Infrared singularities of gauge theory amplitudes

Thomas Becher
Fermilab

Seminar at Fermilab

June 25, 2009

0901.0722, 0903.1126, 0904.1021 with Matthias Neubert

IR singularities

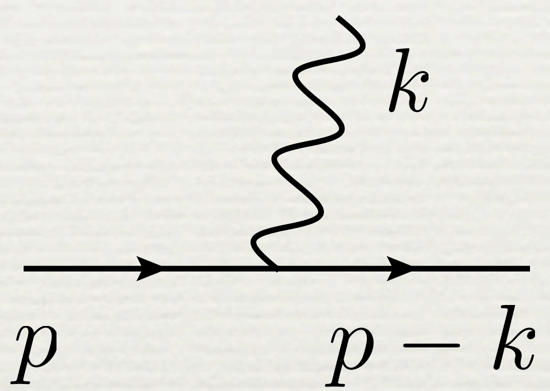
- On-shell parton scattering amplitudes in gauge theories contain IR divergences from soft and collinear loop momenta
- IR singularities cancel between real and virtual contributions

Bloch, Nordsieck 1937
Kinoshita 1962; Lee, Nauenberg 1964

- Nevertheless interesting:
  - resummation of large Sudakov logarithms remaining after cancellation of divergences (relevant for LHC physics!)
  - check on multi-loop calculations

IR singularities in QED

- Singularities arise from soft photon emission (for $m_e \neq 0$); eikonal approximation:



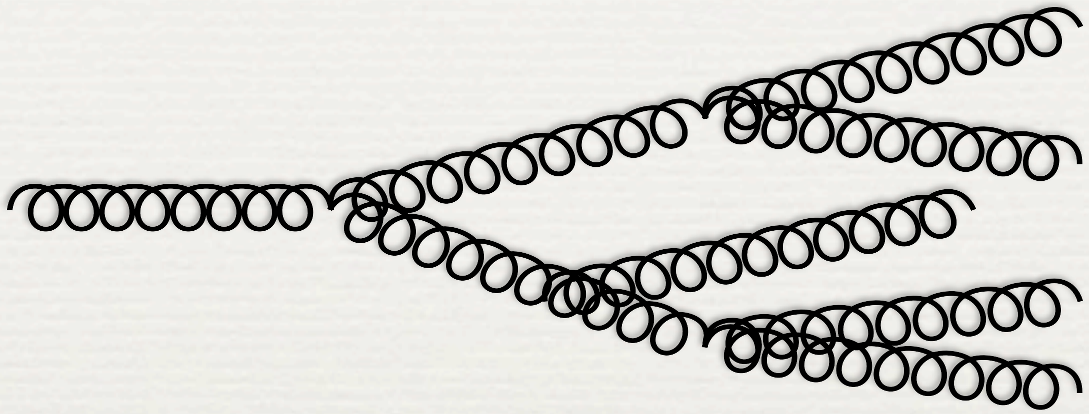
$$\ldots \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2} \gamma_\mu \, u(p)$$

$$\approx \ \ldots \, u(p) \boxed{\frac{p_\mu}{p \cdot k}}$$

- IR divergent part is a multiplicative factor
- Higher-order terms obtained by exponentiating leading-order soft contribution.

Yennie, Frautschi, Suura 1961
Weinberg 1965

IR singularities in QCD



"In [Yang-Mills theory] a soft photon emitted from an external line can itself emit a pair of soft charged massless particles, which themselves emit soft photons, and so on, building up a cascade of soft massless particles each of which contributes an infra-red divergence. The elimination of such complicated interlocking infra-red divergences would certainly be a Herculean task, and might not even be possible."

Weinberg, Phys. Rev. 140B (1965)

IR singularities in QCD

- Much more complicated
  - soft and collinear singularities
  - gluons carry color charge, hence soft emissions do not simply exponentiate
  - but only a restricted set of higher-order contributions can appear (non-abelian exponentiation theorem) Gatheral 1983; Frenkel, Taylor 1984
- For long time, explicit form of IR poles was only understood at two-loop order Catani 1998

Overview of the talk

- IR singularities of gauge theory on-shell amplitudes
  - can be absorbed into multiplicative Z-factor, governed by an anomalous dimension $\mathbf{\Gamma}$
  - conjecture: for massless theories $\mathbf{\Gamma}$ involves only two-parton color-correlations
- Constraints on $\mathbf{\Gamma}$ from non-abelian exponentiation, soft-collinear factorization, collinear limits
- Order-by-order analysis to 3-loops, exclusion of higher Casimir contributions at 4 loops
- Phenomenological application: higher-log resummation for n-jet processes.

Color-space formalism

- Represent amplitudes as vectors in color space:

Catani, Seymour 1996

$$|c_1, c_2, \ldots, c_n\rangle$$

color index of first parton

- Color generator for i$^{\mathrm{th}}$ parton $\boldsymbol{T}_i^a \, |c_1, c_2, \ldots, c_n\rangle$ acts like a matrix:
  - t$^{\mathrm{a}}$ for quarks, f$^{\mathrm{abc}}$ for gluons
  - product $\boldsymbol{T}_i \cdot \boldsymbol{T}_j = \sum_a \boldsymbol{T}_i^a \, \boldsymbol{T}_j^a$ (commutative)
  - charge conservation $\sum_i \boldsymbol{T}_i^a = 0$ implies:

$$\sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j = -\sum_i \boldsymbol{T}_i^2 = -\sum_i C_i$$

i≠j

C$_{\mathrm{F}}$ or C$_{\mathrm{A}}$

Catani's two-loop formula (1998)

("... beautiful, yet mysterious ...")

- Specifies IR singularities of dimensionally regularized n-parton amplitudes at two loops:

$$\left[1 - \frac{\alpha_s}{2\pi} \boldsymbol{I}^{(1)}(\epsilon) - \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon) + \dots \right] |\mathcal{M}_n(\epsilon, \{\underline{p}\})\rangle = \text{finite}$$

amplitude is vector in color space

with

$$\boldsymbol{I}^{(1)}(\epsilon) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} + \frac{g_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right) \sum_{j\neq i} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \left(\frac{\mu^2}{-s_{ij}}\right)^\epsilon$$

$$\begin{aligned}\boldsymbol{I}^{(2)}(\epsilon) &= \frac{e^{-\epsilon\gamma_E}\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(K + \frac{\beta_0}{2\epsilon}\right) \boldsymbol{I}^{(1)}(2\epsilon) \\ &\quad - \frac{1}{2} \boldsymbol{I}^{(1)}(\epsilon) \left(\boldsymbol{I}^{(1)}(\epsilon) + \frac{\beta_0}{\epsilon}\right) + \boldsymbol{H}^{(2)}_{\text{R.S.}}(\epsilon)\end{aligned}$$

$(p_i + p_j)^2$

unspecified

- Later derivation using factorization properties and IR evolution equation for form factor

Sterman, Tejeda-Yeomans 2003

All-order generalization

- IR divergences in d=4-2ε can be absorbed into a multiplicative factor **Z** (a matrix in color space), which derives from an anomalous-dimension matrix:

TB, Neubert 2009

finite

$$|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{\underline{p}\},\mu)\,|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle$$

$$\boldsymbol{Z}(\epsilon,\{\underline{p}\},\mu) = \mathbf{P}\exp\left[\int_\mu^\infty \frac{d\mu'}{\mu'}\,\boldsymbol{\Gamma}(\{\underline{p}\},\mu')\right]$$

- Corresponding RG evolution equation:

$$\frac{d}{d\ln\mu}\,|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle = \boldsymbol{\Gamma}(\{\underline{p}\},\mu)\,|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle$$

⇒ can be used to resum Sudakov logarithms

All-order generalization

- Anomalous dimension is conjectured to be extremely simple:

color charges

anom. dimensions, known to three-loop order

$$\boldsymbol{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2}\, \gamma_{\text{cusp}}(\alpha_s)\, \ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

sum over pairs i≠j of partons

$(p_i + p_j)^2$

  - simple structure, reminiscent of QED
  - IR poles determined by color charges and momenta of external partons
  - color dipole correlations, like at one-loop order

Z factor to three loops

- Explicit result:

d-dimensional β-function

$$\ln \boldsymbol{Z}(\epsilon, \{\underline{p}\}, \mu) = \int_0^{\alpha_s} \frac{d\alpha}{\alpha} \frac{1}{2\epsilon - \beta(\alpha)/\alpha} \left[\boldsymbol{\Gamma}(\{\underline{p}\}, \mu, \alpha) + \int_0^{\alpha} \frac{d\alpha'}{\alpha'} \frac{\Gamma'(\alpha')}{2\epsilon - \beta(\alpha')/\alpha'} \right]$$

where

$$\Gamma'(\alpha_s) \equiv \frac{\partial}{\partial \ln \mu} \boldsymbol{\Gamma}(\{\underline{p}\}, \mu, \alpha_s) = -\gamma_{\text{cusp}}(\alpha_s) \sum_i C_i$$

- Perturbative expansion:

all coefficients known!

$$\begin{aligned}
\ln \boldsymbol{Z} &= \frac{\alpha_s}{4\pi} \left(\frac{\Gamma'_0}{4\epsilon^2} + \frac{\boldsymbol{\Gamma}_0}{2\epsilon} \right) + \left(\frac{\alpha_s}{4\pi} \right)^2 \left[-\frac{3\beta_0 \Gamma'_0}{16\epsilon^3} + \frac{\Gamma'_1 - 4\beta_0 \boldsymbol{\Gamma}_0}{16\epsilon^2} + \frac{\boldsymbol{\Gamma}_1}{4\epsilon} \right] \\
&+ \left(\frac{\alpha_s}{4\pi} \right)^3 \left[\frac{11\beta_0^2 \Gamma'_0}{72\epsilon^4} - \frac{5\beta_0 \Gamma'_1 + 8\beta_1 \Gamma'_0 - 12\beta_0^2 \boldsymbol{\Gamma}_0}{72\epsilon^3} + \frac{\Gamma'_2 - 6\beta_0 \boldsymbol{\Gamma}_1 - 6\beta_1 \boldsymbol{\Gamma}_0}{36\epsilon^2} + \frac{\boldsymbol{\Gamma}_2}{6\epsilon} \right] + \dots
\end{aligned}$$

$\Rightarrow$ exponentiation yields **Z** factor at three loops!

Checks

- Expression for IR pole terms agrees with all known perturbative results:
  - 3-loop quark and gluon form factors, which determine the functions $\gamma^{q,g}(\alpha_s)$ Moch, Vermaseren, Vogt 2005
  - 2-loop 3-jet qqg amplitude Garland, Gehrmann et al. 2002
  - 2-loop 4-jet amplitudes Anastasiou, Glover et al. 2001 Bern, De Freitas, Dixon 2002, 2003
  - 3-loop 4-jet amplitudes in N=4 super Yang-Mills theory in planar limit Bern et al. 2005, 2007

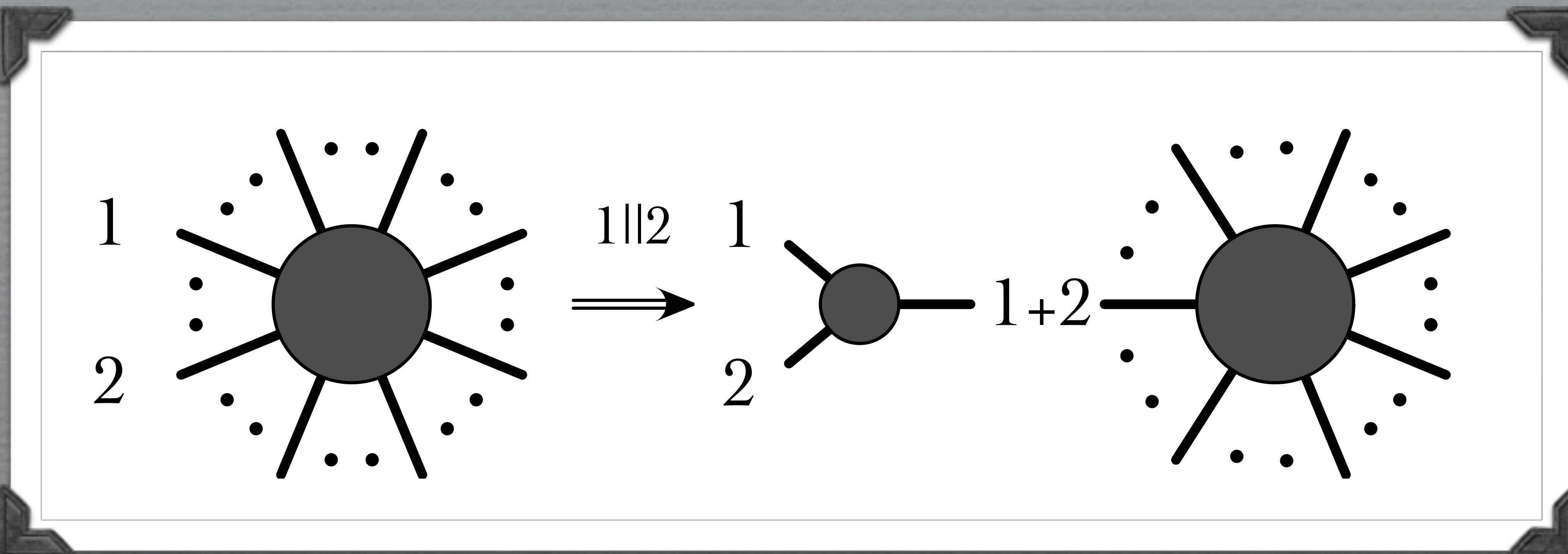
Catani's result

- Comparison with Catani's formula at two loops yields explicit expression for 1/ε pole term:

$$\boldsymbol{H}^{(2)}_{\text{R.S.}}(\epsilon) = \frac{1}{16\epsilon} \sum_i \left(\gamma_1^i - \frac{1}{4}\,\gamma_1^{\text{cusp}}\,\gamma_0^i + \frac{\pi^2}{16}\,\beta_0\,C_i \right)$$

$$+ \frac{i f^{abc}}{24\epsilon} \sum_{(i,j,k)} \boldsymbol{T}_i^a\,\boldsymbol{T}_j^b\,\boldsymbol{T}_k^c \ln\frac{-s_{ij}}{-s_{jk}} \ln\frac{-s_{jk}}{-s_{ki}} \ln\frac{-s_{ki}}{-s_{ij}}$$

- Non-trivial color structure only arises since his operators are not defined in a minimal scheme
- First derived by Mert Aybat, Dixon, Sterman '06 , confirming earlier conjecture Bern, Dixon, Kosower '04



Effective theory analysis and factorization constraints

Misconception

- Conventional thinking is that UV and IR divergences are of totally different nature:
  - UV divergences absorbed into renormalization of parameters of theory; structure constrained by RG equations
  - IR divergences arise in unphysical calculations; cancel between virtual corrections and real emissions
- In fact, IR divergences can be mapped onto UV divergences of operators in effective field theory!

Re-interpretation of IR divergences

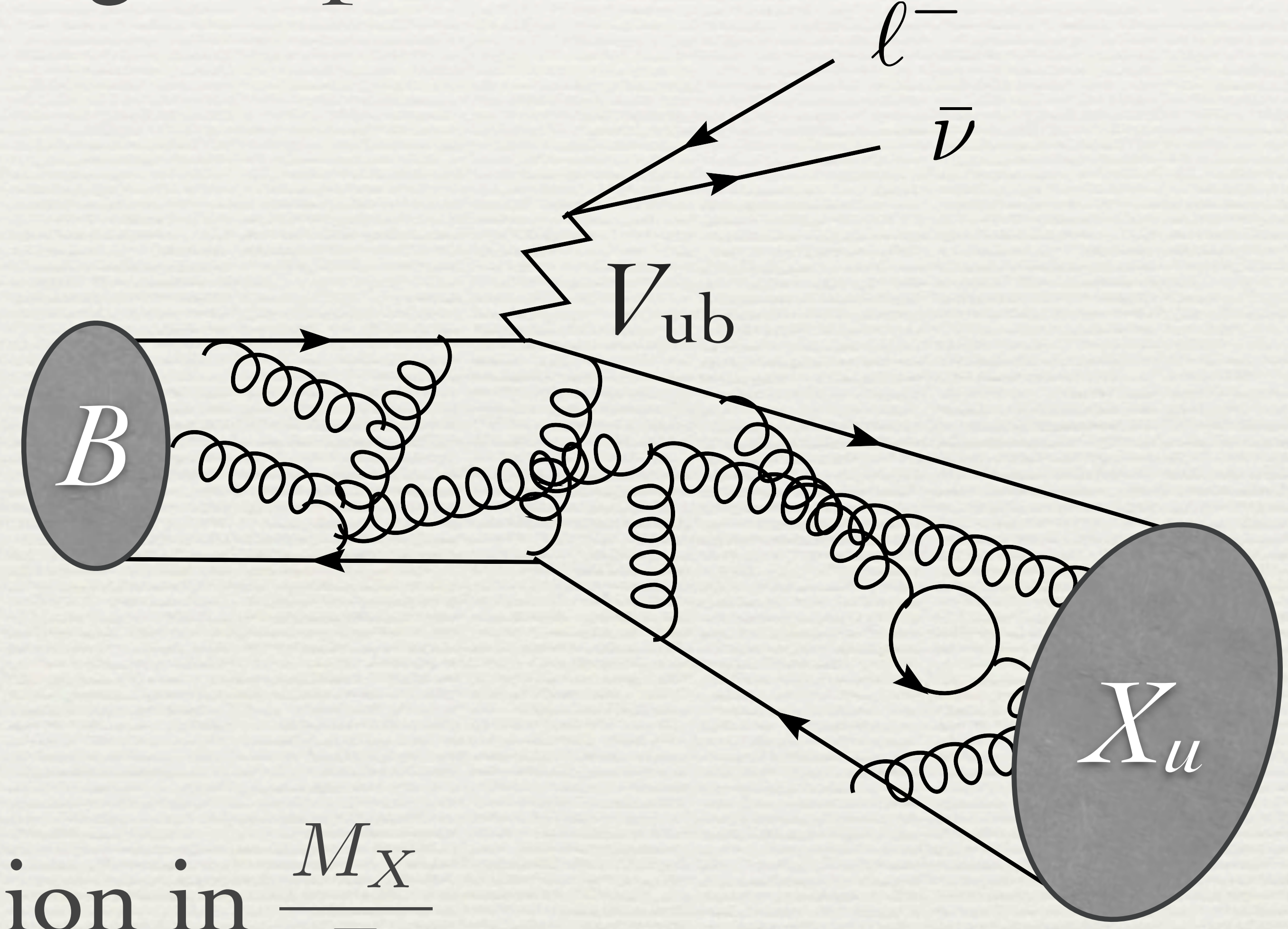
- In our case, $\mathbf{\Gamma}$ is the anomalous-dimension matrix of n-jet operators in SCET, and $\mathbf{Z}$ is the associated matrix of renormalization factors
- Will discuss structure of SCET for n-jet processes and constraints on anomalous dimension $\mathbf{\Gamma}$ arising from
  - charge conservation $\sum_i \boldsymbol{T}_i = 0$
  - soft-collinear factorization
  - non-abelian exponentiation
  - consistency with collinear limits

TB, Neubert, arXiv:0903.1126

Soft-Collinear Effective Theory

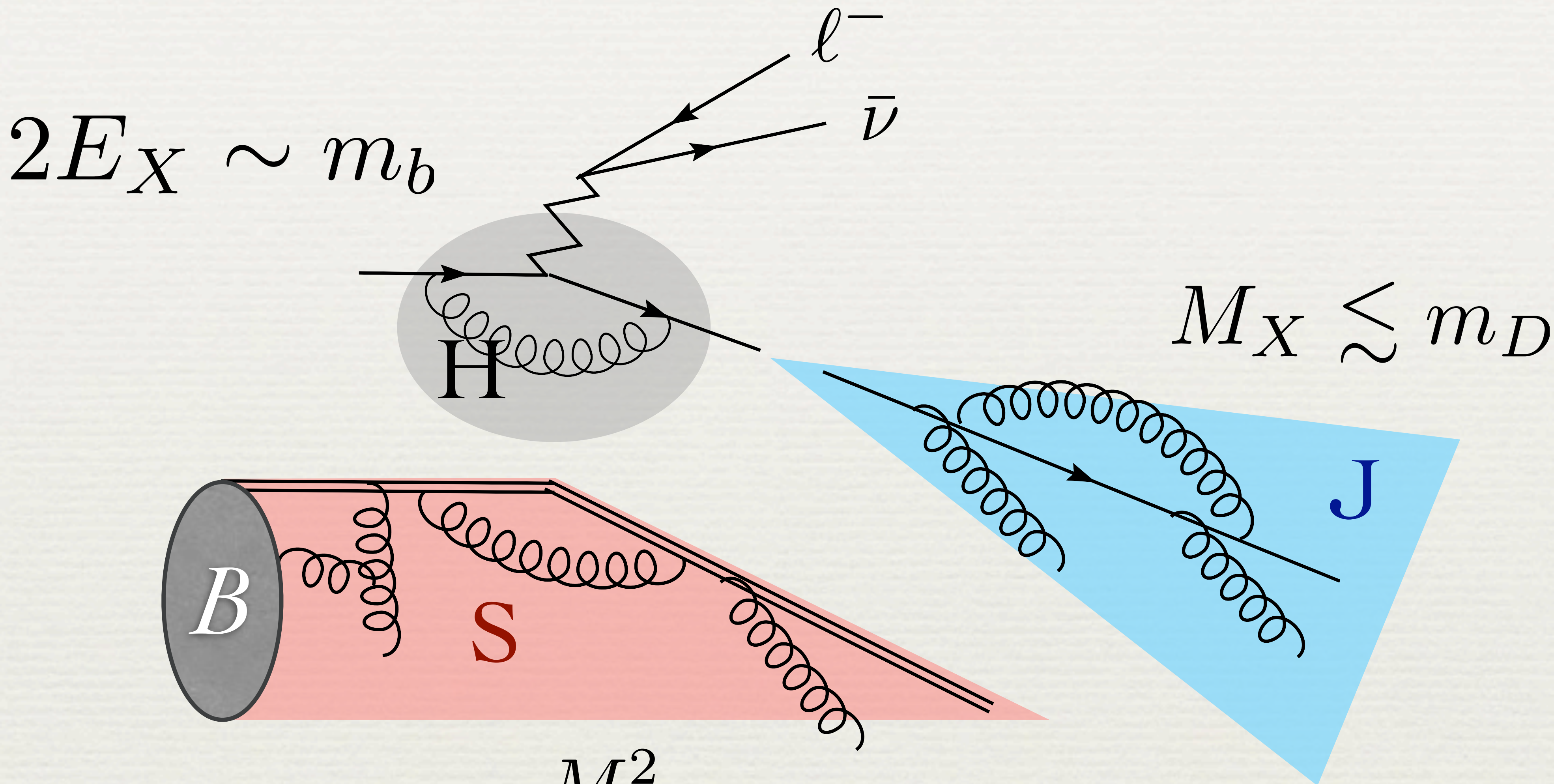
Bauer, Pirjol, Stewart et al. 2001, 2002; Beneke et al. 2002

- An effective theory for processes for processes with energetic particles.



- Expansion in $\frac{M_X}{2E_X}$
- Sudakov resummation $\alpha_s^n \ln^{2n}\left(\frac{M_X}{2\,E_X}\right)$

Soft-Collinear Factorization

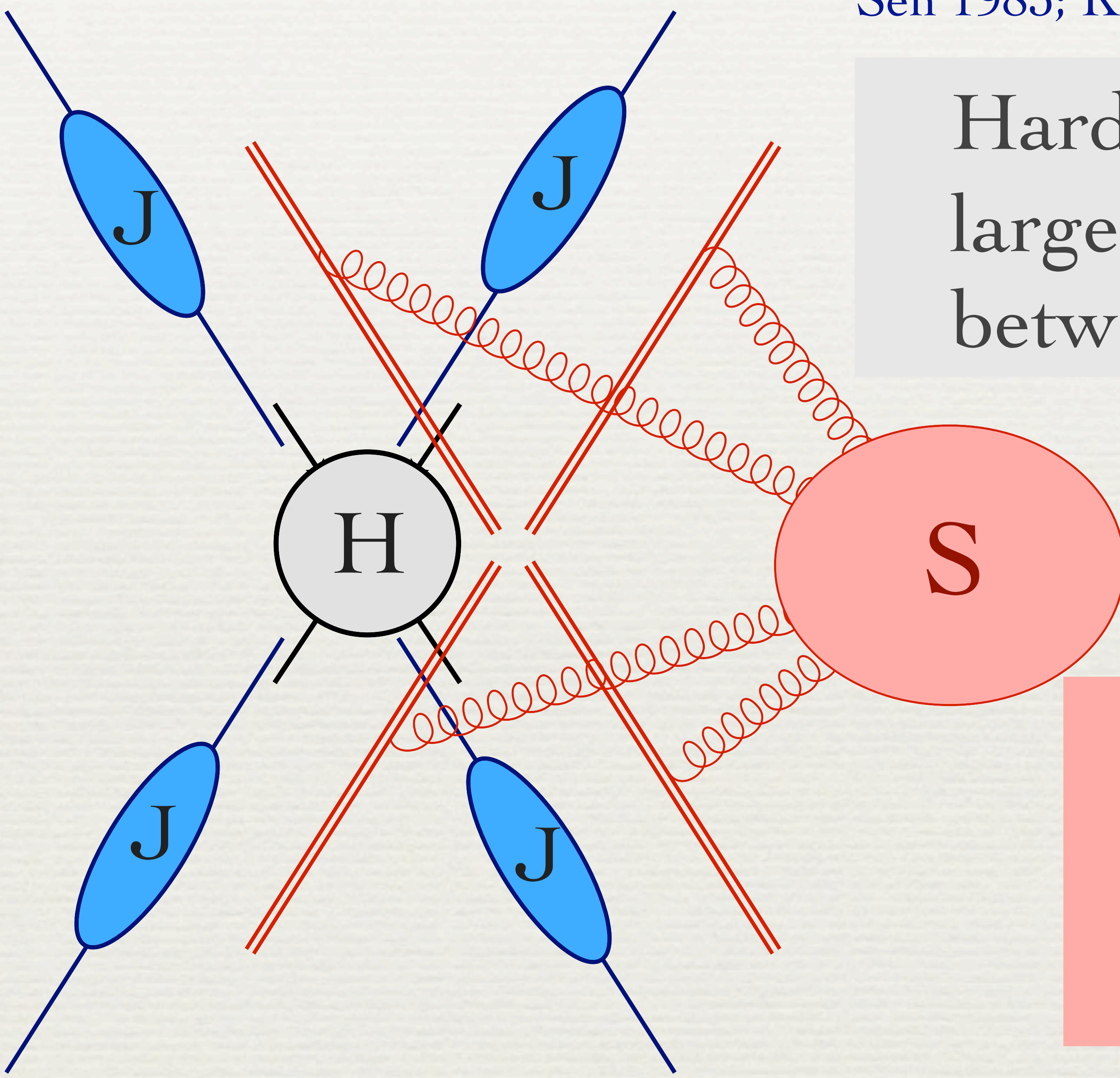


$$\Lambda_S = \frac{M_X^2}{2E_X}$$ nonperturbative!

$$d\Gamma = H \cdot J \otimes S$$

Soft-collinear factorization: *n* jet case

Sen 1983; Kidonakis, Oderda, Sterman 1998



Hard function H depends on large momentum transfers s_{ij} between jets

Soft function S depends on scales $\Lambda_{ij}^2 = \dfrac{M_i^2 M_j^2}{s_{ij}}$

Jet functions $J_i = J_i(M_i^2)$

SCET for *n*-jet processes

- *n* different types of collinear quark and gluon fields ($\to$ jet functions J_i), interacting only via soft fields (soft function S)
  - operator definitions for J_i and S
- Hard contributions ($Q \sim \sqrt{s}$) are integrated out and absorbed into Wilson coefficients:

$$\mathcal{H}_n = \sum_i \mathcal{C}_{n,i}(\mu)\, O_{n,i}^{\text{ren}}(\mu)$$

Bauer, Schwartz 2006

- Scale dependence controlled by RGE:

$$\frac{d}{d\ln\mu}\,|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle = \mathbf{\Gamma}(\mu,\{\underline{p}\})\,|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle$$

anomalous-dimension matrix

On-shell parton scattering amplitudes

- Hard functions $\mathcal{C}_n$ can be obtained by setting the jet masses to zero: jet and soft functions become scaleless, loop corrections vanish.
- One obtains:

renormalization factor
(minimal subtraction of IR poles)

$$|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{\underline{p}\},\mu)\,|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle$$

TB, Neubert 2009

where

$$\boldsymbol{\Gamma} = -\frac{d\ln\boldsymbol{Z}}{d\ln\mu}$$

- IR poles of scattering amplitudes mapped onto UV poles of n-jet SCET operators
- Multiplicative subtraction, controlled by RG

Factorization constraint on $\boldsymbol{\Gamma}$

- Operator matrix elements must evolve in the same way as hard matching coefficients, such that physical observables are scale independent
- Factorization of matrix element then implies (with $\Lambda_{ij}^2 = \frac{M_i^2 M_j^2}{s_{ij}}$):

trivial color structure

$$\boldsymbol{\Gamma}(s_{ij}) = \boldsymbol{\Gamma}_s(\Lambda_{ij}^2) + \sum_i \Gamma_c^i(M_i^2)\,\mathbf{1}$$

M_i dependence must cancel!

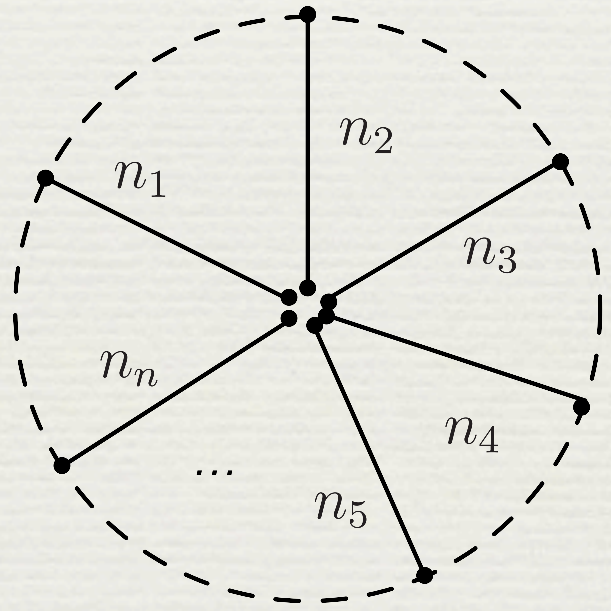
  - suggests logarithmic dependence on s_{ij} and M_i^2
  - $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}_S$ must have same color structure

Soft function

- SCET decoupling transformation removes soft interactions among collinear fields and absorbs them into soft Wilson lines

$n_i \sim p_i$ light-like reference vector

$$\boldsymbol{S}_i = \mathbf{P} \exp\left[ig \int_{-\infty}^{0} dt\, n_i \cdot A_a(tn_i)\, T_i^a \right]$$



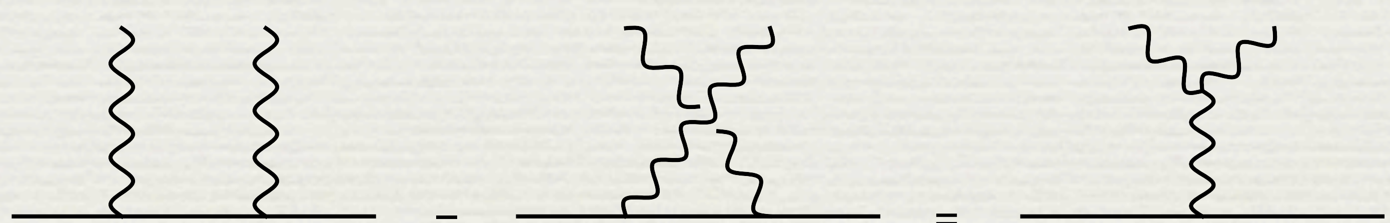
- For n-jet operator one gets:

$$\mathcal{S}(\{\underline{n}\}, \mu) = \langle 0 | \boldsymbol{S}_1(0) \dots \boldsymbol{S}_n(0) | 0 \rangle = \exp(\tilde{\mathcal{S}}(\{\underline{n}\}, \mu))$$

Non-abelian exponentiation

Gatheral 1983; Frenkel and Taylor 1984

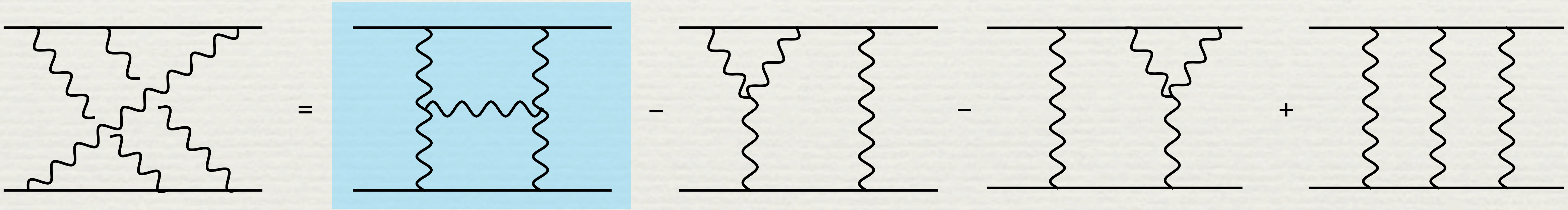
- The exponent $\tilde{\mathcal{S}}$ receives contributions only from Feynman diagrams whose color weights are "color-connected" (or "maximally non-abelian")
- Color-weight graphs associated with each Feynman diagram can be simplified using the Lie commutator relation:



$$\boldsymbol{T}^a\boldsymbol{T}^b \quad - \quad \boldsymbol{T}^b\boldsymbol{T}^a \quad = \quad i f^{abc}\boldsymbol{T}^c$$

Non-abelian exponentiation

- Use this to decompose any color-weight graph into a sum over products of connected webs, defined as a connected set of gluon lines (not counting crossed lines as being connected)



single connected web
"maximally nonabelian"

- Only color structures consisting of a single connected web contribute to the exponent $\tilde{S}$

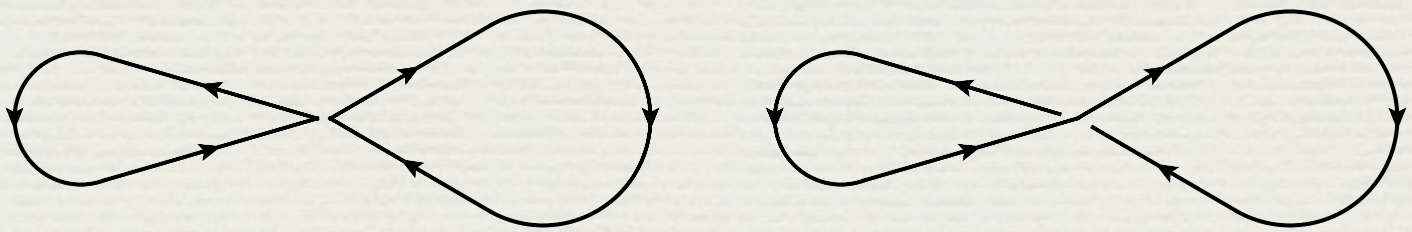
Renormalization of Wilson loops

- Wilson loops containing singular points (cusps or cross points) require UV subtractions

  Polyakov 1980; Brandt, Neri, Sato 1981

- For single cusp formed by tangent vectors n_1 and n_2, renormalization factor depends on cusp angle β_{12} defined as $$\cosh\beta_{12} = \frac{n_1 \cdot n_2}{\sqrt{n_1^2\, n_2^2}}$$

- More generally, sets of related Wilson loops mix under renormalization, with $\mathbf{Z}_s$ matrix depending on all relevant cusp angles



Light-like Wilson lines

- For large values of cusp angle β_{12}, anomalous dimension associated with a cusp or cross point grows linearly with β_{12}, which is then approximately equal to $\ln(2n_1 \cdot n_2/\sqrt{n_1^2\, n_2^2})$

Korchemsky, Radyushkin 1987

- Cusp angle diverges when one or both segments approach the light-cone:

$$\Gamma(\beta_{12}) \overset{n_{1,2}^2 \to 0}{\longrightarrow} \Gamma^i_{\text{cusp}}(\alpha_s)\, \ln \frac{\mu^2}{\Lambda_s^2} + \ldots$$

Korchemskaya, Korchemsky 1992

- Presence of single logarithm characteristic for Sudakov problems (double logs)

Light-like Wilson lines

- Introducing IR regulators $p_i^2 \neq 0$ to define the soft and collinear scales, we obtain:

$$\beta_{ij} = L_i + L_j - \ln \frac{\mu^2}{-s_{ij}}$$

$$\beta_{ij} = \ln \frac{-s_{ij}\,\mu^2}{(-p_i^2)(-p_j^2)}$$

soft log

$$L_i = \ln \frac{\mu^2}{-p_i^2}$$

collinear log

hard log

Soft anomalous-dimension matrix

- Decompositions:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \mathbf{\Gamma}_s(\{\underline{\beta}\},\mu) + \sum_i \Gamma_c^i(L_i,\mu)$$

$$\Gamma_c^i(L_i) = -\Gamma_{\rm cusp}^i(\alpha_s)\,L_i + \gamma_c^i(\alpha_s)$$

- Key equation:

see also: Gardi, Magnea, arXiv:0901.1091

$$\boxed{\frac{\partial\mathbf{\Gamma}_s(\{\underline{s}\},\{\underline{L}\},\mu)}{\partial L_i} = \Gamma_{\rm cusp}^i(\alpha_s)}$$

- Suggests linearity in cusp angles β_{ij} and significantly restricts color structures

Soft anomalous-dimension matrix

- Only exception would be a more complicated dependence on conformal cross ratios, which are independent of collinear scales:

$$\beta_{ijkl} = \beta_{ij} + \beta_{kl} - \beta_{ik} - \beta_{jl} = \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

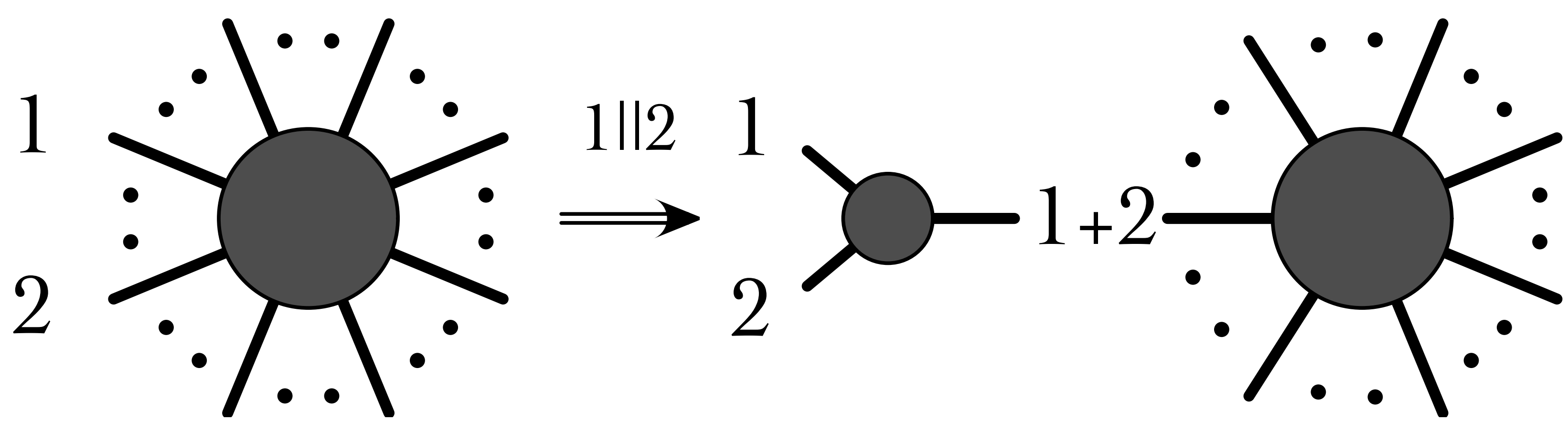
Gardi, Magnea 2009

- Any polynomial dependence on such ratios can be excluded using other arguments, such as consistency with collinear limits

Consistency with collinear limits

- When two partons become collinear, an n-point amplitude M_n reduces to an (n-1)-parton amplitude times a splitting function: Berends, Giele 1989; Mangano, Parke 1991 Kosower 1999; Catani, de Florian, Rodrigo 2003

$$|\mathcal{M}_n(\{p_1,p_2,p_3,\dots,p_n\})\rangle = \mathbf{Sp}(\{p_1,p_2\})\,|\mathcal{M}_{n-1}(\{P,p_3,\dots,p_n\})\rangle + \dots$$



$$\boldsymbol{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) = \boldsymbol{\Gamma}(\{p_1,\dots,p_n\},\mu) - \boldsymbol{\Gamma}(\{P,p_3\dots,p_n\},\mu)\Big|_{\boldsymbol{T}_P\to\boldsymbol{T}_1+\boldsymbol{T}_2}$$

- $\boldsymbol{\Gamma}_{\rm Sp}$ must be independent of momenta and colors of partons 3, ..., n

TB, Neubert 2009

Consistency check

- The form we propose is consistent with factorization in the collinear limit:

$$\mathbf{\Gamma}_{\mathrm{Sp}}(\{p_1,p_2\},\mu) = \mathbf{\Gamma}(\{p_1,\dots,p_n\},\mu) - \mathbf{\Gamma}(\{P,p_3\dots,p_n\},\mu)\big|_{\boldsymbol{T}_P\to\boldsymbol{T}_1+\boldsymbol{T}_2}$$

$$\mathbf{\Gamma}_{\mathrm{Sp}}(\{p_1,p_2\},\mu) = \gamma_{\mathrm{cusp}}\left[\boldsymbol{T}_1\cdot\boldsymbol{T}_2\,\ln\frac{\mu^2}{-s_{12}} + \boldsymbol{T}_1\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln z + \boldsymbol{T}_2\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln(1-z)\right] + \gamma^1 + \gamma^2 - \gamma^P\,,$$

momentum fraction of parton 1

- But this would not work if $\mathbf{\Gamma}$ would involve terms of higher powers in color generators $\boldsymbol{T}_i$ or momentum variables
- A very strong constraint (new)!

$$\boldsymbol{\Gamma}_s(\{\underline{\beta}\},\mu) \stackrel{?}{=} -\sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\text{cusp}}(\alpha_s)\,\beta_{ij} + \sum_i \gamma_s^i(\alpha_s)$$

Diagrammatic analysis of the soft anomalous-dimension matrix

Existing results

- Our conjecture implies for the soft anomalous-dimension matrix:

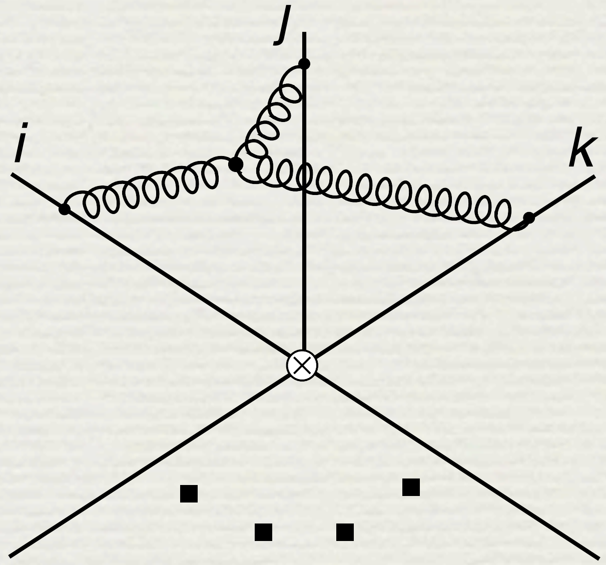
$$\boldsymbol{\Gamma}_s(\{\underline{\beta}\},\mu) = -\sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2}\,\gamma_{\rm cusp}(\alpha_s)\,\beta_{ij} + \sum_i \gamma_s^i(\alpha_s)$$

- This form was confirmed at two loops by showing that diagrams connecting three parton legs vanish

  Mert Aybat, Dixon, Sterman 2006

- Also holds for three-loop fermionic contributions

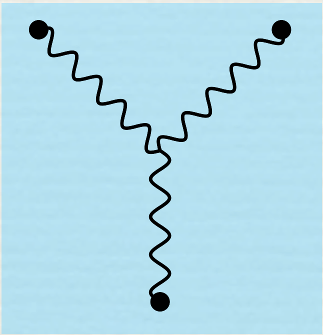
  Dixon 2009



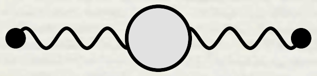
Order-by-order analysis

- One loop (recall $\sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j = -\sum_i \boldsymbol{T}_i^2 = -\sum_i C_i$)
  - one leg: $\boldsymbol{T}_i^2 = C_i$

  - two legs: $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$
- Two loops



(only new structure)



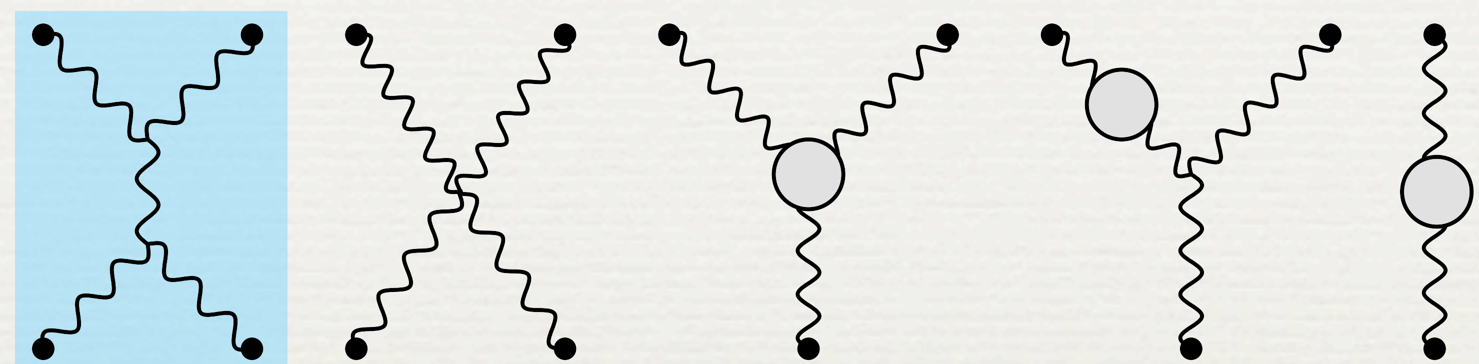
  - one leg: $-if^{abc}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_i^b\,\boldsymbol{T}_i^c = \dfrac{C_A C_i}{2}$
  - two legs: $-if^{abc}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_i^b\,\boldsymbol{T}_j^c = \dfrac{C_A}{2}\boldsymbol{T}_i \cdot \boldsymbol{T}_j$
  - three legs: $-if^{abc}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_j^b\,\boldsymbol{T}_k^c$

$\Rightarrow$ vanishes, since no antisymmetric momentum structure in i,j,k consistent with soft-collinear factorization exists!

explains cancellations observed in:
Mert Aybat, Dixon, Sterman 2006; Dixon 2009

Three-loop order

- Single webs:



(only new structure)

- Six new structures consistent with non-abelian exponentiation exist, two of which are compatible with soft-collinear factorization:

$$\mathbf{\Delta\Gamma}_3(\{\underline{p}\},\mu) = -\frac{\bar{f}_1(\alpha_s)}{4} \sum_{(i,j,k,l)} f^{ade} f^{bce}\, \boldsymbol{T}_i^a\, \boldsymbol{T}_j^b\, \boldsymbol{T}_k^c\, \boldsymbol{T}_l^d\, \ln\frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

$$- \bar{f}_2(\alpha_s) \sum_{(i,j,k)} f^{ade} f^{bce} \left(\boldsymbol{T}_i^a\, \boldsymbol{T}_i^b\right)_+ \boldsymbol{T}_j^c\, \boldsymbol{T}_k^d\,,$$

more generally, arbitrary odd function of conformal cross ratio

Three-loop order

- Neither of these is compatible with collinear limits: the splitting function would depend on colors and momenta of the additional partons
- Consider, e.g., the second term:

$$\boldsymbol{\Delta\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu)\big|_{\bar f_2(\alpha_s)} = 2 f^{ade} f^{bce} \left[\left(\boldsymbol{T}_1^a \boldsymbol{T}_1^b\right)_+ \left(\boldsymbol{T}_2^c \boldsymbol{T}_2^d\right)_+ - \sum_{i\neq 1,2} \left(\boldsymbol{T}_1^a \boldsymbol{T}_2^b + \boldsymbol{T}_2^a \boldsymbol{T}_1^b\right) \left(\boldsymbol{T}_i^c \boldsymbol{T}_i^d\right)_+ \right]$$

$$\boldsymbol{\Delta\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu)\big|_{\bar f_1(\alpha_s)} = f^{ade} f^{bce} \sum_{(i,j)\neq 1,2} \left(\boldsymbol{T}_1^a \boldsymbol{T}_2^b + \boldsymbol{T}_2^a \boldsymbol{T}_1^b\right) \boldsymbol{T}_i^c \boldsymbol{T}_j^d \ln\frac{\mu^2}{-s_{ij}} + \ \ldots$$

dependence on color invariants and momenta of additional partons (i≠1,2)

Arbitrary dependence on conformal cross ratios

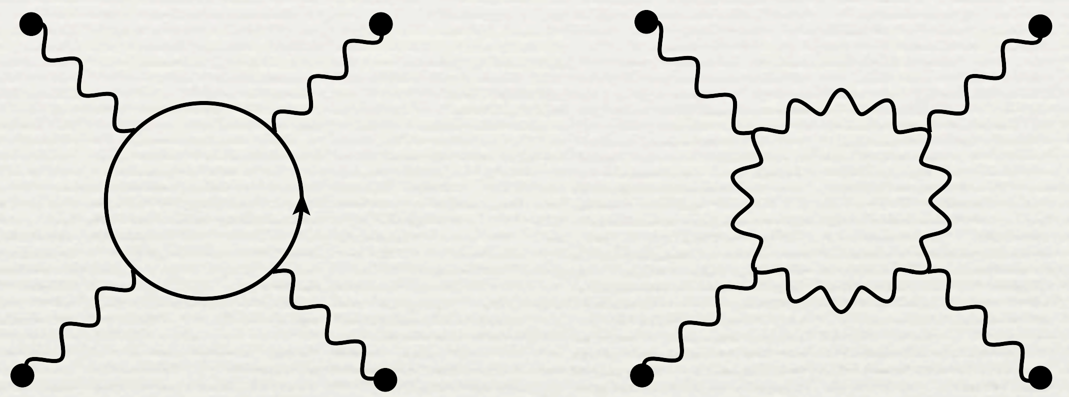
- Most general form [$\beta_{ijkl} = \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$]

$$\mathbf{\Delta\Gamma}_3(\{\underline{p}\},\mu) = \sum_{(i,j,k,l)} f^{ade} f^{bce} \boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d F(\beta_{ijkl}, \beta_{iklj} - \beta_{iljk})$$

  - compatible with soft-collinear factorization
  - inconsistent with collinear limit unless the term vanishes in all collinear limits. (Conformal ratios vanish or diverge in the collinear limit.)
- Unclear whether it appears, but contribution is not excluded by our arguments.

Four-loops and beyond

- Interesting new webs involving higher Casimir invariants first arise at four loops



$$d_F^{abcd}\, \boldsymbol{T}_i^a\, \boldsymbol{T}_j^b\, \boldsymbol{T}_k^c\, \boldsymbol{T}_l^d = d_F^{abcd}\left(\boldsymbol{T}_i^a\, \boldsymbol{T}_j^b\, \boldsymbol{T}_k^c\, \boldsymbol{T}_l^d\right)_+$$

$$d_R^{a_1 a_2 \dots a_n} = \mathrm{tr}\big[\,(\boldsymbol{T}_R^{a_1}\, \boldsymbol{T}_R^{a_2} \dots \boldsymbol{T}_R^{a_n})_+\big]$$

- One linear combination of such terms would be compatible with soft-collinear factorization, but does not have the correct collinear limit

Casimir scaling

- Applied to the two-jet case (form factors), our formula thus implies Casimir scaling of the cusp anomalous dimension:

$$\frac{\Gamma^{q}_{\text{cusp}}(\alpha_s)}{C_F} = \frac{\Gamma^{g}_{\text{cusp}}(\alpha_s)}{C_A} = \gamma_{\text{cusp}}(\alpha_s)$$

- Checked explicitly at three loops Moch, Vermaseren, Vogt 2004
- But contradicts expectations from AdS/CFT correspondence (high-spin operators in strong-coupling limit)

Armoni 2006
Alday, Maldacena 2007

- A real conflict?

Wanted: 3- and 4-loop checks

- Full three-loop 4-jet amplitudes in N=4 super Yang-Mills theory were expressed in terms of small number of scalar integrals Bern et al. 2008
- Once these can be calculated, this will provide stringent test of our arguments (note recent calculation of three-loop form-factor integrals) Baikov et al. 2009; Heinrich, Huber, Kosower, Smirnov 2009
- Calculation of four-loop cusp anomalous dimension would provide non-trivial test of Casimir scaling, which is then no longer guaranteed by non-abelian exponentiation

Heavy particles

- Have extended our analysis to amplitudes which include massive partons
- Effective theory is combination of HQET (for heavy partons) and SCET (massless partons)
- Soft function contains both massless and timelike Wilson lines

$$\mathcal{S}(\{\underline{n}\},\{\underline{v}\},\mu) = \langle 0|\boldsymbol{S}_{n_1}\dots\boldsymbol{S}_{n_k}\,\boldsymbol{S}_{v_{k+1}}\dots\boldsymbol{S}_{v_n}|0\rangle$$

  - v_i are four-velocities of the massive partons
  - n_i are light-cone reference vectors

Anomalous dimension

- Both the full and the effective theory know about the 4-velocities of the massive partons
- Therefore much weaker constraints hold for the massive case:
  - no soft-collinear factorization
  - no constraint from (quasi-)collinear limits
- For the purely massive case, all structures allowed by non-abelian exponentiation at a given order will be present!

Anomalous dimension to two loops

- One- and two-parton terms:

known to two loops

$$\boldsymbol{\Gamma}(\{\underline{p}\},\{\underline{m}\},\mu)\Big|_{2-\text{parton}}$$

massless partons

$$= \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2}\,\gamma_{\text{cusp}}(\alpha_s)\,\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

$$\boxed{\begin{aligned} &- \sum_{(I,J)} \frac{\boldsymbol{T}_I \cdot \boldsymbol{T}_J}{2}\,\gamma_{\text{cusp}}(\beta_{IJ},\alpha_s) + \sum_I \gamma^I(\alpha_s) \\ &+ \sum_{I,j} \boldsymbol{T}_I \cdot \boldsymbol{T}_j\,\gamma_{\text{cusp}}(\alpha_s)\,\ln\frac{m_I\mu}{-s_{Ij}}\,, \end{aligned}}$$

massive partons

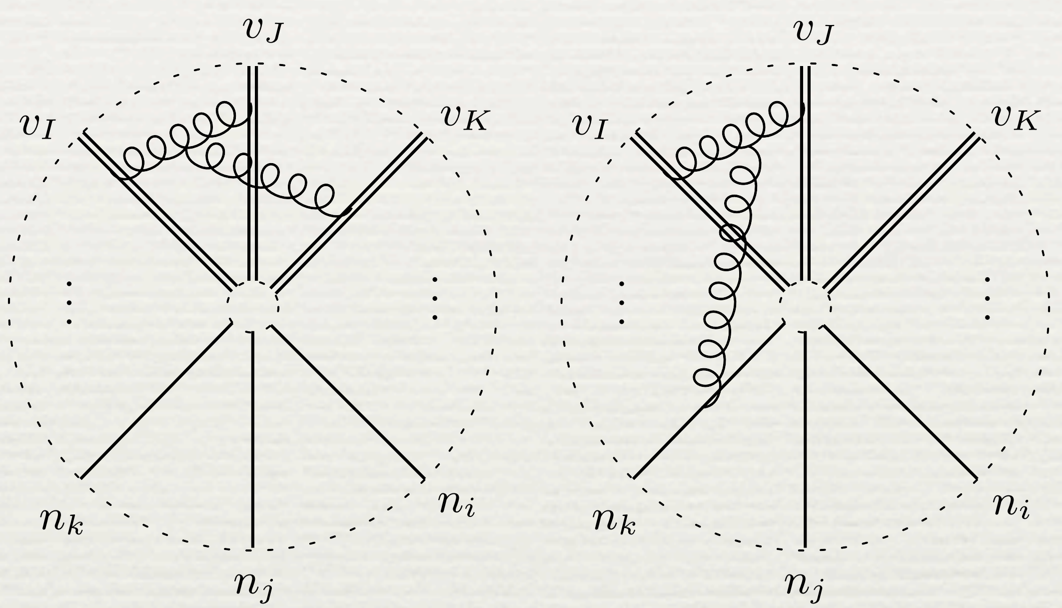
new!

- Generalizes structure found for massless case
- Reproduces IR poles of QCD amplitudes after appropriate matching of coupling constants

Anomalous dimension to two loops

- Also 3-parton correlations appear in massive case!

Mitov, Sterman, Sung 2009



- General structure [with $\beta_{IJ} = \text{arccosh}(v_I \cdot v_J)$]:

$$\begin{aligned}
&\boldsymbol{\Gamma}(\{\underline{p}\},\{\underline{m}\},\mu)\big|_{3-\text{parton}} \\
&= if^{abc} \sum_{(I,J,K)} \boldsymbol{T}_I^a \, \boldsymbol{T}_J^b \, \boldsymbol{T}_K^c \, F_1(\beta_{IJ},\beta_{JK},\beta_{KI}) \\
&\qquad + if^{abc} \sum_{(I,J)} \sum_k \boldsymbol{T}_I^a \, \boldsymbol{T}_J^b \, \boldsymbol{T}_k^c \, f_2\Big(\beta_{IJ}, \ln\frac{-\sigma_{Jk}\, v_J\cdot p_k}{-\sigma_{Ik}\, v_I\cdot p_k}\Big)
\end{aligned}$$

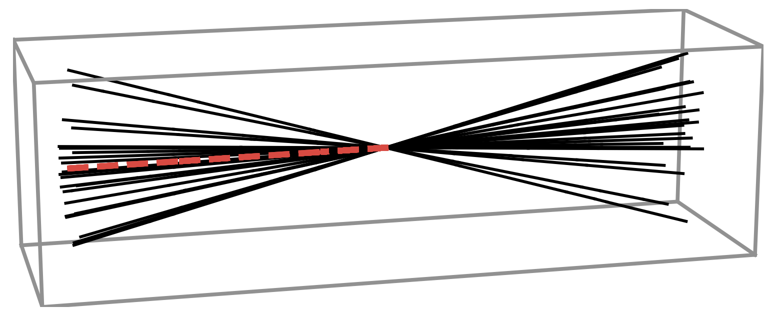
Towards higher-log resummations for n-jet processes

Sudakov resummation with SCET

- Many collider physics applications of SCET in the past few years. Resummations up to N^{3}LL, however only for two jet observables, e.g.
  - thrust distribution in e^+e^- — TB, Schwartz '08
  - Drell-Yan rapidity dist. — TB, Neubert, Xu '07
  - inclusive Higgs production — Idilbi, Ji, Ma and Yuan '06 ; Ahrens, TB, Neubert, Yang '08
- Our result for anomalous dimension allows us to perform higher-log resummations also for more *n*-jet processes

2-jet example: thrust T

$$T = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p}_i \cdot \mathbf{n}|}{\sum_i |\mathbf{p}_i|}$$



$$1 - T \approx \frac{M_1^2 + M_2^2}{Q^2}$$

- Prediction for event-shape variable thrust dominated by perturbative uncertainty. NLO Ellis et al. '81, NNLO corrections Gehrmann et al. '07.
  - Traditional methods allowed resummation to NLL Catani et al. '93 but not beyond.
- Using factorization theorem in SCET we were able to derive NNNLL resummed distribution TB and Schwartz, '08.
  - Need only existing perturbative input. Analytic result, no unphysical Landau-pole singularities. Match to NNLO.

α_s extraction from thrust

- Fit to ALEPH and OPAL data gives

$$\boxed{\begin{aligned}\alpha_s(m_Z) &= 0.1172 \pm 0.0010(\text{stat}) \pm 0.0008(\text{sys}) \pm 0.0012(\text{had}) \pm 0.0012(\text{pert}) \\ &= 0.1172 \pm 0.0022\,.\end{aligned}}$$

from comparing Ariadne Herwig and Pythia

TB and Schwartz, '08

  - Most precise α_s at high energy, pert unc. no longer dominant. Hadronisation uncertainty becomes limiting factor.

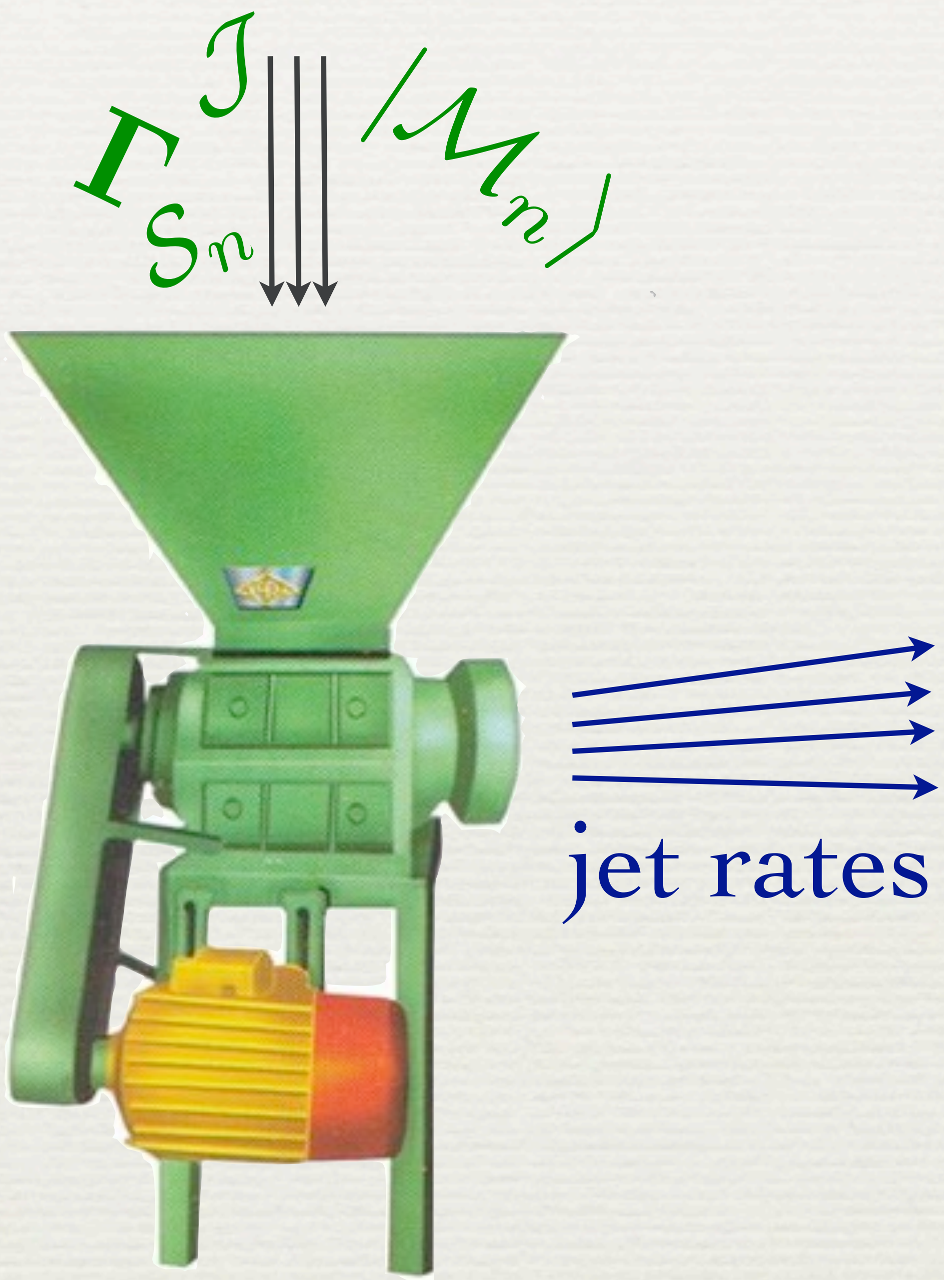
- Abbate, Fickinger, Hoang, Mateu, and Stewart have performed a global fit to all available thrust data using. Extract both α_s and hadronisation corrections. Find *large* hadronisation corrections, preliminary value of α_s $\boxed{\alpha_s(M_Z) = 0.1142 \pm 0.0008 \pm 0.0011}$

(pert) (stat+syst+had)

Beyond LL for *n*-jet processes

- The necessary ingredients are
  - **hard functions**: from fixed-order results for on-shell amplitudes. New unitarity methods allow calculation of one-loop amplitudes with many legs ($\rightarrow$ NNLL resummation)
  - **jet function:** imaginary part of two-point function, inclusive jet function is known to two loops.
  - **soft function:** matrix element of Wilson lines, one-loop calculation is comparatively simple.
- Then resum log's of different scales using RG evolution.

Ultimate goal: automatization



- in the longer term, this will hopefully lead to automated higher-log resummations for jet rates
- goes beyond parton showers, which are only accurate at LL, even after matching
- predicts jets, not individual partons

Conclusions

- IR divergences of arbitrary gauge-theory amplitudes can be derived from SCET anomalous-dimension matrix $\mathbf{\Gamma}$
- Stringent constraints on $\mathbf{\Gamma}$ arise from non-abelian exponentiation (general case), and soft-collinear factorization & collinear limits (massless case only)
- Conjectured form of pure color-dipole correlations demonstrated to hold at 3- and (partial) 4-loop order, assuming polynomial dependence on β_{ijkl}
- In massive case, previously observed properties of 2-loop three-parton correlations understood from symmetry properties in effective theory
- On track to perform higher-log resummations for generic n-jet processes at LHC using RG evolution