B, a_0, and γ if $Bh \leq \gamma < 1$. Let us note that

$$\dim V(-1) = 2(n-3)+2 \qquad \dim V(0) = 3(n-3)+4. \qquad (34)$$

Figure 1: Approximation in $V(0)$ (+), $V(-1)$ (o), and $V_{poly}(*)$



4.3 Numerical Example

In this subsection, we apply the above constructed finite element spaces to a concrete differential equation. We consider

$$\begin{aligned} Lu = -(a(Nx)u')' + bu &= f(x) \qquad \text{on } \Omega = (0,1) \\ u(0) = u(1) &= 0 \end{aligned} \tag{35}$$

where the function a is *1-periodic*, $N \in \mathbb{N}$ large, and the coefficient b is either $b = 0$ or $b = 1$. The right hand side f is taken to be $f = x$ for $b = 0$ and $f = 1$ for $b = 1$. For the 1-periodic function a, we consider two cases:

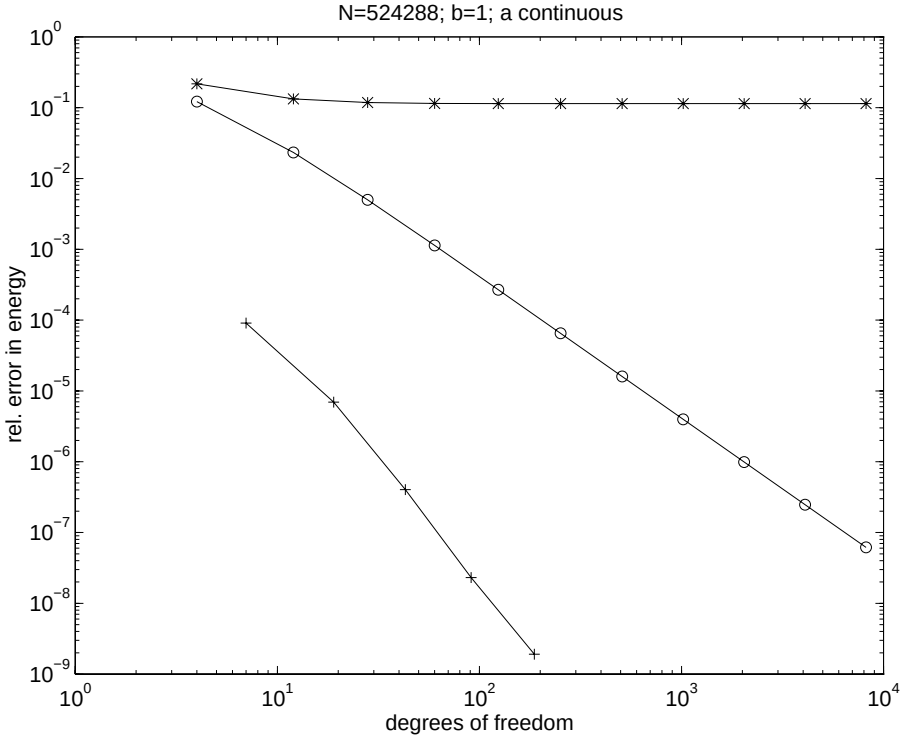
$$\begin{aligned} a_1(x) &= \frac{1}{2 + \cos(2\pi x)} \\ a_2(x) &= \begin{cases} 1 & \text{if } x \in (0, \frac{1}{2}) \\ 2 & \text{if } x \in (\frac{1}{2}, 1). \end{cases} \end{aligned}$$

The solution of (35) is in $H^2(\Omega)$ (even piecewise C^∞) for both choices of the coefficient a. However, the solution is rough in our terminology as is has very large higher derivatives. Associated with this problem is the notion of an "energy"

$$\|u\|_E^2 = \int_0^1 a(Nx)|u'|^2 + b|u|^2 dx$$

and an "energy" norm, which is the square root of the energy.

Figure 2: Approximation in $V(0)$ (+), $V(-1)$ (o), and $V_{poly}(*)$



The typical behavior of the classical piecewise polynomial finite element methods for this particular problem is to converge (in the energy norm) for very small mesh size only, namely when the mesh size h is so small that the finite element space can resolve the oscillation of the coefficient a. The classical finite element methods therefore converge for $h < N^{-1}$ only.

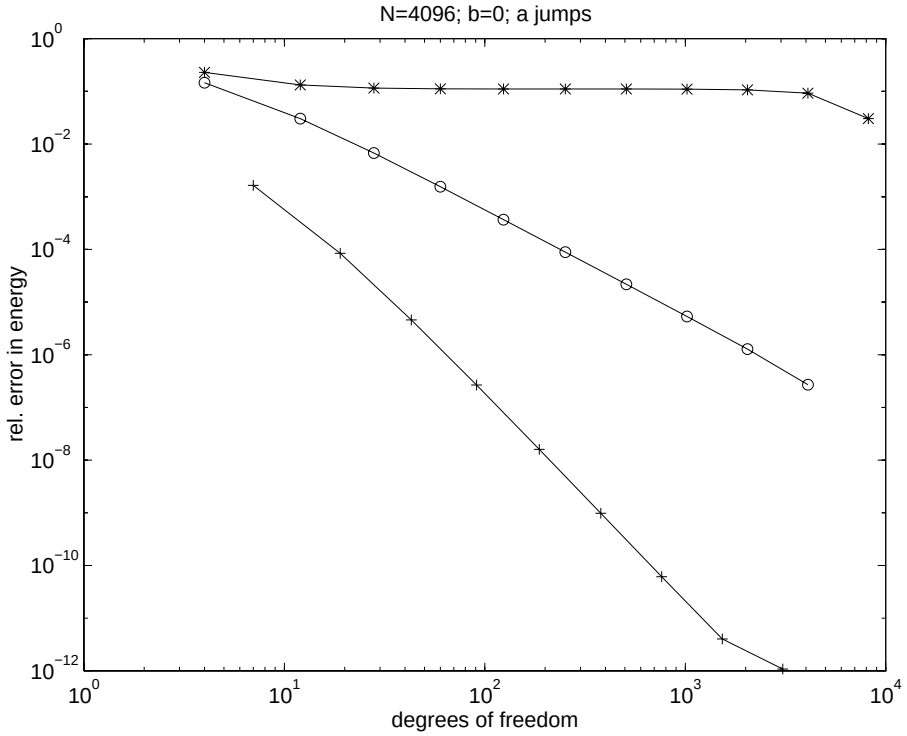
By the method outlined in the preceding subsection, we can create robust approximation spaces of any desired order for the approximation of (35). However, we restrict ourselves here to the two spaces $V(-1)$, $V(0)$ defined in (30), (31). For comparison, let us introduce a third type of spaces, namely, a space where the local approximation spaces consist of polynomials. Using the same partition of unity $\{\varphi_i\}$ as in the construction of $V(-1)$, $V(0)$ (cf. (20)), we define

$$V_{poly} = \text{span}\,\{\varphi_1 \cdot x, \varphi_{n-1} \cdot (x-1), \varphi_i, \varphi_i \cdot (x - x_i)\,|\,i = 2, \ldots, n-2\}. \tag{36}$$

This space V_{poly} contains all piecewise linear functions and is a subset of the usual piecewise quadratic finite element space. It will therefore serve as a comparison of the usual finite element method with our robust spaces.

Fig. 1 and 2 show the performance of the three spaces $V(0)$, $V(-1)$, and V_{poly} for the coefficient a_1 for the cases $b = 0$, $N = 4096$, and $b = 1$, $N = 524288$ whereas fig. 3 and 4 correspond to the coefficient a_2 for the cases $b = 0$, $N = 4096$, and $b = 1$, $N = 524288$. In all the graphs, the mesh size ranges from $h = \frac{1}{4}$ to $h = \frac{1}{4096}$. (34) relates these mesh sizes to the number of degrees of freedom; in particular, the number of degrees of freedom is proportional to $1/h$ for both $V(-1)$ and $V(0)$. Therefore, estimates (32), (33) yield

Figure 3: Approximation in $V(0)$ (+), $V(-1)$ (o), and V_{poly} (*)



bounds of the form

$$\text{rel. error in energy} \leq C\text{dof}^{-2}, C\text{dof}^{-4} \tag{37}$$

for the approximation in $V(-1)$ and $V(0)$, respectively. The size of the constant C is independent of the roughness of the coefficient a, i.e., it is independent of the number N. We can see in fig. 1–4 that these rates of convergence are actually attained and that the method is robust: Estimates (37) hold for very few degrees of freedom and the good behavior of the method is independent of N (the PUFEM performs equally well for the cases $N = 4096$ and $N = 524288$). The spaces V_{poly} behave in a totally different way. Since the graphs only cover the range $h = \frac{1}{4}$ to $h = \frac{1}{4096}$, we still have $h > N^{-1}$ and cannot expect the usual FEM to work. Indeed, the error stays almost constant over the whole range.

We considered two cases $b = 0$ and $b = 1$. The difference between those two cases lies in the fact that for $b = 0$ the spaces $V(-1)$ and $V(0)$ are based on local approximation spaces which contain an exact fundamental system whereas in the case $b = 1$ the local approximation spaces contain only an approximate fundamental system. We see, however, that the approximate fundamental system is accurate enough not to upset the rate of convergence, just as the theory of section 4.1 predicts.

Finally, let us mention that we chose a problem with periodic coefficients for computational convenience. In this particular case, the periodicity could be exploited in such a way that the construction of the stiffness matrix and the evaluation of the right hand side is achieved with an amount of work independent of the number N; the work is – up to a constant – the same as for the usual finite element method for $N = 1$.

Figure 4: Approximation in $V(0)$ (+), $V(-1)$ (o), and V_{poly} (*)

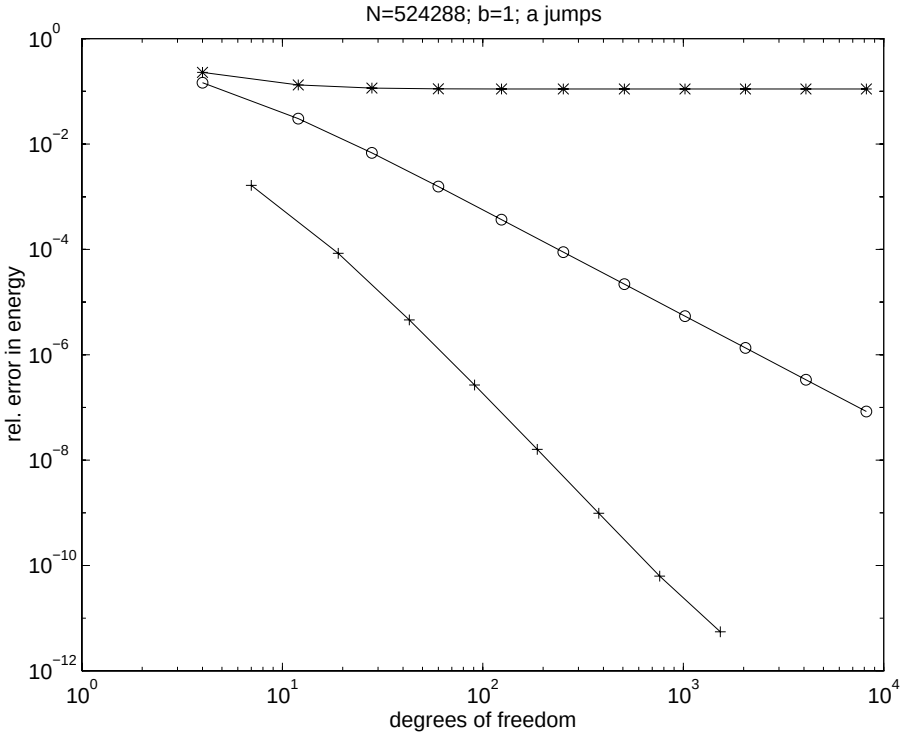


Table 1: DOF necessary to obtain accuracy ϵ in L^2 norm; $k = 100$

ϵ	best p.w. linear approximant	QSFEM	GLSFEM	FEM
30%	2.045D+3	3.969D+3	2.016D+4	7.784D+4
10%	5.041D+3	1.000D+4	6.150D+4	2.352D+5
5%	8.464D+3	1.960D+4	1.274D+5	4.692D+5

This numerical example shows that the PUFEM based on the local approximation spaces constructed in section 4.1 leads to a robust method: The performance of the finite element spaces $V(-1)$, $V(0)$ is independent of the roughness of the coefficients of the differential operator and their performance is comparable to the classical piecewise linear or quadratic finite element spaces for a problem with smooth coefficients.

5 Helmholtz's Equation and Concluding Remarks

5.1 Helmholtz's Equation

In this section, we present an application of the PUFEM to the Helmholtz equation in two dimensions. We consider the problem

$$\begin{aligned} -\Delta u - k^2 u &= 0 \qquad \text{on } \Omega = (0,1)\times(0,1) \subset \mathbb{R}^2 \\ \partial_n u + iku &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{38}$$

Table 2: DOF necessary to achieve various accuracies in L^2 for PUFEM with $n = 4$ and various other methods; $k = 100$

p	L^2 error	PUFEM	best approx.	QSFEM	FEM
26	10.8%	6.50D+2	3.80D+3	7.95D+3	2.08D+5
30	0.69%	7.50D+2	5.89D+4	1.23D+5	3.23D+6
34	0.11%	8.50D+2	3.45D+5	7.23D+5	1.90D+7

Table 3: number of operations using band elimination – the p version of the PUFEM; $n = 4$; $k = 100$; error in L^2

p	L^2 error	PUFEM	QSFEM	FEM
26	10.8%	1.76D+7	6.3D+7	4.3D+11
30	0.69%	2.71D+7	1.5D+10	1.01D+13
34	0.11%	3.94D+7	5.2D+11	3.6D+14

Table 4: number of operations for hp version of PUFEM; $k = 100$; L^2 error

p	n	L^2 error	NOP PUFEM
26	4	10.8%	1.76D+7
18	8	10.6%	5.23D+7
14	16	9.5%	2.75D+8

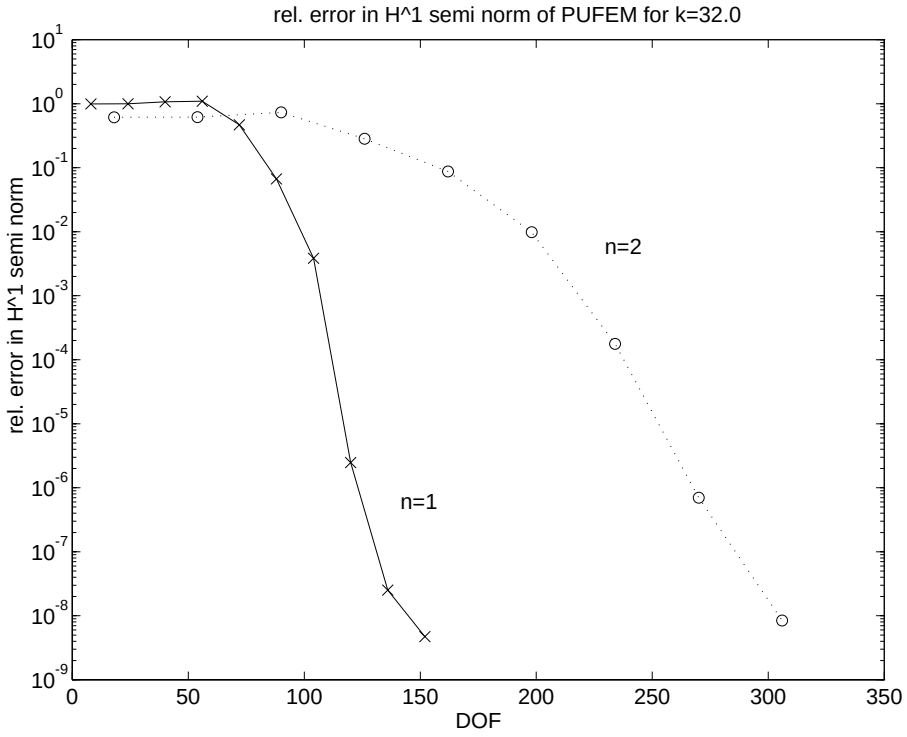
Table 5: operation count for solving linear system; error in H^1 norm;$k = 32$

	Galerkin			QSFEM		
$\sqrt{DOF}$	H^1 error	No. iter	NOP	H^1 error	No. iter	NOP
32	65%	232	4.51D+6	30.5%	272	5.29D+6
64	21.7%	434	3.37D+7	14.3%	492	3.82D+7
128	8.16%	831	2.68D+8	7.02%	953	2.96D+8
256	3.64%	1665	2.07D+9	3.48%	1863	2.31D+9
512	1.72%	3263	1.62D+10	1.69%	3752	1.86D+10

Table 6: operation count for band elimination for PUFEM; $k = 32$, error in H^1; $n = 1$

p	H^1 error	NOP PUFEM
18	46%	1.3D+5
22	6.7%	2.3D+5
26	0.38%	3.8D+5
30	0.00025%	5.9D+5

Figure 5: The p version of the PUFEM



where g is chosen such that the exact solution is a plane wave of the form

$$u(x,y) = \exp\{ik(x\cos\theta + y\sin\theta)\}, \qquad \theta = \frac{\pi}{16}.$$

In section 3.3 we discussed two types of local approximation spaces in for the approximation of solutions of Helmholtz's equation. We could take either the "generalized harmonic polynomials" of (16) or the systems of plane waves (17). In the numerical examples presented here, we concentrate on the systems of plane waves (for a comparison of these two different local spaces, see [8]). The partition of unity for this particular problem is given by piecewise bilinear hat functions: For $n \in \mathbb{N}$, the square Ω is subdivided into $n \times n$ squares of side length $h = \frac{1}{n}$. With each of the $(n+1)^2$ nodes (x_i, y_i) we associate a piecewise bilinear hat function φ_i which vanishes in all nodes except (x_i, y_i). The patches Ω_i are taken to be the supports of these φ_i. The PUFEM is based on this partition of unity and the local approximation spaces V_i are chosen to be the spaces $W(p)$ of (17).
Remark 5.1: In this particular implementation we only used the space $W(p)$ with p of the form $p = 4m + 2$, $m \in \mathbb{N}$, to ensure that the exact solution of problem (38) is not in the PUFEM space.

In this application of the PUFEM, we have thus two parameters which influence the approximation properties of the global finite element space, namely, the mesh size of the partition of unity, which is determined by n, and the size of the local approximation spaces V_i, which is controlled by p. If the parameter p is fixed and the mesh size is variable, we talk about the h version of the PUFEM; if the mesh is fixed and the approximation is

achieved by increasing the size of the local spaces (i.e., by increasing p), we talk about the p version of the PUFEM. If both h and p are varied, we would then talk about the hp version of the PUFEM. The estimates on local approximability of theorem 3.6 let us expect exponential rates of convergence as a p version. This exponential convergence of the p version of the PUFEM can be observed in fig. 5 for the cases $n = 1$ and $n = 2$.
We will discuss the numerical results only briefly; a more detailed analysis can be found in [15]. In tables 1–6 the PUFEM is compared with the usual Galerkin finite element method (FEM), the generalized least squares finite element method (GLSFEM) of [18], and the quasi-stabilized finite element method (QSFEM) of [14]. Since all three methods are based on piecewise linear functions on uniform grids, tables 1 and 2 include the piecewise linear best approximant for reference. In tables 1–4, we use the norm L^2 as the error measure and analyze the case $k = 100$. Tables 5–6 deal with the case $k = 32$ and the H^1 semi norm as the error measure. Tables 1 and 2 show that the p version of the PUFEM needs markedly fewer degrees of freedom to achieve the same accuracy in L^2 as the other methods, which are based on piecewise linear ansatz functions. This reduction in degrees of freedom translates in a reduction of the number of operations when the linear system is solved using Gaussian elimination. This is demonstrated in table 3. In table 4 we list the various combinations of p and n which lead to the same accuracy of 10% in L^2. Since we expect the PUFEM to exhibit exponential rates of convergence as a p version but only algebraic rates as an h version, the number of operations is smallest for the largest mesh size h. In tables 5 and 6 we compare the operation count of the Gaussian elimination for the PUFEM with the operation count of the Galerkin method and the QSFEM. The linear systems in these latter two methods are solved by the iterative method proposed in [4]. We see that here again, the PUFEM performs better than the other two methods.
We have seen that the PUFEM is superior to the other methods both in terms of error versus degrees of freedom and error versus number of operations. Let us point out that the discrepancy between the PUFEM and the other methods becomes larger as the accuracy requirement is increased.
Remark 5.2: We used systems of plane wave as local approximation spaces because their specific structure and the particular form of the partition of unity allowed us to create the stiffness matrix cheaply. Therefore, the overall amount of work for the PUFEM is dominated by the operation count of the Gaussian elimination.

5.2 Concluding Remarks and Open Questions

We presented a new method which allows the user to include a priori knowledge about the problem under consideration in the finite element space. We illustrated this procedure in detail for a one dimensional model problem with rough coefficients. For this one dimensional example, we constructed local approximation spaces which reflect the rough behavior of the solution, and the PUFEM enabled us to build a robust finite element method from these local spaces. A numerical example illustrated the robustness of the method and thereby showed the superiority of the PUFEM over the classical FEM for this particular kind of problem. With an application of the PUFEM to the Helmholtz

equation in two dimensions we demonstrated that the PUFEM can cope with highly oscillatory problems in a very satisfactory fashion.
We mentioned only very briefly the other features of the PUFEM. Among them are the ability to construct smoother space which are necessary for finite element methods for higher order differential equations. Since the regularity of the PUFEM space is governed by the smoothness of the partition of unity, such smoother spaces are easily constructed with the PUFEM. The "meshless" aspect of the PUFEM has also not been addressed in this paper. This is a feature of the PUFEM which can be important for problems which involve frequent remeshing such as the optimal placement of a fastener alluded to in the introduction.
We have seen that the PUFEM offers a new, very promising approach to dealing successfully with non-standard problems where the usual finite element methods fail or are too costly. Since the PUFEM is in still its infancy, there are also many open questions about implementational aspects which need to be addressed. Among them are:

1. The choice of a basis of the PUFEM space. We discussed this topic briefly in section 3.5. It is an important issue because the condition number of the stiffness matrix depends on the choice of the basis.

2. The implementation of essential boundary conditions. We did not discuss this question because we concentrated on a one dimensional model problem where essential boundary conditions can be enforced very easily.

3. The integration of the elements of the stiffness matrix. This is a difficulty which the PUFEM shares with all meshless methods. For the construction of the stiffness matrix, one has to integrate shape functions against each other. Thus, the integrator has to be able to integrate efficiently over the intersection of the supports of the shape functions. Since the shape functions are not necessarily tied to a mesh, the description of these intersections is potentially harder than in the usual FEM. However, specific choices of the partition of unity and/or appropriately designed integrators should be able to cope with the integration issues successfully.

References

[1] A.K. Aziz and I.M. Babuška, editors. *Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations.* Academic Press, New York, 1972.

[2] I. Babuška and J. E. Osborn. private communication.

[3] S. Bergman. *Integral operators in the theory of linear partial differential equations.* Springer Verlag, 1961.

[4] R. W. Freund. A transpose-free quasi-minimal residual algorithm for non-hermitian linear systems. *SIAM J. Sci. Comput.*, 14(2):470–482, 1993.

[5] I.S. Gradshtein. *Table of Integrals, Series and Products.* Academic Press, New York, 1980.

[6] I. Herrera. *Boundary Methods: An Algebraic Theory.* Pitman, Boston, 1984.

[7] R. Kress. *Linear Integral Equations.* Springer Verlag, 1989.

[8] J. M. Melenk. *On Generalized Finite Element Methods.* PhD thesis, University of Maryland, 1995.

[9] S.N. Mergelyan. *Uniform Approximation to Functions of a Complex Variable*, volume 3 of *1*, pages 294–391. AMS, 1962.

[10] N. I. Muskhelishvili. *Some Basic Problems of the Mathematical Theory of Elasticity.* P. Noordhoff, Groningen, 1963.

[11] Hae-Soo Oh and I. Babuška. The p-version of the finite element method for the elliptic boundary value problems with interfaces. *Comput. Meths. Appl. Mech. Engrg.*, 97:211–231, 1992.

[12] Hae-Soo Oh and I. Babuška. The method of auxiliary mapping for the finite element solutions of elasticity problems containing singularities. Technical Report BN-1178, Inst. for Phys. Sci. and Tech., College Park, MD 20742, 1994.

[13] I. Babuška, G. Caloz, and J. Osborn. Special finite element methods for a class of second order elliptic problems with rough coefficients. *SIAM J. Numer. Anal.*, 31:945–981, 1994.

[14] I. Babuška, F. Ihlenburg, E. Paik, and S. Sauter. A generalized finite element method for solving the Helmholtz equation in two dimensions with minimal pollution. *Comput. Meths. Appl. Mech. Engrg.*, 128:325–360, 1995.

[15] I. Babuška and J. M. Melenk. The partition of unity finite element method. Technical Report BN–1185, University of Maryland, Institute for Physical Science and Technology, College Park, MD 20742, 1995.

[16] I. Babuška and Z. Zhang. The partition of unity finite element method for the elastically supported beam. *(to appear).*

[17] G. Szegö. Über Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören. *Mathematische Zeitschrift*, 9:218–270, 1921.

[18] L.L. Thomson and P.M. Pinsky. A Galerkin least squares finite element method for the two-dimensional Helmholtz equation. *Internat. J. Numer. Meths. Engrg.*, 38:371–397, 1995.

[19] I. N. Vekua. *New Methods for Solving Elliptic Equations.* North Holland, 1967.

[20] J.L. Walsh. *Interpolation and approximation by rational functions in the complex domain*, volume 20 of *colloquium publications.* American Mathematical Society, 1960.