TRANSPORT THEORY AND STATISTICAL PHYSICS, 20(2&3), 139-175 (1991)

STATISTICAL RADIATIVE TRANSPORT IN ONE-DIMENSIONAL MEDIA AND ITS APPLICATION TO THE TERRESTRIAL ATMOSPHERE

Graeme L. Stephens, Philip M. Gabriel & Si-Chee Tsay+

Department of Atmospheric Science
Colorado State University
Fort Collins, CO 80523

ABSTRACT

This paper describes radiative transfer through a single cloud layer that is horizontally uniform but statistically distributed in the vertical and compares the radiative transfer in such a statistical cloud to its deterministic counterpart. Specific examples of the derivation of the probability density functions of cloud reflection and transmission by single cloud layers are described for given observed statistical distributions of cloud optical depth and single scattering albedo. Numerical results of the probability density functions for cloud albedo, transmission and absorption are presented, as are the moments of these distributions. Although the computations apply to a hypothetical statistical medium that only approximate clouds in the atmosphere to a limited extent, results of this study demonstrate that radiative transfer in the statistical cloud is substantially different from that of its deterministic counterpart having the same ensemble mean properties. It is also demonstrated that the rate at which the moments of the distributions decay with order, differs for reflection and transmission in a way that is influenced by the absorption of cloud droplets and the asymmetry of droplet scattering. This result suggests that it may be difficult to infer information about the variability of transmitted solar radiation from reflection measurements.

1. INTRODUCTION

One important property of the solar and infrared radiation measured in the terrestrial atmosphere is the extreme variability of these radiation fields. To appreciate this point, satellite images of reflected shortwave and emitted longwave radiation are shown in Fig. 1a and Fig. 1b. Variations of the radiative intensity monitored

---

+ Current affiliation: NASA Goddard Space Flight Center, Greenbelt, MD 20771

(a)

FIG. 1. Satellite images of clouds in the terrestrial atmosphere. (a) The reflected solar radiation (expressed as a reflectivity) and (b) emitted infrared radiation (brightness temperature). The data are from the GMS satellite for June 15, 1987 and centered at 135° E longitude and 15° N latitude. The approximate areal coverage of the images is 3280 x 3280 km. The frequency distributions for (c) reflection and (d) brightness temperature corresponding to these images are also shown.

over the highlighted region which is located over the Pacific Ocean are shown as a frequency distribution in Figs. 1c and 1d. Evident from this simple analyses of the reflected and emitted fields of atmospheric radiation are the large range of radiances (expressed as albedo in Fig. 1a and brightness temperature in Fig. 1b) measured over some volume of atmosphere.

The inhomogeneous nature of atmospheric radiation, portrayed in Figs. 1c and 1d in the form of distribution functions, is largely a result of the spatial and temporal fluctuations in cloudiness. The latter are characterized by time scales on the order of minutes to hours as seen from a point on the surface of the earth, compared to

(b)

Fig. 1 Continued

several hours to days for weather systems as observed for instance in Figs. 1a and 1b. Spatial fluctuations in cloudiness are governed by processes that range in scale from about 10^6 - 10^7 meters for synoptic scale processes (on the scale of the large cloud masses shown in the satellite images of Figs. 1a and 1b), down to the scale of individual clouds (often not resolved in satellite images) and ultimately down to the scale associated with individual cloud droplets (typically a few tens of microns).

One of the most significant and difficult problems that we presently encounter in the field of atmospheric radiative transfer concerns how best to describe the transfer through a medium that is highly inhomogeneous over a vast range of length scales. In the present remote sensing literature, empirical descriptions of the statistical properties of the intensity fields abound[1-3] with no real effort to connect these statistics to the physics described by radiative transfer. In most meteorological applications, the equation of transfer is greatly simplified and inhomogeneities are dealt with by assuming the atmosphere to be plane-parallel and weighting the

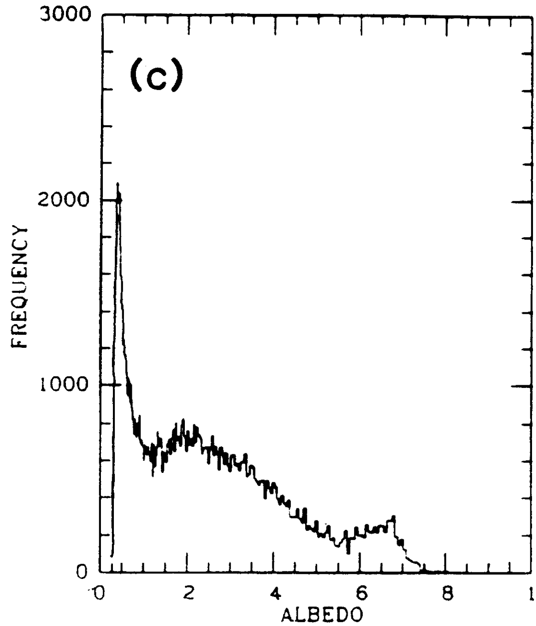


Fig. 1 Continued

transfer processes in cloudy and clear regions appropriately in terms of a factor known as cloud amount. In this way cloud amount enters empirically into radiative transfer. Cloud amount has now assumed major significance in world climate research programs, to an extent that a large body of recent research has focused on the definition of its climatology[4] and its significance to earth's climate.[5] Despite this extensive research, the relationship between cloud amount and radiative transfer lacks a theoretical foundation.[6]

The inadequacy of conventional deterministic approaches to atmospheric radiative transfer can be illustrated in relation to the following problems:

- the determination of the radiation budget of the earth-atmosphere system is crucial to our understanding of climate and climatic change. For instance, the energy budget of the ocean is dominated by the input of solar radiation which,

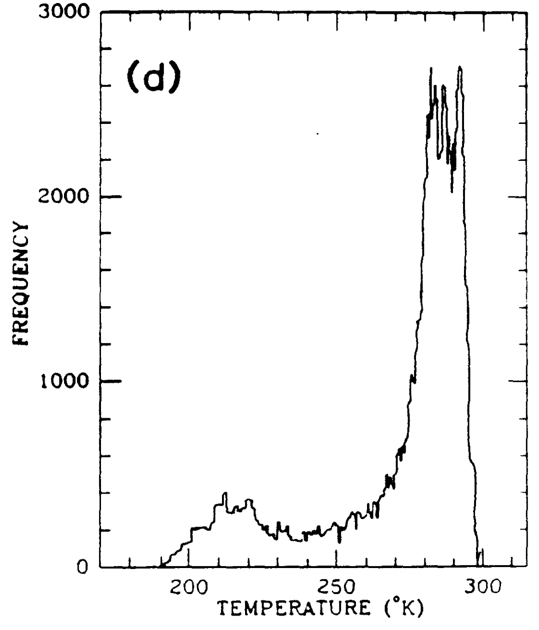


Fig. 1 Continued

in turn, is dominated by cloudiness. This influence is particularly crucial in certain climatically active regions of the globe (such as the Western Pacific[7]). In modeling the energy budget of these regions using oceanic circulation models, the specified solar input is supposed to represent an ensemble average over a spatial scale of hundreds of kilometers and a temporal scale of about a month. However, this poorly represents the real situation in which solar radiation is input in a more episodic manner through large variabilities over smaller space and time scales associated with clouds, and

- the cloud reflection anomaly[8] (sometimes referred to as the albedo paradox[9]) which refers to our inability to match ensemble averaged spectral measurements of radiation reflected by spatially inhomogeneous clouds to a theory based on assumptions of uniformity. In all cases, the clouds appear darker than that predicted by theory at absorbing wavelengths.

These examples help emphasize the need for a coherent theory of radiative transfer through a medium that undergoes large spatial variabilities. We are therefore left to ponder two important questions in this regard:

- what form of equation is appropriate for the description of radiative transfer through a statistically fluctuating medium, and
- how sensitive are the transfer properties, particularly ensemble averaged radiances, to the details of the statistics?

Although recent advanced formulations of deterministic radiative transport in multidimensional media exist,[10–13] little is known about the radiative behavior of cloud systems whose optical properties are subject to spatial and temporal fluctuations. In this paper, we provide a very preliminary attempt to address these broad issues as they relate to clouds in the terrestrial atmosphere. We consider the simplest case of a cloudy medium that is horizontally uniform but statistically distributed in (a) optical depth, and (b) single scattering albedo, and limit our discussion to scattering and absorption processes in an attempt to model the transfer of solar radiation through such a medium. We emphasize that this type of hypothetical cloud does not closely represent reality nor does it represent the most important problems of atmospheric radiative transfer. However, the problem described in this paper does provide a first step towards a resolution of the more general issues posed. Furthermore, the study has limited applicability to stratiform clouds in which variations in optical properties are important.[14]

In the following section, a brief review of some of the literature on stochastic radiative transport is described, followed by a discussion of the statistical distribution of cloud optical depths and the application of these distributions to a two-stream model. Specific examples of the calculation of probability density functions (*pdf*'s) for reflection and transmission are presented in section 4 for single cloud layers. Numerical calculations of the *pdf*'s for the transmission, absorption and reflection by clouds is described in section 5 and the relevance of these results to certain contemporary problems in atmospheric radiation is noted.

2. REVIEW OF LITERATURE

An overwhelming body of literature exists on the radiative transfer in deterministic media (*cf.* ref. 13 for an extensive review up to 1974). By contrast, the analysis of radiative transfer in stochastic media appears to have received generally very little attention in relation to atmospheric radiative transfer. A selective survey of the literature on the subject is summarized in Table 1 as it relates to atmospheric radiative transfer. Early studies were concerned with the interpretation of remotely sensed images of solar radiation reflected from a surface with horizontally variable albedo underlying a uniform atmosphere.[15] The need for a statistical description of the radiative transfer received early recognition by Malkevitch *et al.*[16] and Mullamaa *et al.*[17] To wit:

> *"... attempts to utilize deterministic methods, including the transfer equation for this purpose have proved to be ineffective. At the same time, no statistical theory of transfer exists. At the present time, only the first steps have been taken toward generalization of the transfer theory to media with statistically distributed parameters. This circumstance is to be ascribed to the mathematical difficulties inherent in the solution of the transfer equation for such a medium." Y. Mullamaa et al. (1972)*

Ronholm *et al*[18] considered plane-parallel clouds with small fluctuations about some average of the optical properties. They demonstrated that vertical variability in these properties enhances radiation transmitted through the clouds and reduces the reflection and absorption by these clouds when compared to deterministic calculations. Somewhat consistent with these results, are the findings of Mikhaylov[19] who derived an effective diffusion length for a layer whose optical depth is distributed according to a Gaussian function which was larger than that derived from deterministic theory.

The problem of formulating the statistical theory of transport has also received attention in the nuclear and astrophysical communities in various contexts. For example, in shielding calculations through concrete (or any other heterogeneous substance), the random distribution of the gravel implies a need for a statistical transport treatment to obtain an accurate measure of the shield effectiveness. Yet another example is the boiling water nuclear reactor. The water which acts as both coolant and moderator is a two- fluid random state comprised of liquid and vapor.

Table 1: A summary of selected studies on statistical radiative transfer

Year	Reference	Physical Situation	Method
1964	M. Malkevitch, A. S. Monin, G. Rozenberg[16]	Plane-parallel slab over Lambertian surface whose albedo is a deterministic or random function of position.	Semi-analytical solutions for the emerging radiances based on Fourier decomposition of transfer equation and use of perturbation techniques.
1967	V. I. Drobyshevich[15].	Thin plane-parallel slab over Lambertian surface whose albedo is a random function of position. Illumination by parallel beam of solar radiation.	Equation for the correlation function and spectral densities of outgoing shortwave radiation are derived by means of Friedman-Keller method.
1972	Y. Mullamaa, M. Solev, V. Poldmaa, H. Ohvril, H. J. Niylick, M. I. Allennov, L. G. Tchubakov, A. F. Kuusk[17]	Plane-parallel layer whose upper boundary modulated by Gaussian process. Clouds assumed internally uniform. Extensive experimental investigation of the statistical properties of cumulus and stratiform clouds.	Spectral methods used to relate cloud variability to the statistical properties of the radiation field. A detailed description is given of the results of the experimental studies and a comparison is made with the proposed theoretical model.
1980	K. Ronholm, M. B. Baker, H. Harrison[18]	Bulk optical properties of a medium whose optical properties are stochastic functions of horizontal or vertical position are investigated.	Plane-parallel slab consisting of ten layers was used to calculate numerically the means and standard deviation of the bulk radiative quantities. Horizontal variability was modeled as a single plane-parallel patches each of whose aspect ratio was much less than unity and whose inter- cloud spacing was much greater than the cloud thickness. Modified Gaussian *pdf* was used to distribute ω_o, τ and g.

Year	Reference	Physical Situation	Method
1982	G. A. Mikhaylov[19]	Asymptotic behavior of the average intensity passing through a layer whose optical depth is random is obtained.	Standard probabilistic methods are used to show that in a medium whose extinction is specified by a Gaussian distribution, the effective diffusion length is greater than that of its deterministic counterpart.
1988	G. L. Stephens[6]	Radiative transfer through a layer in which the optical properties are expressed in terms of an ensemble mean and perturbation about that mean.	Formulation of a radiative transfer equation for moments using perturbation techniques. Simplified solutions are presented based on methods of closure (representation of the perturbation properties in terms of ensemble mean).
1990	A. Davis, P. Gabriel, S. Lovejoy, D. Scherter, G. L. Austin[11]	Scattering of radiation energy from mono-dimensional fractal enclosed in a three-dimensional volume with cyclic and open boundary conditions investigated.	Discrete angle radiative transfer in conjunction with scale invariant β-model is employed. clouds.

A proper treatment of the neutron transport must take the statistical nature of the mixture into account. As might be surmised from these examples, much emphasis has been placed on particle transport in random media consisting of two immiscible turbulently mixed materials. While it is not the intent of this section to review literature relevant to these problems, a coherent overview of recent developments of binary statistical mixtures may be found in references(20, 21).

3. FORMULATION OF THE PROBLEM FOR CLOUDS

The optical parameters needed to calculate the bulk radiative properties of clouds are: the optical depth (τ^*), the single scattering albedo ($\tilde{\omega}_o$) and the asymmetry factor (g). We expect that the statistical distribution of these parameters varies from cloud type to cloud type. The optical depth of stratiform clouds, though extremely variable, is apparently subject to a stable statistical description when considered over a sufficiently long period of time (i.e. season, year).[22] Bol'shakov[22] provides an empirical *pdf* for the optical depth τ^* of stratiform clouds sampled over Moscow as

$$f_{\tau^*}(\tau^*) = \frac{2}{3d} e^{-2\sqrt{\tau^*/d}}(1 + 2\sqrt{\tau^*/d}). \quad (1)$$

where the parameter d is set to be 6.625 in this instance. More recently, empirical density functions of cloud optical parameters have been inferred from satellite observations and aircraft measurements.[23,24] Using microphysical data supplied by these investigators, the effects of variabilities in $\tilde{\omega}_o$ on the radiative transfer are addressed below. The effects of fluctuations in τ^* will be addressed using Eq. (1). The variability of g is generally thought to be much smaller than that of either $\tilde{\omega}$ or τ^* and will not be considered[14].

3.1 The two-stream model

We use the two-stream form of the radiative transfer equation and derive the relationship between the bulk radiative properties of clouds and their optical properties from the solution of this equation. Two-stream methods are generally accurate and have been widely used to study many types of transfer problems (e.g., radiation in plant canopies, the insulating properties of fiber glass, the spectral properties of certain paint pigments, the turbidity of oceans and properties of atmospheres among many others) and several reviews of the subject can readily be found.[25–27] The two-stream equations can be written in the form (*cf.* ref. 8 for a detailed derivation)

$$\mp\frac{dF^{\pm}}{d\tau} = -[D(1-\tilde{\omega}_o)+\tilde{\omega}_o b]F^{\pm} + \tilde{\omega}_o b F^{\mp} + F_o\tilde{\omega}_o\chi_{\pm}e^{-\tau/\mu_o}, \tag{2}$$

where D is a measure of the diffuseness of the radiation, μ_o the cosine of zenith angle of the sun such that $\mu_o F_o$ is the flux of collimated solar radiation incident on cloud top and $\chi_+ = b_o, \chi_- = f_o$. The parameters b, χ_+ and χ_- in Eq.(2) are measures of backward and forward scattering respectively and are functions of μ_o, g and $\tilde{\omega}_o$ depending on the particular two-stream method chosen.[8]

Equation (2) defines a linear, two-point boundary value problem with no diffuse radiation entering the cloud top or cloud base. The solution to this system has the general form

$$F^+(\tau) = C_+g_+e^{k\tau} + C_-g_-e^{-k\tau} + F_0Z_+e^{-\tau/\mu_o}$$

$$F^-(\tau) = C_+g_-e^{k\tau} + C_-g_+e^{-k\tau} + F_0Z_-e^{-\tau/\mu_o} \tag{3}$$

where

$$Z_{\pm}(\mu_o) = \tilde{\omega}_o\left[\frac{\chi_{\pm}\tilde{\omega}_o b + \chi_{\pm}[D(1-\tilde{\omega}_o)+\tilde{\omega}_o b \mp \frac{1}{\mu_o}]}{k^2-(\frac{1}{\mu_o})^2}\right], \tag{4}$$

$$k = \{(1-\tilde{\omega}_o)D[(1-\tilde{\omega}_o)D + 2\tilde{\omega}_o b]\}^{1/2}, \tag{5}$$

and

$$g_{\pm} = 1 \pm (1-\tilde{\omega}_o)D/k \tag{6}$$

and where the $C_{\pm}$ represent the boundary conditions.

An isolated plane-parallel cloud layer illuminated only by a collimated source of radiation and subject to the diffuse boundary fluxes

$$F^-(\tau = 0) = 0 \tag{7}$$

$$F^+(\tau = \tau^*) = 0 \tag{8}$$

where the level $\tau = \tau^*$ is used to denote the cloud base, yields

$$C_- = \frac{F_o}{\Delta(\tau^*)}[Z_+(\mu_o)g_- e^{-\tau^*/\mu_o} - Z_-(\mu_o)g_+ e^{k\tau^*}] \tag{9}$$

$$C_+ = \frac{-F_o}{\Delta(\tau^*)}[Z_+(\mu_o)g_+ e^{-\tau^*/\mu_o} - Z_-(\mu_o)g_- e^{-k\tau^*}] \tag{10}$$

where

$$\Delta(\tau^*) = g_+^2 e^{k\tau^*} - g_-^2 e^{-k\tau^*} \tag{11}$$

For the example under consideration, the albedo of the cloud is

$$\mathcal{R} = \frac{F^+(0)}{\mu_o F_o} \tag{12}$$

and the transmission is

$$\mathcal{T} = \frac{F^-(\tau^*)}{\mu_o F_o} + \exp(-\tau^*/\mu_o). \tag{13}$$

These definitions together with Eqs. (4), (9) and (10) in (3) give

$$\mathcal{R} = \frac{1}{\mu_o \Delta(\tau^*)}\{Z_+(\mu_o)\left[\Delta(\tau^*) - \Delta(0)e^{-\tau^*/\mu_o}\right] - Z_-(\mu_o)g_+g_-(e^{k\tau^*} - e^{-k\tau^*})\} \tag{14}$$

and

$$\mathcal{T} = e^{-\tau^*/\mu_o} + \frac{1}{\mu_o \Delta(\tau^*)}\{Z_-(\mu_o)\left[\Delta(\tau^*)e^{-\tau^*/\mu_o} - \Delta(0)\right] - Z_+(\mu_o)g_+g_- e^{-\tau^*/\mu_o}(e^{k\tau^*} - e^{-k\tau^*})\} \tag{15}$$

with cloud absorption defined as

$$\mathcal{A} = 1 - \mathcal{R} - \mathcal{T}. \tag{16}$$

We stress in passing that the potential of two-stream models in describing radiative transfer is limited to horizontally uniform, vertically inhomogeneous media.

3.2 Statistical characterization of the radiation field

Equations (14) through (16) constitute nonlinear transformations mapping a set of input variables, the optical parameters to a set of output functions, the bulk radiative properties. The latter become stochastic variabilities when the input is generated via a random process. We are interested in relating the *pdf* of the optical parameters to the *pdf* of the radiation field and its moments.

If it is only required to characterize the statistical moments (n) of a quantity ϕ given the joint *pdf* $\phi(\tau, g, \omega)$, then it follows from elementary probability theory that

$$< \psi^n >= \int\int\int \phi(\tau, g, \tilde{\omega}_o)\psi^n(\tau, g, \tilde{\omega}_o) dg\, d\tilde{\omega}_o\, d\tau, \tag{17}$$

or equivalently that

$$< \psi^n >= \int \zeta(\psi)\psi^n d\psi, \tag{18}$$

where ψ in this study represents any of the functions $\mathcal{R}$, $\mathcal{T}$. The function $\zeta(\psi)$ is the *pdf* of any of these functions and can be evaluated either numerically or, for certain special cases described below, analytically.

The problem of relating the *pdf* of the radiation field to the *pdf* of the optical properties may be expressed as follows. Given the distribution of l related random variables $x_1, x_2, \cdots, x_l$ (cloud optical properties), what is the joint distribution of y_i (the radiation field) given that $y_i = \eta_i(x_1, x_2, \cdots, x_l)$? The joint *pdf* of y_1, y_2, $\cdots$, y_l can be obtained as[28]

$$f_{\vec{Y}}(\vec{y}) = \sum_i \frac{f_{\vec{X}}(\vec{x}^{(i)})}{|J(\vec{x}^{(i)})|}, \tag{19}$$

where J is the Jacobian of the transformation and $\vec{x}^{(i)} = [x_1^{(i)}, x_2^{(i)}, \cdots, x_l^{(i)}]^{\mathrm{T}}$ is the i^{th} solution to $\vec{y} = \vec{\eta}(\vec{x}) = [\eta_1(\vec{x}), \eta_2(\vec{x}), \cdots, \eta_l(\vec{x})]^{\mathrm{T}}$ where T refers to the vector transpose and the Jacobian J.

In the one-dimensional case, Eq. (19) is simplified as

$$f_Y(y) = \sum_i \frac{f_X(\vec{x}^{(i)})}{|\partial\eta/\partial\vec{x}^{(i)}|}. \tag{20}$$

Selected examples illustrating the use of such transformations are now given.

4. ILLUSTRATIVE EXAMPLES

4.1 Direct transmission of radiation

Consider an atmosphere whose extinction coefficient (β) is observed to vary uniformly in the range $0 \leq \beta \leq 1$. Then the radiance $N(z)$ that is directly transmitted to some point $z = z^*$ from the point $z = 0$ is given by $N(z^*) = N(z = 0)e^{-\beta z^*}$. Now $\frac{dN}{d\beta} = -zN$ and $e^{-z} \leq N \leq 1$. Therefore $f_N(N) = \frac{1}{zN}$. Hence the n^{th} moment is

$$< N^n >= \frac{1}{z}\int_{e^{-z}}^{1} \frac{1}{N} N^n dN = \frac{1}{nz}(1 - e^{-nz}) \tag{21}$$

corresponding to $N(z = 0) = 1$. For the more realistic atmosphere whose optical density is given by Eq.(1), the *pdf* of the directly transmitted radiation N is given by

$$f_N(N) = \frac{2}{3dNz} e^{-2\sqrt{\frac{\ln(N_o/N)}{d}}}(1 + 2\sqrt{\frac{\ln(N_o/N)}{d}}). \tag{22}$$

4.2 Diffuse reflection, transmission, and absorption by a single slab

The statistical properties of the diffuse radiation field may be calculated directly from Eqs. (17) and (20). For example, the probability densities of $\mathcal{R}, \mathcal{T}$ and $\mathcal{A}$ may be calculated using:

$$P(\mathcal{R}_o) = \frac{f(\tau^*)}{\mid \frac{d\mathcal{R}}{d\tau} \mid \tau = \tau^*} \tag{23}$$

where τ^* is the root of $\mathcal{R}(\tau^*) = \mathcal{R}_o$ and $f(\tau^*)$ the optical depth probability density function. Because of the monotonicity of these functions there is at most one root. This root can be found numerically by Newton's method. In fact, the necessity of finding any roots can be obviated by using a table look up procedure. The probability density function is evaluated say for some $\mathcal{R}_o$, corresponding to some τ^*. The required derivative $\frac{d\mathcal{R}}{d\tau}$ evaluated at τ^* can be calculated analytically, but in this paper was evaluated numerically using finite differences.

4.3 Effects of fluctuations in $\tilde{\omega}_o$ on the diffuse radiation for a single layer cloud

The role of fluctuations in the single scattering albedo on the diffuse radiation field in a cloud has received little attention in the literature. This situation is quite understandable because of the difficulty and expense involved in making *in situ* microphysical measurements in cloud from which $\tilde{\omega}_o$ is determined. This information is typically obtained from sampling along one dimensional aircraft flight paths [24]. The most relevant microphysical information about clouds from which $\tilde{\omega}_o, \tau$ and g are inferred is the particle size distribution $\eta(r)$. This distribution defines the number of water droplets in a given size range per unit volume of cloud. In practice, the optical properties of different cloud types can be usefully represented by parameters derived from moments of this distribution. One such parameter is the effective radius defined by:

$$r_{eff} = \frac{\int_{r_{min}}^{r_{max}} r\pi r^2 \eta(r) dr}{\int_{r_{min}}^{r_{max}} \pi r^2 \eta(r) dr} \tag{24}$$

and is often used to calculate $\tilde{\omega}_o$ and other optical parameters.[8] In this study we use the recent aircraft data [24] from which *pdf*'s for r_{eff} were actually derived. It is a common practice to analyze aircraft data by averaging information about cloud microphysics and radiation along the flight path and then ignore the large variabilities associated with these data.

Two examples of *pdf*'s of r_{eff} are shown in Fig. 2a corresponding to measurements performed on two different days. The *pdf*'s for $\tilde{\omega}_o$ were calculated directly from the *pdf*'s of r_{eff} using the parameterization [8,29]

$$\tilde{\omega}_o \cong 1 - .85\kappa r_{eff} \tag{25}$$

which reasonably approximates the absorption by spherical particles near the infrared portion of the solar spectrum. In this study, we used a value of $\kappa = 1.64 \times 10^{-3}/\mu m$ for the bulk absorption coefficient of water corresponding to a wavelength of $2.16\mu m$. The *pdf*'s for $\tilde{\omega}_o$ are illustrated in Fig. 2b. Although the distributions for $\tilde{\omega}_o$ are narrow, it is relevant to note that small changes in $\tilde{\omega}_o$ can significantly alter the albedo, transmission and absorption of the cloud.

5. NUMERICAL RESULTS

In order to investigate the effects of variability in τ^* and $\tilde{\omega}_o$ on the radiation field, two kinds of experiments were performed. The first addressed the sensitivity

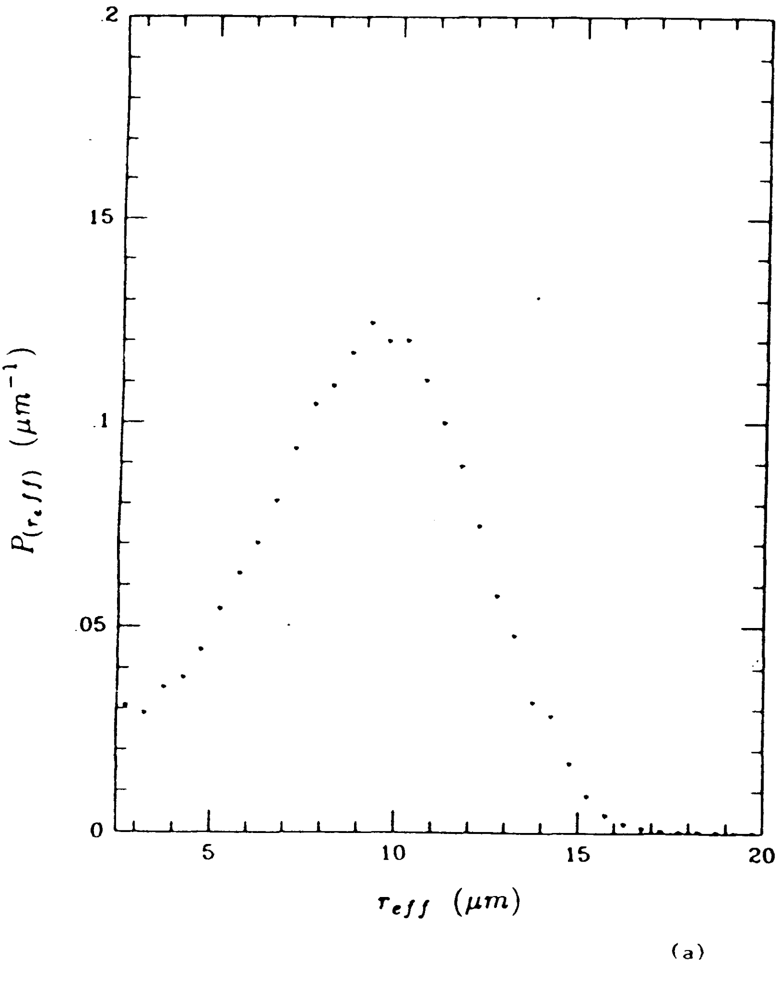


FIG. 2a. Probability density function for the effective radius, r_{eff}, per unit size range (in micrometers) as measured by forward scattering spectrometer on board aircraft on two different days.[24]

of $\mathcal{R}, \mathcal{T}$, and $\mathcal{A}$ to fluctuations in optical depth as generated by Eq. (1) keeping $\tilde{\omega}_o$ fixed; the second assessed the effects of a variable $\tilde{\omega}_o$ (as inferred above using experimental data) on the radiation field, fixing τ^*. These experiments were performed in the manner described in section 3.2.

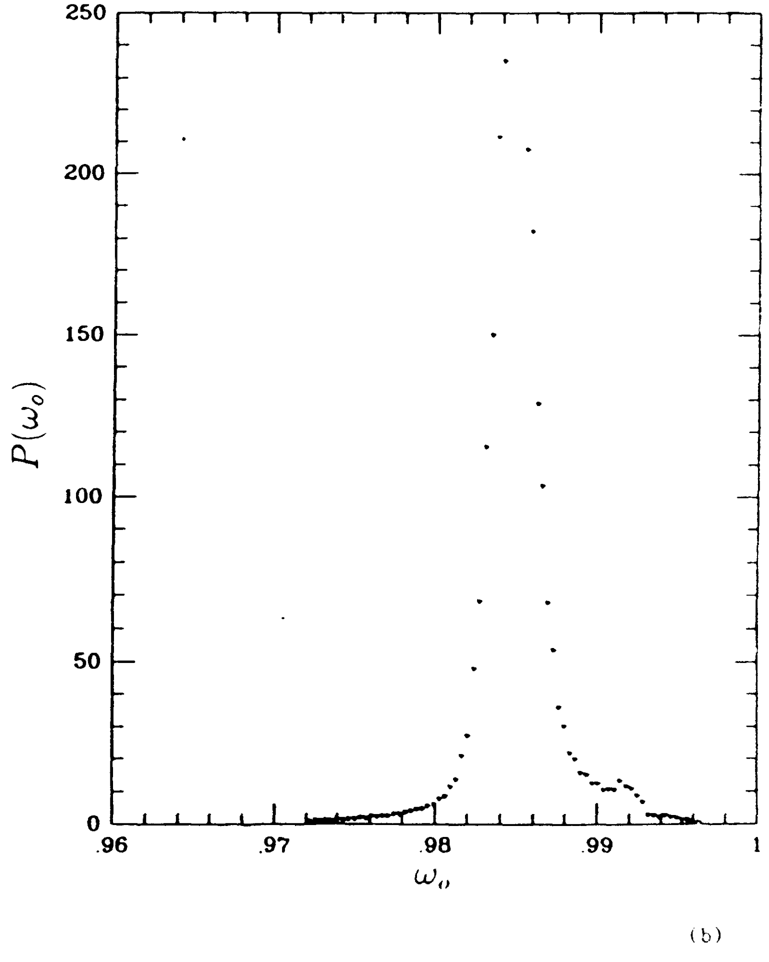


FIG. 2b. Probability density function of $\tilde{\omega}_o$ as derived from Fig. 2a, for $\lambda = 2.16\mu m$.

5.1 Statistical variations in the optical depth

Figures 3 and 4 present the *pdf*'s and statistical moments of the albedo transmission and absorption obtained as the outcome of experiments in which $\tilde{\omega}_o$ and g were fixed but τ^* allowed to fluctuate. These figures show the sensitivity of the bulk radiative properties to variations in the optical properties of the medium. Many

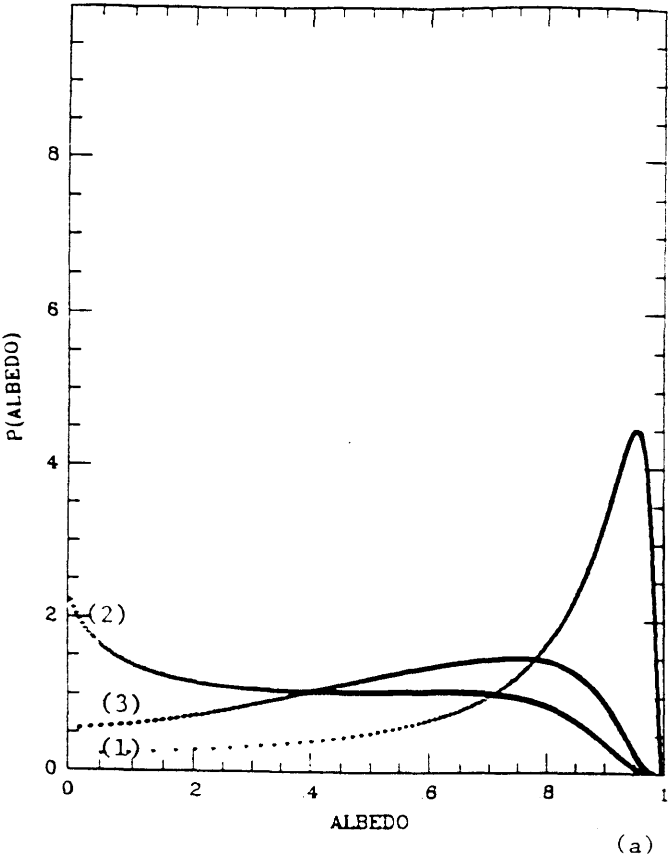


FIG. 3. Probability density function and associated moments of the bulk radiative quantities for conservative scattering ($\bar{\omega}_o = 1$). These figures illustrate the sensitivity of the *pdf's* of these quantities to the asymmetry factor (g) and zenith angle (θ_o) as well as the decay rates of the moments. In all cases $< \tau^* >= 16.53$.

features of these figures can be reasoned from deterministic considerations. For example, in the case of a conservatively scattering plane-parallel atmosphere, the *pdf's* of $\mathcal{R}$ and $\mathcal{T}$ must be mirror images of one another since $\mathcal{R} + \mathcal{T} = 1$. This is borne out in Fig. 3. These figures contrast the large differences between isotropic (g=0) and anisotropic (e.g. g=.85) phase functions for overhead ($\mu_o = 1$) illumination.

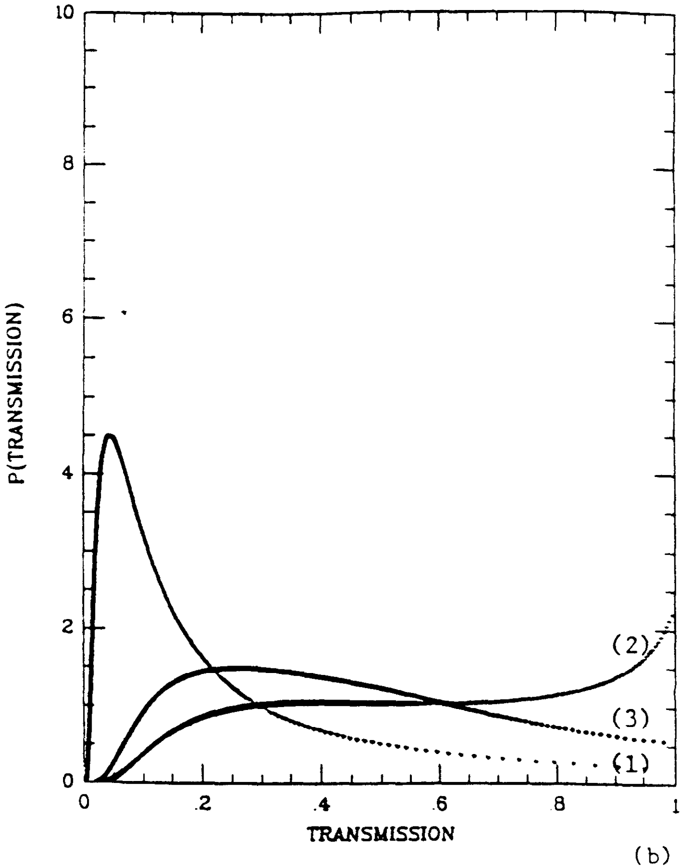


Fig. 3 Continued

As expected these differences are also reflected by the slowly decreasing moments shown in Figs. 3 and 4. As in deterministic transfer, the albedo of a medium which scatters isotropically is greater than one in which the scattering is predominantly in the forward direction at large optical depths. By contrast, the transmission of an anisotropically scattering medium exceeds that of its isotropic counterpart at small optical depths.

Figure 3 also illustrates the effect of the angle of solar elevation on the *pdf's* and the associated statistical moments for conservative scatter with $g = .85$. As expected, the form of the *pdf's* (and the moments) is very nearly the same for the

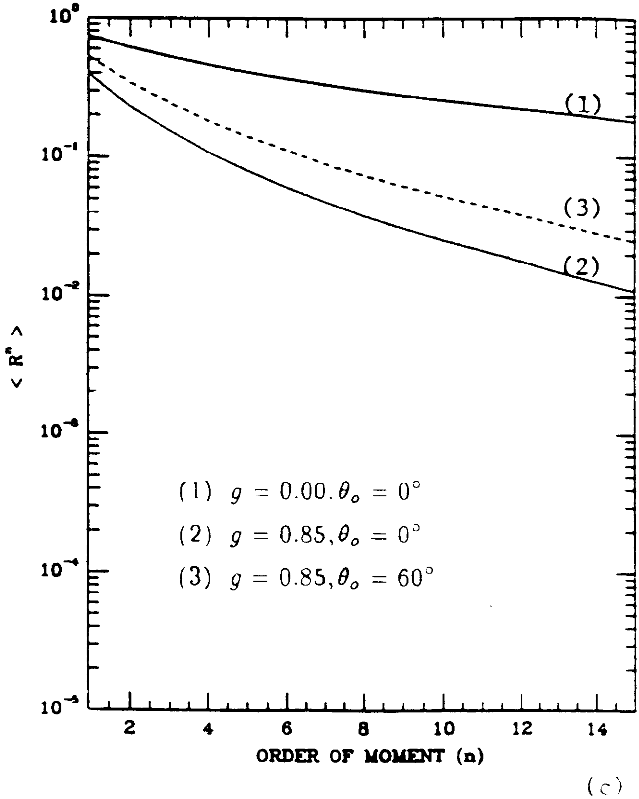
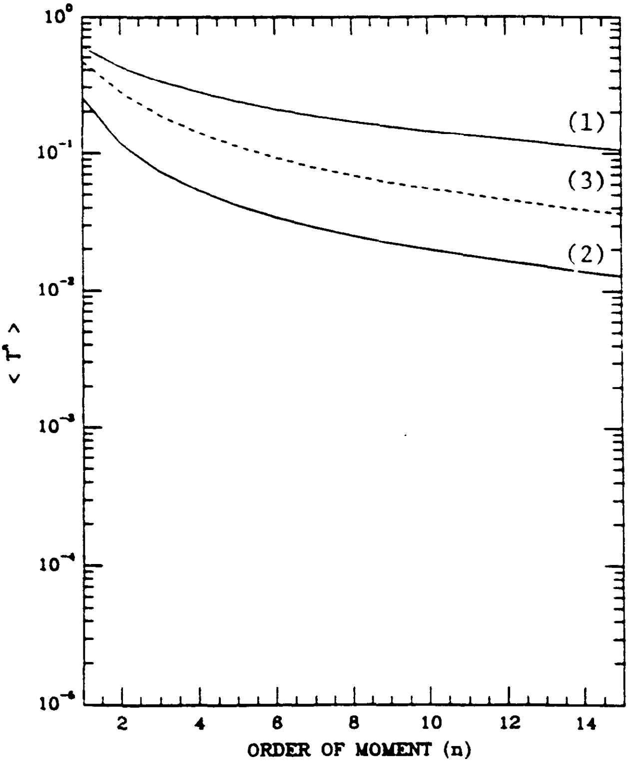


Fig. 3 Continued

two solar zenith angles chosen with the exception of a slightly larger albedo for the 60^o case at large optical depths and less transmission at the other extreme for the same solar elevation. This is due mainly to the larger path lengths traversed by the photon when the sun is low.

When a small amount of absorption is introduced (in this case modeled with $\tilde{\omega}_o = 0.9, 0.95$ and 0.98 which fall in the range for water droplets at near-infrared wavelengths), a limit to the albedo and absorption is established which deterministically corresponds to the largest optical depths sampled. By comparison, the *pdf* for transmission (central panels of Fig. 4) becomes broader as a consequence of the conservation of energy ($\mathcal{R} + \mathcal{T} + \mathcal{A} = 1$).



(d)

Fig. 3 Continued

It is clear then that the only effect exerted by vertical fluctuations is to spread out the bulk radiative quantities about the mean. This dispersion is easily calculated from the *pdf*'s. In many applications, such as in closure models of atmospheric turbulence that include effects of radiative transfer, it is perhaps desirable that the variability of the bulk radiative quantities be expressed in terms of just a few of its moments (the mean and second moment for instance). Evidently, just two moments are not sufficient to capture the full variability as shown in the bottom 3 panels of Fig. 4: higher moments may contribute substantially to the error about the mean. The rates of decay of the higher albedo and absorption moments appear to be particularly sensitive to $\tilde{\omega}_o$ such that these rates of decay increase with increasing

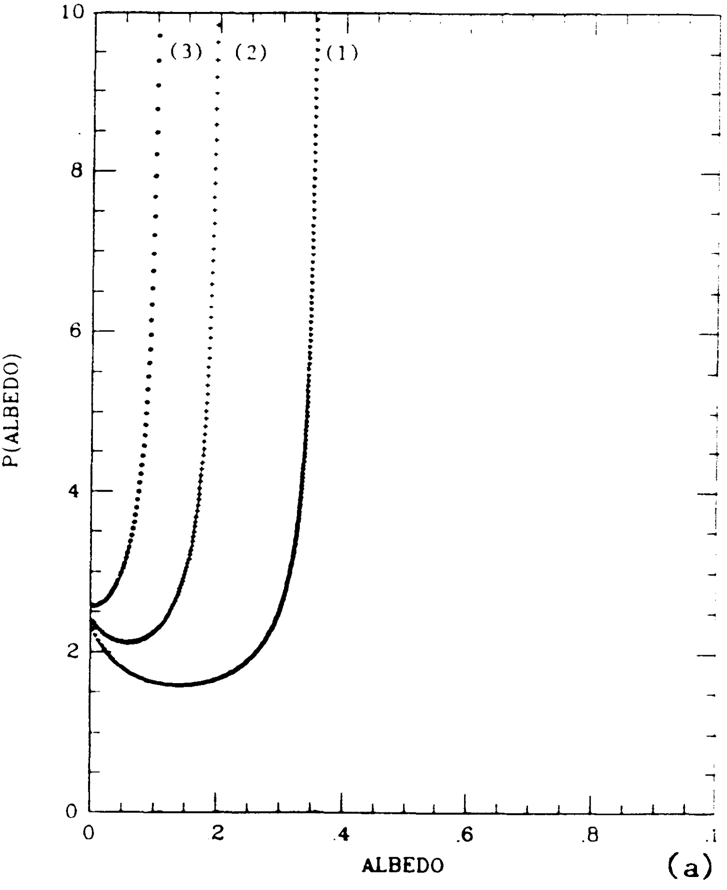


FIG. 4. Probability density function and associated moments of the bulk radiative quantities for non-conservative scattering ($\tilde{\omega}_o < 1$). These figures illustrate the sensitivity of the *pdf*'s and the moments of these quantities to the single scatter albedo for g=0.85 and overhead illumination. The decay rates of the moments are also shown as functions of $\tilde{\omega}_o$. In all cases $< \tau^* >= 16.53$. Curve 1 refers to $\tilde{\omega}_o = .98$, curve 2 to $\tilde{\omega}_o = .95$ and curve 3 to $\tilde{\omega}_o = .90$.

droplet absorption. Table 2 presents a comparison between the statistical averages of the bulk radiative quantities determined from the calculations described above compared to those obtained by directly inserting $< \tau^* >$ into Eqs. (14–16). In

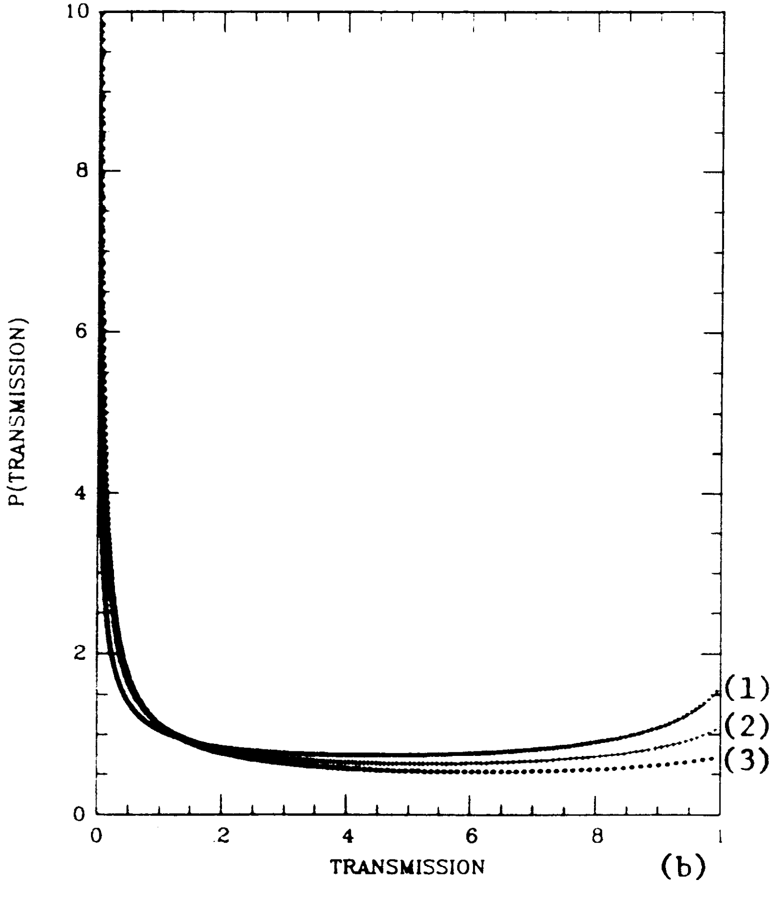


Fig. 4 Continued

general, the effects of vertical variability of the medium on the radiative transfer is to significantly enhance the transmission and reduce both the albedo and absorption.

5.2 Statistical variations in the single scattering albedo

The sensitivity of $\mathcal{R}, \mathcal{T}$, and $\mathcal{A}$ to variations in $\tilde{\omega}_o$ is examined using the *pdf*'s described in section 4.3. The distributions for $\tilde{\omega}_o$ shown in Fig. 2 were used to obtain the *pdf*'s of $\mathcal{R}, \mathcal{T}$, and $\mathcal{A}$ for thin and moderately thick cloud decks having $\tau^* = 4$ and 16.5 respectively. The results of these calculations are shown in Fig.

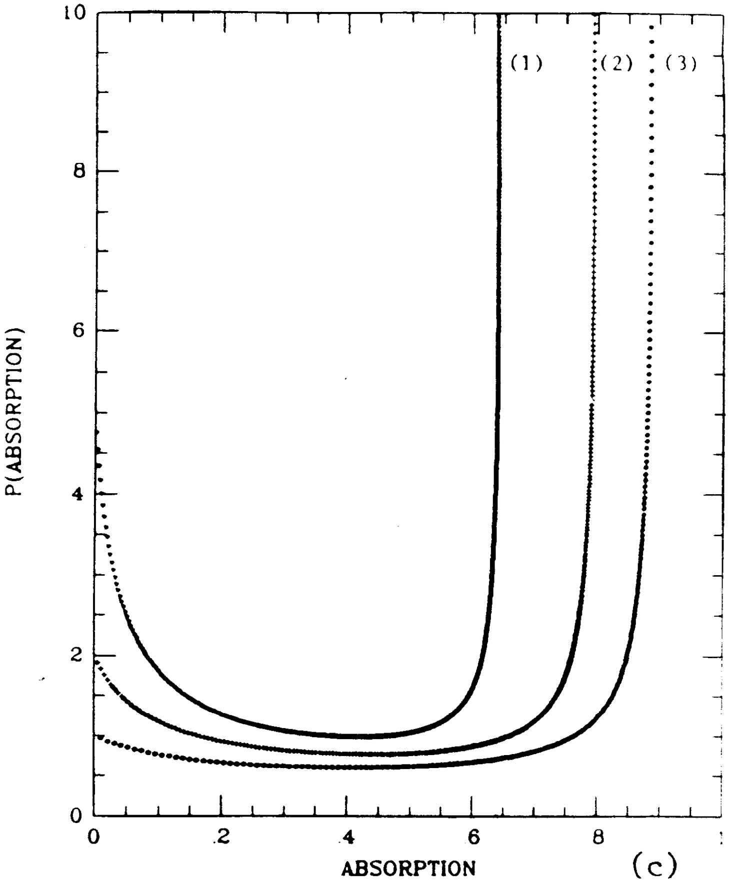


Fig. 4 Continued

5. The ensemble averages of the bulk radiative quantities are compared to those obtained by directly inserting $< \tilde{\omega}_o >$ into Eqs. (17) and (20) in Tables 3a and b. The higher order moments could not be accurately calculated due to the small step size $\Delta\tilde{\omega}_o$ required, but not afforded by the data. Unlike variations in the optical depth, fluctuations in $\tilde{\omega}_o$ suggest a small increase in the albedo and transmission and a reduction in the absorption over the deterministic values.

6. SUMMARY AND CONCLUSION

This paper describes radiative transfer through a single cloud layer that is horizontally uniform but statistically distributed in the vertical and compares the ra-

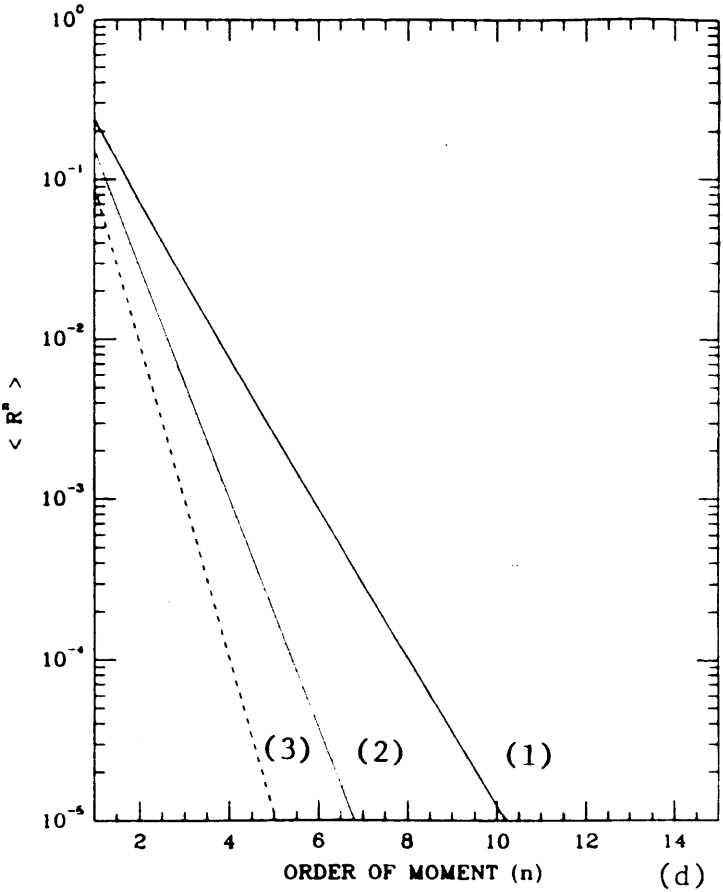


Fig. 4 Continued

diative transfer in such a statistical cloud to its deterministic counterpart. The study is partly motivated by the widespread modeling of clouds as plane-parallel slabs and addresses the errors incurred when ensemble mean optical properties are applied to model vertically uniform clouds. By ascribing a stochastic character to the optical depth and single scattering albedo, the resulting statistical variability of the bulk radiative quantities was investigated. The numerical results presented in this paper are limited to single-layered clouds whose asymmetry factor was fixed in any given simulation. Variations in the optical depth and single scattering albedo

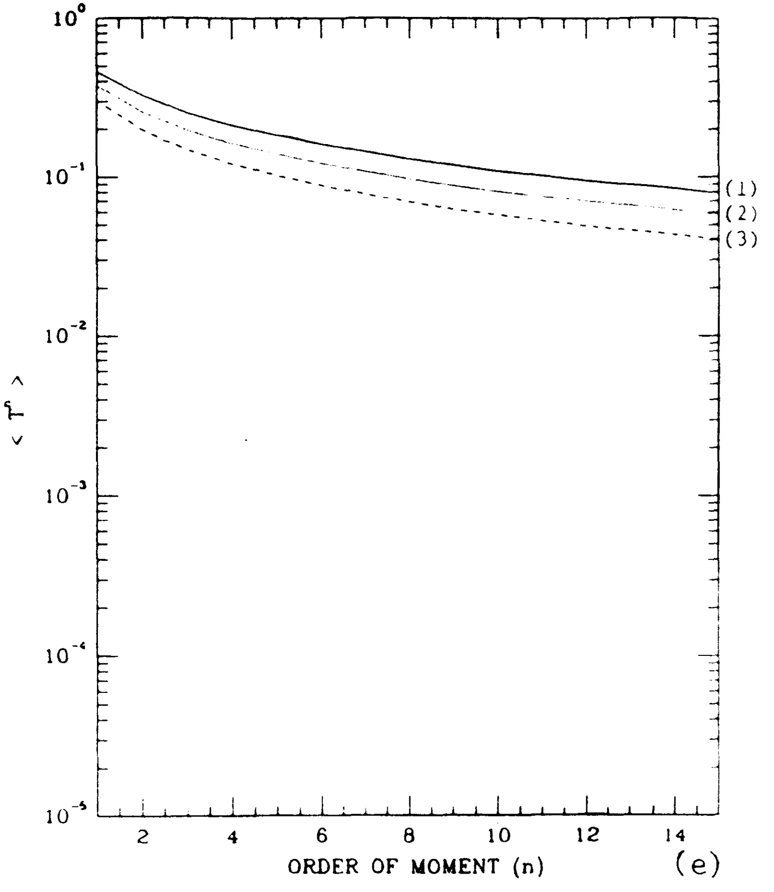


Fig. 4 Continued

were described by empirical *pdf*'s obtained from observations of clouds. The purpose of the simulations described in this paper is to yield quantitative information about the *pdf*'s and moments of the reflection, absorption and transmission of solar radiation through clouds. The results of these numerical experiments are summarized in Tables 2 and 3. According to these results, it is shown how the observed statistical variability of cloud optical depth exerts a substantial effect on the radiative properties of the clouds. It was demonstrated that such variability: (1) reduces the albedo; (2) enhances the diffusely transmitted flux and (3) diminishes the ab-

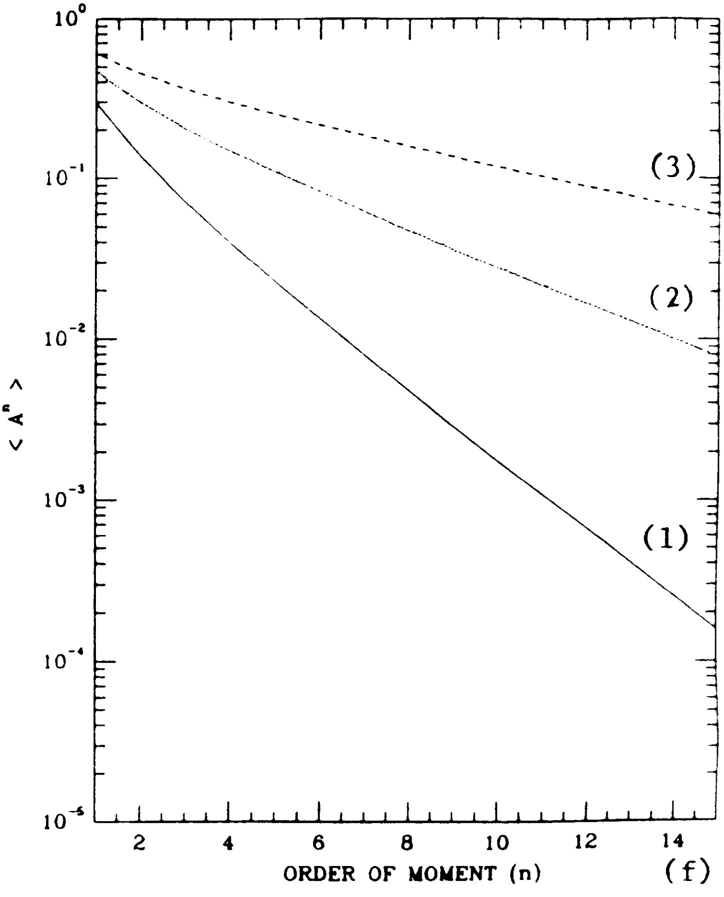


Fig. 4 Continued

sorption within the cloud. These results generally agree with those of reference (18) and others although we show the effects of statistical variability to be much larger than these earlier studies. It is also demonstrated that the rate at which the moments decay differs for reflection and transmission in a way that is influenced by the absorption of cloud droplets and the asymmetry of droplet scattering. Unlike variations in the optical depth, fluctuations in single scattering albedo give rise to small increases in the albedo and transmission and a reduction in the absorption. These results suggest that fluctuations in the single scattering albedo are not as significant as fluctuations in the optical depth.

TABLE 2: Stochastic computations of bulk radiative quantities for a single plane-parallel layer of variable optical depth. The means and standard deviations of the albedo, transmission and absorption are shown. Their deterministic equivalents corresponding to $<\tau>= 16.53$ are indicated in parentheses.

(a) Conservative scattering

g	θ_o	$<\bar{R}>$	$\sigma_{\bar{R}}$	$<\bar{T}>$	$\sigma_{\bar{T}}$
0.00	0.	0.748 (0.907)	±0.235	0.251 (0.093)	±0.237
0.85	0.	0.534 (0.564)	±0.247	0.465 (0.436)	±0.246
0.85	60	0.402 (0.694)	±0.265	0.597 (0.306)	±0.264

(b) Non-conservative scattering with $g = 0.85$ and $\theta_o = 0^o$

$\tilde{\omega}_o$	$<\bar{R}>$	$\sigma_{\bar{R}}$	$<\bar{T}>$	$\sigma_{\bar{T}}$	$<\bar{A}>$	$\sigma_{\bar{A}}$
0.98	0.240 (0.341)	±0.118	0.460 (0.217)	±0.470	0.296 (0.442)	±0.220
0.95	0.154 (0.203)	±0.062	0.378 (0.091)	±0.337	0.467 (0.706)	±0.277
0.90	0.092 (0.111)	±0.030	0.303 (0.027)	±0.325	0.604 (0.862)	±0.301

Despite the limitations of the study, the results point to problems of the retrieval of the solar insolation at the earth's surface using measurements of reflected solar radiation. The computations indicate that the variability of solar radiation transmitted through clouds (and hence of the surface insolation) is different in nature than the variability of radiation reflected from cloud top and monitored by satellites. In all cases discussed, there were differences between statistical and deterministic transfer. In some cases, these differences were substantial. Further theoretical developments in stochastic multidimensional transfer are needed to address these issues.

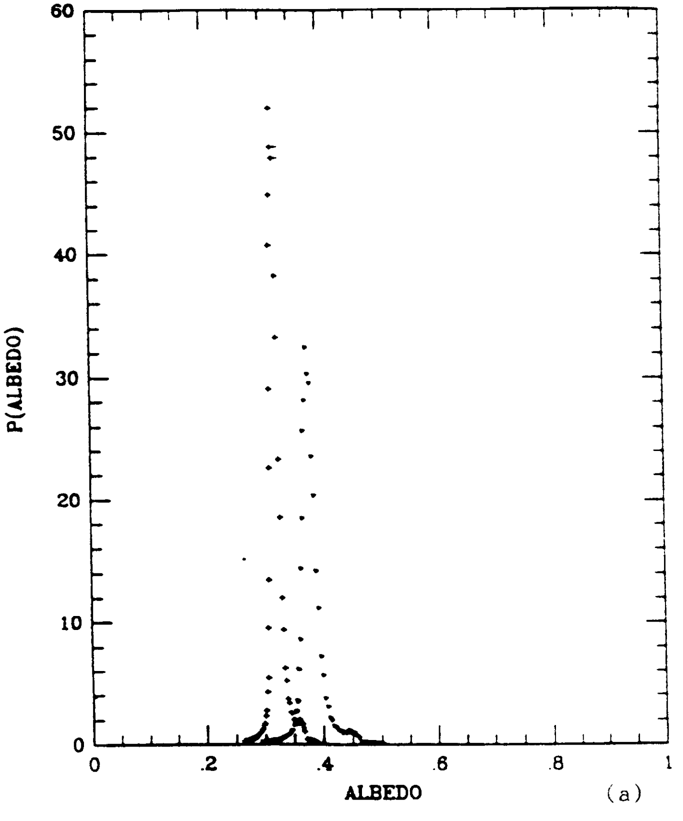


FIG. 5a. Probability density functions of the albedo for $\lambda = 2.16\mu m$. Triangles and crosses serve to distinguish between measurements performed on different days. In this figure, $\tau^* = 4.00$.

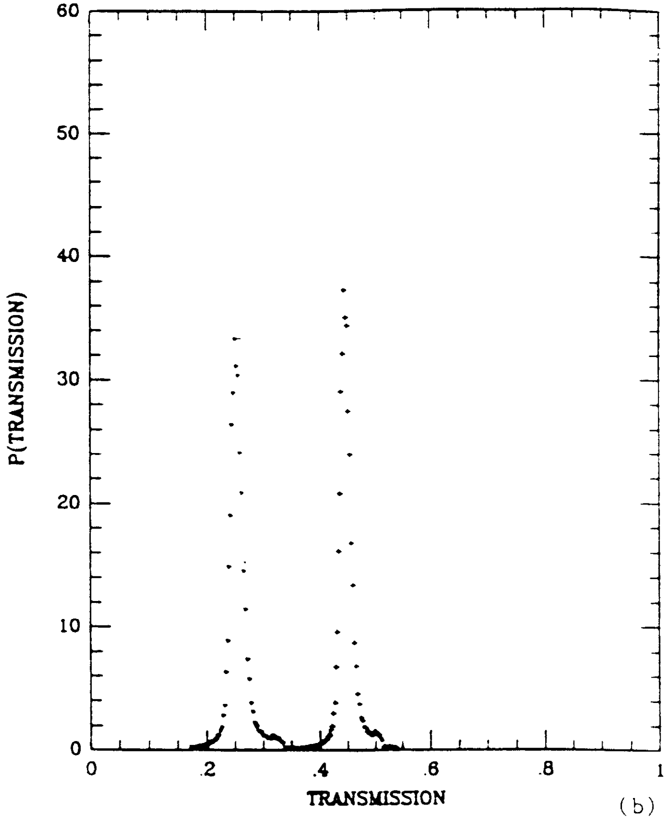


FIG. 5b. Probability density functions of the transmission for $\lambda = 2.16\mu m, \tau^* = 4.00$.

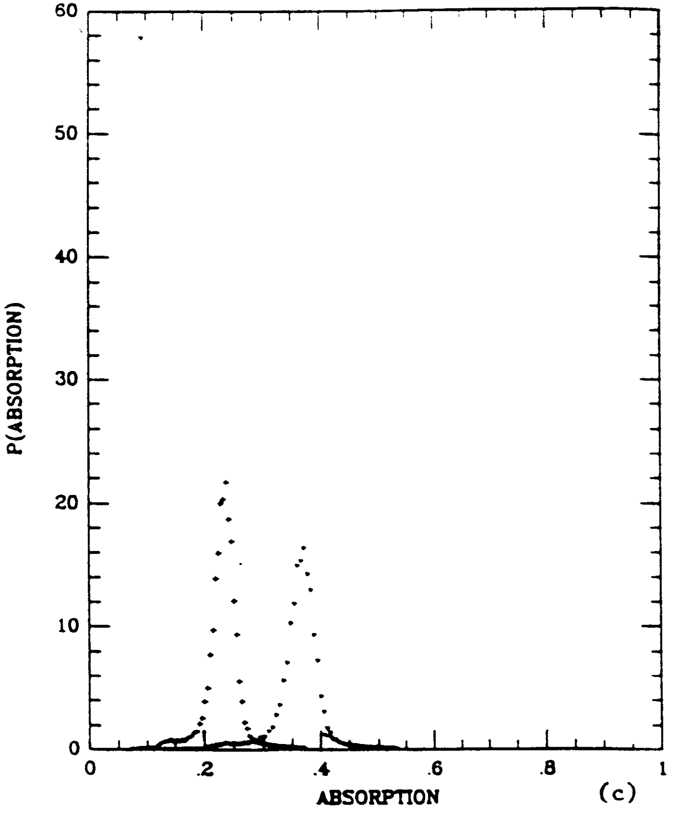


FIG. 5c. Probability density functions of the absorption for $\lambda = 2.16\mu m, \tau^* = 4.00$.

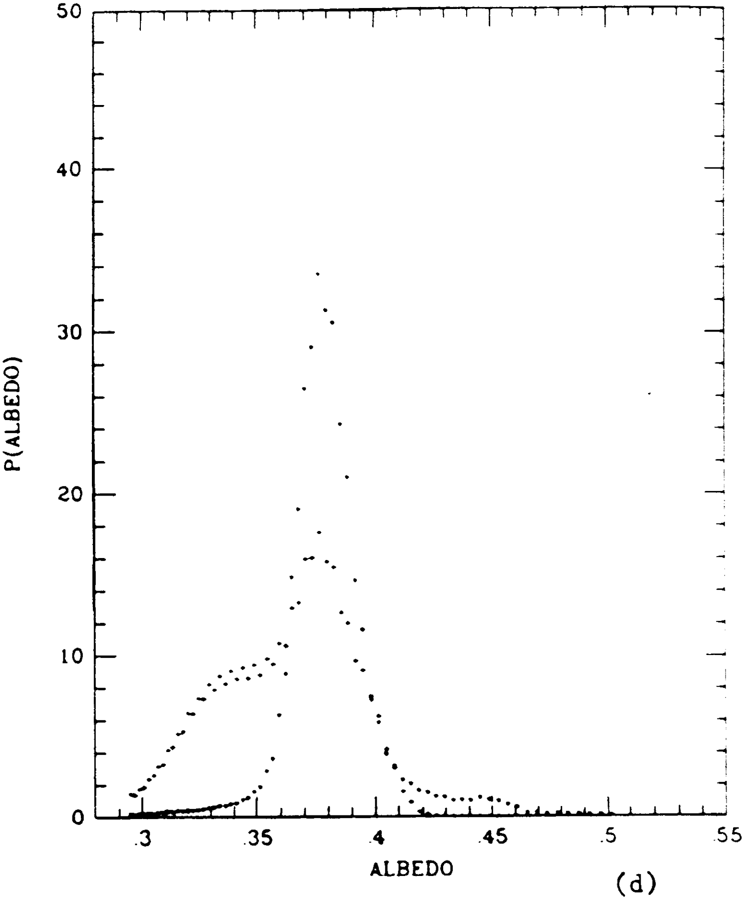


FIG. 5d. As in Figure 5a, probability density functions of the albedo, $\tau^* = 16.53$.

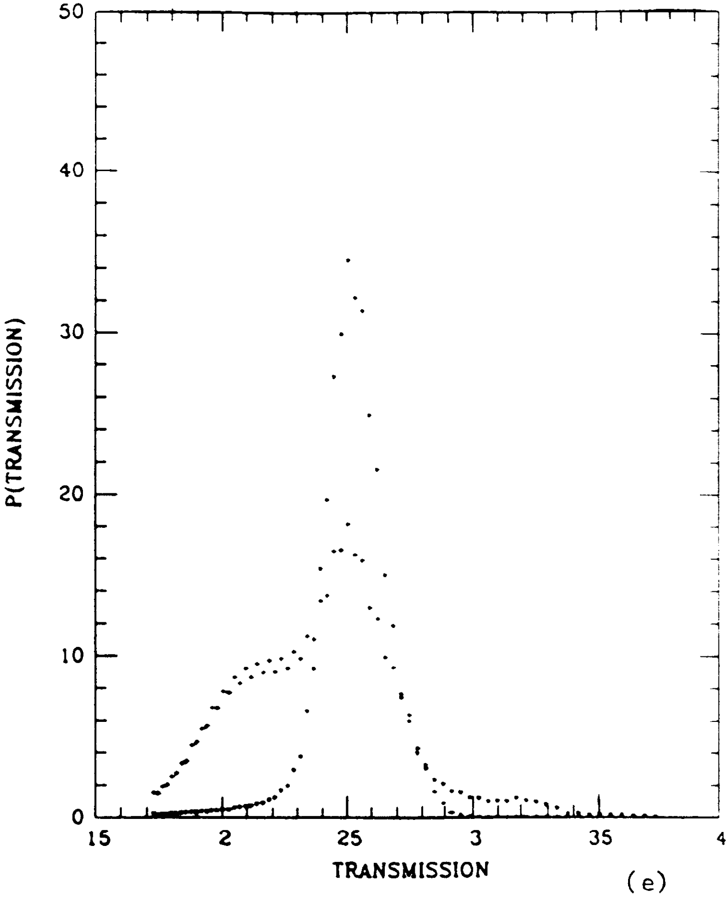


FIG. 5e. As in Figure 5b, probability density functions of the transmission for $\tau^* = 16.53$.

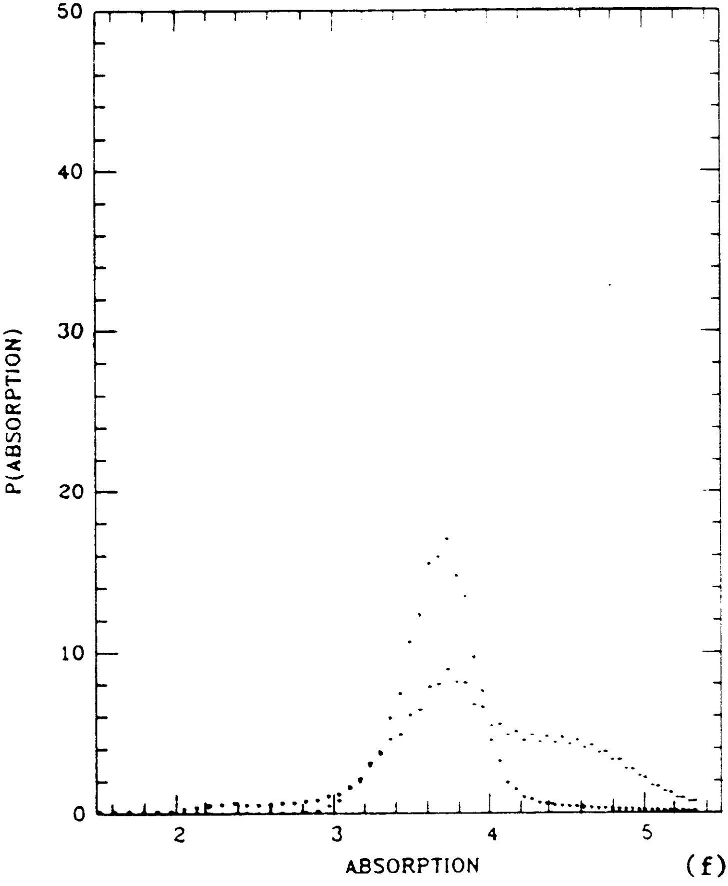


FIG. 5f. As in Figure 5c, probability density functions of the absorption for $\tau^* =$ 16.53.

TABLE 3: Stochastic computations of bulk radiative quantities for a single layer cloud of specified optical depth but statistical $\tilde{\omega}_o$. The deterministic equivalents corresponding to $<\tilde{\omega}_o>= 0.9824$ are indicated in parentheses. This value is associated with $\lambda = 2.16\mu m, g = 0.85$ and $\theta_o = 0°$.

(a)

τ^*	$\mathcal{R}$	$\sigma_{\mathcal{R}}$	$\mathcal{T}$	$\sigma_{\mathcal{T}}$	$\mathcal{A}$	$\sigma_{\mathcal{A}}$
4.00	0.1681 (0.1679)	±0.0043	0.7368 (0.7364)	±0.0086	0.0951 (0.0957)	±0.0181
16.53	0.3617 (0.3597)	±0.0273	0.2362 (0.2341)	±0.0264	0.4021 (0.4062)	±0.0541

(b) As in Table 3a with $\tilde{\omega}_o = 0.9850$.

τ^*	$\mathcal{R}$	$\sigma_{\mathcal{R}}$	$\mathcal{T}$	$\sigma_{\mathcal{T}}$	$\mathcal{A}$	$\sigma_{\mathcal{A}}$
4.00	0.1716 (0.1715)	±0.0035	0.7462 (0.7461)	±0.0067	0.0822 (0.0824)	±0.0122
16.53	0.3821 (0.3811)	±0.0214	0.2557 (0.2548)	±0.0205	0.3622 (0.3641)	±0.0416

Acknowledgments. This study was supported by the National Science Foundation through Grants ATM-8812353, ATM-8907414 and DOD-NAVY-ONR N00014-87-K- 0228/P00001 to Colorado State University. We gratefully acknowledge the assistance of T. Greenwald in the preparation of the satellite images, S. Lini for her help in completing the tables and A. Lini for her typing of the manuscript. We also wish to acknowledge the useful comments of the external referees.

7. REFERENCES

1. R.M. Welch and W.G. Zdunkowski, Contrib. Atmos. Phys., **54**, 258 (1981).

2. R.M. Welch and W.G. Zdunkowski, Contrib. Atmos. Phys., **54**, 273 (1981).

3. R.F. Cahalan, Landsat Observations of Fractal Cloud Structure. In *Scaling, Fractals and Non-Linear Variability in Geophysics*, Eds. D. Schertzer and S. Lovejoy, Kluwer (in Press).

4. R.A. Schiffer and W.B. Rossow, Bull. Amer. Meteor. Soc., **76**, 779 (1983).

5. R.T. Wetherald and S. Manabe, J. Atmos. Sci., **37**, 1485 (1980).

6. G.L. Stephens, J. Atmos. Sci., **45**, 1837 (1988).

7. P.S. Webster, Proceedings of the US TOGA Western Pacific Air-Sea Interaction Workshop, Honolulu, 16-18 Sept., 1987. R. Lucas and P. Webster (Eds.), US TOGA Rept., US-TOGA-8, U. Corp. Atmos. Res., **71** (1988).

8. G.L. Stephens and S.-C. Tsay, On the Cloud Absorption Anomaly, Quart. J. Roy. Meteor. Soc., in press (1990).

9. W.J. Wiscombe, R.M. Welch, W.D. Hall, J. Atmos. Sci., **41**, 1336 (1984).

10. D.J. Diner and J.V. Martonchik, J. Quant. Spectros. Radiat. Transfer, **31**, 97 (1984).

11. A. Davis, P. Gabriel, S. Lovejoy, D. Schertzer, G.L. Austin, Discrete Angle Radiative Angle Radiative Transfer Part III: Numerical Results and Applications. J. Geophys. Res., in press (1990).

12. G.L. Stephens, J. Quant. Spectros. Radiat. Transfer, **36**, 51 (1986).

13. A.L. Crosbie and T.L. Linsenbardt, J. Quant. Spectros. Radiat. Transfer, **19**, 257 (1978).

14. G.L. Stephens and C.M.R. Platt, J. Climate Appl. Meteor., **26**, 1243 (1987).

15. V.I. Drobyshevich, Izv., Atmos. Oceanic Phys., **3**, 1062 (1967).

16. M. Malkevitch, A.S. Monin, G. Rozenberg, Izv., Acad. Sci., USSR, Geophys. Ser., **3**, 237 (1964).

17. Y.R. Mullamaa, M.A. Sulev, V.K. Poldmaa, H.A. Ohvril, H.J. Niylisk, M.I. Allenov, L.G. Tchubakov, NASA Technical Translation TT-F-822 (1972).

18. K. Ronholm, M.B. Baker, H. Harrison, J. Atoms. Sci., **37**, 1279 (1980).

19. G.A. Mikhaylov, Izv., Atmos. Oceanic Phys., **18**, 993 (1982).

20. G. C. Pomraning, C. D. Levermore, and J. Wong, *Lecture Notes in Pure and Applied Mathematics*, **115**, 1 (1989).

21. M. L. Adams, E. W. Larson, and G. C. Pomraning, J. Quant. Spectros. Radiat. Transfer **42**, 253 (1989).

22. V.P. Bol'shakov, Izv. Atmos. Oceanic Phys., **11**, 590 (1975).

23. T. Nakajima and M. D. King, Determination of the Optical Thickness and Effective Particle Radius of Clouds from Reflected Solar Radiation Measurements. Part I: Theory, J. Atmos. Sci., in press (1990).

24. T. Nakajima, M. D. King, and J. D. Spinhirne, Determination of the Optical Thickness and Effective Particle Radius of Clouds from Reflected Solar Radiation Measurements. Part II: Marine Stratocumulus Observations. J. Atmos. Sci. in press (1990).

25. R. M. Welch, S. K. Cox, J. Davies, Meteor. Monogr., **39**, Amer. Meteo. Soc., Boston (1980).

26. W.E. Meador and W.R. Weaver, J. Atmos. Sci., **37**, 630 (1980).

27. M.D. King and Harshvardhan, J. Atmos. Sci., **43**, 784 (1986).

28. A. Papoulis, *Random Variables and Stochastic Processes*, McGraw Hill (1976).

29. S. Twomey and C. F. Bohren, J. Atmos. Sci., **37**, 2086 (1980).

Received: December 7, 1989
Revised: November 19, 1990
Accepted: December 6, 1990