Measuring Goodness of Fit for the Double Bounded Logit Model: Comment

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Researchers estimating a regression model are often concerned about the joint significance of the explanatory variables and goodness of fit among other factors. Goodness of fit measures and statistics for testing the joint significance of explanatory variables are closely related. For example, in ordinary least squares regression analysis both the goodness of fit measure, R^2, and the F statistic, which is used to test the joint significance of all coefficients except the constant, are based on the sum of squared residuals. Similarly, in maximum likelihood estimation both McFadden's R^2 and the likelihood ratio test are based on the logarithm of the likelihood function.

In a recent issue of the *Journal*, Kanninen and Khawaja (1995) have shown that goodness of fit measures, such as McFadden's R^2, cannot be calculated for the double bounded logit model. This arises because the restricted log of the likelihood function is undefined. Not surprisingly, this result implies that the likelihood ratio test cannot be calculated. Kanninen and Khawaja restrict their exposition to the double bounded model. However, the single bounded (SB), double bounded (DB), and spike (SP) (Kristrom 1997) models are special cases of the more general multiple bounded (MB) model (Welsh and Bishop 1993)[1]. Consequently, the computational difficulties noted in Kanninen and Khawaja are, in general, common to all of these models. In this comment, we will build upon Kanninen and Khawaja's work by demonstrating the use of the Wald Statistic for testing joint significance in the family of bounded logit models.

In the MB model, the bids (BID_i) divide the real number line into intervals. An individual's response pattern reveals the interval on the real number line which contains their true willingness to pay (WTP). The respondent's true WTP must be at least equal to the highest value they accepted (which we denote as the lower bid, BID_L) but less than the next higher value (which we term the upper bid, BID_U). From the researcher's point of view, WTP is a random variable. In the multiple bounded logistic model, the probability that a respondent will answer "yes" to any given BID, G(X), is defined as shown in equation (1), where **X** is a vector of explanatory variables including the BID,

$$Prob(WTP > BID) = G(X) = \frac{1}{1 + \exp(\beta' X)}$$

and **β** is a vector of coefficients.

$$\mathrm{Ln}(L) = \sum_{i=1}^{n} \mathrm{Ln}[\, Z_{iU}\, Z_{iL} \,]$$

The log-likelihood function for the MB model is given by (2) where Z=[1-G(X)].

In (2), Z_{iU} and Z_{iL} represent the logistic probabilities for any individual (I) that correspond to a vector of explanatory variables containing BID_U and BID_L respectively.

Because $\ln(L_r)$ is undefined when $L_r=0$, measures of goodness of fit, such as McFadden's R^2 and others, cannot be calculated for the DB, SP, or the MB models. Kanninen and Khawaja describe an alternative measure of goodness of fit. They count the correctly classified cases with respect to the first question alone, then use only the observations that were correctly classified according to the first question to count the correctly classified questions for the second question. The procedure they propose explicitly accounts for the sequential nature of the DB questioning format. The Kanninen and Khawaja procedure appears applicable to the SP model and other cases where the respondent is questioned sequentially. However, it does not appear useful in the MB context when responses are obtained contemporaneously (e.g. Welsh et al. 1995) rather than sequentially (e.g. Loomis and Gonzalez-Caban 1996). While the procedure suggested by Kanninen and Khawaja provides a measure of goodness of fit, it does not provide a means for testing the related hypothesis $\beta_1=\beta_2=\beta_3=...=\beta_n=0$.

The likelihood ratio test, which is analogous to the F-test in linear regression models, is widely used for testing the overall significance of relationships estimated using maximum likelihood methods. The likelihood ratio statistic is given by $\lambda = -2[\ln(L_r)-\ln(L_u)]$. Because $\ln(L_r)$ is undefined except for the SB case, this statistic, like McFadden's R^2, cannot be computed for other bounded models.

While posing an inconvenience, this situation is by no means intractable. Recall that the likelihood ratio statistic (LR) is one of three asymptotically equivalent tests. The other two are the Lagrange multiplier (LM) statistic, and the Wald statistic (W). The small sample properties of all three of these statistics are unknown but their large sample properties are asymptotically χ^2. Although similar in nature, the computational requirements of these three statistics differ. The LR statistic requires computation of both the restricted and unrestricted likelihood function while the LM statistic requires computation of only the restricted likelihood function. In contrast to the LM and LR test statistics, the Wald test statistic is based solely on the value of the unrestricted likelihood function. This feature of the Wald statistic can be exploited in the context of the MB model.

The Wald test is widely used for testing hypotheses about nonlinear restrictions and can also be used for testing hypotheses about linear restrictions. In the case of linear

$$W = [R\beta - r]'[R(V)R'\,]^{-1}\,[R\beta - r] \_ \chi^2_Q$$

restrictions, the Wald test statistic (W) is given by equation (3)

In equation (3), **R** is a matrix of restrictions. The matrix **R** has Q rows and k columns where Q is equal to the number of restrictions and k is equal to the total number of estimated parameters. In this equation, **β** is a k x 1 vector of estimated coefficients, **r** is a Q x 1 vector of constants, and **V** is the estimated variance-covariance matrix of **β**. The

Wald test statistic is distributed asymptotically χ^2 with the degrees of freedom equal to the number of restrictions, Q.

The premise of the Wald test is that, if the set of hypothesized restrictions is valid at least approximately, then W should be near zero. If the set of hypothesized restrictions is erroneous, W should be farther from zero than would be explained by sampling variability alone. The Wald test can be used to determine the significance of the entire family of bounded logit models: SB, SP, DB, and MB. To test the hypothesis that the slope terms $\beta_1=\beta_2=\beta_3=...=\beta_n=0$ are simultaneously equal to zero, the matrix of linear restrictions, **R**, and the vector of constants, **r**, are constructed as shown in (4).

$$R = \begin{bmatrix} 0\,1\,0\,0 \ldots 0 \\ 0\,0\,1\,0 \ldots 0 \\ 0\,0\,0\,1 \ldots 0 \\ \vdots \\ 0\,0\,0\,0 \ldots 1 \end{bmatrix} \qquad r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The Wald test statistic for the hypothesis that all of the slope terms are equal to zero can be calculated by many popular regression packages, either with a few simple commands or by resorting to the matrix manipulation capabilities of the package. Alternatively, the estimated coefficient vector, **β**, and the estimated variance-covariance matrix, **V**, can be retrieved, **R** and **r** can be constructed as shown in (4), and the necessary matrix manipulations can be undertaken using a spreadsheet program.

FOOTNOTES

[1] The MB model was first described by Welsh and Bishop (1993) and has been applied in several recent studies (Gonzales-Caban and Loomis 1997, Loomis and Gonzales-Caban 1997, Loomis and Gonzales-Caban 1996, Loomis and Ekstrand 1997).

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