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OPTIMIZATION OF SOLAR RADIATION INPUT IN FOREST CANOPY AS A TOOL FOR PLANTING/CUTTING OF TREES

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ABSTRACT

The problem of optimal planting and cutting of industrial wood is considered. The criterion for optimization is to maximize the capture of solar energy by a plant stand. The optimization algorithm is based on variation of solar radiation in tree crowns caused by variations in tree density (planting and cutting of trees) and size (tree growth). An equation for optimal value of tree density is derived. Numerical results are presented to illustrate the influence of canopy parameters on the input coefficients of the derived equation.

1. INTRODUCTION

In a plant canopy, the above ground biomass determines the productivity of industrial wood. Solar radiant energy absorbed by trees is converted to woody biomass through photosynthesis. In general, the supply of radiant energy sets a limit to potential production, but if light is not a limiting factor then other variables may be determinant for the actual production.[1] From this viewpoint, a plant stand may be appropriately conceptualized as a solar energy trap – the higher the amount of solar energy it captures, the greater is its photosynthetic capacity. The capacity of a forest canopy for capture depends upon planting pattern of trees as well as cutting of trees (i.e. variations in tree density) in a stand.[2] Under condition of optimal nutrition, soil water availability and temperature, this property gives us a possibility to influence the production of industrial wood. A question then arises as to how the capture of solar radiant energy by a plant stand can be maximized. This is the topic of our research.

The amount of intercepted photosynthetically active radiation (PAR) by a stand as well as canopy photosynthesis can be evaluated once solar radiation field in the tree crowns is known.[2,3,4] Thus, the problem of plant optimization may be reduced to examining the dependence between the distribution of solar radiation inside tree crowns and various schemes of tree planting/cutting. Therefore, we begin with a description of a three-dimensional model for the canopy radiation regime[2] (Section 2). In Section 3, the optimization problem is formulated in terms of optimal control theory. In this section we closely follow the monograph by Pontryagin et al.[5]

In a given phytomedium, the radiation regime depends on the incoming radiation field. Therefore, the description of variations in solar radiation inside the tree crowns can be reduced to examining the changes in the incident radiation at the crown boundary caused by the following three factors – meteorological changes (clouds, sun position, etc.), variations in tree density (cutting and planting of trees) and size (growth of trees). The meteorological changes are assumed known and serve as input for our model. Therefore, we begin with variations in radiation input at the crown boundary caused by variations in tree density and size (Section 4). To describe tree illumination we use the concept of critical arrangement of trees. It is a qualitative characteristic of the forest and establishes the existence of several regimes of incident radiation at the tree crown level. The next step is to introduce a quantitative characteristic of the process to be maximized. For examples, Myneni[2] uses canopy photosynthesis as the value to be maximized and so the objective of the optimization is to maximize canopy photosynthesis. The amount of intercepted radiation by plant stand[4,6–8] is another example of the parameter for maximization.

In spite of the diversity of possible parameters for maximization, they all have certain properties (see Definition in Section 4) in common: sensitivity to changes in tree illumination (or, the critical arrangements of trees) and the amount of trees in a plant stand. It will be shown (Section 5) that only these properties determine the algorithm for plant optimization. Therefore, we shall not discuss the physical sense of the process but focus on examining the above mentioned properties. To achieve this goal, we shall use the concept of the objective functional (See Definition in Section 4).

The dependence of the objective functional on the tree density is investigated in Section 5. It will be shown (Theorem 3) that this dependence consists of a linear and a nonlinear part (see Fig. 3 for illustration). This structure is allowed for different planting patterns, boundary conditions (Fig. 4), soil and leaf optical properties and canopy architecture (See Figs. 5-6), i.e., it may be interpreted as the invariant property of a forest system. The length of the linear part and the angle of its inclination may serve as a quantitative characteristic of such a system.

In Section 6, we present a solution to the optimization problem formulated in Section 3 for the case of a square planted stand. Several numerical results are presented in Section 7.

2. THE THREE–DIMENSIONAL RADIATIVE TRANSFER MODEL

We consider a forest canopy consisting of N identical "mean" trees. The domain, V, where the trees are located is a parallelepiped of dimensions X_S, Y_S and Z_S. The height, Z_S, of the plant stand coincides with the tree height. We idealize this canopy as a turbid medium[9] and follow the formulation of Myneni.[2,10] In this description nonlaminar foliage is disregarded and only flat leaves are considered. For such a leaf canopy, the steady-state monochromatic radiance distribution function $I(\vec{r}, \underline{\Omega})$ in the absence of polarization, frequency shifting interactions, and radiation sources within the canopy, is given by the radiative transfer equation

$$\underline{\Omega} \bullet \vec{\nabla}\, I(\vec{r}, \underline{\Omega}) \;+\; \sigma(\vec{r}, \underline{\Omega})\, I(\vec{r}, \underline{\Omega}) \;=\; \int_{4\pi} d\underline{\Omega}'\; \sigma_s(\vec{r}, \underline{\Omega}' \to \underline{\Omega})\, I(\vec{r}, \underline{\Omega}'). \tag{1}$$

The position vector $\vec{r} = (x, y, z)$ is expressed in Cartesian coordinates with its origin, O, at the top of the forest canopy and the OZ axis directed down towards the ground. The unit direction vector $\underline{\Omega} \sim (\mu, \phi)$ has an azimuthal angle ϕ measured in the (XY) plane from the positive X-axis in a counter-clockwise fashion and a polar angle $\theta = \cos^{-1}(\mu)$ with respect to the OZ axis.

The total interaction cross section, $\sigma(\vec{r}, \underline{\Omega})$, is defined such that the probability that a photon while traveling a distance ds hits a leaf is[11] $\sigma(\vec{r}, \underline{\Omega})\, ds$

$$\sigma(\vec{r}, \underline{\Omega}) \;=\; u_L(\vec{r})\; G(\vec{r}, \underline{\Omega})$$

where $u_L(\vec{r})$ is the leaf area density and $G(\vec{r}, \underline{\Omega})$ is the projection of unit leaf area at $\vec{r}$ onto the direction of photon travel $\underline{\Omega}$, namely[9],

$$G(\vec{r}, \underline{\Omega}) \;=\; \frac{1}{2\pi} \int_{2\pi} d\underline{\Omega}_L\; g_L(\vec{r}, \underline{\Omega}_L)\; |\underline{\Omega} \bullet \underline{\Omega}_L|$$

where $2\pi^{-1}\; g_L(\vec{r}, \underline{\Omega}_L)$ is the probability density of a leaf at $\vec{r}$ with a normal directed outward from its upper surface into a unit solid angle about $\underline{\Omega}_L \sim (\mu_L, \phi_L)$. The functions u_L and g_L characterize the architecture of the forest canopy and can be parameterized with simple models or from empirical data. An example of the leaf area density, $u_L(\vec{r})$, modelled by the quadratic function[9] (also see Section 7) is presented in Fig. 1*a*.

The differential scattering cross section $\sigma_s(\vec{r}, \underline{\Omega}' \to \underline{\Omega})$ may be expressed in terms of a leaf scattering phase function $f(\underline{\Omega}_L, \underline{\Omega}' \to \underline{\Omega})$. For a leaf with outward normal $\underline{\Omega}_L$, this phase function is the fraction of the intercepted energy, from photons initially traveling in direction $\underline{\Omega}'$, that is scattered into a unit solid angle about direction[11] $\underline{\Omega}$. The volumetric rate at which photons traveling in $\underline{\Omega}'$ are scattered into a unit solid angle about $\underline{\Omega}$, by leaves at $\vec{r}$ of all orientations $\underline{\Omega}_L$ equal the differential scattering cross-section; thus, σ_s is

$$\begin{aligned}
\sigma_s(\vec{r}, \underline{\Omega}' \to \underline{\Omega}) \;&=\; u_L(\vec{r})\; \frac{1}{2\pi} \int_{2\pi} d\underline{\Omega}_L\; g_L(\vec{r}, \underline{\Omega}_L)\; |\underline{\Omega}' \bullet \underline{\Omega}_L|\; f(\underline{\Omega}_L, \underline{\Omega}' \to \underline{\Omega}), \\
&=\; u_L(\vec{r})\; \frac{1}{\pi}\; \Gamma(\vec{r}, \underline{\Omega}' \to \underline{\Omega})
\end{aligned}$$

where $\Gamma(\vec{r}, \underline{\Omega}' \to \underline{\Omega})$ is the area scattering phase function originally introduced by Ross[9].

A photon can either be specularly reflected at the surface of the leaf or can undergo reflection and refraction inside the leaf. Specular reflections from the leaves originate at the interface between the air and the cuticular wax layer[12], and its magnitude can be computed from the incidence angle, the index of refraction and characteristics of the surface wax layer. On the other hand, a photon once inside the leaf, will undergo multiple reflections and refractions at the numerous cell wall-air interfaces, and can emerge in any direction with a probability given by Lambert's cosine law. The above picture of photon-leaf interaction is deduced from careful measurements[13,14] and theory[9].

In view of the above discussion, the leaf scattering phase function may be written as

$$f(\underline{\Omega}_L, \underline{\Omega}' \rightarrow \underline{\Omega}) = f_D(\underline{\Omega}_L, \underline{\Omega}' \rightarrow \underline{\Omega}) + f_{SP}(\underline{\Omega}_L, \underline{\Omega}' \rightarrow \underline{\Omega})$$

where f_D and f_{SP} are the leaf phase functions for diffuse scattering in the leaf and specular reflection at the leaf surface, respectively. A simple, but realistic model for f_D was proposed by Ross and Nilson[15]. In this model, a fraction r_{LD} of the intercepted energy is reradiated in a cosine distribution about the leaf normal. Similarly, a fraction t_{LD} is transmitted in a cosine distribution on the opposite side of the leaf. This bi-Lambertian model can be described mathematically as

$$f_D(\underline{\Omega}_L, \underline{\Omega}' \rightarrow \underline{\Omega}) = \begin{cases} \pi^{-1}\, r_{LD}\ |\underline{\Omega} \bullet \underline{\Omega}_L|, & \text{if } (\Omega \bullet \underline{\Omega}_L)(\underline{\Omega}' \bullet \underline{\Omega}_L) < 0. \\ \pi^{-1}\, t_{LD}\ |\underline{\Omega} \bullet \underline{\Omega}_L|, & \text{if } (\underline{\Omega} \bullet \underline{\Omega}_L)(\underline{\Omega}' \bullet \underline{\Omega}_L) > 0. \end{cases}$$

The leaf phase function for specular reflection is determined by the wax layer on the leaf surface and it depends on the following three factors[12]: the angle, α', between the incident ray, $\underline{\Omega}'$, and leaf normal, $\underline{\Omega}_L$, the wax reflection index, ν, and the smoothness of the leaf surface, κ. Hence, the specular component can be written as[16]

$$f_{SP}(\underline{\Omega}_L; \underline{\Omega}' \rightarrow \underline{\Omega}) = K(\kappa, \alpha')\, F(\nu, \alpha')\, \delta(\underline{\Omega} - \underline{\Omega}^*) \,.$$

Here F is the Fresnel parameter, indicating the amount of specularly reflected energy averaged over the polarization states

$$F(\nu,\alpha') = 0.5\left[\frac{\sin^2(\alpha'-\Theta)}{\sin^2(\alpha'+\Theta)} + \frac{\tan^2(\alpha'-\Theta)}{\tan^2(\alpha'+\Theta)}\right]$$

where the reflection angle can be found from Snell's law as $\Theta = \sin^{-1}(\nu^{-1}\ \sin\alpha)$. The function K defines the correction factor for specular reflection $(0 \le K \le 1)$ and the argument $\kappa \ge 0$ characterizes the smoothness of the leaf surface. The vector $\underline{\Omega}^* = \underline{\Omega}^*(\underline{\Omega}', \underline{\Omega}_L)$ defines the direction of specular reflection.

The leaf canopy is illuminated from above by both a direct monodirectional solar component [in direction $\underline{\Omega}_o \sim (\mu_o, \phi_o)$, $\mu_o > 0$; of intensity I_o] as well as by diffuse radiation [in directions $\underline{\Omega} \sim (\mu, \phi)$, $\mu > 0$; of intensity I_d]. Let S be the part of the canopy boundary including the top and lateral surfaces. The edges of the canopy parallelepiped, V, must be excluded[17] from the surface S. The incident radiation field at the surface S is

$$I(\vec{r}_S, \underline{\Omega}) = I_d(\vec{r}_S, \underline{\Omega}) + I_o(\vec{r}_S)\delta(\underline{\Omega} - \underline{\Omega}_o), \quad \underline{\Omega} \bullet \underline{n}_S < 0 \tag{2}$$

where $\underline{n}_S$ is the unit vector of the external normal at the point $\vec{r}_S$ of the surface S; $I_0(\vec{r}_S) = I_0$ if there is no hindrance for direct solar radiation to reach the point $\vec{r}_S$ and $I_0(\vec{r}_S) = 0$ otherwise. To calculate the diffuse component of the incident radiation field, we use the radiative transfer model in broken clouds proposed by Vainikko and Avaste.[18,19]

At the bottom of the canopy, a fraction of the radiation is assumed to be reflected back into the canopy by the ground according to the distribution function γ_b, i.e.

$$I(\vec{r}_b, \underline{\Omega}) = \int_0^{2\pi} d\phi' \int_0^1 d\mu'\ \gamma_b(\underline{\Omega}, \underline{\Omega}')\ \mu'\ I(\vec{r}_b, \underline{\Omega}'), \quad \underline{\Omega} \bullet \underline{n}_b < 0\,, \tag{3}$$

where $\vec{r}_b$ is a point on the ground and $\underline{n}_b$ is the outward unit normal at this point.

The boundary conditions, soil and leaf optical properties and architecture of the forest canopy are assumed known inputs. Thus, the radiative transfer in the forest canopy is fully defined. This model of canopy radiation regime provided good agreement with radiation measurements in a poplar stand.[2]

We represent the solution of the problem (1)-(3) as the sum of two components, viz. $I(\vec{r}, \underline{\Omega}) = I_{dir}(\vec{r}, \underline{\Omega}) + I_{dif}(\vec{r}, \underline{\Omega})$, where I_{dir} and I_{dif} are, respectively, the intensities of direct and diffuse solar radiation. The intensity of direct solar radiation $I_{dir}(\vec{r},\underline{\Omega}) = Q_0(\vec{r})\delta(\underline{\Omega}-\underline{\Omega}_0)$ where $Q_0(\vec{r})$ is the probability density that a photon in the direct solar radiation will arrive at $\vec{r}$ along $\underline{\Omega}_0$ without experiencing a collision, i.e.,

$$Q_0(\vec{r}) = I_0(\vec{r}-l[\vec{r},\underline{\Omega}_0]\underline{\Omega}_0)\exp\left(-\int_0^{l[\vec{r},\underline{\Omega}_0]} \sigma(\vec{r}-s'\underline{\Omega}_0,\underline{\Omega}_0)\,ds'\right).$$

Here $l[\vec{r},\underline{\Omega}]$ denotes the distance between the point $\vec{r}$ and the forest boundary along the direction $-\underline{\Omega}$ (i.e. the point $\vec{r}-l[\vec{r},\underline{\Omega}]\underline{\Omega}$ belongs to the canopy boundary). The intensity of diffuse radiation satisfies the integro-differential equation

$$\begin{aligned}\underline{\Omega}\bullet\vec{\nabla}\, I_{dif}(\vec{r},\underline{\Omega}) + \sigma(\vec{r},\underline{\Omega})\, I_{dif}(\vec{r},\underline{\Omega}) &= \int_{4\pi} d\underline{\Omega}'\ \sigma_s(\vec{r},\underline{\Omega}'\to\underline{\Omega})\, I_{dif}(\vec{r},\underline{\Omega}') + \\ &\quad + \sigma_s(\vec{r},\underline{\Omega}_0\to\underline{\Omega})Q_0(\vec{r}).\end{aligned} \tag{4}$$

and boundary condition

$$\begin{cases} I_{dif}(\vec{r}_S,\underline{\Omega}) = I_d(\vec{r}_S,\underline{\Omega}), & \underline{\Omega}\bullet\underline{n}_S < 0, \\ I_{dif}(\vec{r}_b,\underline{\Omega}) = \displaystyle\int_0^{2\pi} d\phi' \int_0^1 d\mu'\ \gamma_b(\underline{\Omega},\underline{\Omega}')\mu' I_{dif}(\vec{r}_b,\underline{\Omega}') + & \\ \qquad + \gamma_b(\underline{\Omega},\underline{\Omega}_0)\mu_0 Q_0(\vec{r}_b), & \underline{\Omega}\bullet\underline{n}_b < 0. \end{cases} \tag{5}$$

The terms $\sigma_s Q_0$ and $\gamma_b\mu_0 Q_0$ on the right-hand side of Eq. (5) may be interpreted as the external radiation sources created by photons in the direct solar radiation arriving at $\vec{r}$ and $\vec{r}_b$ along $\underline{\Omega}_0$ without experiencing a collision and which is scattered in the direction $\underline{\Omega}$.

We introduce the Banach space, $M(V \times 4\pi)$, of measurable and almost everywhere bounded functions on the set $V\times 4\pi = \{(\vec{r},\underline{\Omega})\,|\,\vec{r}\in V,\ \underline{\Omega}\sim(\mu,\phi),\ -1 \le \mu \le 1,\ 0 \le \phi \le 2\pi\}$, in which the norm is defined by the equality $\|\ I\ \| = \text{ess}\,\sup_{V\times 4\pi}|I(\vec{r},\underline{\Omega})|$. The functions σ and σ_s are assumed to be positive measurable and almost everywhere bounded, respectively, on the sets $V \times 4\pi$ and $V \times 4\pi \times 4\pi$ and they satisfy the following inequality:

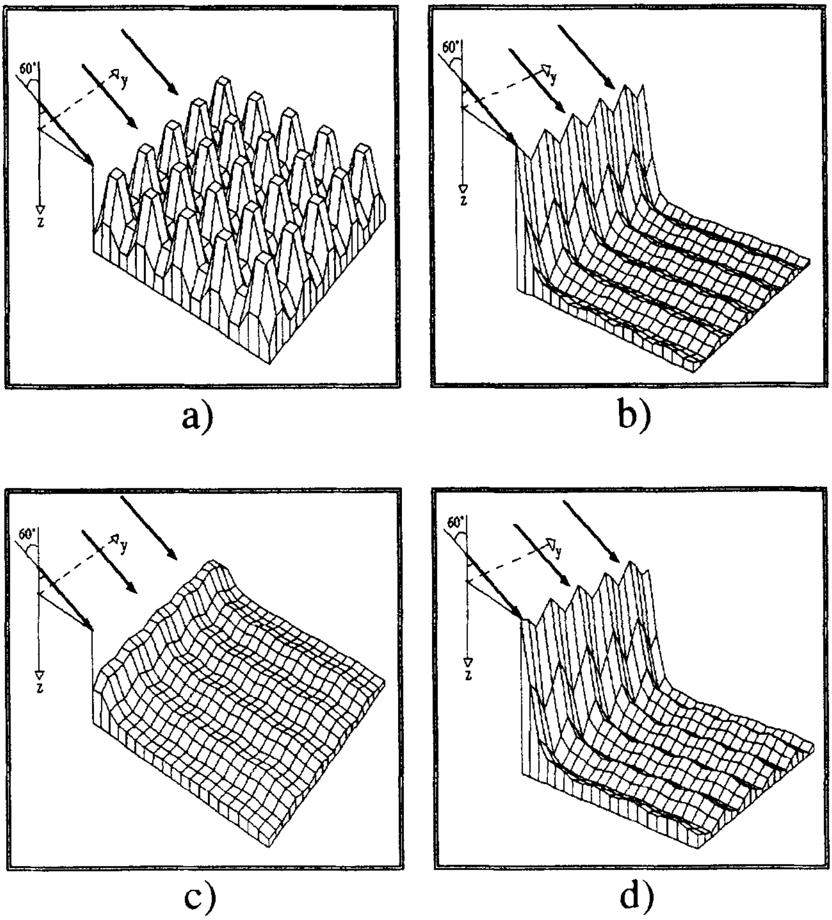

Fig.1 The three-dimensional distribution of leaf area density and radiance distribution function at $z = 1.6$ m. Here $\theta_0 = 60^o$; $\varphi_0 = 0$; $I_0(\vec{r}_S) = I_0$; $X_C = Y_C = 2m$; $Z_C = Z_S = 5m$; $\rho = 2500\frac{tr}{ha}$ and $L = 15$. Other parameters are described in Section 7. a): leaf area density, $u_L(\vec{r})$; b): $J_{dir}(\vec{r}) = \int_{\mu>0} I_{dir}(\vec{r},\Omega)|\mu|\,d\Omega$; c): $J_1(\vec{r}) = \int_{\mu>0} I_1(\vec{r},\Omega)|\mu|\,d\Omega$; d): $J_{one}(\vec{r}) = J_{dir}(\vec{r}), + J_1(\vec{r})$.

$$\int_{4\pi} \sigma_s(\vec{r},\underline{\Omega}'\to\underline{\Omega})\,d\underline{\Omega}' \;\le\; \sigma(\vec{r},\underline{\Omega}).$$

Following the technique proposed by Germogenova[17] it is possible to show that these properties guarantee the existence and uniqueness of the solution to the problem (4)-(5) in the Banach space $M(V\times4\pi)$ and this solution depends continuously on initial data. For numerical solution of the problem (4)-(5) we use the method of successive orders of scattering approximation. The rate of convergence can be evaluated as $\max \omega_L(\Omega_L, \Omega')$, where $\omega_L(\Omega_L, \Omega') = \int_{4\pi} f(\Omega_L, \Omega' \to \Omega)\, d\Omega$ is the leaf albedo.[16] In the photosynthetically active region of the solar spectrum, leaves usually absorb 90% of the intercepted irradiance. Thus, the solution can be approximated reasonably well by single scattering, $I_1(\vec{r}, \Omega)$. An example of the three-dimensional radiance distribution function in a plant canopy integrated over the lower hemisphere with weight $|\mu|$ is shown in Figs. 1b-1d.

3. STATEMENT OF THE OPTIMIZATION PROBLEM

In order to describe the various radiation regimes in a plant stand we introduce the set, $D \subseteq I_{dir} \oplus M(V \times 4\pi) = \{I_{dir} + v |\ v \in M(V \times 4\pi)\}$, of various solutions to the problem (1)-(3) corresponding to the various values of σ, σ_s, I_d, I_0 and Ω_0. The radiation regime of a plant stand changes with time due to changes in I_d, I_0 and Ω_0 (meteorological factors), and σ and σ_s (due to changes in tree density and size). The meteorological changes are assumed known and serve as input data for our optimization algorithm. By varying the tree density it is possibly to influence the course of changes in the plant stand. We introduced the control function, $u(t, x, y)$, whose value is 1 if there is a tree at the point (x, y) on the ground at time t, and 0 otherwise. The set U of the control functions is our admissible controls.[5] Thus, the total interaction and differential scattering cross sections depend on the control function.

Besides tree density, tree growth determines the changes in the total interaction and differential scattering cross sections. Tree growth at any instant of time is assumed characterized by n parameters, $\overline{g}(t)=(g_1(t), g_2(t), \cdots, g_n(t))$ (for example, the functions of stem and crown growth, dry biomass, tangential stresses along the stem counter and so on) and the vector function, $\overline{g}(t)$, satisfies a system of differential equations

$$\frac{dg_i(t)}{dt} = f_i(t, g, u, I),\ u \in U,\ I \in D,\ i = 1, 2, \cdots, n, \tag{6}$$

where f_i is, for each fixed vector $(t, g_1, g_2, \cdots, g_n)$, a functional defined on the set $U \times D$. The functionals and the initial values, $\overline{g}_0$, of the vector function, $\overline{g}(t)$,

$$\overline{g}(0) = \overline{g}_0,\ \ \overline{g}(0) = (g_{1,0}, g_{2,0}, \cdots, g_{n,0}), \tag{7}$$

are assumed to be known and serve as input data for our optimization model. Thus, the total interaction and differential scattering cross sections depend on $u \in U$ and $\overline{g}(t)$.

We idealize our plant stand as a system whose state at any instant of time is characterized by the radiation regime, $I \in D$, and the growth vector $\overline{g} \in R^n$. The set $R^n \times D$ of the variables, $(\overline{g}, I)$, is the space of the plant stand states. The transport equation and the system of differential equation (6) govern changes in the plant stand. We assume that the joint system of equations (1) and (6) with boundary conditions (2), (3) and (7) has a unique solution, $< \overline{g}(t,u), I(t,u) >$, for any admissible control function $u \in U$.

Let $u(t)$ be an admissible control and let $\overline{g}(t)$ and $I(t)$ be the corresponding solution of the joint system described above. We term the couple, $< \overline{g}(t), I(t) >$, a trajectory corresponding to the control $u \in U$. Every trajectory has its initial and final positions. The initial position, $< \overline{g}(0), I(0) >$, $\overline{g}(0) \in R^n$, $I(0) \in D$, satisfies the boundary conditions (7) and (2)-(3), and describes the initial state of the plant canopy. The final position, $< \overline{g}(T), I(T) >$, describes a desired state of our plant canopy which is assumed to be given by a system of relationships

$$\psi_i(g_1(T), g_2(T), \cdots, g_n(T)) = 0, \quad i = 1, 2, \cdots, m,$$

and functionals

$$\Phi_i I = 0, \quad \Phi_i : D \to R^1, \; i = 1, 2, \cdots, l.$$

The functions, $\psi_i : R^n \to R^m$, and functionals, $\Phi_i : D \to R^1$, are assumed given. We introduce the set of final states of the plant canopy, $S_T = R_T \times D_T$, where

$$R_T = \{x \in R^n | \; \psi_i(x) = 0, \; i = 1, 2, \cdots, m\},$$

$$D_T = \{I \in D | \; \Phi_i I = 0, \; i = 1, 2, \cdots, l\}.$$

We say[5] that the admissible control, $u(t)$, transfers state from the position, $< \overline{g}_0, I_0 > \in R^n \times D$, to the final state, S_T, if the trajectory, $< \overline{g}(t), I(t) >$, corresponding to the control, $u \in U$, has initial position, $< \overline{g}(0), I(0) > = < \overline{g}_0, I_0 >$, and its final position satisfies the condition, $< \overline{g}(T), I(T) > \in S_T$.

Let us now suppose that we are given another functional, $f_0(t, u, \overline{g}, I)$ which is defined and is continuous on all of $R^{n+1} \times D \times U$. Then, the optimization problem can be formulated in terms of the theory of optimal processes as follows (The problem with fixed left-hand and variable right-hand endpoints, See Reference 5).

In the space of the plant stand states, $R^n \times D$, the position, $< \overline{g}_0, I_0 >$, and the set, $S_T = R_T \times D_T$, are given. Among all the admissible controls, $u = u(t)$, which transfer the plant canopy state from the position, $< \overline{g}_0, I_0 >$, to the final state, S_T, (if such controls exist), find one for which the integral functional

$$J = \int_0^T f_0(t, u, \overline{g}, I)\, dt, \tag{8}$$

takes on the largest possible value. Here, $< \overline{g}(t), I(t) >$, is the trajectory corresponding to the control, $u \in U$, with initial position $< \overline{g}_0, I_0 >$.

Note the time, T, is not fixed. We only require that the trajectory $< \overline{g}(t), I(t) >$, should be in the position $< \overline{g}_0, I_0 >$, at the initial time, and in the position, S_T, at the final time.

Usually, the functional, f_0, describes a quantitative characteristic of a process to be maximized. Determining it as a continuous functional on the set, $R^{n+1} \times D \times U$, we introduced a rather wide set which does not take into account the features of the problem of plant stand optimization. The next step is to restrict the set of continuous functionals to the one which describes stand states qualitatively and quantitatively.

4. CRITICAL ARRANGEMENTS AND OBJECTIVE FUNCTIONAL

The problem of optimal planting of industrial woods is equivalent to the investigation of the dependence between the distribution of solar radiation inside a plant stand and various schemes of tree planting. The radiation field in a certain stand at a fixed time depends mainly on the incoming radiation field. Therefore, the description of the variations of solar radiation inside tree crowns can be reduced to examining the changes in the incident radiation at the crown boundary.

The total alteration in the incident radiation at the crown boundary is caused by the following three factors – meteorological changes (clouds, sun

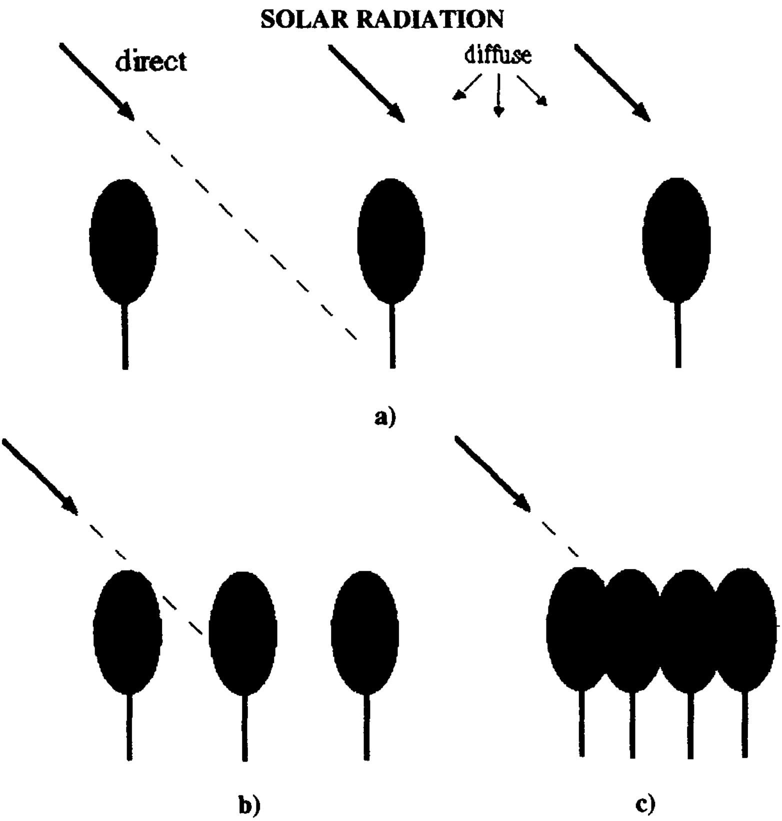

Fig.2 Arrangement of trees. a):*ante* $-$ CAF; b): *post* $-$ CAF and *ante* $-$ AS; c): *post* $-$ AS.

position, etc.), variations in tree density (cutting and planting of trees) and size (growth of trees). The meteorological changes are assumed known and serve as a boundary condition (2) for the radiative transfer equation (1) i.e. input data for our model. Therefore, we exclude it from our analysis in Sections 4 and 5. Thus, the time variable is fixed. At each fixed time, t, the control, $u \in U$, determines the arrangement of trees in the stand. Here, we examine the dependence between incident radiation at the tree crowns and the controls, $u \in U$, and the tree size at fixed time, t. To do so, we introduce the concept of "critical arrangement" of trees.

The incident radiation intensity at the boundary of tree crowns is not dependent on tree density and size provided the trees are far enough apart spatially and when there is no cross-shading between them (See Fig. 2). On the other hand, if the trees are close to one another, the incident radiation changes depending upon the tree density and size. It means there should be some critical tree arrangements at which a transition takes place from the "non-sensitive state" to a "sensitive state". We define such a transition as Critical Arrangement of the First order (CAF).

Let us consider the $post - CAF$ plant stand. We identify two situations in this case (Fig. 2). In the first one, a stand is dense with trees which are sufficiently close to one another ($post - CAF$) but open spaces between the trees are evident. In this case, the radiation field incident on the tree crowns is influenced by tree density and size in a complex manner. It also results in changes in the radiation field inside the crown. The second arrangement of trees is a stand with trees having no open spaces between them. In this case, tree density does not influence the incident solar radiation on tree crowns because only the top level is free for incidence and the total boundary of trees can be idealized as a "rough surface". This arrangement of trees which generates the "rough surface" can be termed as the arrangement of second order (AS). Note we do not identify the tree arrangement at which a transition takes place from the first to the second.

We use the "critical arrangements" for idealizing our plant stand as a system capable of being in three statuses: $ante - CAF$, AS as well as between CAF and AS. The transition from one status to another may take place as a result of tree growth (changes in tree size) as well as variations in tree density (planting and cutting of trees). The $ante - CAF$ status provides the most favourable conditions to capture solar energy. An arrangement of trees between $post - CAF$ and AS corresponds to the situation when tree illumination changes due to tree growth. It leads to competition for capturing more sunlight which, in its turn, may involve the changes in the growth rate, quality of woody biomass as well as death of some trees. The AS status can be interpreted as the final result of tree competition.

With the above in mind, we idealize the plant stand as a system consisting of M subsystems of trees. Each subsystem can be in one of the statuses described above. The transition from one to another may be a result of cutting and/or planting of trees as well as tree growth.

The concept of "critical arrangement" describes qualitatively the status of a forest stand and is not enough for solving the problem of optimization because it is not sensitive enough to variations in planting pattern, i.e., a plant stand may retain the same status while the tree density and size vary simultaneously. Therefore, the next step is to introduce a quantitative characteristic that is sensitive to both the transition from one state to another and the location of trees in the plant stand. For this purpose we use the concept of the objective functional.

Consider a plant stand. Let V be the domain where the trees are located. Let $I(\vec{r}, \Omega)$ be the radiance distribution function in the plant stand. We use the symbols V_i, V_j and V' to denote subdomains of the domain V.

Definition. *The positive functional, $P(I, V')$ is said to be an objective functional if the following properties*

a) $P(I, V') = 0$ *if the subdomain V' has no common points with tree crowns;*

b) $P(I_1, V') \geq P(I_2, V')$ *if* $I_1(\vec{r}, \Omega) \geq I_2(\vec{r}, \Omega)$, $\vec{r} \in V'$, $\Omega \in 4\pi$;

c) $P(I, V') = \sum_{j=1}^{n} P(I, V_j)$ *if* $V' = \cup_{j=1}^{n} V_j$, $V_i \cap V_j = \emptyset$;

hold.

The interpretation of this definition is the following. The first condition allows only the radiation inside the tree crowns. The second describes sensitivity to variations in the radiation field inside the plant canopy. The last condition obliges the objective functional to be sensitive to the number of trees in the plant stand.

Canopy photosynthesis[2] is one example of the objective functional. Its use for optimization is quite natural. Indeed, the higher the value of canopy photosynthesis, the more active is the conversion of solar radiation to woody biomass. But this approach has two major deficiencies. First, it is computationally intensive. Second, the algorithm for solving this problem requires a model of leaf photosynthesis in addition to the parameters described in Section 2.

It has been shown that the growth rate of several vegetation canopies increases linearly with increasing amounts of intercepted photosynthetically active radiation.[6–8] Therefore, the value of total amount, P_a, of radiation absorbed by a plant stand

$$P_a(I,V) = \int_V \int_{4\pi} \sigma_a(\vec{r},\underline{\Omega}) I(\vec{r},\underline{\Omega})\, d\vec{r} d\underline{\Omega}\,.$$

can be taken as the objective functional. Here $\sigma_a(\vec{r},\underline{\Omega})$ is the absorption coefficient. This value satisfies the definition given above.

Two other examples of the objective functional are:

$$P_\sigma(I,V) = \int_V \int_{4\pi} \sigma(\vec{r},\underline{\Omega}) I(\vec{r},\underline{\Omega})\, d\vec{r} d\underline{\Omega}\,, \quad P_K(I,V) = \int_V \int_{4\pi} K(\vec{r}) I(\vec{r},\underline{\Omega})\, d\vec{r} d\underline{\Omega}, \qquad (9)$$

where $\sigma(\vec{r},\underline{\Omega})$ is the interaction cross section and

$$K(\vec{r}) \;=\begin{cases} 1, & \text{if } \vec{r} \text{ belongs to the tree crown;} \\ 0, & \text{otherwise.} \end{cases}$$

The first functional, P_σ, is the total amount of intercepted radiation. Let $\sigma_{s'} = \sigma - \sigma_a$ be the scattering cross section. In the PAR-region of the solar spectrum, leaves usually absorb c.a. 90% of intercepted radiation[10] and so the ratio $\frac{\sigma_{s'}}{\sigma}$ is sufficiently small. It follows from the relationships

$$|P_\sigma - P_a| \;=\; \int_V \int_{4\pi} \sigma_{s'}(\vec{r},\underline{\Omega}) I(\vec{r},\underline{\Omega})\, d\vec{r} d\underline{\Omega} \;=\; \int_V \int_{4\pi} \frac{\sigma_{s'}}{\sigma} \sigma\, I d\vec{r} d\underline{\Omega} \le \left[\sup \frac{\sigma_{s'}(\vec{r},\Omega)}{\sigma(\vec{r},\Omega)}\right] P_\sigma,$$

that the functional P_σ is close to the functional P_a. The second functional, P_K, can be used for optimization of a plant stand with random leaf distribution in the tree crown.

Besides these examples, there is a rather wide set of objective functionals. The choice depends on specific features of the plant stand and the desired accuracy as well as computer time.

5. SOME PROPERTIES OF THE OBJECTIVE FUNCTIONAL

In this Section, we examine the behavior of the plant stand with respect to tree density and examine how best the CAF can be calculated (Theorem 3). Theorems 1 and 2 serve as the auxiliary results.

Consider a leaf canopy consisting of N individual trees. The trees are assumed identical to one another at any fixed time. We denote by V and V_0 the domain of the plant stand and the domain of a single crown, respectively.

Let us consider two identical trees but the first is detached, the second one in a stand. Radiation incident on the crown of the isolated tree and the boundary of the plant stand is assumed equivalent. Let $I_0(\vec{r}, \underline{\Omega})$ and $I_k(\vec{r}, \underline{\Omega})$ be the radiance distribution function in the crown of the detached tree and in the crown of the kth tree in a stand.

Theorem 1. $P(I_0, V_0) \geq P(I_k, V_0)$.

This theorem allows us to compare the radiation field in the crowns of the two trees. Let the canopy photosynthesis be the objective functional. In this case, Theorem 1 has a simple physical interpretation: the photosynthesis of the detached tree is not less than the photosynthesis of an identical tree but which is located in a stand.

Proof. Radiation incident on the crown of the tree in a stand is described by the solution, $I(\vec{r}, \underline{\Omega})$, of transport equations (1), (2), (3) where $\vec{r}$ is on the crown boundary. It follows from the maximum principle[17] that $I(\vec{r}, \underline{\Omega}) \leq I(\vec{r} - l[\vec{r}, \underline{\Omega}]\underline{\Omega}, \underline{\Omega})$ where $l[\vec{r}, \underline{\Omega}]$ denotes the distance between the point $\vec{r}$ and the boundary of the plant stand along the direction $-\underline{\Omega}$. It means[17] that $I_0(\vec{r}, \underline{\Omega}) \geq I_k(\vec{r}, \underline{\Omega})$. Using the definition of the objective functional we derive the desired inequality. *This completes the proof.*

Let $I(\vec{r}, \underline{\Omega})$ be the solution of the radiative transfer problem (1)-(3) and $I'(\vec{r}, \underline{\Omega})$ be the function

$$I'(\vec{r}, \underline{\Omega}) = \begin{cases} I(\vec{r}, \underline{\Omega}), & \text{if } \vec{r} \in V'; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, the function $I'(\vec{r}, \underline{\Omega})$ is the radiance distribution function in the subdomain, V', of the plant stand.

Theorem 2. $P(I, V') = P(I', V')$.

This theorem shows that the objective functional depends on the subdomain, V' (for instance, the crown of a tree), and the radiance distribution function in it only.

Proof. The equality $I(\vec{r}, \underline{\Omega}) = I'(\vec{r}, \underline{\Omega})$, $\vec{r} \in V'$ is equivalent to the system of the inequality: $I(\vec{r}, \underline{\Omega}) \geq I'(\vec{r}, \underline{\Omega})$ and $I(\vec{r}, \underline{\Omega}) \leq I'(\vec{r}, \underline{\Omega})$, $\vec{r} \in V'$. It follows from the property *b)* (See Definition) that both inequalities $P(I, V') \geq P(I', V')$ and $P(I, V') \leq P(I', V')$ are valid. This means that $P(I, V') = P(I', V')$. *This completes the proof.*

Theorem 3. *The plant stand maintains* $ante-CAF$ *status through variations in tree density if and only if the objective functional is linear with respect to the tree density,* $\rho = \frac{N}{X_S Y_S}$, *i.e.*

$$P(I,V) = X_S Y_S P(I,V_0) \cdot \rho, \quad P(I,V_0) = const$$

where V_0 *is the domain of a tree crown.*

This theorem provides the algorithm for finding the critical arrangement (of the first order). For this purpose we evaluate dependence of the objective functional against tree density (Fig. 3, for instance). It follows from Theorem 3 that this dependence consists of two parts; a linear and a nonlinear part. The point at which the linear response changes to a nonlinear response corresponds to the tree density of a CAF stand. Thus, at every fixed time, we can calculate the critical arrangement (of the first order) of trees.

We emphasize that this dependence of the objective functional maintains its structure (linear and nonlinear parts) for different planting patterns, boundary conditions as well as soil and leaf optical properties, i.e., Theorem 3 may be interpreted as an invariant property of a plant stand. The length of the linear part and the angle of its inclination are sensitive to changes in model parameter values (See Section 7).

Proof. We consider $ante-AS$ stand. Let $I_k^N(\vec{r},\Omega)$ be the radiance distribution function in the crown of the kth tree, $1 \le k \le N$. The index N shows that there are N trees in the stand. It follows from the definition of the objective functional [property *c)*] and Theorem 2 that

$$P(I,V) = \sum_{k=1}^{N} P(I_k^N, V_0) = \frac{N}{X_S Y_S} \cdot \frac{X_S Y_S}{N} \sum_{k=1}^{N} P(I_k^N, V_0) = X_S Y_S \psi(N) \rho,$$

where

$$\psi(N) = \frac{1}{N} \sum_{k=1}^{N} P(I_k^N, V_0).$$

Let us consider the $ante-CAF$ stand. It means the tree crowns are illuminated identically. Hence,

$$I_1^N(\vec{r},\Omega) = I_2^N(\vec{r},\Omega) = \ldots = I_0(\vec{r},\Omega).$$

Here $I_0(\vec{r},\Omega)$ is the radiance distribution function in the crown of a tree. Variations in tree density (or N) do not lead to alterations in tree illumination while the plant stand keeps the $ante-CAF$ status. It follows from this that $\psi(N) = P(I_0,V_0) = P(I,V_0) = const.$

Let the objective functional be a linear function with respect to tree density, $\rho = \frac{N}{X_S Y_S}$. Under this condition, the function $\psi(N)$ is constant: $\psi(N) \equiv C$. For $N = 1,2$, we have

$$P(I_1^1, V_0) = C\,,\ \frac{1}{2}\left[P(I_1^2, V_0) + P(I_2^2, V_0)\right] = C\,.$$

Taking into account the inequality $P(I_i^2, V_0) \leq P(I_1^1, V_0)$, $i = 1,2$ (See Theorem 1), it is possible to obtain

$$C = \frac{1}{2}\left[P(I_1^2, V_0) + P(I_2^2, V_0)\right] \leq P(I_1^1, V_0) = C\,.$$

The relationship holds only if $P(I_1^2,V_0) = P(I_2^2,V_0) = P(I_1^1,V_0) = C$. In a similar way, it can be proved that $P(I_i^k, V_0) = C$, $k = 1,2,...,N$, $N = 3,4,...,N_{max}$. It means the plant stand is the $ante-CAF$ state because the objective functional of a tree $(P(I_1^k,V_0))$ is not sensitive to variations in tree density (or N). *This completes the proof.*

Let us consider the $post-CAF$ stand. It follows from Theorem 3 that a change in tree density leads to a nonlinear response of the objective functional. Two features in the behavior of the objective functional can be observed (See Fig. 3) - the considerable and weak response to variations in tree density. The physical meaning of this features is: the first is caused by considerable changes in tree illumination; the second shows that the objective functional changes takes place at the expense of changes in the phytomedium.

In the calculation presented in Fig. 3, we simulate the changes of the phytomedium as follows. The tree size is constant. The leaf area density of a domain obtained with intersection of the trees is the sum of the leaf area densities of intersected trees. Therefore, there are no changes of the phytomedium in the $ante-AS$ stand used in these calculations.

Thus, there are three parts in the behavior of the objective functional with respect to tree density. The first is the linear function and corresponds to the $ante-CAF$ stand. The second is a nonlinear response but "well"-sensitive to variations in tree density. It corresponds to both the $post - CAF$ and $ante - AS$ stand. The last part ("asymptotic region") is weakly sensitive to variations in tree density and denotes the $post - AS$ stand.

6. EQUATION FOR OPTIMAL VALUE OF TREE DENSITY

In this section we give an example for the solution of the optimization problem formulated in Section 3. We suggest additionally our plant stand consists of one ($M = 1$) subsystem of trees that are square planted. Cutting of trees (i.e. the transition from *post* to *ante*-status) as well as planting of trees except the initial ones are excluded from our consideration. With this assumption, the controls do not depend on the time-variable and any control can be completely described with one parameter - tree density, $\rho = \frac{N}{X_S \cdot Y_S}$. Therefore, the set, $U_\omega = \{\rho | \ \rho \leq \omega\}$, can be taken as admissible controls. Here ω is a sufficiently large number. The final states of the plant canopy, S_T, are given as

$$S_T = R_T \times D_T, \quad R_T = R^n, \quad D_T = \left\{ I \in D | \ \frac{dI}{d\rho} = 0 \right\}.$$

We assume that tree growth, $\overline{g}(t)$, does not depend on tree density, i.e.,

$$\frac{dg_i(t)}{dt} = f_i(t, \overline{g}, I), \ I \in D, \ i = 1, 2, \cdots, n.$$

Let us consider the problem: *find an admissible control, $\rho \in U_\omega$, which transfers the plant canopy state from the position $< \overline{g}_0, I_0 >$, to the state, S_T, and which in doing so imparts a maximal value to the functional (8), where f_0 is the objective functional.*

In this example the radiation field in the canopy at the final time, T, does not depend upon the tree density. It means that the set, D_T, contains the $ante - CAF$ plant stand only. Because the transitions from *post* to *ante*-status are excluded from our consideration, this example has a solution if and only if the plant canopy keeps the $ante - CAF$ status for any $t \in [0, T]$. Thus, the problem can be formulated in the following equivalent form: *How should the trees in a stand be initially $ante - CAF$ planted such that the plant stand reaches the CAF-status and the functional (8) attains its maximum.*

The optimally planted trees realize the most favourable conditions in tree illumination. We shall start consideration of our problem without taking into account the meteorological changes, i.e I_d, I_0 and Ω_0 are fixed. Then we discuss how they can be included in the optimization model.

We denote by $\tau_{CAF}(\rho)$ the time for achieving the CAF status by the stand with tree density, ρ, being initially $ante-CAF$ planted. If the $ante-CAF$ stand is unable to transit into a CAF stand (the trees are far enough apart spatially) then the value of $\tau_{CAF}(\rho)$ is assumed to be equal to infinity: $\tau_{CAF}(\rho) = \infty$. For the $post-CAF$ stand, the value of $\tau_{CAF}(\rho)$ is equal to zero: $\tau_{CAF}(\rho) = 0$.

The plant stand keeps the $ante-CAF$ status when $t \le \tau_{CAF}(\rho)$. It means the value of the final time, T, should satisfy the equality: $T = \tau_{CAF}(\rho)$. It follows from Theorem 3 that the objective functional appearing in (8) as the integrand is a linear function with respect to the tree density, ρ, for each fixed t. This allows us to express the problem as

$$\begin{cases} \int_0^t X_S \cdot Y_S \cdot P(t', \overline{g}, I, V_0) \cdot \rho \, dt' \;\rightarrow\; \max, \\ T \;=\; \tau_{CAF}(\rho) \;<\; \infty, \\ g_i'(t) \;=\; f_i(t, \overline{g}, I), \; I \in D, \; i \;=\; 1, 2, \cdots, n. \end{cases} \tag{10}$$

Here we include the variables, t, and $\overline{g}$ in the argument list of the objective functional. Using the standard Lagrange technique, it is possible to obtain the equation for the desired value of the tree density

$$\int_0^{\tau_{CAF}(\rho)} P(t', \overline{g}, I, V_0) \, dt' \;+\; P(\tau_{CAF}(\rho), \overline{g}, I, V_0) \cdot \rho \cdot \frac{\partial \tau_{CAF}(\rho)}{\partial \rho} \;=\; 0\,. \tag{11}$$

There is a special case in the behavior of this equation. It is obvious that the function $\tau(\rho) \equiv 0$ satisfies the expression (11). It means that every value of tree density corresponding to the $past-CAF$ stand is a solution of the equation (11). However, it cannot be taken as the solution of the problem (10) because of losing the transition from the $ante-CAF$ to the CAF status by a plant stand. Therefore, the tree density, ρ, should satisfy both equation (11) and the inequality

$$0 \;<\; \tau_{CAF}(\rho) \;<\; \infty\,. \tag{12}$$

For the purpose of calculating the function $\tau_{CAF}(\rho)$ numerically, we derive another form for problems (11)–(12). To do so, we replace the variable, $t =$

$\tau_{CAF}(\rho)$, in the equation (10). Resolving the expression $t = \tau_{CAF}(\rho)$ with respect to the variable ρ and accounting for the rule of differentiating inverse functions as well as the formula $(\ln \rho)' = \frac{\rho'}{\rho}$, equation (11) for the problem at hand can be transformed to

$$-\frac{\partial \ln \rho(t)}{\partial t} = \frac{P(t, \overline{g}, I, V_0)}{\int_0^t P(t', \overline{g}, I, V_0)\, dt'}. \tag{13}$$

Here $\rho(t) = \tau_{CAF}^{-1}(t)$ is the inverse function to $\tau_{CAF}(\rho)$ (i.e. $\tau_{CAF}(\rho(t)) = t$). The interpretation of the function $\rho(t)$ is the following. Let us consider a detached tree at time t during its growth. The value $\rho(t)$ is the tree density of the CAF plant stand consisting of trees similar to the isolated tree under consideration.

As a result of this we have derived two equations. Both allow determination of the optimal value of tree density. The central question in the problem of radiation optimization is the evaluation of the objective functional, $P(t, I, V)$, and the function $\rho(t)$ as values of time, t. On the basis of this, one can calculate every value occurring in (13) (or (10)-(12)). Indeed, it follows from Theorem 3 that the value of the function $\rho(t) = \tau_{CAF}^{-1}(t)$ is the point, ρ^*, along the tree density axis at which the dependence of the objective functional, $P(t, I, V)$, on tree density changes from the linear part to the nonlinear part, and $P(t, I, V_0) = \frac{P(t,I,V)}{X_S \cdot Y_S \cdot \rho} = const$, $\rho \le \rho^*$.

The solution of the transport equation underlies the calculation of the objective functional. Therefore, the algorithm for solving the radiative transfer equation is the kernel of the algorithm for plant stand optimization. Note that the calculation of the radiation field in a plant canopy is an iterative procedure. Thus, fast methods for solving the transport equation are an extremely important development for the purpose of plant stand optimization.

It will be recalled that the equation for optimal value of tree density is derived without taking into account the meteorological changes. They may be included by means of applying a statistical technique to describe variation of the objective functional due to changing meteorological conditions. We approximate the objective functional, $P(t, I, V)$, by piecewise constant function, $\overline{P}(t, I, V)$, with respect to the time-variable: $\overline{P}(t, I, V) = \overline{P}_i$, $t \in (t_{i-1}, t_i]$. A period of several weeks may be taken as the time-interval $(t_{i-1}, t_i]$. In the capacity of $\overline{P}_i$ we take the mean objective functional

$$\overline{P}_i = \frac{m^i_{\odot}}{m_i} R(I^{\odot}, \rho) + \frac{m^i_c}{m_i} R(I^c, \rho) + \frac{m_i - m^i_{\odot} - m^i_c}{m_i} R(I^b, \rho).$$

Here $R(I, \rho)$ is the mean value of the daytime variation of the objective functional; $I^{\odot}$, I^c and I^b are the daytime variations of the radiance distribution function in plant stand for clear sunny, densely cloudy and broken cloudy days; $m^i_{\odot}$ and m^i_c are the mean number of clear sunny and densely cloudy days in the time-interval $(t_{i-1}, t_i]$; m_i is the total number of days in the time interval of interest. We define the value, $R(I, \rho)$, as a linear function with respect to tree density, ρ. The length of the linear part is equal to the mean length of daytime variation of the linear part of the objective functional, $P(t, I, V)$. The angle of its inclination is the mean angle of daytime variation of the angle, $\arctan\left(\left.\frac{dP}{d\rho}\right|_{\rho\to 0}\right)$. Note that instead of the piecewise approximation discussed above, the piecewise linear approximation of the objective functional

$$\overline{P}(t, I, V) = \frac{\overline{P}_i - \overline{P}_{i-1}}{t_i - t_{i-1}} \cdot t + \frac{t_i \overline{P}_{i-1} - t_{i-1} \overline{P}_i}{t_i - t_{i-1}},$$

$$t_{i-1} \leq t < t_i.$$

may be used as well.

Thus, consideration of meteorological factors into the optimization problem allows us to use the concept of mean objective functional. The mean critical tree arrangement of the first order can be determined as the point at which dependence of the mean objective functional against tree density changes from a linear function into a nonlinear one. The optimal value of tree density can be found by solving equation (13) (or (10)-(12)) once the mean functions $\rho(t)$ and $P(t, I, V_0)$ are known. The problem of specifying a strict definition for the mean objective functional is the topic of a special investigation, and hence, we shall leave it for a detailed analysis at a later time.

7. NUMERICAL EXAMPLES AND DISCUSSION

To illustrate the behavior of the objective functional, we present results of some numerical experiments here. Consider a stand consisting of N identical square planted trees (for instance, a stand of 25 trees was simulated as 5 rows with 5 trees per row). The domain of tree location is the square $X_S = 10m$ and

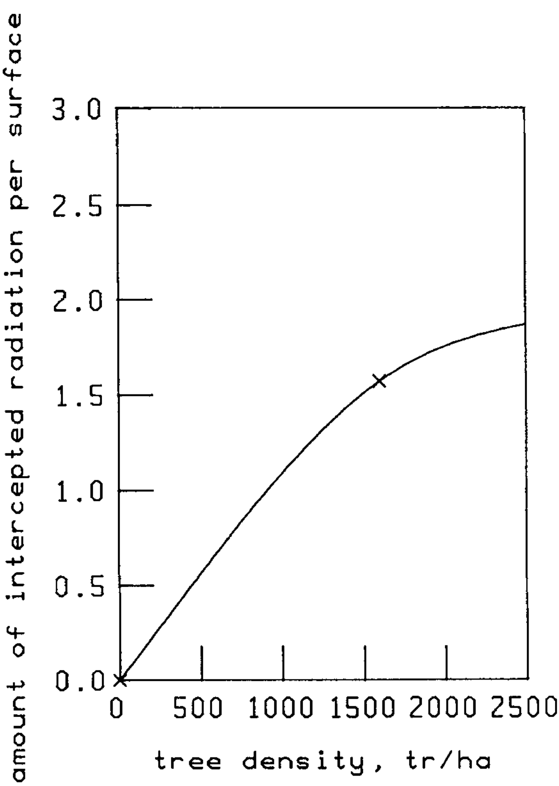

Fig.3 The amount of intercepted radiation per unit surface (in Wm^{-2}) by the plant stand as a function of tree density. Here $X_c = Y_c = 2m$; $Z_C = Z_S = 5m$; $\theta_0 = 60°$; $\varphi_0 = 45^0$; $LAI = 15$.

$Y_S = 10m$. The height of the stand, Z_S, is a variable and is equal to the tree height. We denote by $2X_C$, $2Y_C$ and $2Z_C$ the dimensions of tree crowns. The leaf area density function was modelled using the quadratic function[10] (See Fig. 1a)

$$u_L(\vec{r}) = \frac{1.6875 \cdot L}{Z_S}(1 - Z^2)(1 - Y^2)(1 - Z^2).$$

Here $X = \frac{x-x_0}{X_C}$, $Y = \frac{y-y_0}{Y_C}$ and $Z = \frac{z-z_0}{Z_S}$ where the triplet (x_0, y_0, z_0) denotes the origin of the leaf area distribution and L is the leaf area index of one tree

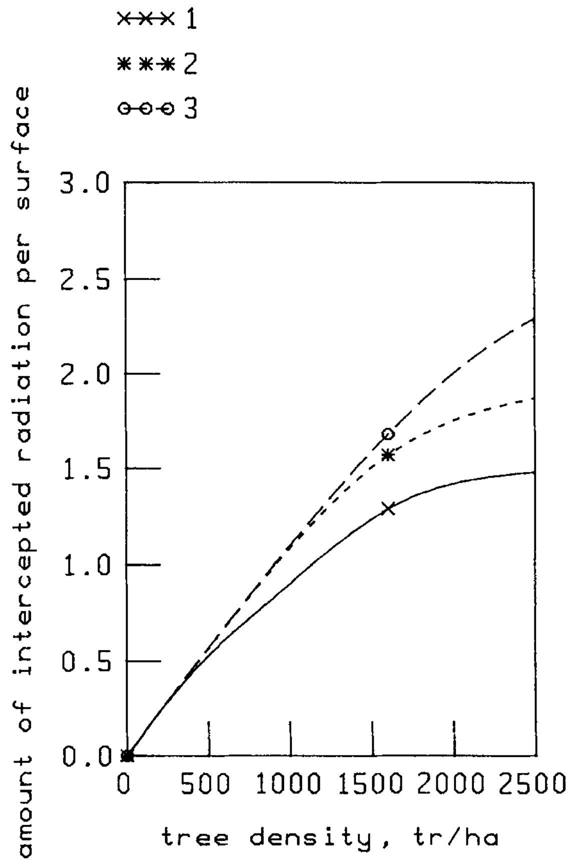

Fig.4 The amount of intercepted radiation per unit surface (in Wm^{-2}) by the plant stand as a function of tree density for various directions, (θ_0, φ_0), of incident solar radiation. 1: $\theta_0 = 80°$; 2: $\theta_0 = 60°$; 3: $\theta_0 = 30°$. Here: $X_C = Y_C = 2m$; $Z_C = Z_S = 5m$; $\varphi_0 = 45^0$; $LAI = 15$.

– leaf area per unit ground area. This model is consistent with a proposal of Ross[9] based on his measured data. The leaf area index will be varied in our calculation. The leaf normal distribution was assumed to be given by an erectophile distribution in the polar angle (i.e. $g_L(\vec{r}, \Omega_L) = \frac{2}{\pi}(1 - \cos(2\theta_L))$ and uniform in azimuth. We ignore specular reflection in these examples. Reflection from the ground is assumed Lambertian: $\gamma_b = \frac{1}{\pi} r_b$. The soil reflectivity r_b was 0.2. These optical properties correspond to average values reported in the literature for PAR wavelengths[10]. Direct solar radiation is incident along

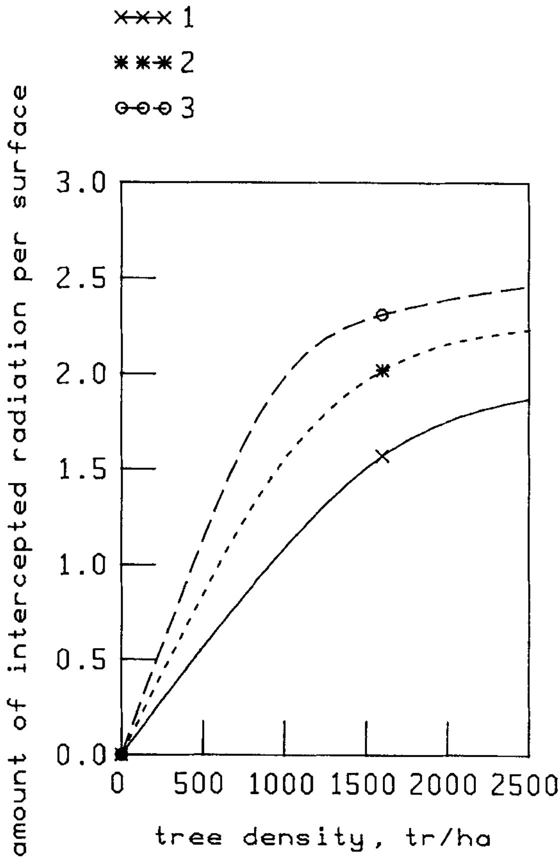

Fig.5 The amount of intercepted radiation per unit surface (in Wm^{-2}) by the plant stand as a function of tree density for various values of tree size. 1: $X_C = Y_C = 2m$, $Z_C = Z_S = 5m$; 2: $X_C = Y_C = 2.5m$, $Z_C = Z_S = 6m$; 3: $X_C = Y_C = 3m$, $Z_C = Z_S = 7m$. Here: $\theta_0 = 60^o$; $\varphi_0 = 45^0$; $LAI = 15$.

polar angle θ_0 and azimuth φ_0 (variables in our calculations). The amount of intercepted radiation per unit surface was considered as the objective functional, i.e., $P(I,V) = \frac{1}{X_S Y_S} P_\sigma$ (in Wm^{-2}) where $P_\sigma(I,V)$ is determined by Eq. (9).

The dependence of the objective functional on tree density is plotted in Fig. 3. One can separate three parts in its shape. The first is a linear function ($0 \leq \rho \leq 866 \frac{tr}{ha}$) and corresponds to the $ante-CAF$ stand. This status provides the

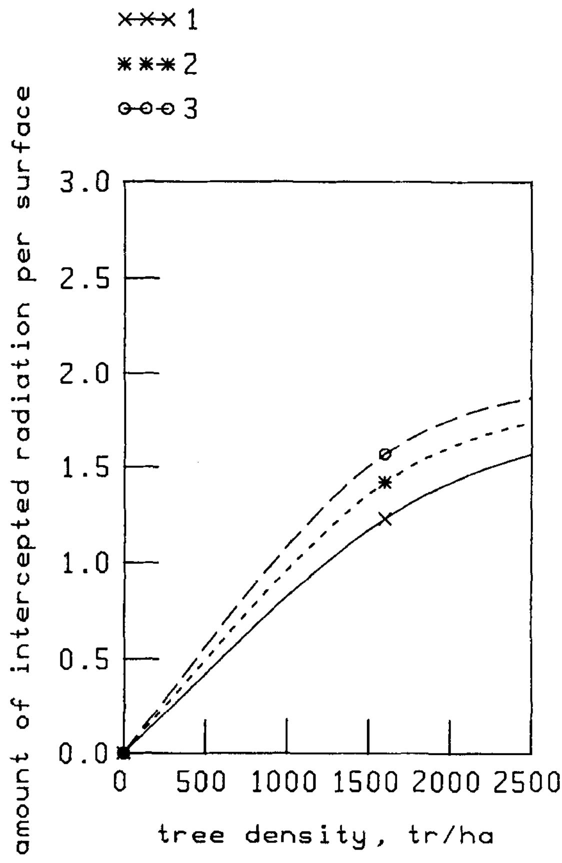

Fig.6 The amount of intercepted radiation per unit surface (in $W m^{-2}$) by the plant stand as a function of tree density for various values of leaf area index. 1: $LAI = 10$; 2: $LAI = 12.5$; 3: $LAI = 15$. Here: $\theta_0 = 60^o$; $\varphi_0 = 45^0$; $X_C = Y_C = 2m$; $Z_C = Z_S = 5m$.

most tree illumination. The amount of intercepted radiation increases linearly with increasing tree density because there are no changes in the radiation field incident on the tree crown. The second part $(866 \le \rho \le \text{c.a.}\, 2500\, \frac{tr}{ha})$ is the nonlinear function and it corresponds to the *post–CAF* and *ante–AS* state. The radiation input in the tree crowns is altered in a complex way with changes in tree density and size. It leads to the situation where the sensitivity of a stand to variations in tree density is less with respect to the *ante – CAF* stand. This state corresponds to the situation where tree illumination changes as a result of

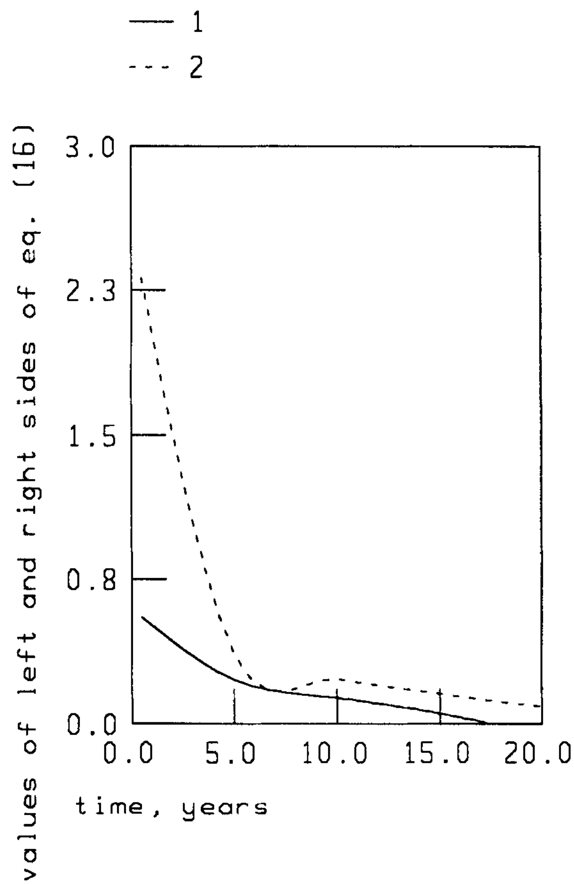

Fig.7 The solution of the equation (13). 1: $-\frac{\partial \ln \rho(t)}{\partial t}$; 2: $\frac{P(t,I,V_0)}{\int_0^t P(t',I,V_0)\,dt'}$

tree growth. It leads to trees competing for more sunlight, which in turn, may involve changes in the growth rate, quality of woody biomass as well as death of some trees. The tree density $\rho \approx 2500\,\frac{tr}{ha}$ corresponds to the stand where trees have no free space between them. It means that variations in tree density do not lead to considerable changes in the solar radiation incident on the crowns because only the top level is open for incidence and the total boundary of trees can be idealized as a rough surface. The sensitivity of $post-AS$ to variations in tree density is much less as compared to $ante-AS$ stand. In the calculation presented in Fig. 3, we simulated the changes in the phytomedium as follows. The tree size was constant. Leaf area density of a domain obtained due to intersection of trees is the sum of the leaf area densities of the intersecting trees.

The dependence of the objective functional on tree density at various values of the polar angle, θ_0, of incident solar radiation is shown in Fig. 4. This example illustrates the sensitivity of the stand to daytime variation in the sun position which underlies the evaluation of the mean value, $R(I,\rho)$, of the daytime variation of an objective functional.

The sensitivity of the objective functional to variations in tree size is presented in Fig. 5. If the i-th curve, $(i = 1,2,3)$ is the dependence of the objective functional on tree density at time, t_i, of tree growth, then the value of the function $\rho(t)$ at the time t_i can be easily calculated. In this case, we have $\rho(t_1) = 866\,\frac{tr}{ha}$, $\rho(t_2) = 766\,\frac{tr}{ha}$, $\rho(t_3) = 667\,\frac{tr}{ha}$.

Fig. 6 demonstrates the sensitivity of the objective functional to the variations in leaf area index, L. One can see that the value of the critical tree density, ρ, is only weakly influenced as leaf area index is varied.

In Fig. 7, the functions $-\frac{\partial \ln \rho(t)}{\partial t}$ and $\frac{P(t,I,V_0)}{\int_0^t P(t',I,V_0)\,dt'}$ are presented. Tree growth is characterized by tree height, $g_1(t) = Z_S(t)$, and dimensions of the tree crowns, $g_2(t) = 2X_C(t)$, $g_3(t) = 2Y_C(t)$ and $g_4(t) = 2Z_C(t)$ (in meter) which satisfy the system of differential equations

$$\begin{cases} g_1'(t) = 1, \\ g_2'(t) = g_3'(t) = g_4'(t) = 0.25, \end{cases}$$

with boundary condition, $g_1(0) = 5\,m$, $g_2(0) = g_3(0) = g_4(0) = 1.25\,m$, i.e. the growth rate of trees is one meter per year (poplar stand) and the crown makes up 25% of the tree height. The position of sun was fixed ($\theta_0 = 30^o$, $\varphi_0 = 0$). The point of intersection of the two curves is the solution of equation (13). In this case, the optimal value of tree density, $\rho^* = \rho(T)_{T=7}$ is $110\frac{tr}{ha}$.

Thus, the examples presented here illustrate the influence of main characteristics of a forest stand – tree growth (variation of tree size and leaf area index), incident radiation and tree density – on the function $\rho(t)$. It means that the equation for optimal tree density derived in Section 6 takes into account all the above mentioned factors, and hence, its solution provides the optimal tree density in accordance with a certain type of plant stand.

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