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ON THE SOLUTION OF THE TRANSPORT EQUATION WITH PERIODIC BOUNDARY CONDITIONS BY THE METHOD OF DISCRETE ORDINATES

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ABSTRACT

The one-dimensional transport equation in slab geometry with periodic boundary conditions is studied. It reduces to the integral equation of the Peierls type. The spectral radius of the integral operator is estimated. We analyze the discrete-ordinates algorithm for estimating the solution. Convergence is proved and methods of estimating the rate of convergence are described. The estimates of some quadrature rules are derived by way of an example. The numerical results confirm the convergence properties of the proposed algorithm.

1. INTRODUCTION

We consider the integro-differential transport equation in an anisotropically scattering plane-parallel slab

$$(1.1) \qquad \mu\frac{\partial\phi}{\partial\tau} + \phi(\tau,\mu) = \frac{\lambda}{2}\int_{-1}^{1} g(\mu,\mu')\phi(\tau,\mu')d\mu' + f(\tau,\mu),$$

where the phase function

$$g(\mu,\mu') \equiv \frac{1}{2\pi}\int_{0}^{2\pi} G(\cos\chi)d\psi'$$

is averaged over the azimuth. Here,

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$$\cos\chi = \mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\,\cos(\psi-\psi').$$

The independent variables are $\tau \in [0,b]$, with b representing the optical slab thickness and $\mu \in [-1,1]$. The dependent variable $\phi(\tau,\mu)$ represents the neutron angular flux, and $f(\tau,\mu)$ is a given nonnegative function which represents the contribution from inner sources. The parameter λ is given.

We add the boundary conditions corresponding to a periodic problem[9]

$$(1.2)\qquad \begin{cases} \phi(0,\mu) = \kappa_1\, \phi(b,\mu) + V_0(\mu), & \mu > 0, \\ \phi(b,\mu) = \kappa_2\, \phi(0,\mu) + V_b(\mu), & \mu < 0, \end{cases}$$

where $0 \leq \kappa_1, \kappa_2 \leq 1$ and V_0, V_b are given nonnegative functions (outer fluxes).

The purpose of this paper is to develop the discrete ordinates method for the boundary-value problem (1.1) and (1.2), to study convergence and to estimate the rate of convergence of this method.

The plan of the paper is as follows: In section 2 we obtain the integral equation corresponding to (1.1) and (1.2) and analyze its solvability. The estimate of the spectral radius of the integral operator is given in section 3. The algorithm of the solution of the boundary-value problem is described in section 4. Sections 5 and 6 are devoted to the questions of convergence and the rate of convergence of the discrete-ordinates method. In section 7, a numerical example confirming the convergence properties of the algorithm proposed in section 4 is considered.

2. AN INTEGRAL EQUATION

We derive the Peierls integral equation by denoting the right-hand part in (1.1) by

$$(2.1)\qquad x(\tau,\mu) = \frac{\lambda}{2}\int_{-1}^{1} g(\mu,\mu')\phi(\tau,\mu')d\mu' + f(\tau,\mu).$$

Then, solving the boundary-value problem, we can easily obtain

$$(2.2)\qquad \phi(\tau,\mu) = \begin{cases} [\kappa_1\phi(b,\mu)+V_0(\mu)]e^{-\frac{\tau}{\mu}} + \frac{1}{\mu}\int_0^\tau x(\tau',\mu)e^{-\frac{\tau-\tau'}{\mu}}\,d\tau', & \mu > 0, \\ [\kappa_2\phi(0,\mu)+V_b(\mu)]e^{\frac{b-\tau}{\mu}} - \frac{1}{\mu}\int_\tau^b x(\tau',\mu)e^{-\frac{\tau-\tau'}{\mu}}\,d\tau', & \mu < 0. \end{cases}$$

From (2.2), it follows that

$$(2.3)\qquad \begin{cases} \phi(b,\mu) = \dfrac{1}{e^{b/\mu}-\kappa_1}[V_0(\mu) + \frac{1}{\mu}\int_0^b x(\tau',\mu)e^{\frac{\tau'}{\mu}}\,d\tau'], & \mu > 0, \\ \phi(0,\mu) = \dfrac{1}{e^{-b/\mu}-\kappa_2}[V_b(\mu) - \frac{1}{\mu}\int_0^b x(\tau',\mu)e^{-\frac{b-\tau'}{\mu}}\,d\tau'], & \mu < 0, \end{cases}$$

and, substituting (2.3) and (2.2) into (2.1), we obtain the integral equation

$$x(\tau,\mu) = \frac{\lambda}{2}\int_0^b\int_0^1 [K_0(\tau,\tau',\mu,\mu')x(\tau',\ \mathrm{sign}(\tau-\tau')\mu')$$

$$(2.4)\qquad + K_1(\tau,\tau',\mu,\mu')x(\tau',\mu') + K_2(\tau,\tau',\mu,\mu')x(\tau',-\mu')]d\mu'd\tau'$$

$$+ f^0(\tau,\mu),$$

where

$$K_0(\tau,\tau',\mu,\mu') = g(\mu,\mathrm{sign}(\tau-\tau')\cdot\mu')\frac{e^{-|\tau-\tau'|/\mu'}}{\mu'},$$

$$K_1(\tau,\tau',\mu,\mu') = \frac{\kappa_1}{\mu'}\, g(\mu,\mu')\frac{e^{-(\tau-\tau')/\mu'}}{e^{b/\mu'}-\kappa_1},$$

$$K_2(\tau,\tau',\mu,\mu') = \frac{\kappa_2}{\mu'}\, g(\mu,-\mu')\frac{e^{-(\tau'-\tau)/\mu'}}{e^{b/\mu'}-\kappa_2},$$

$$f^0(\tau,\mu) = \frac{\lambda}{2}\int_0^1 [g(\mu,\mu')\frac{V_0(\mu')e^{(b-\tau)/\mu'}}{e^{b/\mu'}-\kappa_1}$$

$$+ g(\mu,-\mu')\frac{V_b(-\mu')e^{\tau/\mu'}}{e^{b/\mu'}-\kappa_2}]d\mu' + f(\tau,\mu).$$

In operator notation, (2.4) is written as:

(2.5) $$x = Rx + f^0,$$

where the operator R is defined by

(2.6) $$(Rx)(\tau,\mu) = \frac{\lambda}{2}\int_0^b \sum_{i=0}^{2} T_i^x(\tau,\tau',\mu)d\tau', \quad R:Q^p \to Q^p,$$

with

(2.7) $$T_i^x(\tau,\tau',\mu) = \int_0^1 K_i(\tau,\tau',\mu,\mu')x(\tau',\nu_i(\tau-\tau')\mu')d\mu', \quad i = 0,1,2,$$

and

(2.8) $$\nu_i(t) = \begin{cases} \text{sign } t, & i = 0, \\ 1, & i = 1, \\ -1, & i = 2. \end{cases}$$

Here Q^p is the space of functions $f(\tau,\mu)$ such that $f \in L^p_{[0,b]}$, $1 \le p \le \infty$ with respect to τ and $f \in C_{[-1,1]}$ with respect to μ, with $\| f \|_p = \max_{-1\le\mu\le 1} \| f \|_{L^p}$.

Let the phase function g be continuous on $[-1,1]\mathrm{x}[-1,1]$ and be normalized by the condition,

(2.9) $$\int_{-1}^{1} g(\mu,\mu')d\mu' = 2, \quad \mu \in [-1,1].$$

Then we have

Lemma 2.1. The operator R is compact in $Q^{\infty}\to Q^{\infty}$. If

(2.10) $\quad g(\mu,\mu') \geq q > 0, \quad \mu,\mu' \in [-1,1],$

then

(2.11) $\quad \| R \|_{Q^{\infty}\to Q^{\infty}} < \lambda$

for any κ_1, κ_2 such that

(2.12) $\quad \kappa_1 + \kappa_2 < 2.$

Proof. Let us divide the operator R into two operators $R = R^{+}+R^{-}$, where

$$(R^{+}x)(\tau,\mu) = \frac{\lambda}{2}\Big[\int_0^{\tau} T_0^x(\tau,\tau',\mu)d\tau' + \int_0^b T_1^x(\tau,\tau',\mu)d\tau'\Big]$$

and

$$(R^{-}x)(\tau,\mu) = \frac{\lambda}{2}\Big[\int_{\tau}^{b} T_0^x(\tau,\tau',\mu)d\tau' + \int_0^b T_2^x(\tau,\tau',\mu)d\tau'\Big].$$

We show the compactness of the operator R^{+} (for R^{-} the proof is analogous). Let $\varepsilon > 0$ and

$$(R_{\varepsilon}^{+}x)(\tau,\mu) = \frac{\lambda}{2}\int_{\varepsilon}^{1}\Big[\int_0^{\tau} K_0(\tau,\tau',\mu,\mu')x(\tau',\mu')d\tau'$$
$$+ \int_0^b K_1(\tau,\tau',\mu,\mu')x(\tau',\mu')d\tau'\Big]d\mu'.$$

The operator R_{ε}^{+} is compact, because the set $\{R_{\varepsilon}^{+}x\}$ is uniformly bounded and equicontinuous (kernels K_0 and K_1 are continuous because the singularity at $\mu' = 0$ is absent). But

$$\sup_{\tau,\mu}|R_\varepsilon^+ x - R^+ x| = \sup_{\tau,\mu} \frac{\lambda}{2} \left| \int_0^\varepsilon \left[\int_0^\tau K_0(\tau,\tau',\mu,\mu')x(\tau',\mu')d\tau' \right.\right.$$

$$\left.\left. + \int_0^b K_1(\tau,\tau',\mu,\mu')x(\tau',\mu')d\tau' \right] d\mu' \right|$$

$$\leq \frac{\lambda}{2} g_{max} \|x\|_\infty \sup_\tau \int_0^\varepsilon \frac{1}{\mu'} \left[\int_0^\tau e^{-(\tau-\tau')/\mu'} d\tau' \right.$$

$$\left. + \frac{\kappa_1}{e^{b/\mu'} - \kappa_1} \int_0^b e^{-(\tau-\tau')/\mu'} d\tau' \right] d\mu'$$

$$= \frac{\lambda}{2} g_{max} \|x\|_\infty \sup_\tau \int_0^\varepsilon \left[1 - \frac{e^{-\tau/\mu'}(1-\kappa_1)}{1-\kappa_1 e^{-b/\mu'}}\right] d\mu'$$

$$\leq \frac{\lambda}{2} g_{max} \|x\|_\infty \int_0^\varepsilon d\mu' \to 0,$$

as $\varepsilon \to 0$. From this it follows that R^+ is compact as a uniform limit of the sequence of the compact operators.

Next we estimate its norm. The following inequality is easily obtained

$$\|Rx\|_\infty \leq \frac{\lambda}{2} \|x\|_\infty \sup_{\tau,\mu} \left| \int_0^1 \left\{ g(\mu,\mu')\left[1 - \frac{e^{-\tau/\mu'}(1-\kappa_1)}{1-\kappa_1 e^{-b/\mu'}}\right] \right.\right.$$

(2.13)

$$\left.\left. + g(\mu,\mu')\left[1 - \frac{e^{-(b-\tau)/\mu'}(1-\kappa_2)}{1-\kappa_2 e^{-b/\mu'}}\right] \right\} d\mu' \right|.$$

Taking into account the condition of normalizing the phase function

(2.9), the assumption (2.10), and the fact that if (2.12) holds, the expressions in square brackets are simultaneously equal to 1 only at the point $\mu' = 0$, we arrive at the estimate (2.11). This completes the proof.

Remark 2.1. In the case of $\kappa_1 = \kappa_2 = 1$, one can prove that the equality $Rc = \lambda c$, where c is a constant, is valid. From this and from (2.13) it immediately follows that $\|R\|_{Q^\infty \to Q^\infty} = \lambda$.

3. THE SPECTRUM OF THE TRANSFER OPERATOR AND THE *DIFFERENTIABILITY PROPERTIES OF THE SOLUTIONS*

Following Keller[6], we shall estimate the spectral radius of the integral operator (2.6) by using sufficient conditions for the existence of only trivial solutions to the homogeneous equation corresponding to (2.5).

Taking into account the nonnegativity of kernels K_i, $i = 0,1,2$ we see that the inequality

$$|x(\tau,\mu)| \leq \frac{\lambda}{2} \|x\|_\infty \left| \int_0^b \int_0^1 [K_0+K_1+K_2]d\mu' d\tau' \right|$$

$$= \frac{\lambda}{2} \|x\|_\infty \Big[\int_{-1}^1 g(\mu,\mu')d\mu' - \int_0^1 g(\mu,\mu')e^{-\tau/\mu'}\xi_{\kappa_1}(\mu')d\mu' \tag{3.1}$$

$$- \int_0^1 g(\mu,-\mu')e^{-(b-\tau)/\mu'}\xi_{\kappa_2}(\mu')d\mu' \Big]$$

is valid with

$$\xi_{\kappa_i}(\mu) = \frac{1-\kappa_i}{1-\kappa_i e^{-b/\mu}}, \quad i = 1,2, \quad \mu \in [0,1].$$

It is obvious that $0 \leq \xi_{\kappa_i}(\mu) \leq 1$, $i = 1,2$, $\mu \in [0,1]$ and that

$$\min_{0\le\mu\le 1} \xi_{\kappa_i}(\mu) = \lim_{\mu\to 0+} \xi_{\kappa_i}(\mu) = 1-\kappa_i, \quad i = 1,2.$$

Let $\delta = \min\{1-\kappa_1, 1-\kappa_2\}$; then we obtain from (3.1) that

$$|x(\tau,\mu)| \le \lambda \|x\|_\infty \{1-\delta \int_0^1 [g(\mu,\mu')\frac{e^{-\tau/\mu'}}{2} + g(\mu,-\mu')\frac{e^{-(b-\tau)/\mu'}}{2}]d\mu'\}$$

(3.2)
$$\le \lambda \|x\|_\infty \{1-\delta q \int_0^1 \frac{e^{-\tau/\mu'} + e^{-(b-\tau)/\mu'}}{2} d\mu'\}$$

$$\le \lambda \|x\|_\infty \{1-\delta q \int_0^1 e^{-b/2\mu'} d\mu'\}$$

$$= \lambda \|x\|_\infty \{1-\delta q E_2(b/2)\}.$$

Here we have used the condition (2.9) and the assumption (2.10) about the positivity of the phase function.

Let (2.12) hold; then $\delta > 0$. Since the right-hand part in (3.2) does not depend on τ and μ, we can conclude that

$$\|x\|_\infty \le \lambda[1-\delta q E_2(b/2)] \|x\|_\infty .$$

From this, it follows that the homogeneous equation corresponding to (2.5) has only a trivial solution for $\lambda < \lambda^*$, where

(3.3) $$\lambda^* = [1-\delta q E_2(b/2)]^{-1}.$$

It is clear that $q \le 1$, since, in the opposite case, condition (2.9) would be violated. Further, as $\delta \le 1$ and $E_2(\tau) \le 1$, $\tau \ge 0$ (equality being reached only at the point $\tau = 0$), it follows from (3.3) that $\lambda^* > 1$ (cf. estimate (2.11)).

Let us now assume that (2.12) holds and that, for instance, $\kappa_1 = 1$. Then $\xi_{\kappa_1} \equiv 0$ and $\kappa_2 < 1$. It is not difficult to check that in this case

$$\|x\|_\infty \leq \lambda[1-\tfrac{q}{2}(1-\kappa_2)E_2(b)]\|x\|_\infty$$

and

$$\lambda < \lambda^* = [1-\tfrac{q}{2}(1-\kappa_2)E_2(b)]^{-1}.$$

Lastly, if $\kappa_1 = \kappa_2 = 1$, then the condition $\lambda < 1$ guarantees the existence of only a trivial solution to the homogeneous equation.

Thus we have proved:

Lemma 3.1. Let conditions (2.9) and (2.10) hold. Equation (2.4) has a unique solution for all $\lambda < \lambda^*$, where

$$(3.4) \qquad \lambda^* = \begin{cases} [1-\delta q E_2(b/2)]^{-1}, & 0 \leq \kappa_1, \kappa_2 < 1, \\ [1-q\frac{1-\kappa_i}{2} E_2(b)]^{-1}, & 0 \leq \kappa_i < 1,\ \kappa_j = 1,\ i,j = 1,2,\ i \neq j, \\ 1, & \kappa_1 = \kappa_2 = 1. \end{cases}$$

Here $\delta = \min\{1-\kappa_1, 1-\kappa_2\}$, $q = \inf\limits_{\mu,\mu' \in [-1,1]} g(\mu,\mu')$ and $E_2(b) = \int_0^1 \exp(-b/\mu)d\mu$, $b > 0$.

Corollary 1. The operator R defined in (2.6) has a discrete spectrum with spectral radius estimated by

$$\delta_R \leq 1/\lambda^*,$$

where λ^* was defined in the lemma for different κ_1, κ_2.

The differentiability properties of the solution of the integral transport equation in slab geometry have been studied by many authors.

It suffices to refer to references 1, 4, 5, 16, and 20. Some of these results can be generalized in the setting of the above boundary-value problem (1.1) and (1.2) namely.

Lemma 3.2. Let f be continuously differentiable with respect to $\tau \in [0,b]$ and continuous with respect to $\mu \in [-1,1]$. Let V_0, $V_b \in C_{[-1,1]}$ and $g \in C_{[-1,1]\times[-1,1]}$. Then the solution x^* of equation (2.4) is continuously differentiable with respect to τ within the interval $0 < \tau < b$, its partial derivative $\partial x^*/\partial\tau$ is the unique solution (if the conditions of lemma 3.1 hold) of equation

$$x(\tau,\mu) = (Rx)(\tau,\mu) + \frac{\lambda}{2}\sum_{i=0}^{2}[T_i^{x^*}(\tau,0,\mu) - T_i^{x^*}(\tau,b,\mu)] + \frac{\partial f^0}{\partial\tau}(\tau,\mu),$$

and the estimate

$$(3.5) \qquad \max_{-1\le\mu\le 1}\left|\frac{\partial x^*}{\partial\tau}(\tau,\mu)\right| \le c(|\ln\tau| + |\ln(b-\tau)|),\ 0 < \tau < b,$$

c is a constant,

is valid.

One can carry out the proof of this lemma as in Pedas[15] by using the well-known Lebesgue theorem and the mean value theorem.

In reference 20, Pedas and Vainikko have defined more precisely the class of the solutions for the case $\kappa_1 = \kappa_2 = 0$, $V_0 \equiv V_b \equiv 0$ and derived the estimate for the nth derivative of the solutions to integral equations with weakly singular kernels. Apparently their proof can be extended to the case considered above, but it goes beyond the confines of the present paper.

4. THE DISCRETE-ORDINATES METHOD

Let us replace the integral in the right-hand side of (1.1) by some quadrature rule

$$\int_{-1}^{1} \omega(\mu)d\mu \approx \sum_{|K|=1}^{N} \alpha_K \omega(\mu_K),$$

$$(4.1) \qquad \alpha_K = \alpha_K^{(N)} > 0, \ |K| = 1,2,\ldots,N, \ \sum_{|K|=1}^{N} \alpha_K = 2, \ \mu_K = \mu_K^{(N)}, \quad |K| = 1,2,\ldots,N,$$

$$\mu_K = -\mu_{-K}, \ K = 1,2,\ldots,N, \quad 0 < \mu_1 < \mu_2 < \cdots < \mu_N \le 1.$$

We obtain the following approximate boundary-value problem

$$(4.2) \qquad \begin{cases} \mu_i \dfrac{\partial \phi_i^N}{\partial \tau} + \phi_i^N(\tau) = \dfrac{\lambda}{2} \displaystyle\sum_{|K|=1}^{N} \alpha_K g(\mu_i,\mu_K)\phi_K^N(\tau) + f_i(\tau), \quad |i| = 1,2,\ldots,N, \\ \phi_K^N(0) = \kappa_1 \phi_K^N(b) + V_K^0, \quad K = 1,2,\ldots,N, \\ \phi_K^N(b) = \kappa_2 \phi_K^N(0) + V_K^b, \quad K = -N, -N+1,\ldots,-1, \end{cases}$$

where $\phi_K^N(\tau) = \phi^N(\tau,\mu_K) \approx \phi(\tau,\mu_K)$, $|K| = 1,2,\ldots,N$, $V_K^0 = V_0(\mu_K)$, $V_{-K}^b = V_b(\mu_{-K})$, $K = 1,2,\ldots,N$. From (4.2), by replacing $x_i^N(\tau)$ by

$$(4.3) \qquad x_i^N(\tau) = \frac{\lambda}{2} \sum_{|K|=1}^{N} \alpha_K g(\mu_i,\mu_K) \ \phi_K^N(\tau) + f_i(\tau), \ |i| = 1,2,\ldots,N,$$

we obtain a set of integral equations

$$x_i^N(\tau) = \frac{\lambda}{2} \int_0^b \sum_{K=1}^{N} \alpha_K [K_0(\tau,\tau',\mu_i,\mu_K) x_{\nu_0}(\tau-\tau') K(\tau')$$

$$(4.4) \qquad + K_1(\tau,\tau',\mu_i,\mu_K)x_K^N(\tau') + K_2(\tau,\tau',\mu_i,\mu_K)$$

$$\times x_{-K}^N(\tau')]d\tau' + f_i^N(\tau), \quad |i| = 1,2,\ldots,N.$$

Here

$$f_i^N(\tau) = f_N^0(\tau,\mu_i) = \frac{\lambda}{2}\sum_{K=1}^{N}\alpha_K[g(\mu_i,\mu_K)\frac{V_K^0 e^{(b-\tau)/\mu_i}}{e^{b/\mu_i}-\kappa_1}$$

$$+ g(\mu_i,\mu_{-K})\frac{V_{-K}^b e^{\tau/\mu_i}}{e^{b/\mu_i}-\kappa_2}] + f_i(\tau) \approx f^0(\tau,\mu_i),$$

$$|i| = 1,2,\ldots,N.$$

The solution of the system (4.4) with an arbitrary phase function (provided $\kappa_1 = \kappa_2 = 0$) was studied in detail in reference 8. The averaging of the phase function over the azimuth has not been assumed there; however, the integral with respect to the azimuth was approximated by the Chebyshev quadrature rule. We shall describe a convenient algorithm for the solution of the system (4.4) if the following restrictions are satisfied:

1) the phase function is isotropic:

$$(4.5) \qquad g(\mu,\mu') \equiv 1;$$

2) the inner sources are represented in the form:

$$(4.6) \qquad f(\tau,\mu) = \omega(\mu)e^{\beta(\mu)\tau}, \omega,\beta \in C_{[-1,1]};$$

3) the points of the quadrature rule (4.1) satisfy the condition

$$(4.7) \qquad |\mu_i^{-1}| \neq \beta(\mu_K), \quad |i|, |K| = 1,2,\ldots,N;$$

4) the medium is absorbing, i.e.

$$(4.8) \qquad \lambda < 1.$$

In the particular case $\kappa_1 = \kappa_2 = 0$, $V_0 \equiv V_b \equiv 0$, the proposed algorithm coincides with those in [7] and [19]. Note that in reference 19, the isotropicality of sources (i.e.: ω, β is a constant was additionally assumed.

Theorem 4.1. Let conditions (4.5) - (4.8) hold. Then the solution of (4.4) can be represented in the form

$$x_i^N(\tau) = \sum_{K=1}^{N} [C_K e^{-d_K\tau} + \bar{C}_K e^{(\tau-b)d_K} + a_K^i e^{\beta_K\tau} + \bar{a}_K^i e^{\beta_{-K}\tau}],$$

$$|i| = 1,2,\ldots,N,\ \tau \in [0,b],\ \beta_K = \beta(\mu_K),\ |K| = 1,2,\ldots,N,$$

where the unknowns C_K, $\bar{C}_K$, d_K, a_K^i, $\bar{a}_K^{-i}$, K, $|i| = 1,2,\ldots,N$ are uniquely defined by the following conditions:

$$(4.10)\qquad 1 + \lambda \sum_{j=1}^{N} \frac{\alpha_j}{(d_K\mu_j)^2-1} = 0, \quad K = 1,2,\ldots,N,$$

$$(4.11)\qquad \begin{cases} a_K^i - \dfrac{\lambda}{2} \sum\limits_{j=1}^{N} \alpha_j [\dfrac{a_K^j}{\beta_K\mu_j+1} - \dfrac{a_K^{-j}}{\beta_K\mu_j-1}] = \begin{cases} 0, & i \neq K, \\ \omega_i, & i = K, \end{cases} \\ \\ \bar{a}_K^{-i} - \dfrac{\lambda}{2} \sum\limits_{j=1}^{N} \alpha_j [\dfrac{\bar{a}_K^j}{\beta_{-K}\mu_j+1} - \dfrac{\bar{a}_K^{-j}}{\beta_{-K}\mu_j-1}] = \begin{cases} 0, & i \neq -K, \\ \omega_i, & i = -K, \end{cases} \\ \\ K,\ |i| = 1,2,\ldots,N, \end{cases}$$

$$(4.12)\qquad \begin{cases} \sum\limits_{K=1}^{N} [\dfrac{C_K}{1-d_K\mu_j}(e^{-d_Kb}\kappa_1-1) + \dfrac{\bar{C}_K}{1+d_K\mu_j}(\kappa_1-e^{-d_Kb}) + F_{Kj}] + V_j^0 = 0, \\ \\ \sum\limits_{K=1}^{N} [\dfrac{C_K}{1+d_K\mu_j}(\kappa_2-e^{-d_Kb}) + \dfrac{\bar{C}_K}{1-d_K\mu_j}(e^{-d_Kb}\kappa_2-1) + \bar{F}_{Kj}] + V_{-j}^b = 0, \\ \\ j = 1,2,\ldots,N \end{cases}$$

with

$$F_{Kj} = \frac{a_K^j}{1+\beta_K\mu_j}(e^{\beta_K b}\kappa_1 - 1) + \frac{\bar{a}_K^j}{1+\beta_{-K}\mu_j}(e^{\beta_{-K}b}\kappa_1 - 1),$$

$$\bar{F}_{Kj} = \frac{a_K^{-j}}{1-\beta_K\mu_j}(\kappa_2 - e^{\beta_K b}) + \frac{\bar{a}_K^{-j}}{1-\beta_{-K}\mu_j}(\kappa_2 - e^{\beta_{-K}b}).$$

The following lemmas deal with the solvability of equations (4.10) - (4.12).

Lemma 4.1. Equation (4.10), with $z = d_K^2$, has exactly N pairwise distinct real-valued, nonnegative solutions $z_1, z_2, \ldots, z_N$ lying inside the intervals $(0, \mu_N^{-2})$, $(\mu_N^{-2}, \mu_{N-1}^{-2}), \ldots, (\mu_2^{-2}, \mu_1^{-2})$.

The proof of this assertion is given in reference 19.

Remark 4.1. In (4.9) and (4.12), it is sufficient to substitute the arithmetical values of the roots of the solutions of equation (4.10), i.e. $d_K = \sqrt{z_K}$, $K = 1,2,\ldots,N$. We then see that the exponents in (4.9) and (4.12) have a negative degree; and, if N is enlarged, an overflow does not occur.

Lemma 4.2*. The system (4.11) is uniquely solvable, and its solution a_K^i and $\bar{a}_K^i$ have the form

$$a_K^i = \frac{\lambda}{2}\,\frac{\omega_K\alpha_K/(\beta_K\mu_K+1)}{1+\lambda\sum_{j=1}^{N}\alpha_j/(\beta_K^2\mu_j^2-1)} + \begin{cases} 0, & i \neq K, \\ \omega_K, & i = K; \end{cases} \quad |i|,\ K = 1,2,\ldots,N$$

and

$$\bar{a}_K^i = -\frac{\lambda}{2}\,\frac{\omega_{-K}\alpha_K/(\beta_{-K}\mu_K-1)}{1+\lambda\sum_{j=1}^{N}\alpha_j/(\beta_{-K}^2\mu_j^2-1)} + \begin{cases} 0, & i \neq -K, \\ \omega_{-K}, & i = -K; \end{cases} \quad |i|,\ K = 1,2,\ldots,N.$$

*This lemma was proved by N. Kolesnik--a student of Tartu State University.

Proof. After straightforward, but sufficiently unwieldy transformations of the determinant of (4.11), we determine the solution by Cramer's rule.

Lemma 4.3. System (4.12) is uniquely solvable for any $\lambda < 1$.

Proof. Indeed, in the opposite case it is not difficult to show that if the absolute term is congruent to zero, then system (4.12) has a nontrivial solution. Then the homogeneous equation corresponding to (4.4) also has a nontrivial solution; however, this contradicts (2.11) (see theorem 5.1).

Remark 4.2. The restriction (4.7) can be omitted. In this case the algorithm (4.9) - (4.12) undergoes some changes, namely: the sum in the two last terms of (4.9) is summed up with respect to the set $M = \{K:\beta_K \neq |\mu_i^{-1}|,\ i = 1,2,\ldots,N\}$. Then system (4.11) and the absolute terms F_{Kj} and $\bar{F}_{Kj}$ in (4.12) change. We shall not overburden this paper by writing out all these variations but only refer to reference 10. There, the various modifications of the algorithm were described and justified for an example in a spherical transport equation.

Remark 4.3. Restriction (4.8) can be relaxed, requiring only $\lambda \leq 1$. One of the solutions of equation (4.10) can be equal to zero ($\lambda = 1$, $d_1 = 0$) and then the determinant of system (4.12) will be equal to zero. In order to avoid this, we must look for the solution of (4.4) in the form

$$x_i^N(\tau) = C_1 + \bar{C}_1\tau + \sum_{K=2}^{N} [C_K e^{-d_K\tau} + \bar{C}_K e^{(\tau-b)d_K}]$$

$$+ \sum_{K=1}^{N} [a_K^i e^{\beta_K\tau} + a_K^{-i} e^{\beta_{-K}\tau}],\quad |i| = 1,2,\ldots,N.$$

It is not difficult to obtain new conditions for determining the unknown coefficients.

Proof of theorem 4.1. Substitute (4.9) into the set of equations (4.4) and equalize the coefficients of the linearly independent functions $e^{-d_K\tau}$, $e^{(\tau-b)d_K}$, $e^{\beta_K\tau}$, $e^{\beta_{-K}\tau}$, e^{τ/μ_j}, $e^{-\tau/\mu_j}$, j, K = 1, 2,...,N to zero. This leads precisely to the conditions (4.10) - (4.12) and completes the proof of the theorem.

5. CONVERGENCE

Let us write equation (4.4) in the operator form

$$x^N = P_N R^N x^N + f_N^0, \tag{5.1}$$

where

$$(R^N x)(\tau,\mu) = \frac{\lambda}{2}\int_0^b \sum_{i=0}^{2} T_i^{x,N}(\tau,\tau',\mu)d\tau', \quad R^N : C^{P,N} \to Q^P,$$

$$T_i^{x,N}(\tau,\tau',\mu) = \sum_{j=1}^{N} \alpha_j K_i(\tau,\tau',\mu,\mu_j)x(\tau',\nu_i(\tau-\tau')\mu_j), \quad i = 0,1,2,$$

and $\nu_i(t)$ was defined in (2.8). Here $C^{P,N} = \{f : f = (f_{-N},\ldots,f_{-1}, f_1,\ldots,f_N),\ f_i \in L^P_{[0,b]},\ 1 \le p \le \infty;\ \|f\|_{P,N} = \max_{1\le|i|\le N} \|f_i\|_{L^P}\}$ and the projector $P_N : Q^P \to C^{P,N}$, $(P_N f)_i(\tau) = f(\tau,\mu_i)$, $|i| = 1,2,\ldots,N$.

In the case of the "usual" plane-parallel problem ($\kappa_1 = \kappa_2 = 0$) the convergence $\phi^N \to \phi$ and $x^N \to x$ has been studied rather thoroughly. It is sufficient to refer to references 2, 12, 13, 14. In particular, Nelson and Victory showed the uniform convergence of ϕ^N to ϕ for any Ω from the space of functions continuous and bounded on $[0,b] \times \{[-1,0)\cup(0,1]\}$, provided that

A1) the quadrature rule (4.1) converges for any function continuous on [-1,1], i.e.:

$$\sum_{|j|=1}^{N} \alpha_j \omega(\mu_j) \to \int_{-1}^{1} \omega(\mu)du, \quad N \to \infty,\ \omega \in C_{[-1,1]},$$

and

A2) the phase function g is nonnegative (cf. (2.10)) and continuous on the square $[-1,1]\times[-1,1]$.

Here

$$\Omega(\tau,\mu) = \begin{cases} V_0(\mu)e^{-\tau/\mu} + \frac{1}{\mu}\int_0^\tau f(\tau',\mu)e^{-(\tau-\tau')/\mu}d\tau', & \mu > 0, \\ V_b(\mu)e^{(b-\tau)/\mu} - \frac{1}{\mu}\int_\tau^b f(\tau',\mu)e^{-(\tau-\tau')/\mu}d\tau', & \mu < 0. \end{cases}$$

Thus $\| P_N\phi-\phi^N\|_{\infty,N} \to 0$ results. Then, using (2.1), (4.3) and Nelson's[12] results, we have an analogous convergence for the solutions of the integral equations (2.5) and (5.1), namely $\| P_N x-x^N\|_{\infty,N}\to 0$. It is not difficult to prove convergence for the remaining values of $p \in [1,\infty)$ (see, for example, reference 17 for the isotropic case). Hence, if $\kappa_1=\kappa_2 = 0$ we have $\|P_N x-x^N\|_{P,N}\to 0$ as $N \to \infty$.

Let us represent the operators R and R^N in the form of a sum of the three operators $R = R_1 + R_2 + R_3$ and $R^N = R_1^N + R_2^N + R_3^N$, where

$$(R_i x)(\tau,\mu) = \frac{\lambda}{2}\int_0^b T_i^x(\tau,\tau',\mu)d\tau',$$

$$(R_i^N x)(\tau,\mu) = \frac{\lambda}{2}\int_0^b T_i^{x,N}(\tau,\tau',\mu)d\tau', \quad i = 0,1,2.$$

If assumptions A1) and A2) hold, the uniform convergence $\| P_N R_0 - P_N R_0^N P_N \|_{Q^P\to C^{P,N}} \to 0$ is valid, as $N \to \infty$. In order to draw a similar conclusion about the convergence $R_i^N \to R_i$, $i = 1,2$ it is enough to show that the integrand in (2.7) for $i = 1,2$ "is not worse" than that for $i = 0$.

It is known that the kernel K_0 has a singularity on the diagonal $\tau = \tau'$. Let us consider the kernels K_i, $i = 1,2$. The function

$$A_i(\mu) = (e^{b/\mu} - \kappa_i)^{-1}, \quad b > 0$$

belongs to the space $C^{\infty}_{[0,1]}$ and decreases to 0 at the rate of $e^{-b/\mu}$ if $\mu \to 0$. From this, it follows that the kernels K_1 and K_2 have the same singularity as kernel K_0, but only at the points $\tau - \tau' = -b$ and $\tau - \tau' = b$, respectively. The functions T_i^x defined by (2.7) have a logarithmic singularity (if g is smooth enough) at the points $\tau = \tau'$, if $i = 0$; at $\tau = 0$, $\tau' = b$, if $i = 1$; and at $\tau = b$, $\tau' = 0$, if $i = 2$.

Thus, the convergence $R_i^N \to R_i$ is valid for all $i = 0,1,2$, and $\| P_N R - P_N R^N P_N \|_{Q^P \to C^{P,N}} \to 0$ as $N \to \infty$. It follows from this that, for sufficiently large N, the operator $(I - P_N R^N)^{-1}$ exists and is bounded for all λ, which are not characteristic values of equation (2.5). From the equality

$$(5.2) \qquad P_N x - x^N = (I - P_N R^N)^{-1} [(P_N Rx - P_N R^N P_N x) + (P_N f^0 - f_N^0)]$$

follows the uniform convergence $x^N \to x$ since $f_N^0 \to f^0$ for any continuous V_0, V_b, f. We have thus obtained

<u>Theorem 5.1</u>. Let the conditions A1) and A2) hold and let $V_0, V_b \in C_{[-1,1]}$, $f \in C_{[0,b]\times[-1,1]}$. Then for N suitably large, equation (5.1) has a unique solution for any $\lambda < \lambda^*$ (see (3.4)), and the convergence $\| P_N x - x^N \|_{P,N} \to 0$, $1 \le p \le \infty$ is valid.

<u>Remark 5.1</u>. Let a quadrature rule satisfy the following requirements:

$$\text{B1)} \qquad \left| \int_{-1}^{1} \omega(\mu) d\mu - \sum_{|K|=1}^{N} \alpha_K \omega(\mu_K) \right| \le \frac{C}{N} \int_{-1}^{1} (|\omega(\mu)| + |\omega'(\mu)|) d\mu;$$

B2) $\sum_{K=1}^{j} \alpha_K \leq C\mu_j$, $j = 1,2,\ldots,N$, C is a constant > 0;

B3) phase function $g \in C^1_{[-1,1]\times[-1,1]}$.

Then using the technique of estimation proposed by Pitkäranta and Scott[17] one can prove that there exists a constant C > 0 such that

$$\max_{-1\leq\mu\leq 1} \sum_{i=0}^{2} |T_i^x(\tau,\tau',\mu) - T_i^{x,N}(\tau,\tau',\mu)| \leq \frac{C}{N^\sigma}\Big[1 + |\log|\tau-\tau'||$$

$$+ |\log(b^2-(\tau-\tau')^2)|\Big]^{1-\sigma}\left[\frac{1}{|\tau-\tau'|} + \frac{1}{b^2-(\tau-\tau')^2}\right]^\sigma$$

for any σ in the range $0 \leq \sigma \leq 1$ and τ, $\tau' \in [0,b]$, $\tau \neq \tau' \pm b$; $\tau \neq \tau'$. The following estimate is a consequence of the uniform boundedness of the operators $(I-P_NR^N)^{-1}$ and the preceding inequality,

$$\| P_Nx-x^N \|_{P,N} \leq CN^{-1}(1 + \log N), \quad C \text{ is a constant} > 0.$$

6. THE RATE OF CONVERGENCE

Let us consider the function $(P_NRx-P_NR^NP_Nx)_K(\tau)$, $|K| = 1,2,\ldots,N$. We integrate it by parts. By straightforward but sufficiently unwieldy transformations, we obtain the equality

$$(P_NRx-P_NR^NP_Nx)_K(\tau) =$$

$$= \frac{\lambda}{2}\{[\int_{-1}^{1} g(\mu_K,\mu)x(\tau,\mu)d\mu - \sum_{|j|=1}^{N} \alpha_j g(\mu_K,\mu_j)x(\tau,\mu_j)]$$

$$-[\int_0^1 g(\mu_K,\mu)e^{-\tau/\mu}x(0,\mu)d\mu - \sum_{j=1}^{N} \alpha_j g(\mu_K,\mu_j)e^{-\tau/\mu_j} x(0,\mu_j)]$$

$$(7.1) \qquad \mu \frac{\partial\phi}{\partial\tau} + \phi(\tau,\mu) = \frac{1}{4}\int_{-1}^{1} \phi(\tau,\mu')d\mu' + \left(\frac{1}{3} - \frac{\mu}{2} + \mu^2 - \frac{\mu^3}{2}\right) e^{-\tau/2},$$

with boundary conditions

$$(7.2) \qquad \begin{cases} \phi(0,\mu) = \phi(b,\mu) + (1+\mu^2)(1-e^{-b/2}), & \mu > 0, \\ \phi(b,\mu) = 0.3\phi(0,\mu) + (1+\mu^2)(e^{-b/2}-0.3), & \mu < 0, \end{cases}$$

by the algorithm proposed in section 4. The exact solution of the boundary-value problem (7.1) and (7.2) is $\phi(\tau,\mu) = (1+\mu^2)e^{-\tau/2}$ from which $x(\tau,\mu) = (1-\mu/2+\mu^2-\mu^3/2)e^{-\tau/2}$.

As a quadrature rule we choose the midpoint rectangular rule and compare it with the Gaussian quadrature rule, which yields an exact solution if N = 2 (as ϕ is a polynomial of second order).

The values $x^N(\tau,\mu)$ computed according to algorithm (4.9) - (4.12), where $\alpha_j = N^{-1}, \mu_j = 0.5(2j-1)N^{-1}$, $j = 1,2,\ldots,N$, are given in Table 1. The values computed using the Gaussian rule are given in the row indicated by (*). They are exact and correspond to the exact solution $x(\tau,\mu)$.

It is often necessary to use a quadrature rule to approximate the scalar flux

$$(7.3) \qquad \phi_0(\tau) = \frac{1}{4}\int_{-1}^{1} \phi(\tau,\mu)d\mu.$$

Its values are shown in Table 2.
The values $\phi^N(\tau,\mu_j)$, $|j| = 1,2,\ldots,N$ have been computed by formulae similar to (2.2) - (2.3), and integral (7.3), by the midpoint rectangular rule, i.e.

$$\phi_0(\tau) \approx \phi_0^N(\tau) = \frac{1}{4} \sum_{|K|=1}^{N} \alpha_K \, \phi^N(\tau,\mu_K).$$

(The exact values of ϕ are given in the (*) row.)

$$(6.1)\qquad -\ \kappa_2[\int_0^1 g(\mu_K,-\mu)\,\frac{e^{-(2b-\tau)/\mu}}{1-\kappa_2 e^{-b/\mu}}\,x(b,-\mu)d\mu$$

$$-\ \sum_{j=1}^{N}\alpha_j g(\mu_K,-\mu_j)\,\frac{e^{-(2b-\tau)/\mu_j}}{1-\kappa_2 e^{-b/\mu_j}}\,x(b,-\mu_j)]$$

$$+\ \kappa_2\int_0^b[\int_0^1 g(\mu_K,-\mu)\,\frac{e^{-(b+s-\tau)/\mu}}{1-\kappa_2 e^{-b/\mu}}\,\frac{\partial x}{\partial s}(s,-\mu)d\mu$$

$$-\ \sum_{j=1}^{N}\alpha_j g(\mu_K,-\mu_j)\,\frac{e^{-(b+s-\tau)/\mu_j}}{1-\kappa_2 e^{-b/\mu_j}}\,\frac{\partial x}{\partial s}(s,-\mu_j)]ds\}.$$

Let us make conditions A2) and A3) more rigorous. We assume that

C1) the phase function g has a sufficient number of bounded derivatives of μ (respectively of μ').

Moreover, we also assume that the given functions

C2) V_0, V_b and f have a sufficient number of bounded derivatives (with respect to μ).

Then it is clear that the solution x is such that $\left|\frac{\partial^m x}{\partial \mu^m}(\tau,\mu)\right| \leq C_m$, where C_m are constants independent of τ,μ. Taking into account estimate (3.5), we see that, from equality (6.1), the rate of convergence of $P_N R^N P_N x$ to $P_N Rx$ and correspondingly of x^N to $P_N x$ (see (5.2)) asymptotically depends on, and only on, how well the quadrature rule

$$\sum_{K=1}^{N}\alpha_K\omega(\mu_K) \rightarrow \int_0^1 \omega(\mu)d\mu$$

(cf. (4.1)) integrates the exponential integral E_2. Thus, we have obtained the main result about the rate of convergence.

Theorem 6.1. Let assumptions C1) and C2) hold and suppose x and x^N are the solutions of equations (2.5) and (5.1). Then there is a positive constant C such that

$$\| P_N x - x^N \|_{P,N} \le C \| E_2 - E_2^N \|_{L^P}, \quad 1 \le p \le \infty, \tag{6.2}$$

where $E_2(\tau) = \int_0^1 e^{-\tau/\mu} d\mu$ and $E_2^N(\tau) = \sum_{K=1}^N \alpha_K e^{-\tau/\mu_K}$, $0 \le \tau \le b$.

Some estimates of the rate of convergence obtained in reference 17 follow from (6.2):

1) Let the quadrature rule (4.1) satisfy the conditions B1) and B2), i.e. suppose the quadrature rule is only first-order accurate. Then

$$\int_0^1 (|\psi_\tau(\mu)| + |\psi_\tau'(\mu)|) d\mu = E_2(\tau) + e^{-\tau} \le 2, \ \psi_\tau(\mu) = e^{-\tau/\mu},$$

and we get the following estimate

$$\| P_N x - x^N \|_{\infty,N} \le \frac{C}{N} \int_0^1 (|\psi_\tau(\mu)| + |\psi_\tau'(\mu)|) d\mu \le \frac{C}{N}.$$

2a) Let the quadrature rule (4.1) be Gaussian quadrature applied on [-1,0] ∪ [0,1]. We use the result of De Vore and Scott (proposition 3.2 of reference 17).

$$\left| \int_0^1 \omega(\mu) d\mu - \sum_{K=1}^N \alpha_K \omega(\mu_K) \right| \le C_m N^{-m} \int_0^1 [\mu(1-\mu)]^{m/2} |\omega^{(m)}(\mu)| d\mu, \quad m \le 2N-1, \tag{6.3}$$

where C_m depend only on m. Setting m = 2, $\omega(\mu) = \psi_\tau(\mu)$, we obtain

$$\int_0^1 \mu(1-\mu)|\psi_\tau''(\mu)|d\mu = \int_0^1 \mu(1-\mu)\left|\frac{\tau^2}{\mu^4} - \frac{2\tau}{\mu^3}\right| e^{-\tau/\mu}d\mu$$

$$\leq 2\int_0^1 \frac{\tau}{\mu^2} e^{-\tau/\mu} d\mu + \int_0^1 \frac{\tau^2}{\mu^3} e^{-\tau/\mu}d\mu \leq 3e^{-\tau} + \frac{1}{e},$$

from which it follows $\| P_N x - x^N \|_{\infty,N} \leq CN^{-2}$.

2b) One can obtain the corresponding estimate in the space $C^{1,N}$, estimating, by means of (6.2), the first norm as m = 2 and the second as m = 6 in the following equality

$$\| E_2 - E_2^N \|_{L_{[0,b]}} = \|\cdot\|_{L_{[0,N^{-2}]}} + \|\cdot\|_{L_{[N^{-2},b]}}.$$

It is easily seen that $\|\cdot\|_{L_{[0,N^{-2}]}} \leq CN^{-4}$. Computing directly $\psi_\tau^{(6)}(\mu)$ and substituting it in the right-hand part of (6.3) we obtain the equality

$$\int_0^1 [\mu(1-\mu)]^3|\psi_\tau^{(6)}(\mu)|d\mu = \int_0^1 (1-\mu)^3\left|\frac{e^{-\tau/\mu}}{\mu^3} \sum_{j=1}^{6} (-1)^j \frac{6!5!}{(6-j)!j!(j-1)!}\left(\frac{\tau}{\mu}\right)^j\right|d\mu.$$

But

$$\sum_{j=1}^{6} \int_0^1 \frac{e^{-\tau/\mu}}{\mu^3} \left(\frac{\tau}{\mu}\right)^j d\mu = e^{-\tau} \sum_{K=1}^{8} \kappa_K \tau^{6-K},$$

where κ_K, K = 1,2,...,8 are constants dependent only on K. From that it follows that

$$\| E_2 - E_2^N \|_{L_{[N^{-2},b]}} \leq CN^{-4},$$

and

$$\| P_N x - x^N \|_{1,N} \leq CN^{-4}.$$

In the same way, or using the interpolation theory in reference 3, one can obtain the estimates for the remaining spaces $C^{P,N}$, $1 < P < \infty$ as well as for the Gauss rule on [-1,1].

3) Let us return to inequality (6.2). Following reference 11, we shall construct a composite quadrature rule, namely: we shall divide the interval [0,1] on m+1 subintervals by the points $0 < a_1 < a_2 < \cdots < a_m < a_{m+1} = 1$. On each subinterval $[a_K, a_{K+1}]$, $K = 1,2,\ldots,m$ we apply the n-point Gauss rule. Then provided the points a_K are chosen by the recurrent relation

$$a_1 = 2^{-4n},\ a_{K+1} = a_K^{\,2n/(2n+1)} + a_K,\ K = 1,2,\ldots,m-1,$$

we obtain the estimate

$$\| E_2 - E_2^N \|_{C_{[0,b]}} \leq CN^{1/8}\, 2^{-2\sqrt{N}}\ . \tag{6.4}$$

Here, $N = mn$ and $E_2^N(\tau) = \sum_{K=1}^{m} \sum_{j=1}^{n} \alpha_{Kj}\, e^{-\tau/\mu_{Kj}}$ is an approximation of the exponential integral.

From (6.2) and (6.4), it follows that $\| P_N x - x^N \|_{\infty,N} = \mathcal{O}(N^{1/8} 2^{-2\sqrt{N}})$, which states that the quadrature rule constructed above is significantly more accurate for the discrete-ordinates method than the ones described previously. The initial value of N, where this advantage is obvious, is discussed and numerically pointed out in reference 11.

Note that in this section we let C denote a positive constant which may take on different values upon different usages.

7. A NUMERICAL EXAMPLE

We give in this section a numerical solution of the equation,

$$(7.1) \qquad \mu \frac{\partial\phi}{\partial\tau} + \phi(\tau,\mu) = \frac{1}{4}\int_{-1}^{1} \phi(\tau,\mu')d\mu' + \left(\frac{1}{3} - \frac{\mu}{2} + \mu^2 - \frac{\mu^3}{2}\right) e^{-\tau/2},$$

with boundary conditions

$$(7.2) \qquad \begin{cases} \phi(0,\mu) = \phi(b,\mu) + (1+\mu^2)(1-e^{-b/2}), \ \mu > 0, \\ \phi(b,\mu) = 0.3\phi(0,\mu) + (1+\mu^2)(e^{-b/2}-0.3), \ \mu < 0, \end{cases}$$

by the algorithm proposed in section 4. The exact solution of the boundary-value problem (7.1) and (7.2) is $\phi(\tau,\mu) = (1+\mu^2)e^{-\tau/2}$ from which $x(\tau,\mu) = (1-\mu/2+\mu^2-\mu^3/2)e^{-\tau/2}$.

As a quadrature rule we choose the midpoint rectangular rule and compare it with the Gaussian quadrature rule, which yields an exact solution if N = 2 (as ϕ is a polynomial of second order).

The values $x^N(\tau,\mu)$ computed according to algorithm (4.9) - (4.12), where $\alpha_j = N^{-1}, \mu_j = 0.5(2j-1)N^{-1}$, $j = 1,2,\ldots,N$, are given in Table 1. The values computed using the Gaussian rule are given in the row indicated by (*). They are exact and correspond to the exact solution $x(\tau,\mu)$.

It is often necessary to use a quadrature rule to approximate the scalar flux

$$(7.3) \qquad \phi_0(\tau) = \frac{1}{4}\int_{-1}^{1} \phi(\tau,\mu)d\mu.$$

Its values are shown in Table 2.
The values $\phi^N(\tau,\mu_j)$, $|j| = 1,2,\ldots,N$ have been computed by formulae similar to (2.2) - (2.3), and integral (7.3), by the midpoint rectangular rule, i.e.

$$\phi_0(\tau) \approx \phi_0^N(\tau) = \frac{1}{4} \sum_{|K|=1}^{N} \alpha_K \, \phi^N(\tau,\mu_K).$$

(The exact values of ϕ are given in the (*) row.)

TABLE 1

N \ τ	0.0	0.8	1.2	2.0	μ
*	0.9375	0.6284	0.5145	0.3449	0.5
3	0.9303	0.6227	0.5097	0.3417	
9	0.9367	0.6278	0.5140	0.3445	
15	0.9372	0.6282	0.5143	0.3448	
*	1.5625	1.0473	0.8575	0.5748	-0.5
3	1.5553	1.0416	0.8527	0.5716	
9	1.5612	1.0463	0.8566	0.5742	
15	1.5622	1.0471	0.8573	0.5747	

TABLE 2

N \ τ	0.0	0.8	1.2	2.0
*	0.6667	0.4469	0.3659	0.2453
2	0.6504	0.4339	0.3549	0.2380
5	0.6641	0.4448	0.3641	0.2441
8	0.6656	0.4461	0.3652	0.2448
11	0.6661	0.4465	0.3655	0.2450
14	0.6663	0.4466	0.3657	0.2451

Remark 7.1. In the course of solving the set (4.12), we note that it is worthwhile to multiply the matrix of the system by its conjugate matrix. In spite of an increase of the condition number, all eigenvalues will become positive, and the system is easily solved by any method (for example, by the method of straightforward Gaussian elimination with partial pivoting).

8. CONCLUSION

The present paper deals with the integro-differential transport equation with boundary conditions corresponding to the periodic problem. We have converted this equation to the integral equation of the Peierls type according to the following considerations.

Firstly, in this case, the algorithm of the solution looks more simple and is more easily programmed for the computer. Secondly, the study of convergence and especially the rate of convergence is more convenient in terms of integral operators (in particular, operators with weakly singular kernels). And thirdly, there are integral equations of radiation transfer (for example, in some models of broken clouds) that have no integro-differential analogue.

This paper describes the method which makes it possible to estimate the rate of convergence using any quadrature rule. The question about the optimal quadrature rule remains open, even in the one-dimensional case. This problem, as was shown, is equivalent (with some restriction) to the selection of a quadrature rule which approximates the exponential integral E_2 in the optimum manner. The author would like to point out that the composite quadrature rule constructed in reference 11 surpasses the Gaussian and Clenshaw-Curtis rules recommended; e.g. in references 17 and 18. Unfortunately, the numerical example proposed in this paper does not reflect this advantage because the solution $\phi(\tau,\mu)$ in this example is a polynomial with respect to μ.

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