Four-Stream Spherical Harmonic Expansion Approximation for Solar Radiative Transfer

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ABSTRACT

This paper presents a four-stream extension of the δ-Eddington approximation by considering the higher-order spherical harmonic expansion in radiative intensity. By using the orthogonality relation of the spherical harmonic functions, the derivation of the solution is fairly straightforward. Calculations show that the δ-four-stream spherical harmonic expansion approximation can reduce the errors in reflection, transmission, and absorption substantially in comparison with the δ-Eddington approximation. For the conservative scattering case, the error of the new model is generally less than 1% for optical thicknesses greater than unity except for grazing incident solar beam. For nonconservative scattering cases (single scattering albedo $\omega = 0.9$), the error is less than 5% for optical thicknesses greater than unity, in contrast to errors of up to 20% or more under the δ-Eddington approximation. This model can also predict the azimuthally averaged intensity to a good degree of accuracy. The computational time for this model is not as intensive as for the rigorous numerical methods, owing to the analytical form of the derived solution.

1. Introduction

The fundamentals of atmospheric radiation are based on solving the radiative transfer equation and parameterizing the radiative properties of gases, water vapor, aerosol, and cloud drops. The physical process of radiative transfer is described by a differential–integral equation. The exact solution of the radiative transfer equation in a scattering and absorbing media is difficult to obtain in a computationally efficient manner even for a plane-parallel case; thus approximate methods are necessary. In the last three decades considerable attention has been paid to finding simple and effective methods for solving the radiative transfer equation. The simplest method to determine the radiative flux is the two-stream approximation, which is widely used in climate models. The "two stream" method has a general meaning, according to the discussions by Meador and Weaver (1980), Zdunkowski et al. (1980), and King and Harshvardhan (1986) in that the Eddington approximation, quadrature discrete ordinate method, and hemispheric constant method can all be incorporated into a standard solution form with appropriate choice of parameters.

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The two-stream approximations provide a simple and rapid answer to radiative transfer in a plane-parallel medium. The accuracies of the two-stream methods have been compared by King and Harshvardhan (1986). It is found that the relative error in the radiative quantities can be up to 20% or higher for any of the two-stream methods, over a range of optical thicknesses and solar zenith angles. It follows that improvements to the two-stream approximations are needed if a higher accuracy in the calculations is desired. Generally, the technique for improvement is to extend the two-stream approximations to four-stream approximations or, in general, multistream approximations.

Except for the hemispheric constant method (Coakley and Chýlek 1975), only the first two moments in the expansion of the phase function are kept in the two-stream approximations. The phase function is very poorly represented if only the first two moments are preserved, especially for large and nonspherical ice crystal particles. Generally, the higher-order terms in the expansion of the phase function can be incorporated in the solution only if the corresponding higher-order stream approximation is considered.

In modeling of atmospheric radiative entropy, photochemical interactions and remote sensing, not only the radiative flux but also the radiative intensity is of interest. The radiative flux is a measure of the vertical energy flow in the atmosphere, while the intensity contains the angular information about the radiation field. The evaluation of the radiative intensity in the context of an approximate method is an important issue for atmospheric radiation studies.

Though a high accuracy in calculations of both radiative flux and intensity for a plane-parallel medium has been achieved by rigorous numerical radiative transfer models (for example, Stamnes et al. 1988; Ramaswamy and Freidenreich 1991), such models are cumbersome and very time consuming and cannot be applied directly to present climate models. Here, we are interested in an analytical extension of the two-stream approximations, in which the accuracy is improved but an adequate computational efficiency is retained. In the case of the discrete ordinate method, the extension of the analytical solution for the four-stream approximation has already been considered (Liou 1974; Liou et al. 1988). However, little attention has been paid so far to four-stream extensions of other approximations such as the Eddington approximation and the hemispherical constant method. To perform a general, consistent study of the four-stream extension of these methods is the purpose of this research. Specifically, in this paper, the four-stream extension of the Eddington approximation is investigated.

Zdunkowski and Korb (1974) have considered the four-stream extension of the Eddington approximation. However, the orthogonal relation of the spherical function, the solution of eigen equation, and the boundary condition were not treated appropriately, which resulted in a very complicated solution and generally poor accuracy of the results. There is no similarity between this work and that of Zdunkowski and Korb (1974).

Karp et al. (1980) considered the numerical solution of radiative transfer with the spherical harmonic expansion approximation. In their method, the solution consists of the evaluation of an equation with exponential matrix forms; such exponential matrix equations are diagonalized or transformed to a Jordan form. This process, generally, is very cumbersome even for the four-stream case (see Zdunkowski and Korb 1974). In contrast to the Karp et al. approach, our solution is based on directly solving the linear, constant coefficient, differential equations using a standard method (Ross 1974). The solution here is much simpler in comparison with the method of Karp et al. (1980), involving fewer steps of mathematical derivation. Also, in the Karp et al. study, in order to avoid the singularity problem (conservative scattering case with single scattering albedo $\omega \rightarrow 1$), a number of transformations are introduced. The physics for such transformations is not readily clear. In contrast, there is no such singularity problem in our solution and the exact solution corresponding to the conservative scattering case is obtained. The solution of Karp et al. (1980) appears to be quite complicated and the complexity is not reduced in the context of a lower-order approximate calculation. Our method here is general in the sense that it can be applied to higher-order numerical calculations. However, for the purpose of this paper, we are only interested in the lower-order solutions of the spherical harmonic expansion approximation, because probably only the two- and four-stream cases are worthy of analytical solutions (the analytical solution would become far too complicated for a higher-order case). It may be noted that such an analytical solution for a four-stream spherical harmonic expansion has not been systematically discussed before.

In the second section, the background of the radiative transfer process is provided in a general way. The four-stream extension of the Eddington approximation or, equivalently, the four-stream spherical harmonic expansion method and its solution are discussed in section 3. Finally, the results of the calculations and the comparison with the results from a more rigorous model are shown in section 4.

2. Basic theory of solar radiative transfer equation

The solar radiative transfer equation is

$$\mu \frac{dI}{d\tau} = I - J - J_0, \tag{1}$$

where $I(\tau, \mu, \varphi)$ is the diffuse intensity, τ is the optical thickness, $\mu = \cos\theta$, θ is the local zenith angle, and φ is the local azimuth angle. The internal source term due to multiple scattering is

$$J = \frac{\omega}{4\pi} \int_{-1}^{+1} \int_{0}^{\pi} P(\mu, \varphi; \mu', \varphi') I(\tau, \mu', \varphi') d\mu' d\varphi', \tag{2}$$

where ω is the single scattering albedo. $P(\mu, \varphi; \mu', \varphi')$ is the phase function, which can be expanded as

$$P(\mu, \varphi; \mu', \varphi') = \sum_{l=0} \sum_{m=-l}^{l} \frac{\tilde{\omega}_l}{2l+1} Y_l^m(\mu, \varphi) Y_l^{m*}(\mu', \varphi'), \tag{3}$$

where $Y_l^m(\mu, \varphi)$ and $Y_l^{m*}(\mu, \varphi)$ are the spherical harmonic function and its complex conjugate, respectively.

$$Y_l^m(\mu, \varphi) = \left[(2l+1) \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\mu) e^{im\varphi}, \tag{4}$$

with $P_l^m(\mu)$ being the associated Legendre function.

The moments $\tilde{\omega}_l$ can be obtained by expanding the phase function in terms of the scattering angle $\cos\Theta = \mu\mu' + (1-\mu)^{1/2}(1-\mu')^{1/2}\cos(\varphi - \varphi')$,

$$P(\cos\Theta) = \sum_{l=0} \tilde{\omega}_l P_l(\cos\Theta), \tag{5}$$

which is due to the addition theorem for spherical functions, where

$$P_l(\cos\Theta) = \sum_{m=-l}^{l} \frac{1}{2l+1} Y_l^m(\mu, \varphi) Y_l^{m*}(\mu', \varphi'). \qquad (6)$$

Therefore, the moment $\tilde{\omega}_l$ is determined by the orthogonality relation of the Legendre function

$$\tilde{\omega}_l = \frac{2l+1}{2} \int_{-1}^{1} P(\cos\Theta) P_l(\cos\Theta) d\cos\Theta. \qquad (7)$$

It can be shown that $\tilde{\omega}_0 = 1$ and $\tilde{\omega}_1/3 = g$, the asymmetry factor. The external source term due to the single scattering of the direct solar beam is

$$J_0 = \frac{\omega}{4\pi} P(\mu, \varphi; -\mu_0, \varphi_0) \pi F_0 e^{-\tau/\mu_0}, \qquad (8)$$

where $\mu_0 = \cos\theta_0$, θ_0 is solar zenith angle, φ_0 is solar azimuth angle, and πF_0 is the solar flux.

3. Four-stream extension of the Eddington approximation and solution

The purpose of the spherical harmonic expansion of the intensity is to separate out the angle-dependent factor

$$I = \sum_{l=0} \sum_{m=-l}^{l} (2l+1)^{1/2} I_l^m(\tau) Y_l^m(\mu, \varphi), \qquad (9)$$

substituting Eq. (9) into Eqs. (2) and (8), and using the orthogonality property of the spherical harmonic function, we have the source terms

$$J = \omega \sum_{l=0} \sum_{m=-l}^{l} \frac{\tilde{\omega}_l}{(2l+1)^{1/2}} I_l^m(z) Y_l^m(\mu, \varphi) \qquad (10)$$

and

$$J_0 = \frac{\omega}{4} \sum_{l=0} \sum_{m=-l}^{l} \frac{\tilde{\omega}_l}{2l+1} \times Y_l^m(\mu, \varphi) Y_l^{m*}(-\mu_0, \varphi_0) F_0 e^{-\tau/\mu_0}. \qquad (11)$$

Substituting Eqs. (9), (10), and (11) into Eq. (1), we obtain

$$[(l-m+1)(l+m+1)]^{1/2} \frac{d}{d\tau} I_{l+1}^m + [(l+m)(l-m)]^{1/2} \frac{d}{d\tau} I_{l-1}^m = (2l+1) I_l^m - \omega\tilde{\omega}_l I_l^m - \omega\tilde{\omega}_l Y_l^{m*}(-\mu_0, \varphi_0) \times F_0 e^{-\tau/\mu_0}/4(2l+1)^{1/2}. \qquad (12)$$

We consider a solution with a truncation of order L_t, which means that the spherical harmonic function $Y_l^m(\mu, \varphi)$ is restricted to order $l = 0, 1, 2, \cdots, L_t$. Truncated at $L_t = 1$ for the lowest order case, if only calculations of flux and the azimuthally averaged intensity are desired, we have (for $m = 0$)

$$\left.\begin{aligned} \frac{d}{d\tau} I_1^0 &= a_0 I_0^0 - b_0^0 e^{-\tau/\mu_0} \\ \frac{d}{d\tau} I_0^0 &= a_1 I_1^0 - b_1^0 e^{-\tau/\mu_0} \end{aligned}\right\}, \qquad (13)$$

where $a_l = [(2l+1) - \omega\tilde{\omega}_l]$ and $b_l^m = \omega\tilde{\omega}_l Y_l^{m*} \times (-\mu_0, \varphi_0) F_0/4(2l+1)^{1/2}$.

Equation (13) is the well-known two-stream result and is the same as that obtained in the Eddington approximation (Shettle and Weinman 1970).

$L_t = 2$ corresponds to a degenerate case and will be discussed later. Consider $L_t = 3$, when four terms in the expansion of Eq. (9) are incorporated. Therefore, this corresponds to the four-stream case. Also, for the purpose of obtaining the flux and the azimuthally averaged intensity, only the case of $m = 0$ needs to be considered. Equation (12) yields

$$\left.\begin{aligned} \frac{d}{d\tau} I_1 &= a_0 I_0 - b_0 e^{-\tau/\mu_0} \\ 2\frac{d}{d\tau} I_2 + \frac{d}{d\tau} I_0 &= a_1 I_1 - b_1 e^{-\tau/\mu_0} \\ 3\frac{d}{d\tau} I_3 + 2\frac{d}{d\tau} I_1 &= a_2 I_2 - b_2 e^{-\tau/\mu_0} \\ 3\frac{d}{d\tau} I_2 &= a_3 I_3 - b_3 e^{-\tau/\mu_0} \end{aligned}\right\}. \qquad (14)$$

Since the paper considers only the azimuthally independent case, the superscript "0" in I_i^0 and b_i^0 ($i = 0, 1, 2, 3$) is neglected.

First, consider the homogeneous solution of Eq. (14). Assuming $I_i = G_i e^{-\lambda\tau}$ ($i = 0, 1, 2, 3$) and substituting them in the homogeneous part of Eq. (14) (Ross 1974), we obtain a eigen-determinant with the eigenvalues of λ determined by the nontrivial solution requirement for G_i

$$\begin{vmatrix} a_0 & \lambda & 0 & 0 \\ \lambda & a_1 & 2\lambda & 0 \\ 0 & 2\lambda & a_2 & 3\lambda \\ 0 & 0 & 3\lambda & a_3 \end{vmatrix} = 0, \qquad (15)$$

which leads to

$$f(\lambda) = \lambda^4 - \beta\lambda^2 + \gamma = 0, \qquad (16)$$

where $\beta = a_0 a_1 + \frac{4}{9} a_0 a_3 + \frac{1}{9} a_2 a_3$ and $\gamma = \frac{1}{9} a_0 a_1 a_2 a_3$.

Eigenvalues obtained from Eq. (16) are $\lambda_1 = -\lambda_3 = [\beta + (\beta^2 - 4\gamma)^{1/2}]^{1/2}/\sqrt{2}$, $\lambda_2 = -\lambda_4 = [\beta - (\beta^2 - 4\gamma)^{1/2}]^{1/2}/\sqrt{2}$. The coefficients of G_i are not independent, but are related through the homogeneous part of the equations. By taking the homogeneous parts of the first three equations in Eq. (14), we obtain

$$\left.\begin{aligned} G_1 &= -a_0/\lambda G_0 \\ G_2 &= \frac{1}{2}(a_0a_1/\lambda^2 - 1)G_0 \\ G_3 &= -\frac{3}{2}(a_0a_1/\lambda - \lambda)/a_3G_0 \end{aligned}\right\}. \quad (17)$$

Thus, the other three coefficients can be determined in terms of G_0.

For the particular solution, we assume the solution form $\eta_i e^{-\tau/\mu_0}$ ($i = 0, 1, 2, 3$) for the components of I_i ($i = 0, 1, 2, 3$) (Ross 1974). Substituting the particular solution in Eq. (14), we can obtain the coefficients of η_i by solving a group of linear algebraic equations. The solutions are of a general form $\eta_i = \Delta_i/\Delta$ ($i = 0, 1, 2, 3$), where $\Delta = 9f(1/\mu_0)$, and

$$\Delta_0 = \begin{vmatrix} b_0 & 1/\mu_0 & 0 & 0 \\ b_1 & a_1 & 2/\mu_0 & 0 \\ b_2 & 2/\mu_0 & a_2 & 3/\mu_0 \\ b_3 & 0 & 3/\mu_0 & a_3 \end{vmatrix}$$
$$= (a_1b_0 - b_1/\mu_0)(a_2a_3 - 9/\mu_0^2) + 2(a_3b_2 - 2a_3b_0 - 3b_3/\mu_0)/\mu_0^2. \quad (18)$$

Similarly, Δ_1' can be obtained by replacing b_i in the second column of the eigenvalue determinant, and so on. We have

$$\Delta_1 = (a_0b_1 - b_0/\mu_0)(a_2a_3 - 9/\mu_0^2) - 2a_0(a_3b_2 - 3b_3/\mu_0)/\mu_0, \quad (19)$$

$$\Delta_2 = (a_3b_2 - 3b_3/\mu_0)(a_0a_1 - 1/\mu_0^2) - 2a_3(a_0b_1 - b_0/\mu_0)/\mu_0, \quad (20)$$

$$\Delta_3 = (a_2b_3 - 3b_2/\mu_0)(a_0a_1 - 1/\mu_0^2) + (6a_0b_1 - 4a_0b_3 - 6b_0/\mu_0)/\mu_0^2. \quad (21)$$

The final results are the linear combinations of the homogeneous solutions corresponding to each of the eigenvalues and the particular solution

$$\left.\begin{aligned} I_0 &= C_1e^{-\lambda_1\tau} + D_1e^{\lambda_1\tau} + C_2e^{-\lambda_2\tau} + D_2e^{\lambda_2\tau} + \eta_0e^{-\tau/\mu_0} \\ I_1 &= P_1(C_1e^{-\lambda_1\tau} - D_1e^{\lambda_1\tau}) + P_2(C_2e^{-\lambda_2\tau} - D_2e^{\lambda_2\tau}) + \eta_1e^{-\tau/\mu_0} \\ I_2 &= Q_1(C_1e^{-\lambda_1\tau} + D_1e^{\lambda_1\tau}) + Q_2(C_2e^{-\lambda_2\tau} + D_2e^{\lambda_2\tau}) + \eta_2e^{-\tau/\mu_0} \\ I_3 &= R_1(C_1e^{-\lambda_1\tau} - D_1e^{\lambda_1\tau}) + R_2(C_2e^{-\lambda_2\tau} - D_2e^{\lambda_2\tau}) + \eta_3e^{-\tau/\mu_0} \end{aligned}\right\}, \quad (22)$$

where C_1, D_1, C_2, and D_2 are constants to be determined by the boundary conditions, and the constants $P_{1,2} = -a_0/\lambda_{1,2}$, $Q_{1,2} = \frac{1}{2}(a_0a_1/\lambda_{1,2}^2 - 1)$, and $R_{1,2} = -\frac{3}{2}(a_0a_1/\lambda_{1,2} - \lambda_{1,2})/a_3$ are obtained by Eq. (17).

We use the so-called Marshak boundary condition (Evans 1993). For a considered layer, at the upper boundary (optical depth $\tau = \tau_u$)

$$\int_0^{-1}\int_0^{2\pi} [I(\tau_u, \mu, \varphi) - \Gamma^-(\tau_u, \mu, \varphi)]Y_l^{m*}(\mu, \varphi)d\mu d\varphi = 0$$
$$(l = 1, \cdots, L_t;\ m = \pm 1, \cdots, \pm l), \quad (23a)$$

where $\Gamma^-(\tau_u, \mu, \varphi)$ is the downward diffuse intensity at the upper boundary; at the lower boundary (optical depth $\tau = \tau_l$)

$$\int_0^{1}\int_0^{2\pi} [I(\tau_l, \mu, \varphi) - \Gamma^+(\tau_l, \mu, \varphi)]Y_l^{m*}(\mu, \varphi)d\mu d\varphi = 0$$
$$(l = 1, \cdots, L_t;\ m = \pm 1, \cdots, \pm l), \quad (23b)$$

where $\Gamma^+(\tau_l, \mu, \varphi)$ is the upward diffuse intensity at the lower boundary. The spherical harmonic functions $Y_l^{m*}(\mu, \varphi)$ are restricted to be an odd function of μ ($l + m$ = odd). In our opinion, the boundary condition of Eq. (23) is well defined. Actually, the real boundary condition for upper boundary should be $I(\tau_u, \mu, \varphi) - \Gamma^-(\tau_u, \mu, \varphi) = 0$ for all local angles of μ ($\mu < 0$) and φ, which generates an infinite number of continuity conditions. In the discrete-ordinate method the continuity is superposed on a finite number of angular points (two specified angles in the four-stream case). Also, the continuity condition is true for the hemispheric integral of intensity, weighted by any angular function as shown in Eq. (23a). The restriction to odd functions of μ makes the upper hemisphere and the lower hemisphere separate for the weight function. Therefore, the continuity of intensity is not imposed on the surface at $\mu = 0$, which has no physical meaning. In Eq. (23), the lowest-order case ($l = 1$; $m = 0$) is just the continuity of the vertical energy flow at the boundary.

For a single-layer medium, at the top ($\tau = 0$), there is no downward diffuse intensity,

$$\int_0^{-1}\int_0^{2\pi} I(0, \mu, \varphi)Y_1^{0*}d\mu d\varphi \sim \frac{1}{2}I_0(0) - I_1(0) + \frac{5}{8}I_2(0) = 0 \quad (24)$$

and

$$\int_0^{-1}\int_0^{2\pi} I(0, \mu, \varphi)Y_3^{0*}d\mu d\varphi \sim -\frac{1}{8}I_0(0) + \frac{5}{8}I_2(0) - I_3(0) = 0; \quad (25)$$

at the bottom of the layer ($\tau = \tau_0$) there is no upward diffuse intensity (surface albedo is assumed to be zero),

$$\int_0^1 \int_0^{2\pi} I(\tau_0, \mu, \varphi) Y_1^{0*} d\mu d\varphi \sim \frac{1}{2} I_0(\tau_0) + I_1(\tau_0) + \frac{5}{8} I_2(\tau_0) = 0 \quad (26)$$

and

$$\int_0^1 \int_0^{2\pi} I(\tau_0, \mu, \varphi) Y_3^{0*} d\mu d\varphi \sim -\frac{1}{8} I_0(\tau_0) + \frac{5}{8} I_2(\tau_0) + I_3(\tau_0) = 0. \quad (27)$$

These four linear equations of Eqs. (24)–(27) determine the constants C_1, D_1, C_2, and D_2. The boundary condition with nonzero surface albedo is shown in Eq. (A1).

For the conservative case (single scattering albedo $\omega = 1$), we have $a_0 = 0$. Under this circumstance, $\lambda_2 = -\lambda_4 = 0$. Assume the ansatz for I_0 corresponding to the double zero roots is $C_2 + D_2\tau$, and similar ansatz for other components (Ross 1974). Substituting these ansatzes in the homogeneous part of Eq. (14), two groups of linear equations are generated. The relations between the coefficients in ansatzes can be simply determined by solving the linear equations. We therefore have, for the conservative case,

$$\left.\begin{aligned} I_0 &= C_1 e^{-\lambda_1 \tau} + D_1 e^{\lambda_1 \tau} + C_2 + D_2 \tau + \eta_0 e^{-\tau/\mu_0} \\ I_1 &= D_2 / a_1 + \eta_1 e^{-\tau/\mu_0} \\ I_2 &= Q_1 (C_1 e^{-\lambda_1 \tau} + D_1 e^{\lambda_1 \tau}) + \eta_2 e^{-\tau/\mu_0} \\ I_3 &= R_1 (C_1 e^{-\lambda_1 \tau} - D_1 e^{\lambda_1 \tau}) + \eta_3 e^{-\tau/\mu_0} \end{aligned}\right\}. \quad (28)$$

The coefficients λ_1, Q_1, R_1, and η_i are the same as those above, provided $a_0 = 0$. The constants C_1, D_1, C_2, and D_2 are obtained through the same boundary condition as that of the nonconservative case. Because $\omega = 1$ is a singular point for the nonconservative solution, using a value of ω very close to 1 for performing the conservative scattering calculations can easily cause a numerical instability (e.g., Karp et al. 1980).

The upward flux F^+ and downward flux F^- are obtained as

$$F^{\pm} = \pi\left(I_0 \pm 2I_1 + \frac{5}{4} I_2\right). \quad (29)$$

In the δ-Eddington approximation, the corresponding fluxes are

$$F^{\pm} = \pi(I_0 \pm 2I_1). \quad (30)$$

Thus, one more term is added in the new four-stream model. Note that the I_1 in Eq. (30) is one-third of that in Shettle and Weinman (1970), owing to the normalization coefficients in Eq. (9).

4. Computational results and discussions

In the four-stream extension of the Eddington approximation, because the phase function is truncated for a moment corresponding to $L_t = 3$, the δ-M scaling technique (Wiscombe 1977a) can be applied. The δ-M scaling adjustment is based on the physical consideration of separating out the forward peak in the phase function. Under the δ-M adjustment, the optical parameters are scaled by

$$\tau' = \tau(1 - f\omega), \quad (31)$$

$$\omega' = \omega(1 - f)/(1 - f\omega), \quad (32)$$

$$\tilde{\omega}_l' = [\tilde{\omega}_l - (2l + 1)f]/(1 - f). \quad (33)$$

For the four-stream approximation case, $f = \tilde{\omega}_4/9$ (Wiscombe 1977a). In the following, the δ-M scaling technique is used for all the calculations. Consequently, the two- and four-stream spherical harmonic expansion approximations are termed δ-Eddington approximation and δ-four-stream spherical harmonic expansion approximation (δ-four-stream SHEA), respectively.

a. Flux

We analyze the accuracy of the reflection $r(\tau_0, \mu_0)$, transmission $t(\tau_0, \mu_0)$, and fractional absorption $a(\tau_0, \mu_0)$ predicted by this model. The formal definitions of reflection, transmission, and absorption are

$$r(\tau_0, \mu_0) = F^+(0)/\mu_0 \pi F_0, \quad (34)$$

$$t(\tau_0, \mu_0) = F^-(\tau_0)/\mu_0 \pi F_0 + e^{-\tau_0/\mu_0}, \quad (35)$$

$$a(\tau_0, \mu_0) = 1 - r(\tau_0, \mu_0) - t(\tau_0, \mu_0). \quad (36)$$

The absolute error is defined as the value obtained from the approximation method minus the value obtained using a rigorous method; the relative error is the absolute error divided by the value from the rigorous method. The rigorous standard model used in the following is the discrete-ordinate numerical model of Stamnes et al. (1988). Forty-eight streams are used in the discrete ordinate calculations.

To demonstrate the accuracy of the method, tests for a wide range of optical thicknesses and incoming solar zenith angles are undertaken. As in King and Harshvardhan (1986), the optical thickness varies from 0.1 to 100 (in steps of 0.02 on a logarithmic scale), and the cosine of solar zenith angle varies from 0 (0.02) to 1 (in steps of 0.02). The Henyey–Greenstein phase function is used. This function is easy to handle in the calculation, since a higher order moment in phase function is just a power of the asymmetry factor. To contrast accuracies, the results using the δ-Eddington approximation and δ-four-stream discrete ordinate method will also be shown. Our results using the δ-Eddington approximation are slightly different from that of King and Harshvardhan (1986), because the phase function they employed is from the exact Mie theory; further, their cloud optical parameters correspond to a specific droplet size distribution.

First, we consider the case of a nonabsorbing medium with a single scattering albedo $\omega = 1$. The relative errors for reflection and transmission are plotted in Fig. 1. The top panels represent the error in the δ-Eddington approximation and the lower panels represent the error in the corresponding δ-four-stream SHEA results. The asymmetry factor $g = 0.8$.

The contours of the error in Fig. 1 for the δ-Eddington approximation are quite similar to the results shown in King and Harshvardhan (1986) even though, as noted earlier, the phase functions are somewhat different in these two calculations. For reflection, the results are relatively more accurate in the region of thicker optical depths and smaller solar zenith angles. For transmission, too, the accuracy is greater in the region of smaller solar zenith angles, but is not as sensitive to the optical thickness.

The middle panels show that the δ-four-stream SHEA yields substantially more accurate results. For instance, the error in reflection using the the δ-Eddington approximation is as high as 10% for $\tau_0 > 2$. However, this region has an error of less than 2% in the case of the δ-four-stream SHEA. For transmission, the error using the δ-Eddington approximation is bounded by 10% in the region of $\mu_0 > 0.2$, while the error is suppressed to less than 1% in the four-stream case over the same domain.

The corresponding results of δ-four-stream discrete ordinate method are shown in the bottom panels. For optical thickness less than one, all methods produce large errors; this a region that our method can not be applied. The thin optical thickness case will be further discussed in a subsequent work.

Here $\omega = 0.9$ is used as example of an absorbing case, as in King and Harshvardhan (1986). The asymmetry factor $g = 0.8$. The error comparisons for the δ-Eddington approximation and δ-four-stream SHEA and δ-four-stream discrete ordinate approximation are illustrated in Fig. 2. The upper panels are results from the δ-Eddington approximation, and the middle panels are the results from the δ-four-stream SHEA.

First, we focus on the δ-Eddington results. It is found in Fig. 2 that the absorbing media leads to a larger error in reflection and transmission compared to the nonabsorbing case. In particular, the transmission shows very poor results for large optical thicknesses, the errors being up to 20% or higher. This is also true for other two stream schemes (King and Harshvardhan 1986) and is partly due to the very small value of transmission occurring in these cases, with the absolute error in this region being small (see King and Harshvardhan 1986). The δ-Eddington approximation predicts good results for absorption and also for the large optical thickness cases ($\tau_0 > 10$). For a small value of the optical thickness, a large portion of the figure for absorption is dominated by errors greater than 10%.

In the case of the δ-four-stream SHEA (middle panel), it is found that the relative errors are very much suppressed compared to the δ-Eddington approximation. The relative error in reflection is mostly (for $\mu_0 > 0.1$) bounded by 5% for optical thickness $\tau_0 > 1$; in contrast, it is up to 15% in the δ-Eddington case. Significant improvements occur in the region of small solar zenith angle ($\mu_0 > 0.7$); for $\tau_0 > 1$, the error is bounded by 1%, whereas the error is close to 10% in the two-stream case. A dramatic improvement occurs for transmission. Most of the region has errors less than 1%, even in the thin optical thickness region; this is to be contrasted with the fact that a large region is dominated by errors of up to 10% in the δ-Eddington case. For absorption, the contours of δ-Eddington approximation and the δ-four-stream SHEA are roughly similar. Though there is a marked improvement, the δ-four-stream approximation cannot completely eliminate the region where the errors exceed 10%; however, this domain is much smaller compared to the δ-Eddington case. For $\tau_0 > 2$, most of the regions are bounded by errors in absorption of less than 2%, except for the grazing incident case of $\mu_0 < 0.1$. In contrast, the error is up to 10% in the δ-Eddington case for the same region. The corresponding results for the δ-four-stream discrete ordinate method are shown in the bottom panels and can be compared with the accuracies presented in the other two panels.

We consider a strongly absorbing case in Fig. 3 with single scattering albedo $\omega = 0.5$; again we choose $g = 0.8$. It is found that the relative errors in reflection and transmission increase as ω decreases, while the error in absorption decreases as ω decreases. Again, the δ-four-stream SHEA provides more accurate results in comparison to that from the δ-Eddington approximation.

b. Azimuthally averaged intensity

The spherical harmonic expansion method is based on a specific assumption of the spatial variation of intensity. Thus, the intensity can be obtained even in the two-stream solution. In the case of $m = 0$, only the evaluation of azimuthally averaged intensity is available. In the two-stream case or Eddington approximation, this azimuthally averaged intensity is given by $I = I_0 + 3\mu I_1$. This linear variation of intensity with the cosine of local zenith angle shows a large error in comparison with the exact results (Davies 1980). We examine here the accuracy in the context of the four-stream solution. The azimuthally averaged intensity is given by

$$I = I_0 Y_0^0 + \sqrt{3} I_1 Y_1^0 + \sqrt{5} I_2 Y_2^0 + \sqrt{7} I_3 Y_3^0. \quad (37)$$

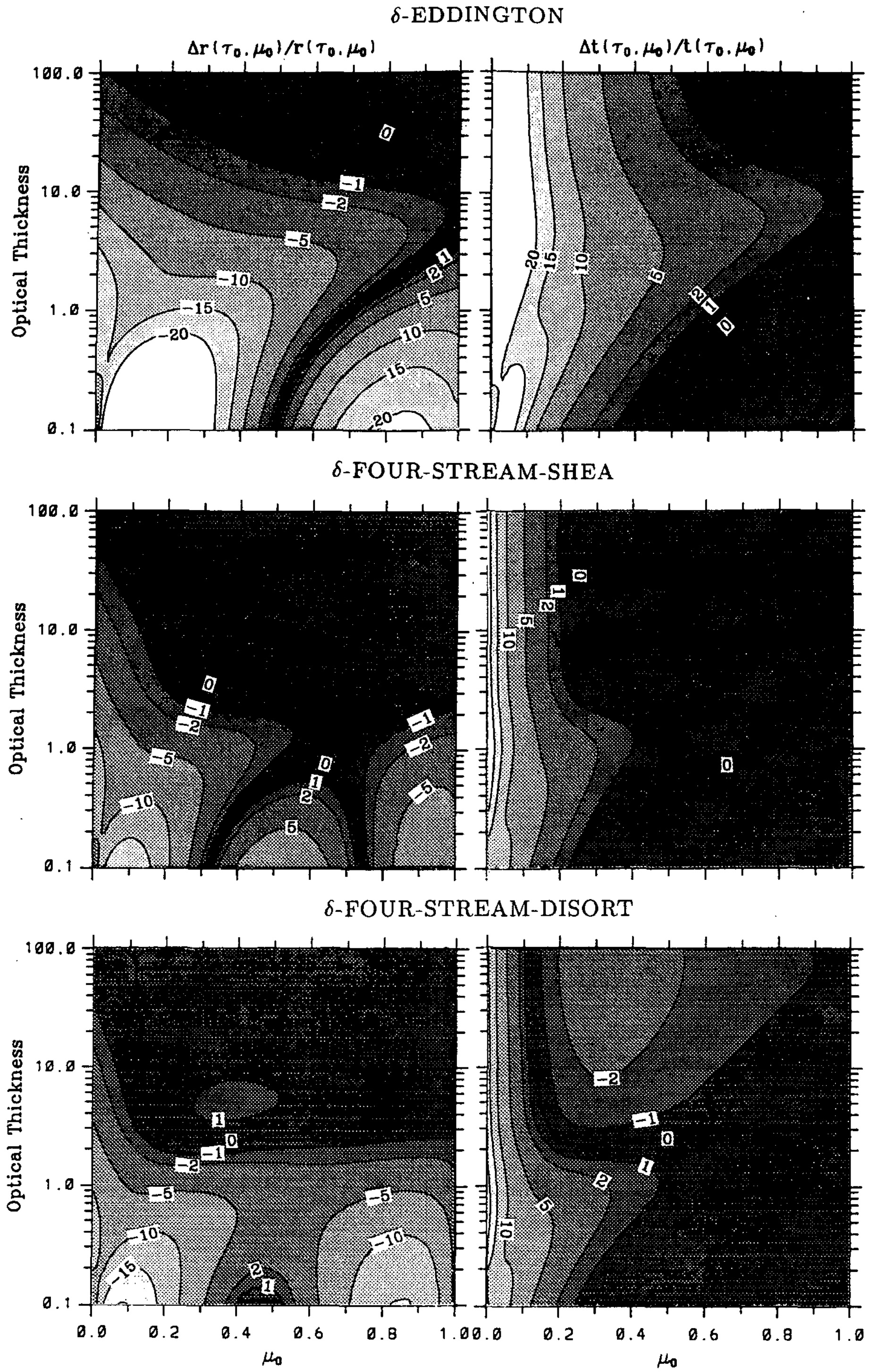

FIG. 1. Relative errors of the two-stream δ-Eddington approximation (top panels), δ-four-stream SHEA (middle panels), and δ-four-stream discrete-ordinate model (bottom panels) for reflection, $\Delta r(\tau_0, \mu_0)/r(\tau_0, \mu_0)$, and transmission, $\Delta t(\tau_0, \mu_0)/t(\tau_0, \mu_0)$. The Henyey–Greenstein phase function is used, with the asymmetry factor $g = 0.8$. The cloud layer is nonabsorbing ($\omega = 1$).

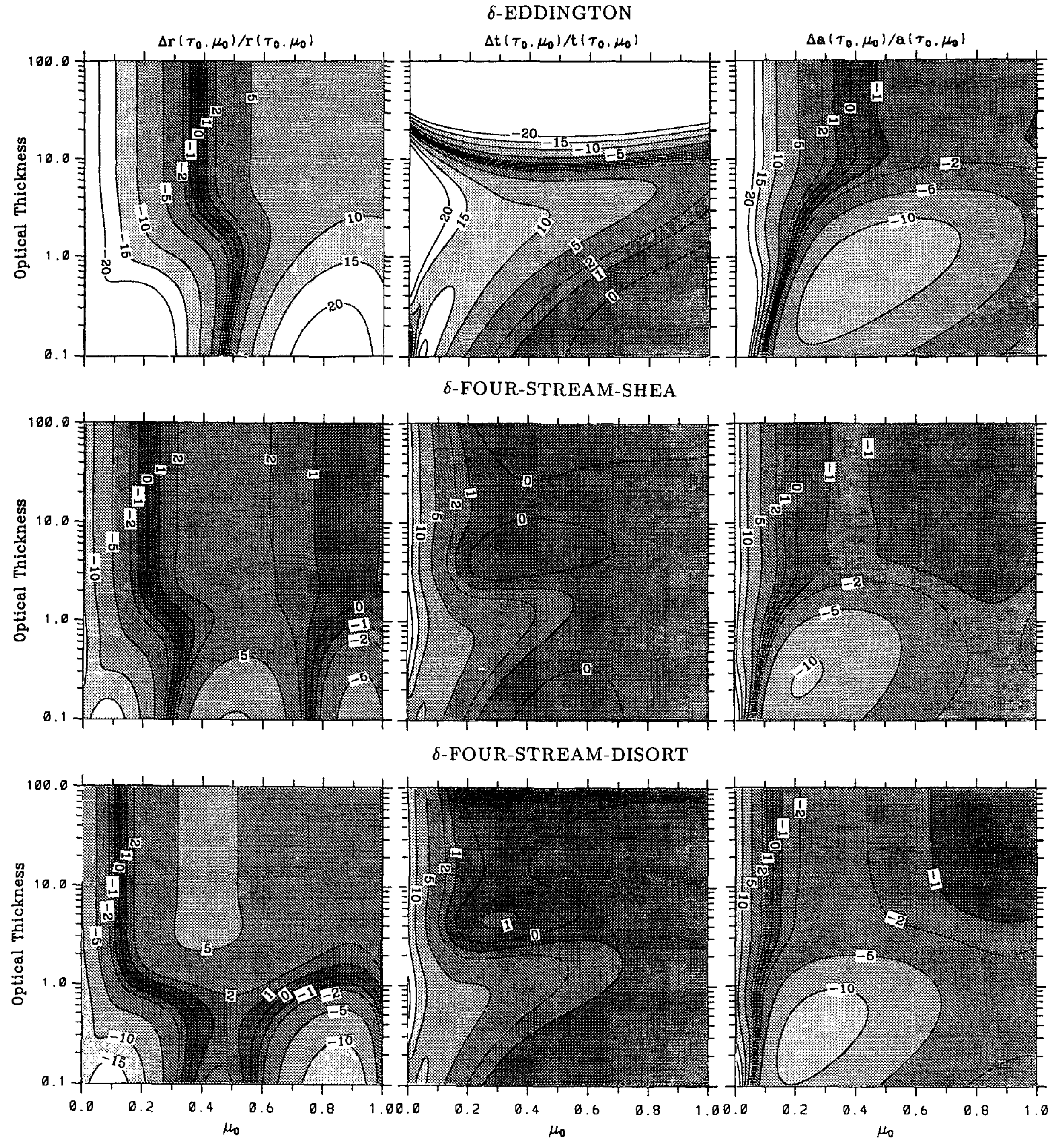

FIG. 2. Same as in Fig. 1 except for absorbing media with $\omega = 0.9$. The relative errors for absorption, $\Delta a(\tau_0, \mu_0)/a(\tau_0, \mu_0)$, are also shown.

We find through numerical calculations that the results of Eq. (37) are still poor compared to the rigorous results in some regions of the parameter space. Therefore, adjustments have to be made to improve the accuracy. Comparing Eq. (37) with Eq. (29), we note that the last term of I_3 does not appear in the equation for flux. Therefore, an adjustment can be made to the last term of Eq. (37) without influencing the results of the flux. Here $Y_3^0(\mu)$ varies quite steeply with μ for μ close to 1, which leads to a poor result in this region. To modulate it, some other spherical function may be added. We simply replace the last term in Eq. (37) by $\sqrt{7}I_3(Y_3^0 + 1.6Y_4^0 + 3(1 - \mu)^4)$. A factor of $3(1 - \mu)^4$ is added to adjust the result in the region of small μ;

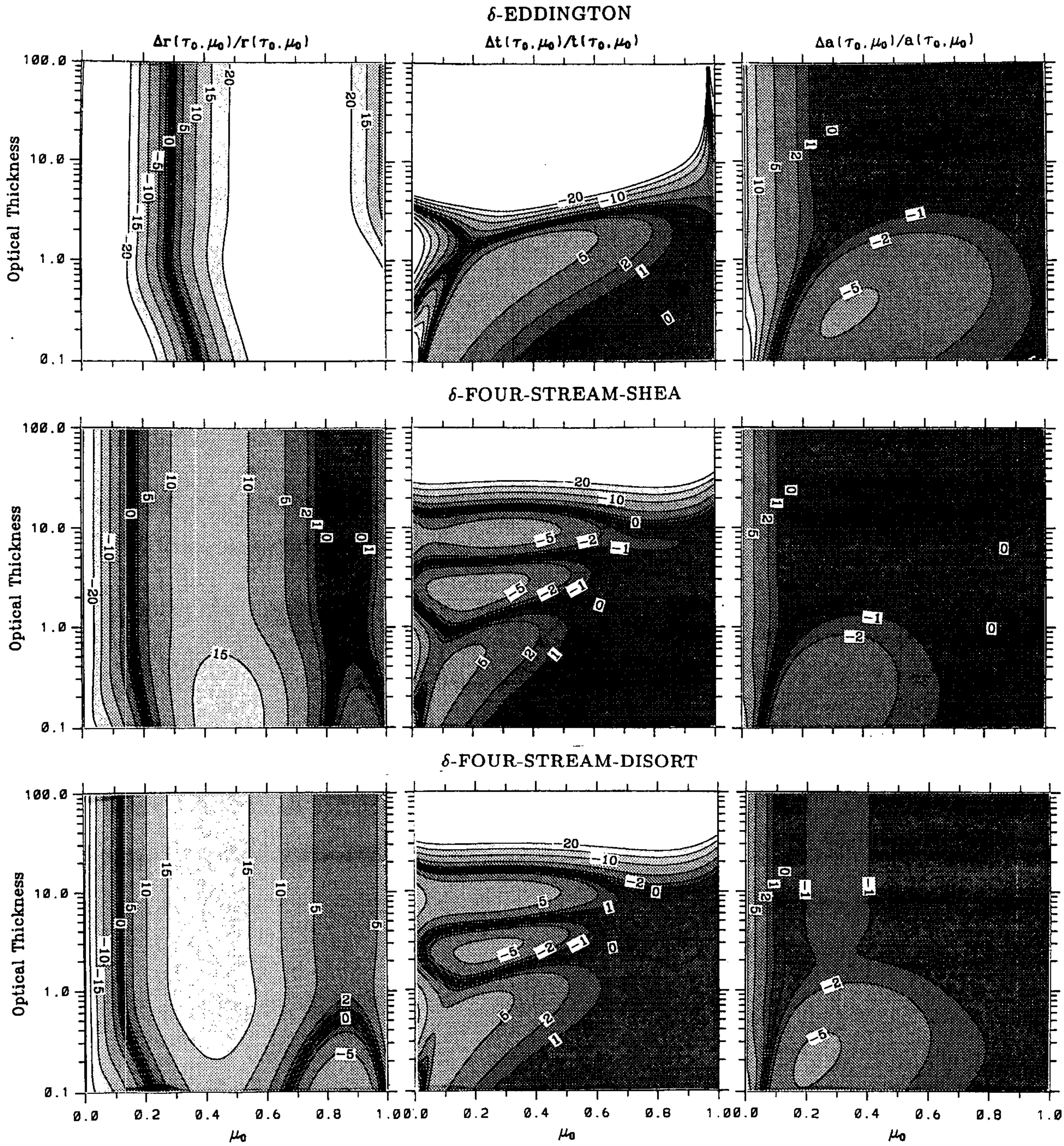

FIG. 3. Same as in Fig. 2 except for $\omega = 0.5$.

otherwise, the values are usually smaller in this region in comparison with the exact results.

By doing this simple adjustment, the computational results become much closer to the rigorous results. The azimuthally averaged intensities for different optical thicknesses are shown in Fig. 4. The cosine of incoming solar zenith angle $\mu_0 = 0.5$, the asymmetry factor of the considered layer $g = 0.8$, and the cloud layer is a nonabsorbing medium. The discrete ordinate model (Stamnes et al. 1988) is used as the standard model for comparison, and forty-eight streams are considered in the rigorous calculations. The Henyey–Greenstein phase function is used again. In Fig. 4, it is found that the intensity obtained by this model is quite close to the rigorous results. The accuracy is higher for the thicker cloud cases, as was the situation for the flux

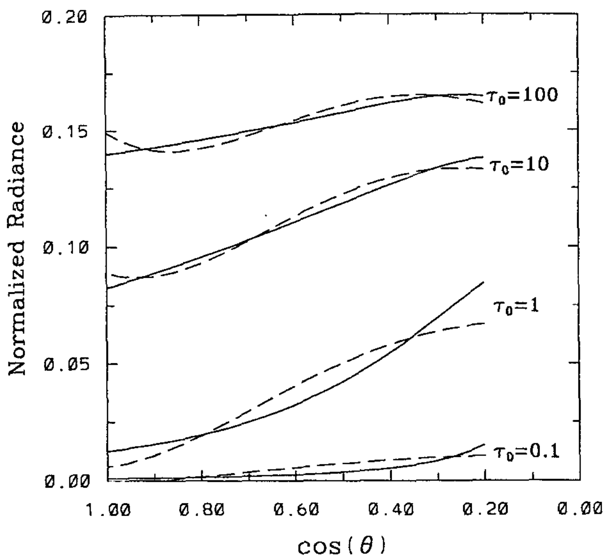

Fig. 4. Azimuthally averaged radiative intensity as a function of viewing angle μ for various optical thickness τ_0: $F_0 = 1$, $\omega = 1$, $g = 0.8$, and $\mu_0 = 0.5$. The solid lines are the results of the rigorous model and the dashed lines are the results of the four-stream SHEA model.

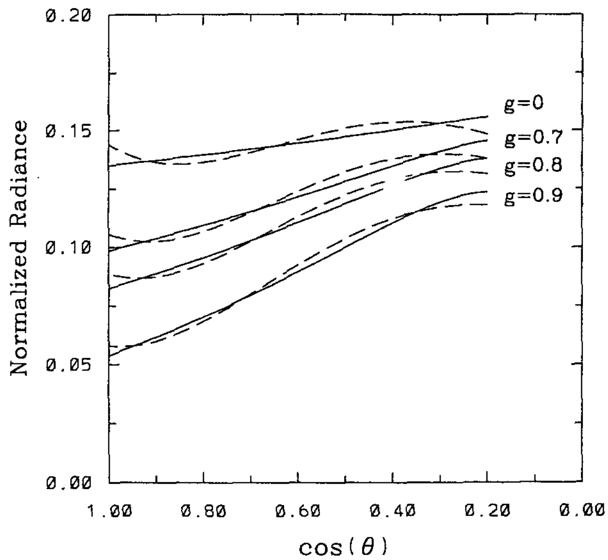

Fig. 6. Azimuthally averaged radiative intensity as a function of viewing angle μ for various values of the asymmetry factor g: $F_0 = 1$, $\tau_0 = 10$, $\omega = 1$, and $\mu_0 = 0.5$. The solid lines are the results of the rigorous model and the dashed lines are the results of the four-stream SHEA model.

results. In Fig. 5, the incoming solar zenith angle $\mu_0 = 1$, while the other parameters are the same as those in Fig. 4. When the solar zenith angle becomes small, we find from Fig. 5 that the accuracy in intensity is improved for the thin cloud cases.

We have found that the accuracy of the intensity predicted by this model is not sensitive to the optical properties of the cloud. The calculations of intensity for different asymmetry factors are shown in Fig. 6. The considered layer is of optical thickness $\tau_0 = 10$, the cosine of incoming solar zenith angle $\mu_0 = 0.5$, and single scattering albedo $\omega = 1$. It is seen that the absolute error increases slightly as the asymmetry factor decreases.

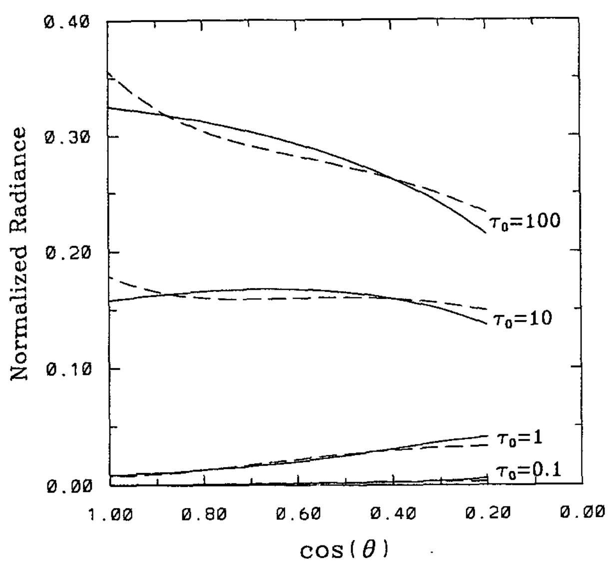

Fig. 5. Same as in Fig. 3 except for incoming solar zenith angle $\mu_0 = 1$.

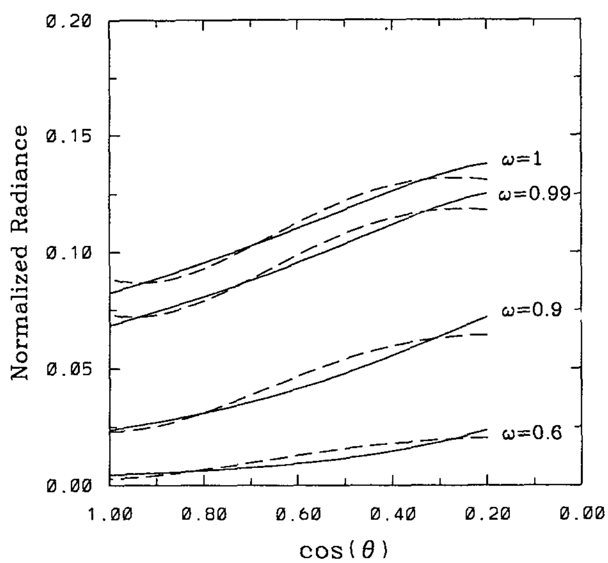

Fig. 7. Azimuthally averaged radiative intensity as a function of viewing angle μ for various values of single scattering albedo ω: $F_0 = 1$, $\tau_0 = 10$, $g = 0.8$, and $\mu_0 = 0.5$. The solid lines are the results of the rigorous model and the dashed lines are the results of the four-stream SHEA model.

The sensitivity to single scattering albedo is shown in Fig. 7. The asymmetry factor is $g = 0.8$, while the other parameters are the same as in Fig. 6. The relative error is larger for the smaller single scattering albedo cases, which is similar to the situation discussed in the last section for the reflected flux.

For azimuthally dependent intensity, the solutions of equations corresponding to $m \neq 0$ are required. This is beyond the scope of this paper. The azimuthally averaged intensity provides the local zenith angle–dependent radiation information. If the four-stream spherical harmonic expansion method is applied to climate models, more accurate values of intensity will result in more accurate radiative entropy evaluations (Li et al. 1994). The evaluation of the radiative intensity in a low-order approximate radiative transfer context has been considered by Davies (1980) and Xiang et al. (1994).

5. Conclusions

The physical essence of the Eddington approximation is that it is the lowest-order ($L_t = 1$) case in the spherical harmonic expansion of the intensity. The mode considered in this paper corresponds to the second lowest-order expansion ($L_t = 3$). There does exist a solution for the case of $L_t = 2$, which can be easily shown to be a degenerate one. Only two of the three components in the intensity expansion are independent. The results of the flux calculations in the case of $L_t = 2$ are even poorer than the corresponding results of the Eddington approximation.

It is well known that, in the calculations of flux and heating rate, the Eddington approximation is poor for the cases of small optical thickness and large solar zenith angle. Calculations show that the accuracy of the reflection, transmission, and absorption are all significantly improved by using the new four-stream extension method. Although we have only considered a single-layer case, the model can be easily extended to multiple-layer cases, just as in the instance of the δ-Eddington approximation (see appendix).

The four-stream spherical harmonic expansion can be extended to three-dimensional radiative transfer problems, which ought to lead to more accurate results in studies of finite clouds (Davies 1978) and internally inhomogeneous clouds (Li et al. 1995).

The solution of the four-stream SHEA has an elegant form. Since we only consider the azimuth-independent case here, the spherical harmonic function reduces to Legendre function and no complex quantities appear. The four-stream extension presented in this work is an analytical method that makes for computational efficiency. All calculations show that the CPU time for this four-stream extension method represents a modest increase over that for the δ-Eddington approximation. Therefore, this method should be applicable in climate models, where a fast but accurate result is required. In particular, the derivation here is an appropriate higher-order substitute for the δ-Eddington approximation that is currently used in several general circulation models of weather and climate.

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APPENDIX

Boundary Connection for Multiple Layers

We consider a vertically inhomogeneous column of n layers, with each layer being internally homogeneous (Fig. 8). There are generally two methods to formulate the radiative transfer process involving the single homogeneous layers. One could be termed as the "direct connection." We denote the intensity in ith layer by $I^{(i)} = \sum_{l=0}^{n} \sqrt{2l+1} I_l^{(i)}(\tau) Y_l^0$. Assuming no downward scattered radiation on the top of the system and a Lambertian surface albedo α for the bottom of the system, we directly have the following relations from Eq. (23).

$$\frac{1}{2} I_0^{(1)}(0) - I_1^{(1)}(0) + \frac{5}{8} I_2^{(1)}(0) = 0$$

$$-\frac{1}{8} I_0^{(1)}(0) + \frac{5}{8} I_2^{(1)}(0) - I_3^{(1)}(0) = 0$$

$$\frac{1}{2} I_0^{(1)}(\tau_1) + I_1^{(1)}(\tau_1) + \frac{5}{8} I_2^{(1)}(\tau_1) = \frac{1}{2} I_0^{(2)}(\tau_1) + I_1^{(2)}(\tau_1) + \frac{5}{8} I_2^{(2)}(\tau_1)$$

$$-\frac{1}{8} I_0^{(1)}(\tau_1) + \frac{5}{8} I_2^{(1)}(\tau_1) + I_3^{(1)}(\tau_1) = -\frac{1}{8} I_0^{(2)}(\tau_1) + \frac{5}{8} I_2^{(2)}(\tau_1) + I_3^{(2)}(\tau_1)$$

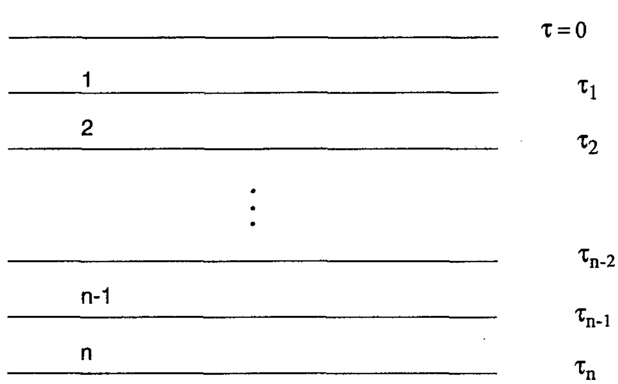

FIG. 8. Schematic of n homogenous layers for radiative transfer in a vertically inhomogeneous atmosphere.

$$\frac{1}{2}I_0^{(n-1)}(\tau_{n-1}) - I_1^{(n-1)}(\tau_{n-1}) + \frac{5}{8}I_2^{(n-1)}(\tau_{n-1})$$
$$= \frac{1}{2}I_0^{(n)}(\tau_{n-1}) - I_1^{(n)}(\tau_{n-1}) + \frac{5}{8}I_2^{(n)}(\tau_{n-1})$$

$$-\frac{1}{8}I_0^{(n-1)}(\tau_{n-1}) + \frac{5}{8}I_2^{(n-1)}(\tau_{n-1}) - I_3^{(n-1)}(\tau_{n-1})$$
$$= -\frac{1}{8}I_0^{(n)}(\tau_{n-1}) + \frac{5}{8}I_2^{(n)}(\tau_{n-1}) - I_3^{(n)}(\tau_{n-1})$$

$$\frac{1}{2}I_0^{(n)}(\tau_n) + I_1^{(n)}(\tau_n) + \frac{5}{8}I_2^{(n)}(\tau_n)$$
$$= \frac{1}{2}\alpha\left\{I_0^{(n)}(\tau_n) - 2I_1^{(n)}(\tau_n) + \frac{5}{4}I_2^{(n)}(\tau_n) + \mu_0F_0e^{-\tau_n/\mu_0}\right\}$$

$$-\frac{1}{8}I_0^{(n)}(\tau_n) + \frac{5}{8}I_2^{(n)}(\tau_n) + I_3^{(n)}(\tau_n)$$
$$= -\frac{1}{8}\alpha\left\{I_0^{(n)}(\tau_n) - 2I_1^{(n)}(\tau_n) + \frac{5}{4}I_2^{(n)}(\tau_n) + \mu_0F_0e^{-\tau_n/\mu_0}\right\}. \quad \text{(A1)}$$

There are $4n$ linear algebra equations that determine the $4n$ constants in the radiative intensities (flux) for the n layers. Note that $2n$ linear algebra equations are needed (Wiscombe 1977b) for the two-stream δ-Eddington case. In the process of solving the linear algebra equation of Eq. (A1), an ill-conditioning may occur. The exponential factor $e^{\pm\lambda_i\tau}$ (i = 1, 2) could reach values beyond the range of available computer capacity for a large value of τ. This can generally be avoided by using scaling transformations developed by Stamnes et al. (1984), which has been used in the numerical DISORT model (Stamnes et al. 1988).

Another method is a version of the so-called "adding" method, in which the fluxes across the layer interfaces are added in an appropriate fashion. The optical thickness of ith single layer is $\tau_i^* = \tau_i - \tau_{i-1}$ (see Fig. 8). For each layer, the direct and diffuse reflections and transmissions are distinguished. The direct reflection and transmission can be obtained in the same way as shown in this paper with the boundary condition of Eqs. (24)–(27), provided τ_0 is replaced by τ_i^* for the ith layer. For the diffuse reflection and transmission, we have the same solutions as Eqs. (22) and (28), provided the particular solution $\eta_i = 0$ (i = 0, 1, 2, 3), since there is no direct solar beam contribution. Assume the diffuse beam to have an isotropic distribution (Coakley et al. 1983) and let the downward diffuse flux to the ith layer be $\bar{F}_0^{(i)}$; thus, the downward diffuse intensity is $\bar{F}_0^{(i)}/\pi$. Therefore, the boundary conditions for the diffuse radiation are

$$\frac{1}{2}\bar{I}_0^{(i)}(0) - \bar{I}_1^{(i)}(0) + \frac{5}{8}\bar{I}_2^{(i)}(0) = \frac{1}{2\pi}\bar{F}_0^{(i)}$$
$$-\frac{1}{8}\bar{I}_0^{(i)}(0) + \frac{5}{8}\bar{I}_2^{(i)}(0) - \bar{I}_3^{(i)}(0) = -\frac{1}{8\pi}\bar{F}_0^{(i)}$$
$$\frac{1}{2}\bar{I}_0^{(i)}(\tau_i^*) + \bar{I}_1^{(i)}(\tau_i^*) + \frac{5}{8}\bar{I}_2^{(i)}(\tau_i^*) = 0$$
$$-\frac{1}{8}\bar{I}_0^{(i)}(\tau_i^*) + \frac{5}{8}\bar{I}_2^{(i)}(\tau_i^*) + \bar{I}_3^{(i)}(\tau_i^*) = 0, \quad \text{(A2)}$$

where an overbar is used to distinguish the diffuse radiation. The diffuse upward (downward) flux is

$$\bar{F}^{(i)\pm} = \pi\left(\bar{I}_0^{(i)} \pm 2\bar{I}_1^{(i)} + \frac{5}{4}\bar{I}_2^{(i)}\right). \quad \text{(A3)}$$

Finally, we have the diffuse reflection and transmission for the ith layer

$$\bar{r}_i = \bar{F}^{(i)+}(0)/\bar{F}_0^{(i)}, \quad \text{(A4)}$$

$$\bar{t}_i = \bar{F}^{(i)-}(\tau_i^*)/\bar{F}_0^{(i)}. \quad \text{(A5)}$$

In the two-stream δ-Eddington approximation case, this process leads to analytical results for the diffuse reflection and transmission as shown in Coakley et al. (1983).

For two layers designated as 1 and 2, the reflection and transmission for the combined system (radiation incident on layer 2) are (Coakley et al. 1983)

$$r_{12}(\mu_0) = r_1(\mu_0) + \frac{\bar{t}_1\{t_1(\mu_0) - e^{-\tau_1^*/\mu_0})\bar{r}_1 + e^{-\tau_1^*/\mu_0}r_2(\mu_0)\}}{1 - \bar{r}_1\bar{r}_2}, \quad \text{(A6)}$$

$$t_{12}(\mu_0) = e^{-\tau_1^*/\mu_0}t_2(\mu_0) + \frac{\bar{t}_2\{t_1(\mu_0) - e^{-\tau_1^*/\mu_0}) + e^{-\tau_1^*/\mu_0}r_2(\mu_0)\bar{r}_1\}}{1 - \bar{r}_1\bar{r}_2}, \quad \text{(A7)}$$

$$\bar{r}_{12} = \bar{r}_1 + \frac{\bar{t}_1\bar{r}_2\bar{t}_1}{1 - \bar{r}_1\bar{r}_2}, \quad \text{(A8)}$$

$$\bar{t}_{12} = \frac{\bar{t}_1\bar{t}_2}{1 - \bar{r}_1\bar{r}_2}, \quad \text{(A9)}$$

where $r_{1,2}(\mu_0)$ and $t_{1,2}(\mu_0)$ are the direct reflection and transmission, respectively, for layer 1 or 2.

To combine the layers over the entire column, two passes are made through the layers: one starting from the top and proceeding downward and other starting from surface and proceeding upward. The upward and downward fluxes are therefore obtained at each interface of the column (Coakley et al. 1983; Briegleb 1992). Alternatively, a matrix form of the up and down fluxes across the layer interfaces can be set up and solved (Ramaswamy and Bowen 1994).

REFERENCES

Briegleb, B. P., 1992: Delta-Eddington approximation for solar radiation in the NCAR community climate model. *J. Geophys. Res.*, **97**, 7603–7612.

Coakley, J. A., and P. Chýlek, 1975: The two-stream approximation in radiative transfer: Including the angle of the incident radiation. *J. Atmos. Sci.*, **32**, 409–418.

——, R. D. Cess, and F. B. Yurevich, 1983: The effect of tropospheric aerosols on the Earth's radiation budget: A parameterization for climate models. *J. Atmos. Sci.*, **40**, 116–138.

Davies, R., 1978: The effect of finite geometry on the three-dimensional transfer of solar irradiance in clouds. *J. Atmos. Sci.*, **35**, 1712–1725.

——, 1980: Fast azimuthally dependent model of the reflection of solar radiation by plane-parallel clouds. *Appl. Opt.*, **19**, 250–255.

Evans, K. F., 1993: Two-dimensional radiative transfer in cloudy atmosphere: The spherical harmonic spatial grid method. *J. Atmos. Sci.*, **50**, 3111–3124.

Karp, A. H., J. J. Greenstadt, and J. A. Filmore, 1980: Radiative transfer through an arbitrarily thick, scattering atmosphere. *J. Quant. Spectrosc. Radiat. Transfer*, **24**, 391–406.

King, M. D., and Harshvardhan, 1986: Comparative accuracy of selected multiple scattering approximations. *J. Atmos. Sci.*, **43**, 784–801.

Li, J., P. Chýlek, and G. B. Lesins, 1994: Entropy in climate models. Part I: Vertical structure of atmospheric entropy production. *J. Atmos. Sci.*, **51**, 1691–1701.

——, D. J. W. Geldart, and P. Chýlek, 1995: Second order perturbation solution for radiative transfer in clouds with a horizontally arbitrary periodic inhomogeneity. *J. Quant. Spectrosc. Radiat. Transfer*, **53**, 445–456.

Liou, K. N., 1974: Analytic two-stream and four-stream solutions for radiative transfer. *J. Atmos. Sci.*, **31**, 1473–1475.

——, Q. Fu, and T. P. Ackerman, 1988: A simple formulation of the delta-four-stream approximation for radiative transfer parameterization. *J. Atmos. Sci.*, **45**, 1940–1947.

Meador, W. E., and W. R. Weaver, 1980: Two-stream approximations to radiative transfer in planetary atmospheres: A unified description of existing methods and a new improvement. *J. Atmos. Sci.*, **37**, 630–643.

Ramaswamy, V., and S. M. Freidenreich, 1991: Solar radiative line-by-line determination of water vapor absorption and water cloud extinction in inhomogeneous atmospheres. *J. Geophys. Res.*, **96**, 9133–9157.

——, and M. M. Bowen, 1994: Effect of changes in radiatively active species upon the lower stratospheric temperatures. *J. Geophys. Res.*, **99**, 18 909–18 921.

Ross, S. L., 1974: *Differential Equation.* Xerox College, 712 pp.

Shettle, E. P., and J. A. Weinman, 1970: The transfer of solar irradiance through inhomogeneous turbid atmospheres evaluated by Eddington's approximation. *J. Atmos. Sci.*, **27**, 1048–1054.

Stamnes, K., S. C. Tsay, and P. Conklin, 1984: A new multi-layer discrete ordinate approach to radiative transfer in vertical inhomogeneous atmosphere. *J. Quant. Spectrosc. Radiat. Transfer*, **31**, 273–282.

——, ——, W. Wiscombe, and K. Jayaweera, 1988: A numerically stable algorithm for discrete-ordinate-method radiative transfer in multiple scattering and emitting layered media. *Appl. Opt.*, **27**, 2502–2509.

Wiscombe, W., 1977a: The delta-M method: Rapid yet accurate radiative flux calculations. *J. Atmos. Sci.*, **34**, 1408–1422.

——, 1977b: The delta-Eddington approximation for a vertically inhomogeneous atmosphere. NCAR Tech. Note NCAR/TN-121+STR, 66 pp. [NTIS PB-270 618/2GI.]

Xiang, X., E. A. Smith, and C. G. Justus, 1994: A rapid radiative transfer model for reflection of solar radiation. *J. Atmos. Sci.*, **51**, 1978–1988.

Zdunkowski, W. G., and G. Korb, 1974: An approximative method for the determination of short-wave radiative fluxes in scattering and absorbing media. *Contrib. Atmos. Phys.*, **47**, 129–144.

——, R. M. Welch, and G. Korb, 1980: An investigation of the structure of typical two-stream-methods for the calculation of solar fluxes and heating rate in clouds. *Contrib. Atmos. Phys.*, **53**, 147–166.