# Heavy quark current correlators for precise quark masses and strong coupling constant

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# Introduction

## Motivation

Precise determination of quark masses and strong coupling:

- Quark masses/strong coupling are fundamental parameters of the Standard Model
- Quark masses play an important role in Higgs physics: e.g. decay:

  $$\Gamma(H \to b\bar{b}) = \frac{G_f M_h}{4\sqrt{2}\pi} m_b^2 \left(1 + \mathcal{O}(\alpha_s) + \ldots\right), \quad \Gamma(H \to c\bar{c}) \sim m_c^2$$

- Flavor physics
- Experiments at different energies allow tests of energy dependence of $\alpha_s(\mu)$ based on the RGE
- Convergence of the three gauge couplings to common value might reveal possibilities of embedding the SM in the framework of GUT
- Comparison with other methods

# Methods

Here:

2 Methods:

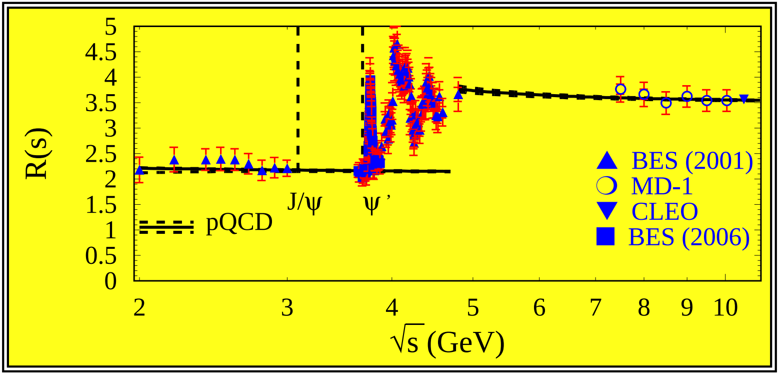
I. Method: $m_c$ and $m_b$ from measured $R$-ratio
II. Method: $m_c$ and $\alpha_s$ from lattice simulations

Both methods require the perturbative computation of heavy quark correlators

# Method I: *R*-ratio

## Experiment

$$R(s) = \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



# Method I

## Theory

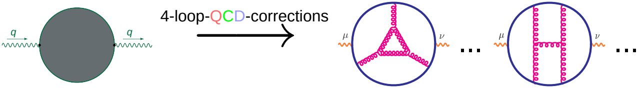
- Heavy quark correlator

$$\Pi^{\mu\nu}(q,j) = i \int dx\, e^{iqx} \langle 0|T j^{\mu}(x) j^{\nu}(0)|0\rangle$$

<u>Here:</u> $j^{\mu}(x)$ electromagnetic heavy quark <u>vector current</u>

- Diagrammatically:

$\Pi^{\mu\nu}(q,j) =$



- Relation to polarization function $\Pi(q^2)$:

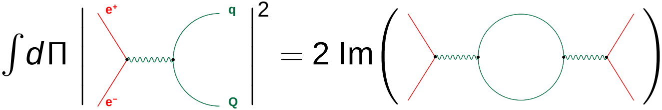
$$\Pi^{\mu\nu}(q) = \left(-g^{\mu\nu} q^2 + q^{\mu} q^{\nu}\right) \Pi(q^2)$$

# Method I

## Relation: Experiment ⟺ Theory

- Optical theorem

$$R(s) = \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 12\pi \text{Im}\left[\Pi(q^2 = s + i\varepsilon)\right]$$



- With the help of dispersion-relations

$$\boxed{\Pi(q^2) = \Pi(q^2 = 0) + \frac{q^2}{12\pi^2} \int ds \frac{R(s)}{s(s - q^2)}}$$

# Method I

## Relation: Theory $\Longleftrightarrow$ Experiment

- Exp. moments are related to derivatives of $\Pi(q^2)$ at $q^2 = 0$:

$$\frac{12\pi}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\bigg|_{q^2=0} = \boxed{\mathcal{M}_n^{\text{exp}} = \int \frac{ds}{s^{n+1}} R^{\text{exp}}(s)}$$

- In terms of expansion coefficients:

$$\Pi(q^2) = \frac{3Q_f^2}{16\pi^2}\sum_n \overline{C}_n^v \left(\frac{q^2}{4m^2}\right)^n,$$

$Q_f$: charge of quark

$m = m(\mu)$ : $\overline{\text{MS}}$ mass

$\overline{C}_n^v$ can be calculated perturbatively

- First and higher derivatives of $\Pi(q^2)$ allow direct determination of the $\overline{\text{MS}}$ charm- and bottom-quark mass:

$$\boxed{\overline{m}(\mu) = \frac{1}{2}\left(Q_f^2 \frac{9}{4}\frac{\overline{C}_n^v}{\mathcal{M}_n^{\text{exp}}}\right)^{1/(2n)}}$$

← Theory

← Experiment

c-quarks: Novikov et al. ’78; b-quarks: Reinders et al. ’85

$\overline{C}_n^v$ depend on the quark mass through $\log(m(\mu)^2/\mu^2)$

# Method II: Data from Lattice

I. Allison, E. Dalgic, C.T.H. Davies, E. Follana, R.R. Horgan, K. Hornbostel, G.P. Lepage, C. McNeile, J. Shigemitsu, H. Trottier, R.M. Woloshyn(HPQCD), K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, C.S.

Idea: Replace moments obtained from *R*-ratio by computation of correlator through lattice simulations

- Allows to substitute electromagnetic current by pseudoscalar operator $r_{2k+2} = (\overline{C}^{p}_{k}/\overline{C}^{p,(0)}_{k})^{\frac{1}{2k-2}}$, $k = 2, 3, \dots$
  $\overline{C}^{p}_{k}$: expansion coeff. of pseudoscalar correlator
- Quark mass:

$$\boxed{\overline{m}_c(\mu) = \frac{m^{exp}_{\eta_c}}{2}\,\frac{r^{pQCD}_{2k+2}}{\mathcal{R}^{LQCD}_{2k+2}}}$$

← Pert. theory

← Lattice Sim. HPQCD

- Strong coupling:

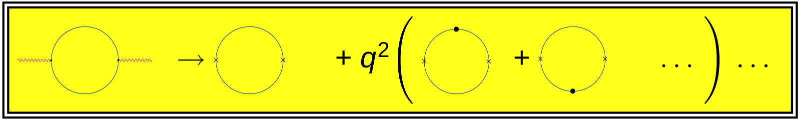
$$\boxed{r_4(\alpha_s, \mu/m_c) = \mathcal{R}^{LQCD}_{4}}$$

+ ratios of moments,
weak dependence on quark mass

# Pert. calculation of expansion coefficients

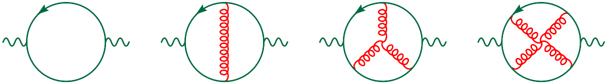
## Theory

- Expansion diagrammatically:



$\hookrightarrow$ One-scale multi-loop integrals in pQCD

- Sample diagrams



- 3-loop(order $\alpha_s^2$) coefficients $\overline{C}_n$ up to n=8 Chetyrkin,Kühn,Steinhauser 96
  up to higher moments $n \sim 30$ Czakon et al. 06; Maierhöfer, Maier, Marquard 07
  for correlators $VV$, $AA$, $PP$, $SS$

# Calculation

## Techniques

Integration-by-parts (IBP): K.G. Chetyrkin, F.V. Tkachov

$$0 = \int [d^D \ell_1] \dots [d^D \ell_4] \quad \partial_{(\ell_j)_\mu} \left( \ell_l^\mu \, I_{\alpha\beta} \right) \; , \qquad j, l = 1, \dots, \text{loops=4}$$

$I_{\alpha\beta}$: Generic integrand with propagator powers $\alpha = \{\alpha_1, \dots\}$

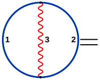
and scalar-product powers $\beta = \{\beta_1, \dots\}$

Laporta-Algorithm: S. Laporta, E. Remiddi

Idea:
- – IBP-identities for explicit numerical values of $\alpha, \beta$
- – Introduction of an order among the integrals
- – Solving a linear system of equations

Automation: Generation & solution of the system of lin. equations with:
- – Implementation based on `FORM3` J.A.M. Vermaseren
- – Simplification of rational functions in $d$ by `FERMAT` R.H. Lewis

## ... an example for integration-by-parts:



$$= \int d\ell_1 \int d\ell_2 \, \frac{1}{(\ell_1^2+m^2)(\ell_2^2+m^2)(\ell_1+\ell_2)^2} = \int d\ell_1 \int d\ell_2 \, \frac{1}{D_1^1 D_2^1 D_3^1} = f(1,1,1)$$

IBP-identities:

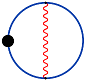
I) $0 = \int d\ell_1 \int d\ell_2 \;\; \partial_{\ell_1} \ell_2 \;\; \frac{1}{D_1 D_2 D_3}$ II) $0 = \int d\ell_1 \int d\ell_2 \;\; \partial_{\ell_1} \ell_1 \;\; \frac{1}{D_1 D_2 D_3}$

$$0 = f(1,1,1) - f(2,1,0) \\ -2m^2 f(2,1,1) - f(1,1,1)$$

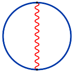
$$0 = d f(1,1,1) \\ -2f(1,1,1) + 2m^2 f(2,1,1) \\ -f(1,1,1)$$

$$\Rightarrow f(2,1,1) = -\frac{1}{2m^2} f(2,1,0)$$

$$\Rightarrow f(1,1,1) = \frac{1}{d-3} f(2,1,0)$$

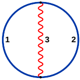


$$= \frac{-1}{2m^2}$$



$$= \frac{1}{d-3}$$

## ... an example for integration-by-parts:



$$= \int d\ell_1 \int d\ell_2 \frac{1}{(\ell_1^2+m^2)(\ell_2^2+m^2)(\ell_1+\ell_2)^2} = \int d\ell_1 \int d\ell_2 \frac{1}{D_1^1 D_2^1 D_3^1} = f(1,1,1)$$

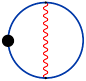
IBP-identities:

$$\text{I)}\ 0 = \int d\ell_1 \int d\ell_2\ \partial_{\ell_1} \ell_2\ \frac{1}{D_1 D_2 D_3} \qquad \text{II)}\ 0 = \int d\ell_1 \int d\ell_2\ \partial_{\ell_1} \ell_1\ \frac{1}{D_1 D_2 D_3}$$

$$0 = f(1,1,1) - f(2,1,0) - 2m^2 f(2,1,1) - f(1,1,1)$$

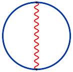
$$0 = d f(1,1,1) - 2f(1,1,1) + 2m^2 f(2,1,1) - f(1,1,1)$$

$$\Rightarrow f(2,1,1) = -\frac{1}{2m^2} f(2,1,0) \qquad \Rightarrow f(1,1,1) = \frac{1}{d-3} f(2,1,0)$$

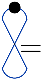


$= \frac{-1}{2m^2}$

$= \frac{d-2}{4m^4}$



$= \frac{1}{d-3}$



$= \frac{(2-d)}{(d-3)2m^2}$

# Techniques

Reducible: Diagrams which can be mapped on diagrams with less lines

At 4-loop:

Problem: Dramatic growth of number of equations

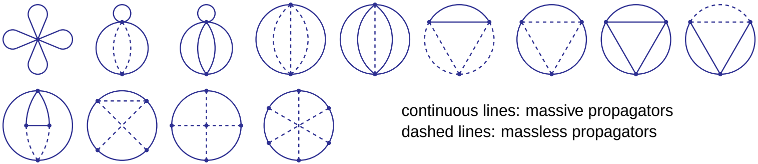
Here: >30 million IBP-identities generated and solved
⇝ About 6 Gbyte of integral-tables with solutions for around 5 million integrals, expressed in terms of 13 masters

Important: Consider all symmetries of diagrams
⇝ Smaller number of IBP-equations,
⇝ Keep size of integral-tables under control

# Master Integrals

## in the Standard Basis

13 Master integrals:


continuous lines: massive propagators
dashed lines: massless propagators

Standard basis: – "Minimal number of lines",
– "Minimal number of dots",
– "Lowest power of irred. scalar products"

Solution with high precision numerics Y. Schröder, A. Vuorinen
with difference equation (DE) method S. Laporta
Subsequently with independent method: $\varepsilon$-finite basis
K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov

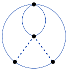
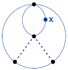
other contributions: D.J. Broadhurst; S. Laporta; B.A. Kniehl, A.V. Kotikov; Y. Schröder, M. Steinhauser

# Master integrals

## with difference equations

S. Laporta

Raise one propagator to symbolic power $x$:

e. g. $M =$  $\rightarrow$ $M(x) =$ 

$$= \int \frac{[dk_1]\ldots[dk_4]}{D_1^x D_2 \ldots D_8}$$

Use IPB and Laporta alg. to construct difference equations

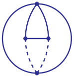
$$\sum_{j=0}^{R} p_j(d,x)\, M(x-j) = \text{combination of subtopologies of } M(x)$$

Ansatz: factorial series $M(x) = \mu^x \sum_{s=0}^{\infty} a_s \frac{\Gamma(x+1)}{\Gamma(x+1+s-K)}$

- recursion formula for $a_s$ → sum up to specified $s_{max} \sim 1000-2000$
- better convergence for large x → use DE to get M(1)

# Master Integrals

## Example result for DE M. Faisst, P. Maierhöfer, C.S.



$$
\begin{aligned}
&= -0.16666666666666666666666666666666666666666666666667\,\epsilon^{-4}\\
&- 1.50000000000000000000000000000000000000000000000000\,\epsilon^{-3}\\
&- 9.86872356982626095206640482817811665743165295900717\,\epsilon^{-2}\\
&- 46.6303411944320215719397994165165851567120807356798\,\epsilon^{-1}\\
&- 137.332112842523179736741159027274031745616189046337\\
&- 929.858212294016976382457232976107077779091868024097\,\epsilon^{1}\\
&- 1698.90639250653423244336742929239614618805875415910\,\epsilon^{2}\\
&+ \mathcal{O}(\epsilon^{3})
\end{aligned}
$$

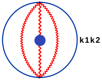
- High numerical precision (usually $>$ 30 digits)
- However, construction of difference equations increasingly tedious for masters with many lines
- Pole part analytically

# Master Integrals

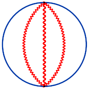
## in the $\varepsilon$-finite basis K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov

**Problem:** Coefficient functions $c_{ij}$ in $I_i = \sum_j c_{ij} M_j$ can have “spurious” poles

Example:



$$= \frac{-(d-3)(d-2)}{2(d-4)}$$



Arise while solving IBP-identities through division by $(d-4)$.

⇝ Master integrals with spurious poles as coefficient need to be known in higher order in $\varepsilon$

But: Choice of master integrals is not unique

Idea: Select a new basis with finite coefficient functions

Solution: “$\varepsilon$-finite basis”

⇝ Advantage: Members need only be evaluated up to order $\varepsilon^0$

# Master Integrals

## Constructing an $\varepsilon$-finite basis K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov

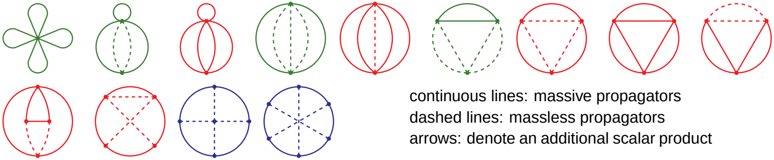
Members of $\varepsilon$-finite basis can be found among the set of initial integrals $I_i$

Algorithm:

1. Express all initial integrals in terms of standard masters $I_i = \sum_j c_{ij} M_j$
2. Choose equation with highest $\epsilon$-pole in a coefficient $c_{ij}$
3. Solve it for $M_j$
   $\rightarrow$ All coefficients in the new equation are finite
4. Replace $M_j$ in all equations and treat $I_i$ as new master integral
5. Repeat steps 2–4 until all coefficients are finite

# Master Integrals

## $\varepsilon$-finite basis K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov

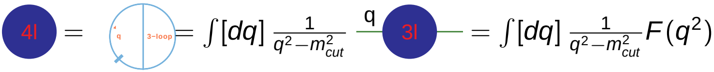


continuous lines: massive propagators
dashed lines: massless propagators
arrows: denote an additional scalar product

### Computation with Padé-method

Fleischer, Tarasov; Broadhurst, Baikov

Idea: Perform integration over 3 loop momenta "semi-analytically" and the $4^{th}$ numerically:



$$= \int [dq] \frac{1}{q^2 - m_{cut}^2} \quad = \int [dq] \frac{1}{q^2 - m_{cut}^2} F(q^2)$$

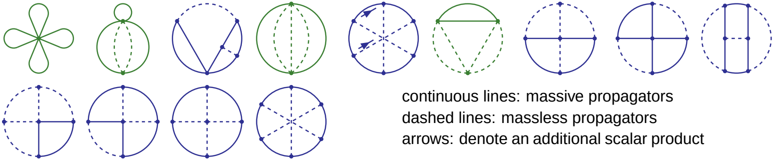
- Perform low and high-energy expansion of $F(q^2)$
- Reconstruction of $F(q^2)$ through Padé-Approximation

$$F(q^2) \longrightarrow [i/j](q^2) = \frac{a_0 + a_1 q^2 \ldots a_n q^{2n}}{b_0 + b_1 q^2 \ldots b_m q^{2m}}$$

$[i/j](q^2)$ : Same low- and high-energy behavior like $F(p^2)$

# Master Integrals

## $\varepsilon$-finite basis K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov



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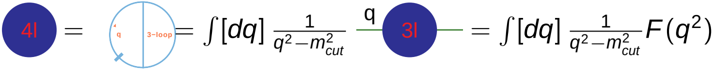

$$= \int [dq] \frac{1}{q^2 - m_{cut}^2} \qquad = \int [dq] \frac{1}{q^2 - m_{cut}^2} F(q^2)$$

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$[i/j](q^2)$ : Same low- and high-energy behavior like $F(p^2)$

# Example result M. Faisst, P. Maierhöfer, C.S.

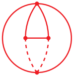
- Pole part analytically

  $\Rightarrow$ Allows analytical cancellation of (UV-)$\epsilon$-poles
- Express standard basis through $\varepsilon$-finite one (and vice versa):

$$\boxed{\text{standard basis}} \overset{\text{IBP-relations}}{\Longleftrightarrow} \boxed{\varepsilon\text{-finite basis}}$$

$\rightsquigarrow$ Can be used to get analytical information also for standard basis:

Example:



$$= -\frac{1}{6}\,\epsilon^{-4} - \frac{3}{2}\,\epsilon^{-3} - \left(\frac{26}{3} + \zeta_3\right)\,\epsilon^{-2} - \left(41 + \frac{\pi^4}{60} + \frac{10}{3}\zeta_3\right)\,\epsilon^{-1}$$
$$- \frac{1039}{6} - \frac{7}{30}\pi^4 + 3\zeta_3 + 53\zeta_5$$
$$- 929.858212294016976382457232976107077779091868024097\,\epsilon^1$$
$$- 1698.90639250653423244336742929239614618805875415910\,\epsilon^2$$
$$+ \mathcal{O}(\epsilon^3)$$

# Results at 4-loops

## Theory

- Vector case:
  - first moments $\overline{C}_0$, $\overline{C}_1$
    K. G. Chetyrkin, J. H. Kühn, C.S.'06; R. Boughezal, M. Czakon, T. Schutzmeier'06
  - second moment $\overline{C}_2$ A. Maier, P. Maierhöfer, P. Marquard'08
- Pseudoscalar case:
  - first moments $\overline{C}_0$, $\overline{C}_1$, $\overline{C}_2$ I. Allison, E. Dalgic, C.T.H. Davies, E. Follana, R.R. Horgan, K. Hornbostel, G.P. Lepage, C. McNeile, J. Shigemitsu, H. Trottier, R.M. Woloshyn, K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, C.S. 08
  - third moment $\overline{C}_3$ A. Maier, P. Maierhöfer, P. Marquard'08
- Axial-vector and scalar case:
  - first moments $\overline{C}_0$, $\overline{C}_1$ C. S.'08

# Result

$$
\begin{aligned}
\overline{C}_n = \overline{C}_n^{(0)} \quad &+ \quad \left(\frac{\alpha_s}{\pi}\right)\left(\overline{C}_n^{(10)} + \overline{C}_n^{(11)} l_{m_c}\right) \\
&+ \quad \left(\frac{\alpha_s}{\pi}\right)^2\left(\overline{C}_n^{(20)} + \overline{C}_n^{(21)} l_{m_c} + \overline{C}_n^{(22)} l_{m_c}^2\right) \\
&+ \quad \left(\frac{\alpha_s}{\pi}\right)^3\left(\overline{C}_n^{(30)} + \overline{C}_n^{(31)} l_{m_c} + \overline{C}_n^{(32)} l_{m_c}^2 + \overline{C}_n^{(33)} l_{m_c}^3\right) \\
&+ \quad \ldots, \text{with } l_{m_c} = \log(m_c^2/\mu^2)
\end{aligned}
$$

Pseudoscalar case($n_f = 4$):

| | 1-loop | 2-loop | | 3-loop | | | 4-loop | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n | $\overline{C}_n^{(0)}$ | $\overline{C}_n^{(10)}$ | $\overline{C}_n^{(11)}$ | $\overline{C}_n^{(20)}$ | $\overline{C}_n^{(21)}$ | $\overline{C}_n^{(22)}$ | $\overline{C}_n^{(30)}$ | $\overline{C}_n^{(31)}$ | $\overline{C}_n^{(32)}$ | $\overline{C}_n^{(33)}$ |
| 1 | 1.3333 | 3.1111 | 0.0000 | 0.1154 | -6.4815 | 0.0000 | -1.2224 | 2.5008 | 13.5031 | 0.0000 |
| 2 | 0.5333 | 2.0642 | 1.0667 | 7.2362 | 1.5909 | -0.0444 | 7.0659 | -7.5852 | 0.5505 | 0.0321 |
| 3 | 0.3048 | 1.2117 | 1.2190 | 5.9992 | 4.3373 | 1.1683 | 14.5789 | 7.3626 | 4.2523 | -0.0649 |
| 4 | 0.2032 | 0.7128 | 1.2190 | 4.2670 | 4.8064 | 2.3873 | — | 14.7645 | 11.0345 | 1.4589 |

Result also available completely analytically

# Result

$$\begin{aligned}\overline{C}_n = \overline{C}_n^{(0)} &+ \left(\frac{\alpha_s}{\pi}\right)\left(\overline{C}_n^{(10)} + \overline{C}_n^{(11)} l_{m_c}\right)\\ &+ \left(\frac{\alpha_s}{\pi}\right)^2\left(\overline{C}_n^{(20)} + \overline{C}_n^{(21)} l_{m_c} + \overline{C}_n^{(22)} l_{m_c}^2\right)\\ &+ \left(\frac{\alpha_s}{\pi}\right)^3\left(\overline{C}_n^{(30)} + \overline{C}_n^{(31)} l_{m_c} + \overline{C}_n^{(32)} l_{m_c}^2 + \overline{C}_n^{(33)} l_{m_c}^3\right)\\ &+ \ldots, \text{with } l_{m_c} = \log(m_c^2/\mu^2)\end{aligned}$$

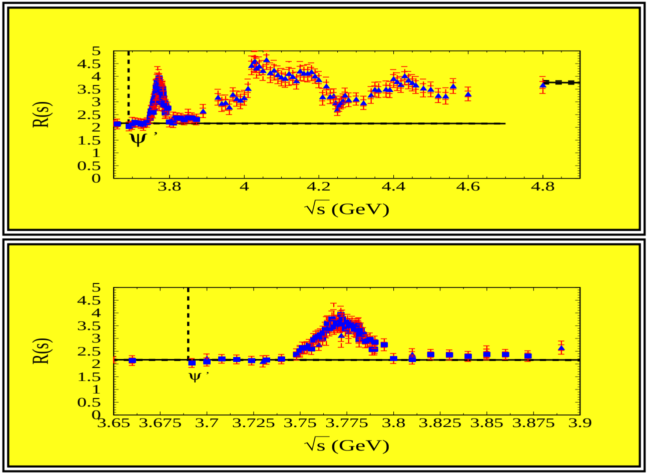
Vector case($n_f = 4$):

| | 1-loop | 2-loop | | 3-loop | | | 4-loop | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n | $\overline{C}_n^{(0)}$ | $\overline{C}_n^{(10)}$ | $\overline{C}_n^{(11)}$ | $\overline{C}_n^{(20)}$ | $\overline{C}_n^{(21)}$ | $\overline{C}_n^{(22)}$ | $\overline{C}_n^{(30)}$ | $\overline{C}_n^{(31)}$ | $\overline{C}_n^{(32)}$ | $\overline{C}_n^{(33)}$ |
| 1 | 1.0667 | 2.5547 | 2.1333 | 2.4967 | 3.3130 | -0.0889 | -7.7624 | -0.0599 | 1.5851 | -0.0543 |
| 2 | 0.4571 | 1.1096 | 1.8286 | 3.2319 | 5.0798 | 1.9048 | -3.4937 | 4.0100 | 7.2551 | 0.1058 |
| 3 | 0.2709 | 0.5194 | 1.6254 | 2.0677 | 4.5815 | 3.3185 | – | 5.6496 | 13.4967 | 2.3967 |
| 4 | 0.1847 | 0.2031 | 1.4776 | 1.2204 | 3.4726 | 4.4945 | – | 3.9381 | 17.2292 | 6.2423 |

Result also available completely analytically

# Method I: $R$-ratio

Determine: $\mathcal{M}_n^{exp} = \int \frac{ds}{s^{n+1}} R^{\text{exp}}(s)$



# Analysis

## Extraction of the exp. moments from $R(s)$ (charm quark case) J.H. Kühn, M. Steinhauser, C.S.

Determine: $\mathcal{M}_n^{exp} = \int \frac{ds}{s^{n+1}} R^{\text{exp}}(s) = \mathcal{M}_n^{res} + \mathcal{M}_n^{thr} + \mathcal{M}_n^{cont}$

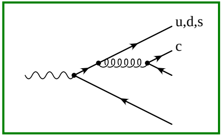
For charm quarks:

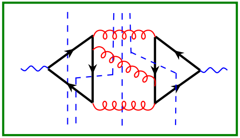
$\mathcal{M}_n^{res}$: Contains: $J/\Psi, \Psi(2S)$ treated in narrow width approximation

$$R^{res}(s) = \frac{9\pi M_R \Gamma_{ee}}{\alpha^2} \left(\frac{\alpha}{\alpha(s)}\right)^2 \delta(s - M_R^2)$$

| | $J/\Psi$ | $\Psi(2S)$ |
|---|---|---|
| $M_\Psi$(GeV) | 3.096916(11) | 3.686093(34) |
| $\Gamma_{ee}$(keV) | 5.55(14) | 2.48(6) |
| $(\alpha/\alpha(M_\Psi))^2$ | 0.957785 | 0.95554 |

$\mathcal{M}_n^{thr}$: BES data ($\sqrt{s} \geq 3.73$ GeV) subtract background from $R_{uds}$,





$\bar{R}$ from data below 3.73 GeV, $\sqrt{s}$-dependence from theory

# Analysis

## Extraction of the exp. moments from $R(s)$ (charm quark case) J.H. Kühn, M. Steinhauser, C.S.

$\mathcal{M}_n^{cont}$: pQCD above $\sqrt{s} \geq 4.8$ GeV ,

no data,

$R(s)$ with full quark mass dependence rhad: R. Harlander, M. Steinhauser '02

$\mathcal{M}_n^{exp}$:

| $n$ | $\mathcal{M}_n^{\text{res}}$ $\times 10^{(n-1)}$ | $\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(n-1)}$ | $\mathcal{M}_n^{\text{cont}}$ $\times 10^{(n-1)}$ | $\mathcal{M}_n^{\text{exp}}$ $\times 10^{(n-1)}$ | $\mathcal{M}_n^{\text{np}}$ $\times 10^{(n-1)}$ |
|---|---|---|---|---|---|
| 1 | 0.1201(25) | 0.0318(15) | 0.0646(11) | 0.2166(31) | −0.0001(2) |
| 2 | 0.1176(25) | 0.0178(8) | 0.0144(3) | 0.1497(27) | 0.0000(0) |
| 3 | 0.1169(26) | 0.0101(5) | 0.0042(1) | 0.1312(27) | 0.0007(14) |
| 4 | 0.1177(27) | 0.0058(3) | 0.0014(0) | 0.1249(27) | 0.0027(54) |

$$\delta\mathcal{M}_n^{\text{np}} = \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left(1 + \frac{\alpha_s}{\pi} \bar{b}_n\right)$$

D.J. Broadhurst, P.A. Baikov, V.A. Ilyin,
J. Fleischer, O.V. Tarasov, V.A. Smirnov

# Analysis

## Determination of the charm quark mass from $R(s)$ J.H. Kühn, M. Steinhauser, C.S.

$$\mathcal{M}_n^{th} + \mathcal{M}_n^{np} = \mathcal{M}_n^{exp} \quad \text{with} \quad \mathcal{M}_n^{th} = \frac{9}{4} Q_q^2 \left( \frac{1}{4m_c^2} \right)^n \overline{C}_n$$

$$m(\mu) = \frac{1}{2} \left( Q_f^2 \frac{9}{4} \frac{\overline{C}_n}{\mathcal{M}_n^{exp} - \mathcal{M}_n^{np}} \right)^{1/(2n)}$$

$\mu = (3 \pm 1)$ GeV $\quad \alpha_s(M_Z) = 0.1189 \pm 0.002$

| $n$ | $m_c(3\text{ GeV})$ | exp | $\alpha_s$ | $\mu$ | np | total | $\delta\bar{C}_n^{(30)}$ | $m_c(m_c)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.986 | 0.009 | 0.009 | 0.002 | 0.001 | 0.013 | — | 1.286 |
| 2 | 0.976 | 0.006 | 0.014 | 0.005 | 0.000 | 0.016 | — | 1.277 |
| 3 | 0.982 | 0.005 | 0.014 | 0.007 | 0.002 | 0.016 | 0.010 | 1.282 |
| 4 | 1.012 | 0.003 | 0.008 | 0.030 | 0.007 | 0.032 | 0.016 | 1.309 |

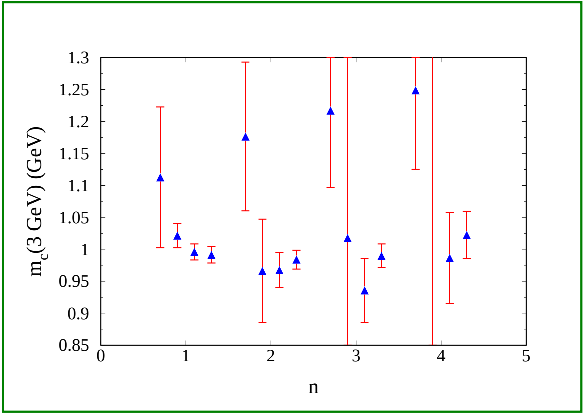
$$-6.0 \leq \overline{C}_3^{(30)} \leq 5.2, \; -6.0 \leq \overline{C}_4^{(30)} \leq 3.1$$

n=2: old $m_c(3\text{ GeV}) = 0.979$ GeV; estimated $-6.0 \leq \overline{C}_2^{(30)} \leq 7.0$;
new: $m_c(3\text{ GeV}) = 0.976$ GeV; $\overline{C}_2^{(30)} = -3.4937$ A. Maier, P. Maierhöfer, P. Marquard

# Analysis

## Determination of the charm quark mass from $R(s)$ J.H. Kühn, M. Steinhauser, C.S.

Charm-quarks



$$m_c(3\ \text{GeV}) = 0.986(13)\ \text{GeV}$$

# Analysis

## Extraction of the exp. moments from $R(s)$ (bottom quark case) J.H. Kühn, M. Steinhauser, C.S.

$\mathcal{M}_n^{th}$: analog to charm case, only $n_f = 5$

$\mathcal{M}_n^{np}$: negligible

$\mathcal{M}_n^{res}$: $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$, $\Upsilon(4S)$

$\mathcal{M}_n^{thr.}$: CLEO data up to 11.24 GeV

$\mathcal{M}_n^{cont}$: pQCD above 11.24 GeV

$\mathcal{M}_n^{exp}$:

| $n$ | $\mathcal{M}_n^{\rm res}$ $\times 10^{(2n+1)}$ | $\mathcal{M}_n^{\rm thresh}$ $\times 10^{(2n+1)}$ | $\mathcal{M}_n^{\rm cont}$ $\times 10^{(2n+1)}$ | $\mathcal{M}_n^{\rm exp}$ $\times 10^{(2n+1)}$ |
|---|---|---|---|---|
| 1 | 1.394(23) | 0.296(32) | 2.911(18) | 4.601(43) |
| 2 | 1.459(23) | 0.249(27) | 1.173(11) | 2.881(37) |
| 3 | 1.538(24) | 0.209(22) | 0.624(7) | 2.370(34) |
| 4 | 1.630(25) | 0.175(19) | 0.372(5) | 2.178(32) |

# Analysis

## Determination of the bottom quark mass from $R(s)$ J.H. Kühn, M. Steinhauser, C.S.

$\mu = (10 \pm 5)$ GeV; $\alpha_s(M_Z) = 0.1189 \pm 0.002$

| $n$ | $m_b(10\text{ GeV})$ | exp | $\alpha_s$ | $\mu$ | total | $\delta\bar{C}_n^{(30)}$ | $m_b(m_b)$ |
|---|---|---|---|---|---|---|---|
| 1 | 3.593 | 0.020 | 0.007 | 0.002 | 0.021 | — | 4.149 |
| 2 | 3.607 | 0.014 | 0.012 | 0.003 | 0.019 | — | 4.162 |
| 3 | 3.618 | 0.010 | 0.014 | 0.006 | 0.019 | 0.008 | 4.173 |
| 4 | 3.631 | 0.008 | 0.015 | 0.021 | 0.027 | 0.012 | 4.185 |

$$-6.0 \leq \overline{C}_3^{(30)} \leq 5.2, \; -6.0 \leq \overline{C}_4^{(30)} \leq 3.1$$

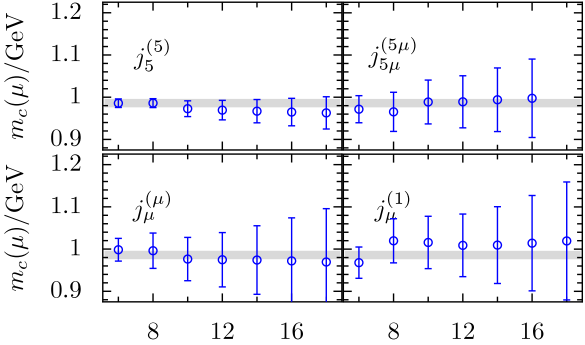
n=2: old $m_b(10\text{ GeV}) = 3.609(25)$ GeV; estimated $-6.0 \leq \overline{C}_2^{(30)} \leq 7.0$;
new: $m_b(10\text{ GeV}) = 3.607(19)$ GeV; $\overline{C}_2^{(30)} = -2.6438$ A. Maier, P. Maierhöfer, P. Marquard

# Method II

## Charm mass

I. Allison, E. Dalgic, C.T.H. Davies, E. Follana, R.R. Horgan, K. Hornbostel, G.P. Lepage, C. McNeile, J. Shigemitsu, H. Trottier, R.M. Woloshyn(HPQCD), K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, C.S.

- Lattice simulation of correlator moments $\mathcal{R}_{2k+2}$ for different currents



<u>Result:</u> $m_c(3\text{ GeV}) = 0.986(10)$ GeV lattice + pQCD
( compared to $m_c(3\text{ GeV}) = 0.986(13)$ GeV $e^+e^-$ + pQCD)

# Method I ↔ Method II

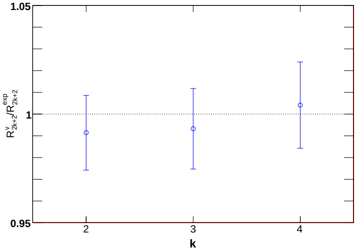
I. Allison, E. Dalgic, C.T.H. Davies, E. Follana, R.R. Horgan, K. Hornbostel, G.P. Lepage, C. McNeile,
J. Shigemitsu, H. Trottier, R.M. Woloshyn(HPQCD), K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, C.S.

- Reduced moments for VV correlator

$$\mathcal{R}^{(v)}_{2k+2} = \frac{2m_c(\mu)}{m_\Psi}\,\frac{\overline{C}^{\,v}_k}{\overline{C}^{\,v,(0)}_k}\,\frac{\overline{C}^{\,v,(0)}_{k-1}}{\overline{C}^{\,v}_{k-1}}$$

can be used for comparison:

$$\mathcal{R}^{v}_{2k+2} \leftrightarrow \mathcal{R}^{exp}_{2k+2}$$



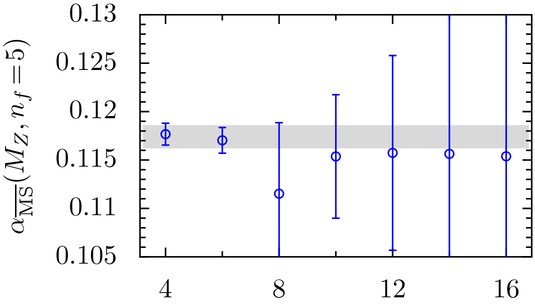
- Within combined errors experimental and simulation results agree within $\sim 2\%$

# Method II

## Strong coupling constant

I. Allison, E. Dalgic, C.T.H. Davies, E. Follana, R.R. Horgan, K. Hornbostel, G.P. Lepage, C. McNeile, J. Shigemitsu, H. Trottier, R.M. Woloshyn(HPQCD), K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, C.S.

- Lowest moment & ratios of moments $\rightsquigarrow$ weak dependence on quark mass $\rightsquigarrow$ extract $\alpha_s$
- First step extract $\alpha_s(3\text{GeV}, n_f = 4)$, than run to $\alpha_s(M_z, n_f = 5)$



**Result:**

$\bar{\alpha}_s(M_z) = 0.1174(12)$

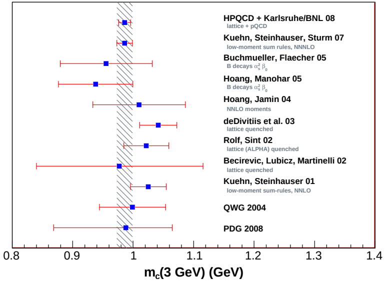
lattice + pQCD

(compared to

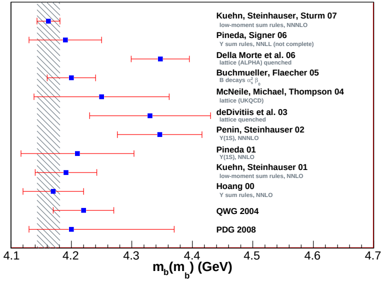
$\bar{\alpha}_s(M_z) = 0.1176(20)$ PDG)

# Comparison with other methods

charm-quarks



bottom-quarks



# Summary & Conclusion

- Heavy quark current correlators can be used to perform a precise quark mass determination in combination with experimentally measured $R$-ratio and with lattice simulations
- Extraction of charm- and bottom-quark masses from $R$-ratio including NNNLO results in pQCD
- Charm-quark mass and strong coupling from lattice simulations including NNNLO results in pQCD
- Quark masses & strong coupling:
  - Charm-mass: $m_c(3\text{ GeV}) = 0.986(13)$ GeV $e^+e^-$ + pQCD
    $m_c(3\text{ GeV}) = 0.986(10)$ GeV lattice + pQCD
  - Bottom-mass:
    $m_b(10\text{ GeV}) = 3.607(19)$ GeV $e^+e^-$ + pQCD + $\overline{C}_2^{(30)}$
  - strong coupling: $\bar{\alpha}_s(M_z) = 0.1174(12)$ lattice + pQCD