Exact lattice supersymmetry

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Lattice SUSY

- Old problem.
- Difficult. SUSY extends Poincaré – broken by discretization.
- Folklore: Impossible to put SUSY on lattice exactly.
- Leads to (very) difficult fine tuning – lots of relevant SUSY breaking counterterms...

Way out!

Motivations ?

- Rigorous definition of SUSY QFT - like lattice QCD.
- Dynamical SUSY breaking. Predicting soft terms in MSSM ...
- Gauge-gravity duality ? Eg. large N strongly coupled $\mathcal{N} = 4$ SYM and type II string theory in 5d AdS.

New ideas

- Topological twisting
- Orbifolding/deconstruction (D. B. Kaplan, M. Unsäl, A. Cohen, ...)
- Focus on former. Emphasizes geometry. Continuum limit clear.
- Warning: Tricks work only for no. SUSYs Q multiple 2^D ... In $D=4$ unique theory: $\mathcal{N}=4$ SYM

Example: Twisting in 2D

Simplest theory contains 2 fermions λ^i_α
Global symmetry: $SO_{\rm Lorenz}(2) \times SO_{\rm R}(2)$
Twist: decompose under diagonal subgroup
Consider fermions as matrix

$$\lambda^i_\alpha \longrightarrow \Psi_{\alpha\beta}$$

Natural to expand:

$$\Psi = \frac{\eta}{2} I + \psi_\mu \gamma_\mu + \chi_{12}\gamma_1\gamma_2$$

scalar, vector and tensor (twisted) components!

Twisted supersymmetry

- Twisted theory has scalar SUSY $\mathcal{Q}$.
- $\{Q, \overline{Q}\} = \gamma_\mu p_\mu$ implies:
  - $\mathcal{Q}^2 = 0$
  - $\{\mathcal{Q}, \mathcal{Q}_\mu\} = p_\mu$
- Plausible: $S = \mathcal{Q}\Lambda(\Phi, \Psi)$

Basic idea of lattice theory: discretize twisted formulation, exact (scalar) SUSY only requires $\mathcal{Q}^2 = 0$

Example: $\mathcal{Q} = 4$ SYM in 2D

In twisted form (adjoint fields AH generators)

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\,\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta d\right)$$

$$\begin{aligned}
\mathcal{Q}\,\mathcal{A}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}$$

Note: complexified gauge field $\mathcal{A}_\mu = A_\mu + iB_\mu$

Action

$\mathcal{Q}$-variation, integrate d:

$$S = \frac{1}{g^2}\int \mathrm{Tr}\, \left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu \right)$$

Rewrite as

$$S = \frac{1}{g^2}\int \mathrm{Tr}\, \left(-F_{\mu\nu}^2 + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2 + L_F \right)$$

where

$$L_F = \begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix} \begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Moral

- Twisting changes spins fields:
  - Scalars become vectors. Naturally embedded in complexified connection
  - Fermions integer spins. Form components of Kähler-Dirac field.
- Twisted entire Lorentz symmetry with R-symmetry – maximal twist. Necessary for lattice.
- Flat space - twisting just change of variables.

Lattice ?

- $\mathcal{A}_\mu(x) \to \mathcal{U}_\mu(n)$. Complexified Wilson links.
- Natural fermion assignment – η on sites, ψ_μ links, χ_{12} diagonal links of cubic lattice.
- Fields pick up non-standard $U(N)$ gauge transformations:

$$
\begin{aligned}
\eta(\mathbf{x}) &\to G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x}) \\
\psi_\mu(\mathbf{x}) &\to G(\mathbf{x})\psi_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\mu) \\
\chi_{\mu\nu}(\mathbf{x}) &\to G(\mathbf{x}+\mu+\nu)\chi_{\mu\nu}(\mathbf{x})G^\dagger(\mathbf{x}) \\
\mathcal{U}_\mu(\mathbf{x}) &\to G(\mathbf{x})\mathcal{U}_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\mu) \\
\overline{\mathcal{U}}_\mu(\mathbf{x}) &\to G(\mathbf{x}+\mu)\overline{\mathcal{U}}_\mu(\mathbf{x})G^\dagger(\mathbf{x})
\end{aligned}
$$

- Choice of orientations ensure G.I

Lattice supersymmetry

As in continuum:

$$
\begin{aligned}
\mathcal{Q}\,\mathcal{U}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{U}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= \mathcal{F}^{L\dagger}_{\mu\nu} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}
$$

Note: $\mathcal{Q}^2 = 0$ still.

Derivatives

$$\mathcal{D}^{(+)}_\mu f_\nu(\mathbf{x}) = \mathcal{U}_\mu(\mathbf{x}) f_\nu(\mathbf{x}+\mu) - f_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\nu)$$

$$\overline{\mathcal{D}}^{(-)}_\mu f_\mu(\mathbf{x}) = f_\mu(\mathbf{x})\overline{\mathcal{U}}_\mu(\mathbf{x}) - \overline{\mathcal{U}}_\mu(\mathbf{x}-\mu) f_\mu(\mathbf{x}-\mu)$$

- For $U_\mu(x) = 1 + A_\mu(x) + \ldots$ reduce to adjoint covariant derivatives
- $\mathcal{F}_{\mu\nu} = \mathcal{D}^{(+)}_\mu \mathcal{U}_\nu(\mathbf{x}) = \mathcal{U}_\mu(\mathbf{x})\mathcal{U}_\nu(\mathbf{x}+\mu) - \mathcal{U}_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\nu)$
- Remarkably satisfy exact Bianchi identity: $\epsilon_{\mu\nu\rho\lambda} D^{(}+)_\nu \mathcal{F}_{\rho\lambda} = 0$

Recap

- Discretize twisted version of continuum SYM
- Need subgroup of R-symmetry to match $SO(D)$.
- Ensures all fermions represented by integer spin forms. Natural map to lattice.
- In flat space: twisted formulation completely equivalent to usual theory
- Absence of fermion doubling – twisted fermions fill out Kähler-Dirac field (like staggered quarks)
- Lattice theory G.I, possesses exact $\mathcal{Q}$ and a point group symmetry which is subgroup of twisted rotational symmetry.

Bonuses

- Topological subsector:
  $< O(x_1)\ldots O(x_N) >$ independent of coupling g^2, and points $x_1 \ldots x_N$ if $\mathcal{Q}O = 0$. Eg

$$\frac{\partial < O >}{\partial g^2} =< \mathcal{Q}(\Lambda O) >= 0$$

- Novel gauge invariance properties of lattice theory strongly constrains possible counter terms – reduces substantially fine tuning needed to get full SUSY in continuum limit.

$\mathcal{Q} = 16$ **SYM in 4D**

- Twist: diagonal subgroup of $SO_{\rm Lorenz}(4) \times SO_{\rm R}(4)$
- Again after twisting regard fermions as 4×4 matrix.
- To represent 10 bosons of $\mathcal{N} = 4$ theory with complex connections is most natural in five dimensions.
- Fermion counting requires multiplet $(\eta, \psi_a, \chi_{ab})$ where $a, b = 1 \ldots 5$
- Action contains same $\mathcal{Q}$-exact term as for $\mathcal{Q} = 4$ plus new $\mathcal{Q}$-closed piece.

Details

- Dimensional reduction to 4D – $\mathcal{A}_5$ plus imag parts of $\mathcal{A}_\mu, \mu = 1 \ldots 4$ yield 6 scalars of $\mathcal{N} = 4$
- Fermions: $\chi_{ab} \rightarrow \chi_{\mu\nu} \oplus \overline{\psi}_\mu$, $\psi_a \rightarrow \psi_\mu \oplus \overline{\eta}$
- $S = \mathcal{Q}\Lambda - \frac{1}{8}\int \epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c\chi_{ab}$
- Twisted action reduces to Marcus topological twist of $\mathcal{N} = 4$ (GL-twist). Equivalent to usual theory in flat space.
- Identical to $\mathcal{Q} = 16$ orbifold action (Kaplan, Unsäl)

Transition to lattice

- Introduce cubic lattice with unit vectors $\mu_a^i = \delta_a^i, a = 1 \ldots 4$. Additional vector $\mu_5 = (-1, -1, -1, -1)$.
- Notice: $\sum_a \mu_a = 0$. Needed for G.I.
- Assign fields to links in cubic lattice (plus diagonals). Eg $\chi_{ab}(\mathbf{x})$ lives on link from $(\mathbf{x} + \mu_a + \mu_b) \to \mathbf{x}$.
- Derivatives similar to $\mathcal{Q} = 4$. eg $D_a^{(+)} f(\mathbf{x}) = \mathcal{U}_a(\mathbf{x}) f(\mathbf{x} + \mathbf{a}) - f(\mathbf{x}) \mathcal{U}_a(\mathbf{x})$

Simulations

- Integrate out fermions. Resulting $\mathrm{Pf}\,[M_F(\mathcal{A})]$ simulated using RHMC alg. (lattice QCD)
- Use pbc – SUSY exact. $Z = W$ Witten index - $\mathcal{Q}$-invariance exhibits topological invariance W.
- Preliminary results from single core code. Parallel code now finished..
- Test SUSY, I.R divergences, check sign problems. $D = 2$ with $\mathcal{Q} = 4$ and $D = 4$ with $\mathcal{Q} = 16$.

Supersymmetric Ward identity

$\mathcal{Q}$-exactness ensures that $\frac{\partial \ln Z_{\rm pbc}}{\partial \kappa} = 0$

where $\kappa = \frac{1}{g^2}\frac{L^D}{V^{4-D}}$

Ensures: $< \kappa S_B >= \frac{1}{2}V(N^2-1)(n_{\rm bosons}-1)$

Example: $D=0$ $SU(2)$

κ	κS_B	exact
1.0	4.40(2)	4.5
10.0	4.47(2)	4.5
100.0	4.49(1)	4.5

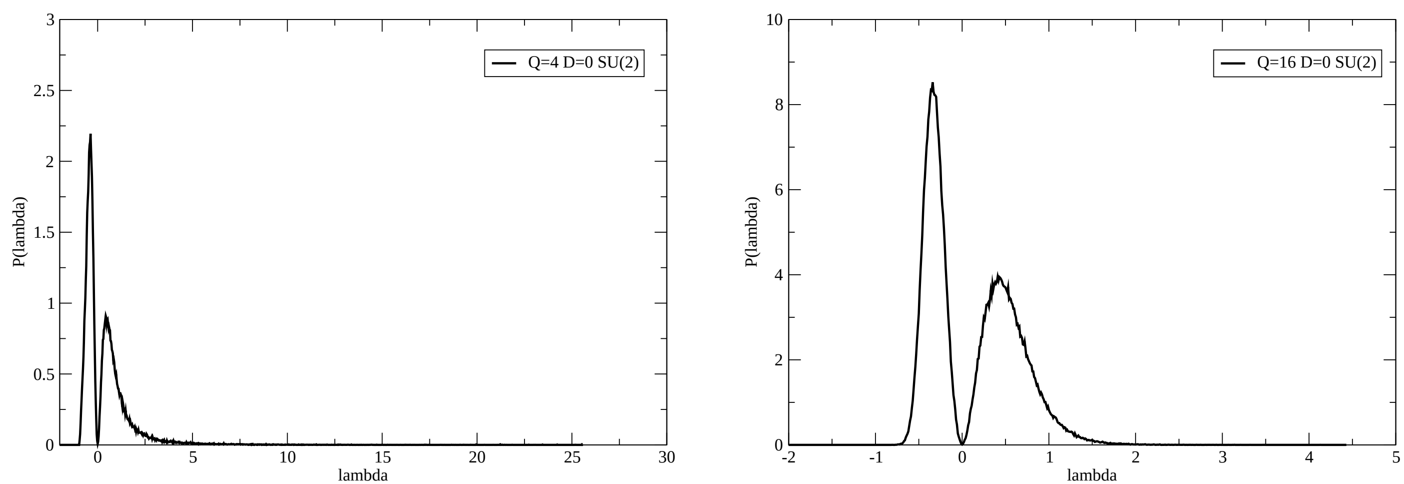
$\mathcal{Q}=4$

κ	κS_B	exact
1.0	13.67(4)	13.5
10.0	13.52(2)	13.5
100.0	13.48(2)	13.5

$\mathcal{Q}=16$

Vacuum stability - flat directions

Is integration over moduli space $[B_\mu, B_\nu] = 0$ divergent ?



$\mathcal{Q} = 4$ $\mathcal{Q} = 16$

$D = 0$. $SU(2)$. Periodic bcs. Eigenvalues of $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - 1$
Scalars localized close to origin. Power law tails.
$p(\mathcal{Q} = 4) \sim 3$, $p(\mathcal{Q} = 16) \sim 15$ (Staudacher et al.)

Pfaffian phase

Simulation uses $|Pf(\mathcal{U})|$. Measure phase $\alpha(\mathcal{U})$.

$$< O >= \frac{< Oe^{\alpha} >_{\text{phase quenched}}}{< e^{\alpha} >_{\text{phase quenched}}}$$

$SU(2)$ $D = 2$: 4^2.

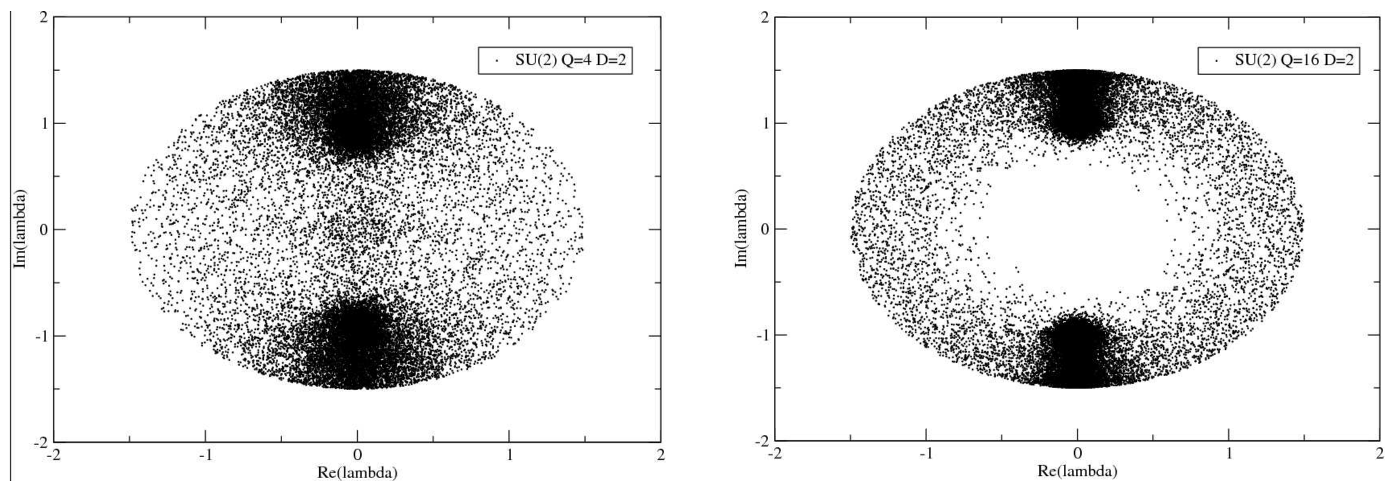
$\mathcal{Q}$	S_B^q	S_B	S_B^e	$\cos\alpha$
4	70.61(4)	65(5)	72.0	-0.016(6)
16	214.7(4)	214.6(3)	216.0	0.999994(3)

$< e^{i\alpha(\mathcal{U}_\mu)} >_{\text{phase quenched pbc}} = W = 0$ for $\mathcal{Q} = 4$?

SUSY breaking (Tong et. al) ?

Fermion eigenvalue distribution

$SU(2)$ $D = 2$: 2^2



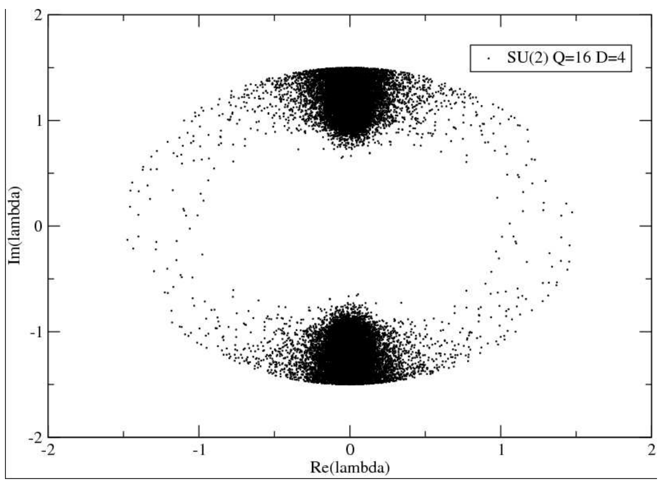
$\mathcal{Q} = 4$ $\mathcal{Q} = 16$

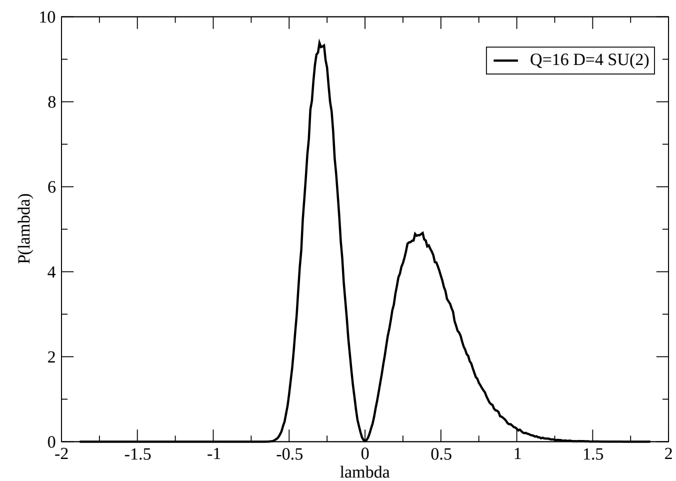
Non-zero density for $\mathcal{Q} = 4$ close to origin – linked to log divergence of $< \delta\lambda^2 >$?
Potential Goldstino ?

$\mathcal{N} = 4$ SYM in four dimensions

Initial results encouraging: 6000 trajs on $SU(2)$ 2^4 lattice (1000 hrs)

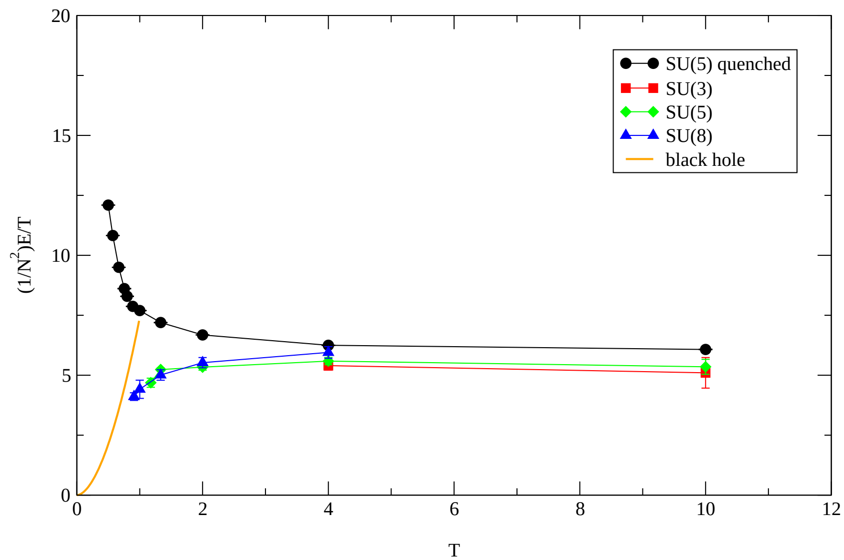
$S_B/S_B^{\rm exact} = 0.98 < \cos(\alpha) >= 0.98(1)$





Larger lattices currently under study using parallel code.

Applications: holography



Example of gauge-gravity duality: Thermodynamics of $N \to \infty, T \to 0$ AdS_5 black hole reproduced by $\mathcal{N} = 4$ SYM theory reduced to $D = 1$

Renormalization

Lattice symmetries:

- Gauge invariance
- $\mathcal{Q}$-symmetry.
- Point group symmetry - eg. natural lattice for $\mathcal{N} = 4$ is A_4^*.
- Exact fermionic shift symmetry.

Conclusion: Renormalized action contains same operators as bare theory except for SUSY mass term.
Examine flows at 1-loop - in progress (with J. Giedt)

Future

- Nonperturbative exploration $\mathcal{N}=4$ YM. Tests of AdSCFT. Supersymmetric Wilson loops.
- But – what residual fine tuning needed to get full SUSY as $a\to 0$?
- Dimensional reductions – duality between strings with Dp-branes and $(p+1)$-SYM ?
- Add fermions in fundamental .. (Matsuura, Sugino in $D=2$ recently).
- Break $\mathcal{N}=4$ to $\mathcal{N}=1$ a la Strassler ..

Marcus twist

Reduces to:

$$
\begin{aligned}
S \quad = \quad & \int \mathrm{Tr}\, \Bigg(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\left[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu\right]^2 + \frac{1}{2}\left[\overline{\phi}, \phi\right]^2 + (\mathcal{D}_\mu\phi)^\dagger(\mathcal{D}_\mu\phi) \\
- \quad & \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \overline{\psi}_\mu\mathcal{D}_\mu\overline{\eta} - \overline{\psi}_\mu\left[\phi, \psi_\mu\right] \\
- \quad & \eta\overline{\mathcal{D}}_\mu\psi_\mu - \eta\left[\overline{\phi}, \overline{\eta}\right] - \chi^*_{\mu\nu}\overline{\mathcal{D}}_\mu\overline{\psi}_\nu - \chi^*_{\mu\nu}\left[\overline{\phi}, \chi_{\mu\nu}\right]\Bigg)
\end{aligned}
$$

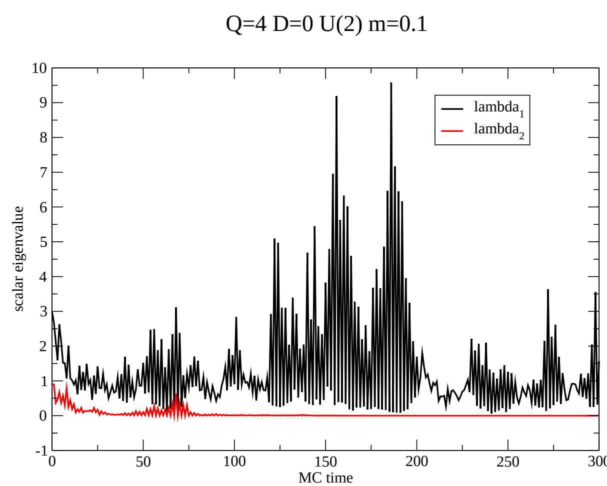
Vacuum stability - trace mode

Correspondance to continuum requires $\mathcal{U}_\mu = 1 + aA_\mu + O(a^2)$.

For $U(N)$ this is not true $< \frac{1}{N}\mathrm{Tr}\,\mathcal{U}_\mu^\dagger(x)\mathcal{U}_\mu(x) >\sim 0.5$

$\det(\mathcal{U}_\mu^\dagger(x)\mathcal{U}_\mu(x)) \rightarrow 0$!

Vacuum instability – $\det(\mathcal{U}_\mu^\dagger\mathcal{U}_\mu) \sim e^{B_\mu^0}$ implies $B_\mu^0 \rightarrow -\infty$



Truncation

Cannot cure with mass $m^2 \sum \mathrm{Tr}\,(\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - I)^2$

m	$< \mathcal{U}_\mu^\dagger \mathcal{U}_\mu >$
0.01	0.45(2)
0.1	0.57(6)
0.5	0.38(2)

$S_B(e^{-\delta B_\mu^0}\mathcal{U}) \sim e^{-4\delta B_\mu^0} S(\mathcal{U}_\mu)$ any $\{\mathcal{U}_\mu\}$

Exponential effective potential for B_μ^0.

Fix ? - truncate to $SU(N)$ – $\delta S \sim \frac{1}{N^2} O(a)$

Also removes exact 0 mode in fermion op.