France, Japan and the United States. Several of these groups were successful in instituting national scientific projects with the help of SEDI. Also, SEDI endorsed several new scientific projects, including ISOP, INTERMAGNET, and the Canadian GGP. All these activities were a great boon for the deep-earth geoscientific community and helped these studies maintain their visibility and viability. This success was due in large part to Ned's steady hand at the helm during SEDI's crucial formative years.

Ned received a number of honors for his scientific research and service, including NASA's Group Achievement Award in 1983, a scholarship from the Cecil H. and Ida M. Green Foundation for Earth Sciences, and election as Fellow of the American Geophysical Union shortly before his untimely death. The loss of his scientific talents, leadership ability, and companionship is still felt strongly by his many colleagues.

David Loper

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Benton E.R. 1979b. Magnetic contour maps at the core-mantle boundary. *Journal of Geomagnetism and Geoelectricity*, **31**: 615–626.

Cross-references

IAGA, International Association of Geomagnetism and Aeronomy
SEDI, Study of the Earth's Deep Interior

BINGHAM STATISTICS

When describing the dispersion of paleomagnetic directions expected to have antipodal symmetry, it is a standard practice within paleomagnetism to employ Bingham (1964, 1974) statistics. The Bingham distribution that forms the basis of the theory is derived from the intersection of a zero-mean, trivariate Gaussian distribution with the unit sphere. For full-vector Cartesian data $\mathbf{x} = (x_1, x_2, x_3)$ the Gaussian density function is

$$p_g(\mathbf{x}|\mathbf{C}) = \frac{1}{(2\pi)^{3/2}|\mathbf{C}|^{1/2}} \exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}\right], \quad \text{(Eq. 1)}$$

where $\mathbf{C}$ is a covariance matrix. But for directional data the Cartesian vectors $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ are of unit length, and the corresponding density function is Bingham's distribution

$$p_b\left(\hat{\mathbf{x}}|\mathbf{EKE}^T\right) = \frac{1}{F(\kappa_1, \kappa_2, \kappa_3)} \exp\left[\hat{\mathbf{x}}^T\mathbf{EKE}^T\hat{\mathbf{x}}\right]. \quad \text{(Eq. 2)}$$

The matrix $\mathbf{E}$ is defined by the eigenvectors $\mathbf{e}^m$ of the covariance matrix $\mathbf{C}$ (see *Principal component analysis for paleomagnetism*). The diagonal concentration matrix

$$\mathbf{K} = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}, \quad \text{(Eq. 3)}$$

is formed from the Bingham concentration parameters κ_m. Normalization of (Eq. 2) is obtained through the confluent hypergeometric function, represented here as a triple sum (see Abramowitz and Stegan, 1965):

$$F(\kappa_1, \kappa_2, \kappa_3) = \sum_{i,j,k=0}^{\infty} \frac{\Gamma\left(i+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(i+j+k+\frac{3}{2}\right)} \frac{\kappa_1^i\kappa_2^j\kappa_3^k}{i!j!k!}, \quad \text{(Eq. 4)}$$

Let us now examine some of the properties of the Bingham distribution. The density function (Eq. 2) is clearly antipodally symmetric:

$$p_b(\hat{\mathbf{x}}) = p_b(-\hat{\mathbf{x}}). \quad \text{(Eq. 5)}$$

The distribution is also invariant for any change in the sum of the concentration parameters,

$$\sum_m \kappa_m \to \sum_m \kappa_m + \kappa', \quad \text{(Eq. 6)}$$

where κ' is an arbitrary real number. Therefore, for specificity, we set the largest parameter equal to zero, and we choose an order for the other two, so that

$$\kappa_1 \le \kappa_2 \le \kappa_3 = 0. \quad \text{(Eq. 7)}$$

Thus, the Bingham distribution is a two-parameter distribution. For all possible values of κ_1 and κ_2, the density (Eq. 2) describes a wide range of distributions on the sphere (see Figure B10). In spherical coordinates of inclination I and declination D the Bingham density function reduces to

$$p_b(\hat{\mathbf{x}}|\kappa_1, \kappa_2, \mathbf{e}^1, \mathbf{e}^2) = \frac{1}{F(\kappa_1, \kappa_2)} \exp\left[\kappa_1(\hat{\mathbf{x}}\cdot\mathbf{e}^1)^2 + \kappa_2(\hat{\mathbf{x}}\cdot\mathbf{e}^2)^2\right] \cos I, \quad \text{(Eq. 8)}$$

where the unit data vectors are given by

$$\hat{x}_1 = \cos I \cos D, \quad \hat{x}_2 = \cos I \sin D, \quad \hat{x}_3 = \sin I, \quad \text{(Eq. 9)}$$

where the unit eigenvectors are given by

$$\hat{e}_1^m = \cos I^m \cos D^m, \quad \hat{e}_2^m = \cos I^m \sin D^m, \quad \hat{e}_3^m = \sin I^m, \quad \text{(Eq. 10)}$$

and where the normalization function is

$$F(\kappa_1, \kappa_2) = \sqrt{\pi} \sum_{i,j=0}^{\infty} \frac{\Gamma(i+\frac{1}{2})\Gamma(j+\frac{1}{2})}{\Gamma(i+j+\frac{3}{2})} \frac{\kappa_1^i\kappa_2^i}{i!j!}. \quad \text{(Eq. 11)}$$

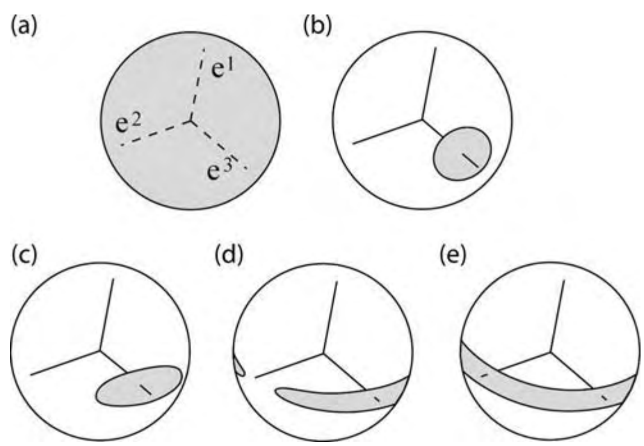

Figure B10 Bingham density function, with representative contours for a (a) uniform density $\kappa_1 = \kappa_2 = 0$, (b) symmetric bipolar density $\kappa_1 < \kappa_2 \ll 0$, (c) asymmetric bipolar density $\kappa_1 < \kappa_2 \ll 0$, (d) asymmetric girdle $\kappa_1 \ll \kappa_2 < 0$, and (e) symmetric girdle density $\kappa_1 \ll \kappa_2 = 0$. (After Collins and Weiss, 1990).

Three important special cases are worthy of attention, which are most clearly illustrated in the coordinate system determined by the eigenvectors $\mathbf{e}^m$. First, for $\kappa_1 = \kappa_2$ we have a axially symmetric bipolar distribution, with density function

$$p_3(\theta|\kappa) = \frac{1}{F(-\kappa)} \exp\left[-\kappa \cos^2\theta\right] \sin\theta, \quad \text{for} \quad \kappa \le 0, \qquad \text{(Eq. 12)}$$

(see (Eq. 19) of *Statistical methods for paleomagnetic vector analysis*.) The angle θ is defined by the directional datum $\hat{x}$ and the principal axis determined by the eigenvector with the largest eigenvalue $\mathbf{e}^3$

$$\cos\theta = \hat{\mathbf{x}} \cdot \mathbf{e}^3, \qquad \text{(Eq. 13)}$$

normalization is given by

$$F(\kappa) = \sum_{i=0}^{\infty} \frac{\Gamma(i+\frac{1}{2})}{\Gamma(i+\frac{3}{2})} \frac{\kappa^i}{i!}. \qquad \text{(Eq. 14)}$$

Second, for $\kappa_1 < \kappa_2 = 0$ the distribution is a axially symmetric girdle, with density function (Dimroth 1962; Watson 1965)

$$p_1(\theta|\kappa) = \frac{1}{F(\kappa)} \exp\left[\kappa \cos^2\theta\right] \sin\theta, \quad \text{for} \quad \kappa \le 0 \qquad \text{(Eq. 15)}$$

The angle θ is defined by the directional datum $\hat{x}$ and the principal axis determined by the eigenvector with the smallest eigenvalue $\mathbf{e}^1$

$$\cos\theta = \hat{\mathbf{x}} \cdot \mathbf{e}^1. \qquad \text{(Eq. 16)}$$

These two special cases of the more general Bingham distribution are useful for describing the dispersion (Eq. 12) of bipolar data about some mean pole, and the dispersion (Eq. 15) of bipolar data about some mean plane. And, finally, the third symmetric distribution is uniform, obtained for $\kappa_1 = \kappa_2 = 0$. In spherical coordinates this is just

$$p_0(\theta|\kappa) = \sin\theta, \quad \text{for} \quad \kappa = 0. \qquad \text{(Eq. 17)}$$

The uniform distribution has no preferred direction and so θ can be measured from an arbitrary axis. Of course, if we now allow κ to be positive or negative (or zero), then the full range of axially symmetric density functions is available (Mardia, 1972, p. 234):

$$p_{1,3}(\theta|\kappa) = \frac{1}{F(\kappa)} \exp\left[\kappa \cos^2\theta\right] \sin\theta, \quad \text{for} \quad -\infty \le \kappa \le \infty. \qquad \text{(Eq. 18)}$$

The off-axis angle θ is then defined by the axis of symmetry of the distribution. Symmetric density functions, ranging from bipolar to girdle, obtained for a variety of values of κ, are illustrated in Figure B11.

Maximum-likelihood estimation

In fitting a Bingham distribution to paleomagnetic directional data a convenient method is that of maximum-likelihood; for a general review see Stuart *et al*. (1999). With this formalism, the likelihood function is constructed from the joint probability-density function for the existing data set. Using the general form of the Bingham density function (Eq. 8) the likelihood for N data is just

$$L(\kappa_1, \kappa_2, \mathbf{e}^1, \mathbf{e}^2) = \prod_{j=1}^{N} pb(I_j, D_j|\kappa_1, \kappa_2, \mathbf{e}^1, \mathbf{e}^2). \qquad \text{(Eq. 19)}$$

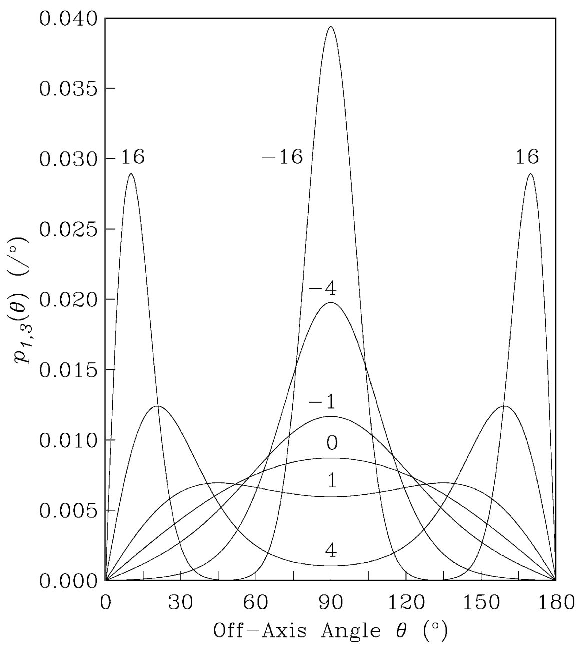

Figure B11 Examples of the axially symmetric Bingham probability density function $p_{1,3}(\theta)$, (Eq. 18) for a variety of κ concentration parameters: $0, \pm1, \pm4, \pm16$. Note that as $|\kappa|$ is increased the dispersion decreases.

Maximizing L is accomplished numerically (Press *et al*., 1992), an exercise yielding a pair of eigenvectors $\mathbf{e}^1$ and $\mathbf{e}^2$ and their corresponding concentration parameters κ_1 and κ_2. The third eigenvector $\mathbf{e}^3$ is determined by orthogonality and its concentration parameter by convention (Eq. 7) is zero. Some investigators (e.g., Onstott, 1980; Tanaka, 1999) prefer a two-step estimation method, where the eigenvectors are determined by principal component analysis, but these vectors are identical to those found by maximizing (Eq. 19). In any case, obtaining the eigenvalues of the data set through principal component analysis is still required for establishing confidence limits.

Confidence limits

It is unfortunate that the relationship between the concentration parameters κ_m, determined (usually) by maximum likelihood, and the eigenvalues of the covariance matrix λ_m, determined (usually) by principal component analysis, is very complicated. This fact makes the establishment of confidence limits on the eigenvectors difficult. However, Bingham (1974, p. 1220) has discovered an approximate formula for the confidence limit valid under certain circumstances. The confidence ellipse, within which a specified percentage (%) of estimated eigenvectors $\mathbf{e}^m$ can be expected to be realized from a statistically identical data set, is given by the elliptical axes

$$\alpha_{\%}^{mn} = \left[\frac{\chi^2_{\%}}{2N\boldsymbol{\Delta}_{mn}}\right]^{1/2}, \qquad \text{(Eq. 20)}$$

for

$$\Delta_{mn} = (\kappa_m - \kappa_n)(\lambda_m - \lambda_n) \ll 1, \quad \text{and} \quad N \to \infty, \qquad \text{(Eq. 21)}$$

Table B1 Principal component and maximum likelihood analysis of the Hawaiian paleomagnetic directional data recording a mixture of normal and reverse polarities over the past 5 Ma

Eigenvalue	Eigen direction		Bingham κ_m	Confidence axes		
	I(o)	D(o)		α_{95}^{m1}	α_{95}^{m2}	α_{95}^{m3}
0.0304	−14.6	87.6	−9.8749		>1	0.0765
0.0601	−58.9	−48.4	−9.3490	>1		0.0801
0.9088	31.1	−1.5	0.0000	0.0765	0.0801	

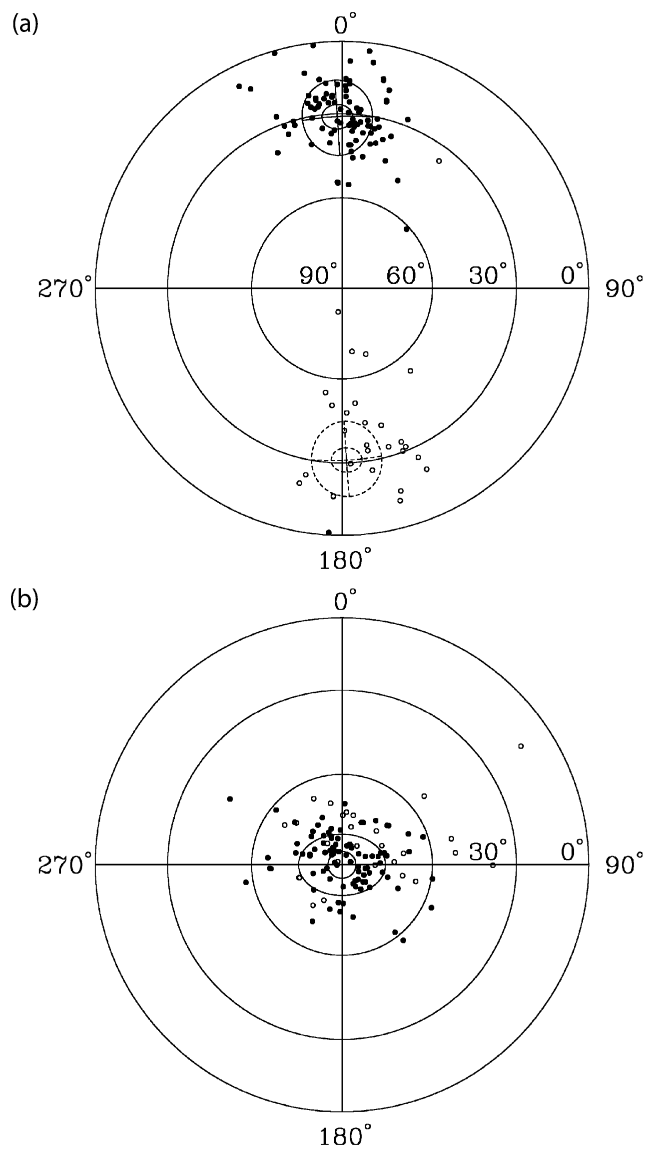

Figure B12 Equal-area projection of Hawaiian directional data, defined in (a) geographic coordinates and (b) eigen coordinates. Also shown are the projections of the variance minor ellipse, defined by λ_1 and λ_2, and, inside of that, the α_{95}^{3n} confidence ellipse. As is conventional, the azimuthal coordinate is declination (clockwise positive, 0° to 360°), and the radial coordinate is inclination (from 90° in the center to 0° on the circular edge).

and where $\chi^2_{\%}$ is the usual chi-squared value for two degrees of freedom. Alternative methods have been proposed for establishing confidence limits, most notably the bootstrap method popularized within paleomagnetism by Tauxe (1998).

Using the mixed-polarities directional from Hawaii covering the past 5 Ma (see *Principal component analysis*), in Table B1 we summarize the statistical parameters, and in Figure B12 we show the 95% confidence limit α_{95}^{3n} about the eigenvector defining the mean bipolar direction ($\mathbf{e}^3$). Note that the confidence limit is much smaller than the variance of the data.

Jeffrey J. Love

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Cross-references

Fisher Statistics
Principal Component Analysis in Paleomagnetism
Statistical Methods for Paleo Vector Analysis