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Asymptotic theory for optically thick layers: application to the discrete ordinates method

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Asymptotic expressions for the reflected, transmitted, and internal scattered radiation field in optically thick, vertically homogeneous, plane-parallel media are derived from first principles by using the discrete ordinates method of radiative transfer. Compact matrix equations are derived for computing the escape function, diffusion pattern, diffusion exponent, and the reflection function of a semi-infinite atmosphere in terms of the matrices, eigenvectors, and eigenvalues that occur in the discrete ordinates method. These matrix equations are suitable for numerical computations and are valid throughout the full range of single scattering albedos. The present formulations are validated by comparing them with established methods of radiative transfer.

Key words: Multiple scattering, radiative transfer, asymptotic theory, discrete ordinates method.

1. Introduction

The study of multiple scattering in optically thick atmospheres has a long history of development, largely as a result of the simplicity of the asymptotic form of the radiation field deep within the medium. Within this region of a scattering and absorbing medium, the radiative energy density follows a diffusion equation.[1] Theoretical studies of radiative transfer in plane-parallel atmospheres have shown that the radiative intensity field can be expressed in especially simple functional forms.[2–7] For example, the reflected $u(0; -\mu, \mu_0, \phi)$ and transmitted $u(\tau_c; +\mu, \mu_0, \phi)$ intensities from a nonconservative and vertically homogeneous plane-parallel layer of sufficient optical thickness τ_c can be written as

$$u(0; -\mu, \mu_0, \phi) = u_\infty(-\mu, \mu_0, \phi) - \frac{ml \exp(-2k\tau_c)}{1 - l^2 \exp(-2k\tau_c)} \times K(\mu)K(\mu_0)\frac{\mu_0 F_0}{\pi}, \tag{1}$$

$$u(\tau_c; +\mu, \mu_0, \phi) = \frac{m \exp(-k\tau_c)}{1 - l^2 \exp(-2k\tau_c)} \times K(\mu)K(\mu_0)\frac{\mu_0 F_0}{\pi}. \tag{2}$$

In these expressions μ_0 is the cosine of the solar zenith angle, μ is the cosine of the emerging zenith angle (in which the positive sign denotes downward propagating radiation and the negative sign denotes upward propagating radiation), ϕ is the azimuth angle measured from the solar plane, F_0 is the incident solar flux, k is the diffusion exponent, $K(\mu)$ is the escape function, l and m are scalar constants determined by the optical properties of the medium, and $u_\infty(-\mu, \mu_0, \phi)$ is the reflected intensity from a semi-infinite layer having the same optical properties as the finite layer. In addition, the intensity field deep within the layer at optical depth τ can be written as

$$u(\tau; \pm\mu, \mu_0, \phi) = \frac{\exp(-k\tau)}{1 - l^2 \exp(-2k\tau_c)} \times \{P(\pm\mu) - l \exp[-2k(\tau_c - \tau)]P(\mp\mu)\}K(\mu_0)\frac{\mu_0 F_0}{\pi}, \tag{3}$$

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where $P(\pm\mu)$ is the diffusion pattern. These equations show that the intensity field reflected, transmitted, and deep within an optically thick medium can be expressed in terms of simple functions and constants that depend on angle as well as optical properties of the layer.

Recently King[8,9] and Nakajima and King[10] developed methods for retrieving the single scattering albedo, optical thickness, and effective particle radius of clouds by applying asymptotic theory for plane-parallel atmospheres. In these methods, Eqs. (1)–(3), together with their conservative atmosphere equivalents, were used to derive relationships between observed quantities (reflected or internal scattered radiation) and the inherent optical properties of the medium (optical thickness, cloud droplet radius, and single scattering albedo). Furthermore, King *et al.*[11] showed that Eq. (3) agrees well with measurements of the internal scattered radiation field within a horizontally extensive and optically thick marine stratocumulus cloud layer. Asymptotic theory for thick layers also plays an important role in simplifying solutions of the radiative transfer equation in vertically inhomogeneous atmospheres[12,13] and in geometrically complicated fractal clouds.[14]

In spite of the advantages and interesting features of asymptotic theory for multiple scattering problems in optically thick atmospheres, access to asymptotic theory has been difficult, even for plane-parallel atmospheres. This is because it is first necessary to compute various functions and constants that appear in Eqs. (1)–(3) before one can use these expressions to obtain the desired radiation fields. Lenoble[7] suggests an iterative method to solve a characteristic equation for the eigenvalue k and eigenfunction $P(\pm\mu)$. Sobolev[5] uses a recurrence formula to solve this characteristic equation by expanding the diffusion pattern in a Legendre polynomial series. Both Sobolev and Lenoble suggest essentially the same method to solve the integral equation for $u_\infty(-\mu, \mu_0$ $\phi)$. Once k, $P(\pm\mu)$, and $u_\infty(-\mu, \mu_0, \phi)$ have been determined, $K(\mu)$ can be obtained by iteration of an integral equation for $K(\mu)$.

As an alternative method of solution, van de Hulst[6,15] suggested using an asymptotic fitting method whereby computational results from the doubling method are fit to known general forms of the asymptotic equations [such as Eqs. (1)–(2) and Eq. (3) at the midlevel $\tau = \tau_c/2$]. Duracz and McCormick[16] derived expansions of the diffusion pattern as well as other asymptotic functions and constants in terms of the similarity parameter and the coefficients of the Legendre polynomial expansion of the phase function. Yi *et al.*[17] further developed a parameterization for $K(\mu)$ as a function of s that is applicable to water clouds when $\mu \geq 0.5$ and $\omega_0 \geq 0.8$. Since these series were expanded in terms of the similarity parameter $s = [(1 - \omega_0)/(1 - \omega_0 g)]^{1/2}$, a function of single scattering albedo ω_0 and asymmetry factor g, they are the most accurate for small values of s (weak absorption). King[8] and King and Harshvardhan[18] presented similarity relations for asymptotic constants l, m, k, and other constants not appearing in Eqs. (1)–(3), as a function of the similarity parameter for the whole range of single scattering albedo ($0 \leq \omega_0 \leq 1$). These parameterizations, however, do not extend to the functions $u_\infty(-\mu, \mu_0, \phi)$, $K(\mu)$, and $P(\pm\mu)$ that appear in Eq. (1), thereby requiring a full radiative transfer code to be employed to perform the calculations for arbitrary optical parameters.

The intent of this paper is to present efficient numerical algorithms for deriving the asymptotic functions and constants that are valid for any single scattering albedo without resorting to numerical fittings. These algorithms are based on recent matrix formulations of the discrete ordinates method (DOM)[19,20] of solving the radiative transfer equation. Although many studies directed toward obtaining the asymptotic functions exist, as reviewed above, it is useful to present such an algorithm in a systematic way at this stage, since there is a renewed interest in DOM computer codes that are fast and stable for any plane-parallel atmosphere.[21–23] Matrix formulations of the theory are more suitable than traditional functional analysis methods for numerical calculations because of recently improved computer capability and large memory now available. In this paper we show that all the asymptotic functions and constants may be expressed in terms of eigenvalues and eigenvectors of one basic eigenvalue problem. Knowing the asymptotic limit of the DOM is also useful for improving the efficiency of DOM computer codes since asymptotic theory permits one to bypass some numerical procedures that are unnecessary for optically thick atmospheres. Although asymptotic formulas for vertically inhomogeneous stratification exist, it is useful to have a more general transfer code such as a DOM with a built-in asymptotic routine that automatically works when the sublayer becomes thick. The purpose of this paper is to address these points.

Since the structure of the matrices in the eigenspace takes on an important role in the present study, the formulations of the DOM method from Nakajima and Tanaka[20,23] are rearranged and summarized in Section 2. The asymptotic limit of the DOM is derived in Section 3, followed by a discussion of a numerically efficient algorithm for obtaining the asymptotic functions and constants, presented in Sections 4 and 5.

2. Matrix Formulation of the Discrete Ordinates Method

A. Basic Equations

The equation describing the transfer of solar radiation through a plane-parallel and vertically homogeneous medium can be written as[24]

$$\mu \frac{du(\tau; \mu, \mu_0, \phi)}{d\tau} = -u(\tau; \mu, \mu_0, \phi) + \frac{\omega_0}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \Phi(\mu, \phi; \mu', \phi') u(\tau; \mu', \mu_0, \phi') d\phi' d\mu' + \frac{\omega_0}{4\pi} \Phi(\mu, \phi; \mu_0, \phi_0) F_0 \exp(-\tau/\mu_0), \quad (4)$$

where $\Phi(\mu, \phi; \mu', \phi')$ is the single scattering phase function normalized such that

$$\frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \Phi(\mu, \phi; \mu', \phi') d\phi' d\mu' = 1. \quad (5)$$

If $\Phi(\mu, \phi; \mu', \phi')$ is a function only of the cosine of the scattering angle, then the product of the single scattering albedo and the phase function $\Phi(\cos\Theta)$ can be expressed as a finite expansion in Legendre polynomials of the form

$$\omega_0 \Phi(\cos\Theta) = \sum_{l=0}^{L} \omega_l P_l(\cos\Theta), \quad (6)$$

where Θ is the scattering angle and $P_l(\cos\Theta)$ is a Legendre polynomial of order l. By making use of the addition theorem for spherical harmonics, we can express the phase function as

$$\omega_0 \Phi(\mu, \phi; \mu', \phi') = h^0(\mu, \mu') + 2 \sum_{m=1}^{L} h^m(\mu, \mu') \cos m(\phi - \phi'), \quad (7)$$

where the azimuth-dependent redistribution functions $h^m(\mu, \mu')$ are given by[6]

$$h^m(\mu, \mu') = \sum_{l=m}^{L} \omega_l Y_l^m(\mu) Y_l^m(\mu'), \quad (8)$$

with the renormalized associated Legendre polynomials $Y_l^m(\mu)$ expressible in terms of the associated Legendre polynomials $P_l^m(\mu)$ by[25]

$$Y_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\mu). \quad (9)$$

By further expressing the intensity as a finite Fourier series of the form

$$u(\tau; \mu, \mu_0, \phi) = u^0(\tau; \mu, \mu_0) + 2 \sum_{m=1}^{L} u^m(\tau; \mu, \mu_0) \cos m(\phi - \phi_0), \quad (10)$$

and making use of the orthogonality property of the cosine function, we can rewrite Eq. (4) as L independent equations (one for each Fourier component) as

$$\mu \frac{du^m(\tau; \mu, \mu_0)}{d\tau} = -u^m(\tau; \mu, \mu_0) + \tfrac{1}{2} \int_{-1}^{1} h^m(\mu, \mu') u^m(\tau; \mu', \mu_0) d\mu' + \frac{h^m(\mu, \mu_0)}{4\pi} F_0 \exp(-\tau/\mu_0). \quad (11)$$

B. Matrix Formulation

When multiple scattering calculations with either the adding–doubling[26] or discrete ordinates[21–23] methods are performed, it is advantageous to subdivide the angular interval $[0, 1]$ into N Gaussian quadrature points $0 < \mu_1 < \ldots < \mu_N < 1$ with mirror symmetric points on the interval $[-1, 0]$ for a total of $2N$ streams. Then, if the Gaussian weights are $w_1, \ldots, w_N$, Eq. (11) can be rewritten as

$$\pm\mu_i \frac{du^m(\tau; \pm\mu_i, \mu_{0j})}{d\tau} = -u^m(\tau; \pm\mu_i, \mu_{0j}) + \frac{h^m(\pm\mu_i, \mu_{0j})}{4\pi} F_0 \exp(-\tau/\mu_{0j}) + \tfrac{1}{2} \sum_{n=1}^{N} [h^m(\pm\mu_i, \mu_n) u^m(\tau; \mu_n, \mu_{0j}) + h^m(\pm\mu_i, -\mu_n) u^m(\tau; -\mu_n, \mu_{0j})] w_n \quad (12)$$

for each of M solar incident directions μ_{0j}, $j = 1, \ldots, M$. This expression can be compactly written in matrix form for each Fourier component as

$$\pm\mathbf{M} \frac{d\mathbf{u}^{\pm}(\tau)}{d\tau} = -\mathbf{u}^{\pm}(\tau) + \mathbf{h}^{\pm}\mathbf{W}\mathbf{u}^{+}(\tau) + \mathbf{h}^{\mp}\mathbf{W}\mathbf{u}^{-}(\tau) + \mathbf{S}^{\pm}\mathbf{E}_0(\tau), \quad (13)$$

where

$$\begin{aligned}
\mathbf{u}^{\pm}(\tau) &= [u^m(\tau; \pm\mu_i, \mu_{0j})], & i &= 1, \ldots, N, \\
& & j &= 1, \ldots, M; \\
\mathbf{h}^{\pm} &= [\tfrac{1}{2} h^m(\pm\mu_i, \mu_j)], & i,j &= 1, \ldots, N; \\
\mathbf{S}^{\pm} &= \left[\frac{F_0}{4\pi} h^m(\pm\mu_i, \mu_j)\right], & i &= 1, \ldots, N, \\
& & j &= 1, \ldots, M; \\
\mathbf{M} &= [\mu_i \delta_{ij}], & i,j &= 1, \ldots, N; \\
\mathbf{W} &= [w_i \delta_{ij}], & i,j &= 1, \ldots, N; \\
\mathbf{E}_0(\tau) &= [\exp(-\tau/\mu_{0j})\delta_{ij}], & i,j &= 1, \ldots, M.
\end{aligned}$$

In these expressions $\mathbf{u}^{\pm}(\tau)$ represents the $N \times M$ downwelling $(+)$ and upwelling $(-)$ diffuse intensity matrices for the mth Fourier frequency at optical depth τ; $\mathbf{h}^{\pm}$ represents the redistribution (phase) matrices for transmission (+) and reflection (−); $\mathbf{S}^{\pm}$ represents the redistribution matrices arising from single scattering out of the direct solar beam, and $\mathbf{M}$, $\mathbf{W}$, and $\mathbf{E}_0(\tau)$ are diagonal matrices. Illustrations of the $\mathbf{h}^-$ and $\mathbf{S}^-$ matrices for a Henyey–Greenstein phase function and a phase function representative of clouds at visible wavelengths can be found in Ref. 26 for $m = 0$ and 1, for which both phase functions have the asymmetry factor $g = 0.841$.

C. Basic Solution

In order to proceed further it is convenient to define a scaled intensity matrix $\hat{\mathbf{u}}^{\pm}(\tau)$ such that

$$\hat{\mathbf{u}}^{\pm}(\tau) = \mathbf{W}^{+}\mathbf{u}^{\pm}(\tau), \tag{14}$$

where

$$\mathbf{W}^{+} = \sqrt{\mathbf{WM}}, \qquad \mathbf{W}^{-} = \sqrt{\mathbf{WM}^{-1}}.$$

In this notation the square root of a diagonal matrix represents that matrix whose diagonal elements are the square roots of the diagonal elements of the original matrix. By using these definitions, Eq. (13) can be rewritten in the form

$$\pm\frac{d\hat{\mathbf{u}}^{\pm}(\tau)}{d\tau} = -\mathbf{M}^{-1}\hat{\mathbf{u}}^{\pm}(\tau) + \hat{\mathbf{h}}^{\pm}\hat{\mathbf{u}}^{+}(\tau) + \hat{\mathbf{h}}^{\mp}\hat{\mathbf{u}}^{-}(\tau) + \hat{\mathbf{S}}^{\pm}\mathbf{E}_0(\tau), \tag{15}$$

with

$$\hat{\mathbf{h}}^{\pm} = \mathbf{W}^{-}\mathbf{h}^{\pm}\mathbf{W}^{-}, \qquad \hat{\mathbf{S}}^{\pm} = \mathbf{W}^{-}\mathbf{S}^{\pm}.$$

With this scaling we obtain the following equation for the combinations $\hat{\psi}^{\pm}(\tau) = \hat{\mathbf{u}}^{+}(\tau) \pm \hat{\mathbf{u}}^{-}(\tau)$:

$$\frac{d\hat{\psi}^{\mp}(\tau)}{d\tau} = -\mathbf{X}^{\pm}\hat{\psi}^{\pm}(\tau) + \hat{\sigma}^{\pm}\mathbf{E}_0(\tau), \tag{16}$$

where $\mathbf{X}^{\pm}$ is the symmetric $N \times N$ matrix defined by

$$\mathbf{X}^{\pm} = \mathbf{M}^{-1} - \mathbf{W}^{-}(\mathbf{h}^{+} \pm \mathbf{h}^{-})\mathbf{W}^{-}, \tag{17a}$$

$$\hat{\sigma}^{\pm} = \mathbf{W}^{-}(\mathbf{S}^{+} \pm \mathbf{S}^{-}). \tag{17b}$$

By eliminating $\hat{\psi}^{-}(\tau)$ from Eq. (16) we obtain the following ordinary differential equation for $\hat{\psi}^{+}(\tau)$:

$$\frac{d^2\hat{\psi}^{+}(\tau)}{d\tau^2} = \mathbf{G}\hat{\psi}^{+}(\tau) + \mathbf{g}\mathbf{E}_0(\tau), \tag{18}$$

where

$$\mathbf{G} = \mathbf{X}^{-}\mathbf{X}^{+}, \tag{19a}$$

$$\mathbf{g} = -\mathbf{X}^{-}\hat{\sigma}^{+} - \hat{\sigma}^{-}\mathbf{M}_0^{-1}, \tag{19b}$$

$$\mathbf{M}_0 = [\mu_{0j}\delta_{ij}], \qquad i,j = 1, \ldots, M. \tag{19c}$$

Equation (18) can readily be solved by eigenvalue decomposition of $\mathbf{G}$ if we diagonalize the asymmetric $\mathbf{G}$ matrix as follows:

$$\mathbf{G} = \mathbf{QL}^2\mathbf{Q}^{-1}, \tag{20a}$$

$$\mathbf{L} = [\lambda_i\delta_{ij}], \qquad i,j = 1, \ldots, N, \tag{20b}$$

where $\mathbf{Q}$ is the matrix containing the N eigenvectors and λ_i are the nonnegative square-root eigenvalues of $\mathbf{G}$. The decomposition of Eq. (20a) can be obtained by one of several methods, namely, direct decomposition of the asymmetric matrix $\mathbf{G}$,[19,22] square-root decomposition,[20] or Cholesky decomposition.[27]

The solution of Eq. (18) can be obtained as a linear combination of the following basis functions:[20]

$$\mathbf{C}(\tau) = (\tfrac{1}{2}\{\exp[-\lambda_i(\tau_c - \tau)] + \exp(-\lambda_i\tau)\}\delta_{ij}), \qquad i,j = 1, \ldots, N, \tag{21a}$$

$$\mathbf{S}(\tau) = (\tfrac{1}{2}\{\exp[-\lambda_i(\tau_c - \tau)] - \exp(-\lambda_i\tau)\}\delta_{ij}), \qquad i,j = 1, \ldots, N, \tag{21b}$$

$$\gamma = \left[\frac{(\mathbf{Q}^{-1}\mathbf{g})_{ij}}{\dfrac{1}{\mu_{0j}{}^2} - \lambda_i{}^2}\right], \qquad i = 1, \ldots, N; j = 1, \ldots, M. \tag{21c}$$

The offset τ_c is necessary in order to stabilize the system of linear equations numerically for large values of τ,[28] and results in all exponentials having negative arguments as required to avoid fatal overflows for large values of τ_c. Finally, the solution of Eq. (15) can be expressed as

$$\hat{\mathbf{u}}^{\pm}(\tau) = \mathbf{A}^{\pm}(\tau)\boldsymbol{\alpha} + \mathbf{B}^{\pm}(\tau)\boldsymbol{\beta} + \mathbf{V}^{\pm}\mathbf{E}_0(\tau), \tag{22}$$

where $\mathbf{A}^{\pm}(\tau)$ and $\mathbf{B}^{\pm}(\tau)$ are $N \times N$ matrices defined by

$$\mathbf{A}^{\pm}(\tau) = \mathbf{QC}(\tau) \mp \tilde{\mathbf{Q}}\mathbf{LS}(\tau), \tag{23a}$$

$$\mathbf{B}^{\pm}(\tau) = \mathbf{QL}^{-1}\mathbf{S}(\tau) \mp \tilde{\mathbf{Q}}\mathbf{C}(\tau), \tag{23b}$$

$\mathbf{V}^{\pm}$ are $N \times M$ matrices defined by

$$\mathbf{V}^{\pm} = \tfrac{1}{2}[\mathbf{Q}\gamma \pm \tilde{\mathbf{Q}}\gamma\mathbf{M}_0{}^{-1} \pm (\mathbf{X}^{-})^{-1}\hat{\sigma}^{-}], \tag{23c}$$

and $\tilde{\mathbf{Q}} = (\mathbf{Q}^T)^{-1}$, where $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$. The matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $N \times M$ matrices consisting of integral constants to be determined from the boundary conditions. Instead of the traditional way of specifying downward and upward propagating intensities, we have separated the solution into two sets of functions, $\mathbf{A}^{\pm}(\tau)$ and $\mathbf{B}^{\pm}(\tau)$, which consist of symmetric and antisymmetric fields with respect to the optical center of the layer.

D. Boundary Conditions

The boundary value problem for a homogeneous layer of total optical thickness τ_c can be obtained by inversion of Eq. (22) and can be shown to reduce to the form[20]

$$\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} = \frac{1}{2}\begin{bmatrix}(\mathbf{A}^-)^{-1} & (\mathbf{A}^-)^{-1}\\ -(\mathbf{B}^-)^{-1} & (\mathbf{B}^-)^{-1}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}^+(0) - \mathbf{V}^+\\ \hat{\mathbf{u}}^-(\tau_c) - \mathbf{V}^-\mathbf{E}_0(\tau_c)\end{bmatrix}, \quad (24)$$

where

$$\mathbf{A}^\pm = \mathbf{A}^\pm(\tau_c) = \mathbf{A}^\mp(0), \quad (25a)$$

$$\mathbf{B}^\pm = \mathbf{B}^\pm(\tau_c) = -\mathbf{B}^\mp(0). \quad (25b)$$

When the $N \times N$ scaled reflection and transmission matrices are defined as

$$\hat{\mathbf{R}} = \tfrac{1}{2}[\mathbf{A}^+(\mathbf{A}^-)^{-1} + \mathbf{B}^+(\mathbf{B}^-)^{-1}], \quad (26)$$

$$\hat{\mathbf{T}} = \tfrac{1}{2}[\mathbf{A}^+(\mathbf{A}^-)^{-1} - \mathbf{B}^+(\mathbf{B}^-)^{-1}], \quad (27)$$

it follows from Eqs. (22) and (24) that the $N \times M$ scaled reflected and transmitted intensities can be reduced to the form

$$\begin{bmatrix}\hat{\mathbf{u}}^-(0)\\ \hat{\mathbf{u}}^+(\tau_c)\end{bmatrix} = \begin{pmatrix}\hat{\mathbf{R}} & \hat{\mathbf{T}}\\ \hat{\mathbf{T}} & \hat{\mathbf{R}}\end{pmatrix}\begin{bmatrix}\hat{\mathbf{u}}^+(0) - \mathbf{V}^+\\ \hat{\mathbf{u}}^-(\tau_c) - \mathbf{V}^-\mathbf{E}_0(\tau_c)\end{bmatrix} + \begin{pmatrix}\mathbf{V}^-\\ \mathbf{V}^+\mathbf{E}_0(\tau_c)\end{pmatrix}, \quad (28)$$

which is an expression of the interaction principle in the discrete ordinates method. Note further that the scaled reflection and transmission matrices are symmetric, since $\mathbf{A}^+(\mathbf{A}^-)^{-1}$ and $\mathbf{B}^+(\mathbf{B}^-)^{-1}$ are both symmetric matrices.

3. Asymptotic Limits of the Matrix Formulations

A. Reflection and Transmission Matrices

When τ_c is sufficiently large, the reflection and transmission matrices tend to analytically simple expressions known as asymptotic theory for thick layers. This can be shown by decomposing $\mathbf{A}^+(\mathbf{A}^-)^{-1}$ as follows:

$$\begin{aligned}\mathbf{A}^+(\mathbf{A}^-)^{-1} &= 2\mathbf{Q}\mathbf{C}(\tau_c)(\mathbf{A}^-)^{-1} - \mathbf{I},\\ &= 2[\mathbf{I} + \tilde{\mathbf{Q}}\mathbf{S}(\tau_c)\mathbf{L}\mathbf{C}(\tau_c)^{-1}\tilde{\mathbf{Q}}^T]^{-1} - \mathbf{I},\\ &= (\mathbf{A} - \tilde{\mathbf{Q}}\mathbf{a}^+\tilde{\mathbf{Q}}^T)^{-1} - \mathbf{I},\\ &= \mathbf{A}^{-1}(\mathbf{I} - \tilde{\mathbf{Q}}\mathbf{a}^+\tilde{\mathbf{Q}}^T\mathbf{A}^{-1})^{-1} - \mathbf{I},\\ &= \mathbf{A}^{-1} - \mathbf{I} + \mathbf{A}^{-1}\tilde{\mathbf{Q}}(\mathbf{I} - \mathbf{a}^+\mathbf{q})^{-1}\mathbf{a}^+\tilde{\mathbf{Q}}^T\mathbf{A}^{-1},\end{aligned} \quad (29)$$

and, in a similar manner,

$$\mathbf{B}^+(\mathbf{B}^-)^{-1} = \mathbf{A}^{-1} - \mathbf{I} + \mathbf{A}^{-1}\tilde{\mathbf{Q}}(\mathbf{I} - \mathbf{a}^-\mathbf{q})^{-1}\mathbf{a}^-\tilde{\mathbf{Q}}^T\mathbf{A}^{-1}, \quad (30)$$

where $\mathbf{I}$ is the identity matrix and

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{I} + \tilde{\mathbf{Q}}\mathbf{L}\tilde{\mathbf{Q}}^T), \quad (31a)$$

$$\mathbf{q} = \tilde{\mathbf{Q}}^T\mathbf{A}^{-1}\tilde{\mathbf{Q}} \quad (31b)$$

$$\mathbf{a}^+ = \tfrac{1}{2}[\mathbf{L} - \mathbf{S}(\tau_c)\mathbf{L}\mathbf{C}(\tau_c)^{-1}], \quad (31c)$$

$$\mathbf{a}^- = \tfrac{1}{2}[\mathbf{L} - \mathbf{C}(\tau_c)\mathbf{L}\mathbf{S}(\tau_c)^{-1}]. \quad (31d)$$

In deriving these expressions we have made use of the well-known matrix identities

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}, \quad (\mathbf{I} - \mathbf{B})^{-1} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \cdots,$$

together with Eqs. (23) and (25). The advantage of decomposing $\mathbf{A}^+(\mathbf{A}^-)^{-1}$ and $\mathbf{B}^+(\mathbf{B}^-)^{-1}$ as in Eqs. (29) and (30) arises from the fact that these matrices are now separated into τ_c-independent $(\mathbf{A}^{-1} - \mathbf{I})$ and τ_c-dependent terms, where the diagonal matrices $\mathbf{a}^\pm$ contain all the dependence on optical thickness.

If we denote the minimum eigenvalue λ_N by k and take the limit as τ_c approaches infinity, the matrices $\mathbf{a}^\pm$ tend to the following limit:

$$\mathbf{a}_{ij}^\pm \to \begin{cases}\dfrac{\pm k\exp(-k\tau_c)}{1 \pm \exp(-k\tau_c)}, & \text{if } i = j = N\\ 0, & \text{otherwise}\end{cases}. \quad (32)$$

The minimum eigenvalue k that appears in this expression is the same diffusion exponent that appears in Eqs. (1)–(3). Expression (32) further shows that all the elements of $\mathbf{a}^\pm$ except for the Nth diagonal one become vanishingly small as the optical thickness increases.

These results show that the diffusion exponent plays an important role in multiple scattering problems involving optically thick atmospheres. This finding is not surprising in light of Eqs. (21a) and (21b), which clearly show that only the smallest eigenvalue contributes to the diffuse radiation field deep within an optically thick medium. This parameter can readily be determined as the minimum positive square-root eigenvalue of $\mathbf{G}$, defined by Eq. (19a). As such it is seen to depend solely on the single scattering phase function. In Table 1 we summarize values of the diffusion exponent obtained by solving Eq. (20a) for selected Fourier frequencies and for a Henyey–Greenstein phase function having an asymmetry factor $g = 0.85$, in which the various columns of this table apply to specified values of the single scattering albedo ω_0. These results show, for example, that the τ_c-dependent terms of the reflection and transmission matrices become increasingly independent of azimuth angle as τ_c increases. This is a result of the fact that k monotonically increases as Fourier frequency increases, which leads in turn to

Table 1. Minimum Eigenvalues for Several Fourier Frequencies and for Various Values of the Single Scattering Albedo ω_0[a]

Fourier Frequency	$\omega_0 = 1.0$	$\omega_0 = 0.999$	$\omega_0 = 0.9$	$\omega_0 = 0.6$
0	0.0000	0.0212	0.2371	0.5695
1	0.3658	0.3670	0.4731	0.7256
2	0.5794	0.5802	0.6558	0.8402

[a]Henyey–Greenstein phase function with $g = 0.85$.

greater damping of terms such as those appearing in expression (32). Table 1 further shows that the minimum eigenvalues for all the Fourier frequencies increase significantly as ω_0 decreases. This implies that the radiation field becomes increasingly azimuth independent as absorption of the medium increases.

We have compared the eigenvalues obtained by using our method with corresponding values obtained by Garcia and Siewert[29] for various Fourier frequencies and single scattering albedos. In all cases for which the eigenvalues were less than unity, our values agreed with theirs to at least five significant figures by using FORTRAN-77 single-precision calculations. Since our method subdivides the angular interval [0, 1] into N streams with mirror symmetric points on the interval [−1, 0] for a total of $2N$ streams, it necessarily follows that the DOM method leads to N discrete nonnegative eigenvalues of the symmetric matrix given by Eq. (20b). The principal difference between our method and that of Garcia and Siewert is that they obtain discrete eigenvalues only when the eigenvalues are less than unity. Since we have compared our radiation calculations with corresponding ones obtained with the adding–doubling method, which makes use of a totally different algorithm for calculating the intensity field, we are convinced that our algorithm is sufficiently accurate to permit multiple scattering calculations to be performed for most applications of interest in atmospheric physics.

Returning to Eqs. (29) and (30) and noting the asymptotic limit of $\mathbf{a}^{\pm}$, we can show that

$$[(\mathbf{I} - \mathbf{a}^{\pm}\mathbf{q})^{-1}\mathbf{a}^{\pm}]_{ij} \rightarrow \begin{cases} \dfrac{\pm k \exp(-k\tau_c)}{1 \pm (1 - kq)\exp(-k\tau_c)}, & \text{if } i = j = N \\ 0, & \text{otherwise} \end{cases}. \tag{33}$$

where

$$q = q_{NN}. \tag{34}$$

When we substitute expression (33) into Eqs. (26), (27), (29), and (30), it readily follows that the asymptotic form of the reflection and transmission matrices in the limit of large optical thickness is given by

$$\hat{\mathbf{R}} = \hat{\mathbf{R}}_{\infty} - \frac{kl \exp(-2k\tau_c)}{1 - l^2 \exp(-2k\tau_c)} \hat{\mathbf{K}} * \hat{\mathbf{K}}^T, \tag{35}$$

$$\hat{\mathbf{T}} = \frac{k \exp(-k\tau_c)}{1 - l^2 \exp(-2k\tau_c)} \hat{\mathbf{K}} * \hat{\mathbf{K}}^T, \tag{36}$$

$$\hat{\mathbf{R}}_{\infty} = \mathbf{A}^{-1} - \mathbf{I}, \tag{37}$$

where

$$l = 1 - kq, \tag{38a}$$

$$\hat{\mathbf{K}} = \mathbf{A}^{-1}\tilde{\mathbf{Q}}_N, \tag{38b}$$

and $\tilde{\mathbf{Q}}_N$ is the Nth column of the $\tilde{\mathbf{Q}}$ matrix. The matrix operation (∗) denotes the dyadic, defined by

$$\hat{\mathbf{K}} * \hat{\mathbf{K}}^T = \{\hat{\mathbf{K}}_i \hat{\mathbf{K}}_j{}^T\},$$

and $\hat{\mathbf{K}}$ is a column vector of length N. Note that Eqs. (35) and (36) have removable singular points for the 0th Fourier frequency when $\omega_0 = 1$ ($k = 0$) such that energy is conserved according to

$$\hat{\mathbf{R}} = \hat{\mathbf{R}}_{\infty} - \hat{\mathbf{T}}, \tag{39}$$

$$\hat{\mathbf{T}} = \frac{1}{2(\tau_c + q)} \hat{\mathbf{K}} * \hat{\mathbf{K}}^T. \tag{40}$$

It is thus clear from Eq. (40) why the scalar q is referred to by van de Hulst[6] as the extrapolation length, for it denotes an extrapolation of the optical thickness to a larger value appropriate for multiple scattering in optically thick layers.

Comparing Eqs. (1) and (2) with Eqs. (35)–(37) we note that the escape function and asymptotic constants may be obtained in terms of matrices associated with the eigenvalue problem of Eq. (20a). The column vector $\hat{\mathbf{K}}$ is henceforth referred to as the scaled escape function. From Eqs. (35) and (36) and Table 1 we conclude that $\hat{\mathbf{R}}_{\infty}$ contains most of the azimuthal dependence of the reflection and transmission matrices for large values of the optical thickness, since the optical thickness-dependent terms in these expressions are rapidly damped for large values of τ_c when the Fourier frequency increases. The formulation presented in Section 2 is nevertheless valid for all Fourier frequencies. Since the transmission matrix results from a small difference between two matrices having nearly the same values [namely, $\mathbf{A}^{+}(\mathbf{A}^{-})^{-1}$ and $\mathbf{B}^{+}(\mathbf{B}^{-})^{-1}$], the computer code of Nakajima and Tanaka[20] will fail to calculate the transmission matrix for extremely large values of τ_c ($\gtrsim 10^6$) by using the basis functions defined by Eqs. (21a) and (21b). The value of the critical optical thickness depends on the optical properties of the layer as well as the accuracy of a computer's floating point calculations. However, this condition does not often occur for realistic atmospheric conditions, since it arises only when $\hat{\mathbf{T}}$ becomes negligibly small compared to $\hat{\mathbf{R}}_{\infty}$. When such small values of the transmission matrix elements are required, $\hat{\mathbf{T}}$ can best be calculated by using the asymptotic expression given by Eq. (36).

This numerical ill-conditioning, characteristic of many discrete ordinates implementations for large values of the optical thickness, can readily be avoided by using the scaling transformations introduced by Stamnes and Conklin[30] and incorporated in the computer code of Stamnes *et al.*[22] In this investigation, however, we have demonstrated analytically that, when the discrete ordinates formulation of Nakajima and Tanaka[20] is used, it is possible to derive the well-known asymptotic formulas for the reflection and transmission functions of optically thick layers that were previously derived by using rather different approaches.[5,6] This necessarily leads to alternative methods of computing the asymptotic functions and constants that arise in these formulas. This procedure is especially useful in remote sensing applications in which the use of asymptotic formulas permits the analytic inversion of remotely sensed data without the need for large table lookups that are characteristic of conventional methods.

B. Internal Scattered Radiation Field

Another result of considerable importance in asymptotic theory is the angular and vertical distributions of the intensity field deep within an optically thick, multiple scattering medium. The angular distribution of the intensity field can be obtained by using the present matrix formulations by making further use of the interaction principle and the principles of invariance.[31] Each azimuthal component of the internal intensity field at optical depth τ within an optically thick layer of total optical thickness τ_c can be obtained from the expressions

$$\hat{\mathbf{u}}^{+}(\tau) = (\mathbf{I} - \hat{\mathbf{R}}_a\hat{\mathbf{R}}_b)^{-1}\hat{\mathbf{T}}_a\hat{\mathbf{u}}^{+}(0), \tag{41}$$

$$\hat{\mathbf{u}}^{-}(\tau) = \hat{\mathbf{R}}_b\hat{\mathbf{u}}^{+}(\tau), \tag{42}$$

where $\hat{\mathbf{R}}_a$ and $\hat{\mathbf{T}}_a$ correspond to the scaled reflection and transmission matrices of a layer of optical thickness $\tau_a = \tau$ and $\hat{\mathbf{R}}_b$ corresponds to the scaled reflection matrix of a layer of optical thickness $\tau_b = \tau_c - \tau$, for which these matrices can be obtained from Eqs. (35)–(37) by letting $\tau_c = \tau$ and $\tau_c - \tau$, respectively.

Substituting Eq. (36) into Eq. (41) leads to the following expression for the downward propagating intensity field at optical depth τ within an optically thick medium of total optical thickness τ_c:

$$\hat{\mathbf{u}}^{+}(\tau) = \frac{k\exp(-k\tau)}{1 - l^2\exp(-2k\tau)} \times (\mathbf{I} - \hat{\mathbf{R}}_a\hat{\mathbf{R}}_b)^{-1}\hat{\mathbf{K}} * \hat{\mathbf{K}}^T\hat{\mathbf{u}}^{+}(0). \tag{43}$$

When we use Eq. (35) for both $\hat{\mathbf{R}}_a$ and $\hat{\mathbf{R}}_b$, it can be shown that

$$\begin{aligned}\mathbf{v}(\tau) &= (\mathbf{I} - \hat{\mathbf{R}}_a\hat{\mathbf{R}}_b)^{-1}\hat{\mathbf{K}},\\ &= [(1 - c_1 + c_3) - c_2\hat{\mathbf{R}}_\infty](\mathbf{I} - \hat{\mathbf{R}}_\infty^{2})^{-1}\hat{\mathbf{K}},\end{aligned} \tag{44}$$

where $\mathbf{v}(\tau)$ is a column vector of length N and the scalars c_1, c_2, and c_3 are defined by

$$c_1 = \gamma_a\hat{\mathbf{K}}^T\hat{\mathbf{R}}_\infty\mathbf{v}(\tau), \tag{45a}$$

$$c_2 = \gamma_b\hat{\mathbf{K}}^T\mathbf{v}(\tau), \tag{45b}$$

$$c_3 = \gamma_a\gamma_b\hat{\mathbf{K}}^T\hat{\mathbf{K}} * \hat{\mathbf{K}}^T\mathbf{v}(\tau), \tag{45c}$$

with

$$\gamma_a = \frac{kl\exp(-2k\tau)}{1 - l^2\exp(-2k\tau)}, \tag{46a}$$

$$\gamma_b = \frac{kl\exp[-2k(\tau_c - \tau)]}{1 - l^2\exp[-2k(\tau_c - \tau)]}. \tag{46b}$$

In order to proceed further, it is useful if we define the scaled diffusion pattern vectors

$$\hat{\mathbf{P}}^{+} = k(\mathbf{I} - \hat{\mathbf{R}}_\infty^{2})^{-1}\hat{\mathbf{K}}, \tag{47a}$$

$$\hat{\mathbf{P}}^{-} = \hat{\mathbf{R}}_\infty\hat{\mathbf{P}}^{+}, \tag{47b}$$

which are both vectors of length N, independent of optical thickness. These vectors represent the angular distribution of scattered radiation in the downward $(+)$ and upward $(-)$ propagating directions within the diffusion domain of a semi-infinite atmosphere. By making further use of Eqs. (31a), (37), and (38b), it is straightforward to obtain the following expressions for the scaled diffusion pattern vectors:

$$\begin{aligned}\hat{\mathbf{P}}^{+} &= k\mathbf{A}(2\mathbf{A} - \mathbf{I})^{-1}\tilde{\mathbf{Q}}_N,\\ &= \frac{k}{2}[(\tilde{\mathbf{Q}}\mathbf{L}\tilde{\mathbf{Q}}^T)^{-1} + \mathbf{I}]\tilde{\mathbf{Q}}_N,\\ &= \tfrac{1}{2}(\mathbf{Q}_N + k\tilde{\mathbf{Q}}_N),\end{aligned} \tag{48a}$$

$$\begin{aligned}\hat{\mathbf{P}}^{-} &= k(\mathbf{I} - \mathbf{I} + \hat{\mathbf{R}}_\infty)(\mathbf{I} - \hat{\mathbf{R}}_\infty^{2})^{-1}\hat{\mathbf{K}},\\ &= \hat{\mathbf{P}}^{+} - k(\mathbf{I} + \hat{\mathbf{R}}_\infty)^{-1}\hat{\mathbf{K}},\\ &= \hat{\mathbf{P}}^{+} - k\tilde{\mathbf{Q}}_N.\end{aligned} \tag{48b}$$

The scaled diffusion pattern vectors must themselves satisfy the following normalization conditions:

$$\begin{aligned}\hat{\mathbf{K}}^T\hat{\mathbf{P}}^{+} &= k\tilde{\mathbf{Q}}_N^{T}(2\mathbf{A} - \mathbf{I})^{-1}\tilde{\mathbf{Q}}_N,\\ &= k\tilde{\mathbf{Q}}_N^{T}(\tilde{\mathbf{Q}}\mathbf{L}\tilde{\mathbf{Q}}^T)^{-1}\tilde{\mathbf{Q}}_N,\\ &= 1,\end{aligned} \tag{49a}$$

$$\begin{aligned}\hat{\mathbf{K}}^T\hat{\mathbf{P}}^{-} &= \hat{\mathbf{K}}^T(\hat{\mathbf{P}}^{+} - k\tilde{\mathbf{Q}}_N),\\ &= 1 - kq,\\ &= l.\end{aligned} \tag{49b}$$

Substituting Eq. (44) into Eqs. (45) and making further use of Eqs. (47) and (49) permits the system of linear equations to be solved for the three scalars c_1, c_2, and c_3, as outlined in Appendix A. The solution thus obtained can be written as

$$(\mathbf{I} - \hat{\mathbf{R}}_a\hat{\mathbf{R}}_b)^{-1}\hat{\mathbf{K}} = \frac{1 - l^2\exp(-2k\tau)}{k - kl^2\exp(-2k\tau_c)} \times \{\hat{\mathbf{P}}^{+} - l\exp[-2k(\tau_c - \tau)]\hat{\mathbf{P}}^{-}\}. \tag{50}$$

Finally, substituting Eq. (50) back into Eqs. (43) and (42) leads to the following asymptotic solution for the downward and upward propagating intensities deep within an optically thick medium (see Appendix A for further details):

$$\hat{\mathbf{u}}^{\pm}(\tau) = \frac{\exp(-k\tau)}{1 - l^2 \exp(-2k\tau_c)} \times \{\hat{\mathbf{P}}^{\pm} - l \exp[-2k(\tau_c - \tau)]\hat{\mathbf{P}}^{\mp}\} * \hat{\mathbf{K}}^T\hat{\mathbf{u}}^+(0). \tag{51}$$

Equation (51) is the matrix equivalent of Eq. (3) and has been derived from first principles of the radiative transfer equation. We see from Eq. (51) that the scaled diffusion pattern vectors $\hat{\mathbf{P}}^{\pm}$ defined by Eq. (47) represent the angular distribution of scattered radiation for downward (+) and upward (−) propagating radiation deep within an optically thick atmosphere in the limit $\tau_c \to \infty$. The functions and constants that appear in asymptotic theory can now be obtained from the matrices, eigenvectors, and eigenvalues that occur in the DOM.

C. Asymptotic Functions and Constants

For an atmosphere in which radiation is incident only from the top and for which there are no embedded sources, we can write the interaction principle for reflected radiation as[6]

$$u(0; -\mu, \mu_0, \phi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R(\tau_c; \mu, \phi; \mu', \phi') \times u(0; \mu', \mu_0, \phi')\mu' d\mu' d\phi', \tag{52}$$

where $R(\tau_c; \mu, \phi; \mu', \phi')$ is the reflection function for radiation incident from direction (μ', ϕ') and scattered into direction (μ, ϕ). By further expressing the reflection function as a Fourier series analogous to Eq. (10) and making use of the orthogonality properties of the cosine function, it can be shown that

$$u^m(0; -\mu, \mu_0) = 2\int_0^1 R^m(\tau_c; \mu, \mu')u^m(0; \mu', \mu_0)\mu' d\mu', \tag{53}$$

which, in terms of matrix notation, can be written as

$$\mathbf{u}^-(0) = 2\mathbf{RWMu}^+(0). \tag{54}$$

By multiplying both sides of this expression by $\mathbf{W}^+$ and comparing the resulting expression with Eq. (26), we find

$$\hat{\mathbf{R}} = 2\mathbf{W}^+\mathbf{R}\mathbf{W}^+, \tag{55a}$$

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{W}^+)^{-1}\hat{\mathbf{R}}(\mathbf{W}^+)^{-1}. \tag{55b}$$

A similar expression results for $\mathbf{T}$ if we neglect the role of direct (unscattered) radiation in comparison with the role of diffuse radiation.

Finally, by noting that the Fourier decomposition of the incident solar beam can be written in matrix notation as

$$\mathbf{u}^+(0) = \left[\frac{F_0}{2\pi w_i}\delta_{ij}\right], \qquad i = 1, \ldots, N; j = 1, \ldots, M, \tag{56}$$

it is relatively straightforward to transform Eqs. (35), (36), and (51) to a form that can be compared with Eqs. (1)–(3). Making further use of Eq. (55b) we obtain the following expressions for the functions that occur in asymptotic theory:

$$R_\infty = \tfrac{1}{2}(\mathbf{W}^+)^{-1}\hat{\mathbf{R}}_\infty(\mathbf{W}^+)^{-1}, \tag{57a}$$

$$\mathbf{K} = \sqrt{\frac{k}{2m}}(\mathbf{W}^+)^{-1}\hat{\mathbf{K}}, \tag{57b}$$

$$\mathbf{P}^{\pm} = \sqrt{\frac{m}{2k}}(\mathbf{W}^+)^{-1}\hat{\mathbf{P}}^{\pm}. \tag{57c}$$

In these expressions m is a scalar constant that depends on the single scattering phase function. Its value can readily be determined by normalizing the diffusion pattern as follows:

$$\begin{aligned}\frac{1}{2}\int_{-1}^1 P(\mu)d\mu &= \frac{1}{2}\sqrt{\frac{m}{2k}}\,\mathbf{1}^T\mathbf{W}(\mathbf{W}^+)^{-1}(\hat{\mathbf{P}}^+ - \hat{\mathbf{P}}^-),\\ &= \frac{1}{2}\sqrt{\frac{m}{2k}}\,\mathbf{1}^T\mathbf{W}^-\mathbf{Q}_N,\\ &= 1,\end{aligned} \tag{58}$$

where $\mathbf{1}$ is a unit column vector of length N. In deriving this expression we have made use of Eqs. (48) and (57c). From this normalization condition, together with Eqs. (48) and (49), it can readily be shown that the asymptotic functions and constants must satisfy the well-known normalization conditions:

$$2\int_0^1 K(\mu)P(\mu)\mu d\mu = 1, \tag{59a}$$

$$2\int_0^1 K(\mu)P(-\mu)\mu d\mu = l, \tag{59b}$$

$$2\int_{-1}^1 [P(\mu)]^2\mu d\mu = m. \tag{59c}$$

Finally, the asymptotic constant n, which occurs in calculations of the reflected and transmitted flux, can be obtained from the following definition:

$$2\int_0^1 K(\mu)\mu d\mu = n. \tag{60}$$

4. Further Considerations for a Practical Method

Although the formulations presented in Section 3 are sufficient for obtaining the asymptotic functions and constants for optically thick and vertically homogeneous plane-parallel atmospheres, they are inefficient because they require the computation of the eigenvalues and eigenvectors of a large ($N \times N$) matrix if one needs to obtain a solution using a finite angular resolution with a large number of discrete quadrature streams N. Alternatively, the escape function can be obtained from an expression for the transmitted intensities for M arbitrary solar incident directions μ_{0j}, $j = 1, \ldots, M$ without the need to increase the value of N. If we consider the situation in which there is no incident radiation from the layer boundaries, namely, $\hat{\mathbf{u}}^+(0) = \hat{\mathbf{u}}^-(\tau_c) = 0$, it readily follows from Eq. (28) that the scaled intensity matrices can be expressed as

$$\hat{\mathbf{u}}^+(\tau_c) = -\hat{\mathbf{T}}\mathbf{V}^+ - \hat{\mathbf{R}}\mathbf{V}^-\mathbf{E}_0(\tau_c) + \mathbf{V}^+\mathbf{E}_0(\tau_c), \quad (61a)$$

$$\hat{\mathbf{u}}^-(0) = -\hat{\mathbf{R}}\mathbf{V}^+ - \hat{\mathbf{T}}\mathbf{V}^-\mathbf{E}_0(\tau_c) + \mathbf{V}^-. \quad (61b)$$

When τ_c is sufficiently large, $\hat{\mathbf{u}}^+(\tau_c)$ reduces to $-\hat{\mathbf{T}}\mathbf{V}^+$. Making use of Eqs. (36) and (57b) and comparing the resulting expression with Eq. (2) leads to

$$\mathbf{K}_0 = -\frac{2\pi}{F_0}\sqrt{\frac{k}{2m}}\,\mathbf{M}_0^{-1}(\mathbf{V}^+)^T\hat{\mathbf{K}}, \quad (62)$$

where $\mathbf{K}_0$ is a column vector of length M.

Once the diffusion exponent has been obtained from Eqs. (20), the simplest way to obtain the diffusion pattern (for azimuth-independent radiation) is to expand it as a finite series in Legendre polynomials of the form (van de Hulst,[6] p. 97)

$$P(\mu) = \sum_{l=0}^{L+1} (2l + 1)g_l P_l(\mu), \quad (63)$$

where the coefficients g_l are themselves polynomials in k^{-1}—the so-called Kuščer polynomials. These coefficients can be obtained by downward recurrence of the following relation (van de Hulst,[6] p. 94):

$$(l + 1)g_{l+1} - (2l + 1 - \omega_l)k^{-1}g_l + lg_{l-1} = 0, \quad (64)$$

where $g_0 = 1$, $g_1 = (1 - \omega_0)/k$, and ω_l are the Legendre coefficients of the phase function defined by Eqs. (6) and (7). The asymptotic constants l, m, and n can readily be determined from the normalization conditions given in Eqs. (59) and (60).

In order to compute the reflected intensity field in a semi-infinite atmosphere, denoted $u_\infty(-\mu, \mu_0, \phi)$ in Eq. (1), we can make use of Eqs. (37) and (57a), together with the definition of the semi-infinite reflection function:

$$R_\infty(\mu, \mu_0, \phi) = \frac{\pi u_\infty(-\mu, \mu_0, \phi)}{\mu_0 F_0}. \quad (65)$$

In this case, however, it is necessary to interpolate the $N \times N$ values of the symmetric reflection matrix $\mathbf{R}_\infty$ at quadrature points μ_i, $i = 1, \ldots, N$ to obtain values $\{\mu_i'\}$ and $\{\mu_{0j}\}$. Such an interpolation is especially important when $N \le 10$. An alternative approach is to make use of the Stamnes and Dale interpolation[32] method based on the formal solution of the radiative transfer equation in the limit $\tau_c \to \infty$:

$$\mathbf{u}_\infty^-(0) = \lim_{\tau_c\to\infty} \mathbf{M}^{-1}\int_0^{\tau_c} \mathbf{E}(\tau')\mathbf{J}^-(\tau')d\tau', \quad (66)$$

where the source matrix $\mathbf{J}^-(\tau)$ follows from the right-hand side of Eq. (13) and is given by

$$\mathbf{J}^-(\tau) = \mathbf{h}^-\mathbf{W}\mathbf{u}^+(\tau) + \mathbf{h}^+\mathbf{W}\mathbf{u}^-(\tau) + \mathbf{S}^-\mathbf{E}_0(\tau). \quad (67)$$

All the matrices appearing in these expressions have been defined previously, except that they apply to emergent directions μ_i', $i = 1, \ldots, N'$ rather than the more numerous quadrature points μ_i, $i = 1, \ldots, N$. Making further use of Eqs. (22) and (23) we can rewrite the $\mathbf{N}' \times \mathbf{M}$ source matrix as

$$\mathbf{J}^-(\tau) = [\mathbf{H}\mathbf{C}(\tau) + \tilde{\mathbf{H}}\mathbf{L}\mathbf{S}(\tau)]\boldsymbol{\alpha} + [\mathbf{H}\mathbf{L}^{-1}\mathbf{S}(\tau) + \tilde{\mathbf{H}}\mathbf{C}(\tau)]\boldsymbol{\beta} + \mathbf{J}_0^-\mathbf{E}_0(\tau), \quad (68)$$

where $\mathbf{H}$ and $\tilde{\mathbf{H}}$ are $N' \times N'$ matrices and $\mathbf{J}_0^-$ is an $N' \times M$ matrix defined by

$$\mathbf{H} = (\mathbf{h}^+ + \mathbf{h}^-)\mathbf{W}^-\mathbf{Q}, \quad (69a)$$

$$\tilde{\mathbf{H}} = (\mathbf{h}^+ - \mathbf{h}^-)\mathbf{W}^-\tilde{\mathbf{Q}}, \quad (69b)$$

$$\mathbf{J}_0^- = \mathbf{h}^-\mathbf{W}^-\mathbf{V}^+ + \mathbf{h}^+\mathbf{W}^-\mathbf{V}^- + \mathbf{S}^-. \quad (69c)$$

Substituting Eq. (68) back into Eq. (66) and taking the limit $\tau_c \to \infty$ leads to an expression for the Fourier-dependent intensity field reflected from a semi-infinite atmosphere for arbitrary direction cosines μ_i', $i = 1, \ldots, N'$ and μ_{0j}, $j = 1, \ldots, M$ as follows:

$$\mathbf{u}_\infty^-(0) = \tfrac{1}{2}(-\mathbf{H} + \tilde{\mathbf{H}}\mathbf{L}) \bullet (\mathbf{M}, \mathbf{L})\tilde{\mathbf{Q}}^T\mathbf{A}^{-1}\mathbf{V}^+ + \mathbf{J}_0^- \bullet (\mathbf{M}, \mathbf{M}_0)\mathbf{M}_0, \quad (70)$$

where the operator $(\bullet)$ is defined such that the matrices

$$\mathbf{H} \bullet (\mathbf{M}, \mathbf{L}) = \left[\frac{\mathbf{H}_{ij}}{\mu_i' + \lambda_j}\right], \quad i, j = 1, \ldots, N', \quad (71a)$$

$$\mathbf{J}_0^- \bullet (\mathbf{M}, \mathbf{M}_0) = \left[\frac{\mathbf{J}_{0ij}^-}{\mu_i' + \mu_{0j}}\right], \quad i = 1, \ldots, N'; \; j = 1, \ldots, M. \quad (71b)$$

5. Numerical Validation of the Formulations

In order to test the validity of the matrix formulations presented in Sections 2–4, we have computed the asymptotic functions and constants for a Henyey–Greenstein phase function having an asymmetry

Table 2. Diffusion Exponent k Derived by Several Different Methods[a]

$\omega_0 = 0.999$	$\omega_0 = 0.9$	$\omega_0 = 0.6$	N	Method
0.02124	0.23713	0.56979	59	Asymptotic fitting
0.02124	0.23713	0.56950	10	Recurrence
0.02124	0.23713	0.56950	10	DOM
0.02124	0.23713	0.56951	5	DOM
0.02124	0.23718	0.57588	3	DOM
0.02122	0.24100	0.61243	2	DOM
0.02703	0.32477	0.90255	1	DOM

[a]Azimuth-independent radiation for a Henyey–Greenstein phase function with $g = 0.85$.

factor $g = 0.85$ and single scattering albedos $\omega_0 = 1.0$, 0.999, 0.9, and 0.6. In Table 2 we summarize values of the diffusion exponent k obtained by using several different methods for the 0th Fourier frequency. In the present method, referred to as the DOM, the minimum eigenvalue $k = \lambda_N$ has been obtained by solving Eq. (20a) for a Gaussian quadrature on the μ interval [0, 1] of order $N = 1$, 2, 3, 5, and 10. Furthermore, we have made use of the delta-M truncation method[33] in which the redistribution matrices $\mathbf{h}^{\pm}$, and hence $\mathbf{X}^{\pm}$ and $\mathbf{G}$, have been altered by modifying the Legendre coefficients of the phase function according to

$$\omega_l^* = \frac{\omega_l - \omega_0 f(2l + 1)}{1 - f}, \qquad l = 0, \ldots, L, \tag{72}$$

where $L = 2N - 1$ and the truncation factor f is defined by

$$f = \frac{\omega_{2N}}{\omega_0(4N + 1)}. \tag{73}$$

When a truncation method is used, such as the delta-M method, we must transform the resultant diffusion exponent as follows:

$$k = (1 - \omega_0 f)k^{\text{truncated}}. \tag{74}$$

Asymptotic constant l is not affected by truncation, and thus the extrapolation length q must be scaled according to

$$q = q^{\text{truncated}}/(1 - \omega_0 f). \tag{75}$$

In addition to the eigenvector/eigenvalue method outlined above, the diffusion exponent as well as other asymptotic functions and constants appearing in Eqs. (1)–(3) can be obtained by applying the asymptotic fitting method of van de Hulst.[15,6] In this method, numerical computations from the doubling method are fit to known asymptotic expressions for the plane albedo, diffuse transmission, and internal intensity field as a function of optical depth for optically thick layers. In Table 2 we summarize values of the diffusion exponent obtained by the asymptotic fitting method by using a doubling code having $N = 59$ Gaussian quadrature points on the interval [0, 1] and without truncation.[26,8]

Finally, we have computed the diffusion exponent k by using the recurrence method described by Sobolev.[5] In this method a characteristic equation is solved that leads to the following continued fraction:

$$1 - \omega_0 = \cfrac{k^2}{a_1 - \cfrac{4k^2}{a_2 - \cfrac{9k^2}{a_3 - \cdots}}}, \tag{76}$$

where

$$a_l = (2l + 1 - \omega_l). \tag{77}$$

The sequence of positive minimum solutions of Eq. (76), $k_0, k_1, k_2, \ldots, k_N$ obtained by truncating to order N, is in general a decreasing sequence indicating that the transmitted flux through optically thick atmospheres will always be underestimated when a low-order radiative transfer algorithm is used. Using this feature we can calculate the minimum diffusion pattern with Newton's method. The range of the search can readily be estimated from k_0, k_1, and k_2 since this is a rapidly converging series.

In Table 2 we show that the DOM with $N = 5$ is sufficiently accurate for most applications of radiative transfer in optically thick atmospheres. Furthermore, we find that this method provides the same solution for k as in the recurrence method. We have also checked that our solution has converged by comparing these computations with the DOM result obtained for $N = 40$. Even for a strongly absorbing medium, such as the ocean, the DOM solution with $N = 5$ leads to a more accurate estimate of the diffusion exponent than the asymptotic fitting method with $N = 59$. This is because the asymptotic fitting method is based on ratios of the global transmission obtained from numerical computations at three doubled optical thicknesses (namely, $\tau_c = 8$, 16, and 32). As ω_0 decreases, this method becomes increasingly less accurate because of the small values of the transmission in a highly absorbing medium. Although not presented in Table 2, we have also found that the delta-M method enhances the convergence of k as well as other asymptotic functions and constants when compared with corresponding results obtained in the absence of truncation. As pointed out by King and Harshvardhan,[18] the DOM with $N = 1$ corresponds to the delta-Eddington approximation, which is known to have large errors in the diffusion exponent when the single scattering albedo is small.

In Table 3 we summarize values of asymptotic constants l, m, and n derived by using three different methods for a Henyey–Greenstein phase function ($g = 0.85$) and for single scattering albedos $\omega_0 = 0.999$, 0.9, and 0.6, for which all the computations apply to azimuth-independent radiation. In addi-

Table 3. Asymptotic Constants l, m and n Derived by Several Different Methods[a]

ω_0	l	m	n	N	Method
0.999	0.81708	0.37769	0.90635	59	Asymptotic fitting
	0.81708	0.37769	0.90635	10	DOM
	0.81708	0.37769	0.90635	5	DOM
	0.81708	0.37769	0.90635	10	Hybrid
	0.81708	0.37769	0.90636	5	Hybrid
	0.81710	0.37766	0.90637	3	Hybrid
	0.81742	0.37724	0.90654	2	Hybrid
0.9	0.12494	4.32592	0.44738	59	Asymptotic fitting
	0.12494	4.33017	0.44716	10	DOM
	0.12493	4.33036	0.44714	5	DOM
	0.12494	4.33017	0.44716	10	Hybrid
	0.12494	4.33010	0.44716	5	Hybrid
	0.12495	4.33212	0.44737	3	Hybrid
	0.11852	4.45945	0.44365	2	Hybrid
0.6	0.01150	12.0149	0.23125	59	Asymptotic fitting
	0.01134	13.0016	0.22125	10	DOM
	0.01132	12.9873	0.22149	5	DOM
	0.01134	13.0016	0.22125	10	Hybrid
	0.01134	13.0035	0.22132	5	Hybrid
	0.00587	13.9176	0.21766	3	Hybrid
	−0.02659	15.3458	0.20242	2	Hybrid

[a]Azimuth-independent radiation for a Henyey–Greenstein phase function with $g = 0.85$.

tion to the asymptotic fitting method, these constants have been determined by using the discrete ordinates method for which l can be obtained from Eqs. (38a) and (34) and m and n from the normalization conditions of Eqs. (58) and (60). In addition to the DOM results for various values of N, in Table 3 we present computational results obtained by using the hybrid method based on Eq. (62) for the escape function and Eq. (63) for the diffusion pattern, with the so-called Kuščer polynomials obtained from the recurrence relation given in Eq. (64). In the hybrid method a large number (μ_{0j}, $j = 1, \ldots, M = 40$) of Gaussian quadrature points on the half-range [0, 1] was used for angular integration of the normalization conditions, keeping the order of the Gaussian quadrature N for solving the eigenvalue problem of Eq. (20a) much reduced. In this way we were able to obtain a good estimate of l, m, and n by using a quadraturization as small as $N = 3$ without a noticeable increase in the computational time required. As was found in Table 2, $N = 5$ is sufficiently accurate for both the DOM and hybrid methods.

Values of the escape function, diffusion pattern, and plane albedo of a semi-infinite layer are summarized in Tables 4–6, in which the plane albedo of a semi-infinite layer is defined by

$$r_\infty(\mu_0) = \frac{1}{\mu_0 F_0}\int_0^{2\pi}\int_0^1 u_\infty(-\mu, \mu_0, \phi)\mu d\mu d\phi,$$
$$= 2\int_0^1 R_\infty{}^0(\mu, \mu_0)\mu d\mu. \qquad (78)$$

Table 4. Escape Function $K(\mu)$ Derived by Several Different Methods[a]

ω_0	$\mu = 1.0$	$\mu = 0.5$	$\mu = 0.1$	N	Method
1.0	1.27141	0.86871	0.46733	59	Asymptotic fitting
	1.27141	0.86870	0.46733	40	DOM
	1.27141	0.86871	0.46743	20	DOM
	1.27143	0.86866	0.46861	10	DOM
	1.27155	0.86888	0.46251	5	DOM
	1.27141	0.86870	0.46733	20	Hybrid
	1.27140	0.86872	0.46706	10	Hybrid
	1.27144	0.86887	0.46847	5	Hybrid
	1.27036	0.86857	0.47501	3	Hybrid
	1.26375	0.86900	0.48605	2	Hybrid
0.999	1.15505	0.78586	0.42189	59	Asymptotic fitting
	1.15580	0.78568	0.42182	40	DOM
	1.15580	0.78568	0.42191	20	DOM
	1.15581	0.78563	0.42297	10	DOM
	1.15583	0.78584	0.41749	5	DOM
	1.15580	0.78568	0.42182	20	Hybrid
	1.15579	0.78569	0.42158	10	Hybrid
	1.15583	0.78583	0.42285	5	Hybrid
	1.15477	0.78556	0.42872	3	Hybrid
	1.14868	0.78629	0.43892	2	Hybrid
0.9	0.72565	0.31740	0.14140	59	Asymptotic fitting
	0.72877	0.31663	0.14113	40	DOM
	0.72872	0.31663	0.14116	20	DOM
	0.72809	0.31662	0.14146	10	DOM
	0.72125	0.31674	0.14020	5	DOM
	0.72877	0.31663	0.14113	20	Hybrid
	0.72877	0.31663	0.14106	10	Hybrid
	0.72828	0.31675	0.14130	5	Hybrid
	0.72059	0.31738	0.14191	3	Hybrid
	0.70157	0.32253	0.15028	2	Hybrid
0.6	0.54111	0.09442	0.02543	59	Asymptotic fitting
	0.63289	0.07999	0.02259	40	DOM
	0.63252	0.07999	0.02259	20	DOM
	0.62764	0.08001	0.02262	10	DOM
	0.58402	0.08023	0.02226	5	DOM
	0.63289	0.07999	0.02259	20	Hybrid
	0.63287	0.07999	0.02258	10	Hybrid
	0.62613	0.08017	0.02241	5	Hybrid
	0.58912	0.08302	0.02150	3	Hybrid
	0.70172	0.07366	0.02261	2	Hybrid

[a]Azimuth-independent radiation for a Henyey–Greenstein phase function with $g = 0.85$.

Since the DOM method of Section 3 provides solutions for these functions at Gaussian quadrature points $\{\mu_i, i = 1, \ldots, N\}$, we have made use of cubic spline interpolation to interpolate the calculated values to the direction cosines presented in the tables. It is likewise possible to use an analytic interpolation, such as the iteration of the source function technique employed by Stamnes and Swanson,[19] although we utilized a simple spline interpolation here simply to intercompare results obtained by several different methods. In the hybrid method we recalculated the values at $\mu_0 = 0.1, 0.5$, and 1.0 after we obtained l, m, and n as in Table 3. Note that the results obtained for the diffusion pattern by using the hybrid method with $M = 10$ are nearly the same as the corresponding results obtained with the DOM by using $N = 40$.

Table 5. Diffusion Pattern $P(\mu)$ Derived by Several Different Methods[a]

ω_0	$\mu = 1.0$	$\mu = 0.5$	$\mu = 0.0$	$\mu = -0.5$	$\mu = -1.0$	N	Method
0.999	1.14862	1.06962	0.99639	0.92858	0.86576	59	Asymptotic fitting
	1.14867	1.06961	0.99640	0.92859	0.86574	40	DOM
	1.14866	1.06961	0.99640	0.92859	0.86573	10	DOM
	1.14858	1.06961	0.99640	0.92859	0.86566	5	DOM
	1.14867	1.06961	0.99640	0.92859	0.86574	10	Hybrid
	1.14867	1.06961	0.99640	0.92859	0.86574	5	Hybrid
	1.14866	1.06960	0.99640	0.92859	0.86575	3	Hybrid
	1.14848	1.06953	0.99641	0.92868	0.86588	2	Hybrid
0.9	3.18110	1.44537	0.70379	0.36524	0.20067	59	Asymptotic fitting
	3.18906	1.44447	0.70363	0.36525	0.20071	40	DOM
	3.18612	1.44449	0.70363	0.36525	0.20061	10	DOM
	3.15646	1.44487	0.70349	0.36502	0.19909	5	DOM
	3.18908	1.44448	0.70363	0.36525	0.20071	10	Hybrid
	3.18904	1.44447	0.70363	0.36526	0.20071	5	Hybrid
	3.18621	1.44358	0.70423	0.36438	0.19871	3	Hybrid
	3.16600	1.47242	0.67105	0.36311	0.14979	2	Hybrid
0.6	6.96337	1.12056	0.25374	0.08563	0.03745	59	Asymptotic fitting
	8.22991	1.04426	0.24591	0.08452	0.03729	40	DOM
	8.16156	1.04457	0.24592	0.08451	0.03723	10	DOM
	7.58614	1.04623	0.24446	0.08330	0.03496	5	DOM
	8.23026	1.04426	0.24591	0.08452	0.03729	10	Hybrid
	8.21849	1.04546	0.24693	0.08510	0.03454	5	Hybrid
	8.13160	0.93351	0.25586	0.02012	−0.07255	3	Hybrid
	7.31692	1.30951	−0.21699	0.08200	−0.44897	2	Hybrid

[a]Azimuth-independent radiation for a Henyey–Greenstein phase function with $g = 0.85$.

This suggests that it is important to increase M to obtain greater accuracy while at the same time keeping N to a value as small as 10. Tables 4–6 show that it is necessary to use a quadraturization with $N \gtrsim 5$ to calculate these asymptotic functions for $\omega_0 \geq 0.9$ ($N \geq 10$ for $\omega_0 = 0.6$) in order to guarantee accurate solutions.

Table 6. Plane Albedo of a Semi-Infinite Atmosphere Derived by Several Different Methods[a]

ω_0	$\mu_0 = 1.0$	$\mu_0 = 0.5$	$\mu_0 = 0.1$	N	Method
0.999	0.78729	0.84940	0.91683	59	Asymptotic fitting
	0.78729	0.84940	0.91683	20	Hybrid
	0.78730	0.84940	0.91688	10	Hybrid
	0.78728	0.84937	0.91662	5	Hybrid
	0.78738	0.84939	0.91541	3	Hybrid
	0.78796	0.84932	0.91344	2	Hybrid
0.9	0.10387	0.20846	0.44872	59	Asymptotic fitting
	0.10387	0.20846	0.44871	20	Hybrid
	0.10388	0.20845	0.44903	10	Hybrid
	0.10374	0.20830	0.44772	5	Hybrid
	0.10341	0.20814	0.43987	3	Hybrid
	0.10534	0.21258	0.42810	2	Hybrid
0.6	0.01607	0.04523	0.17886	59	Asymptotic fitting
	0.01607	0.04523	0.17886	20	Hybrid
	0.01608	0.04521	0.17924	10	Hybrid
	0.01591	0.04511	0.17867	5	Hybrid
	0.01552	0.04484	0.17199	3	Hybrid
	0.01834	0.04853	0.16050	2	Hybrid

[a]Henyey–Greenstein phase function with $g = 0.85$.

Finally in Table 7 we present values of the reflection function of a semi-infinite layer for overhead Sun ($\mu_0 = 1$) and for the same Henyey–Greenstein phase function used previously. For comparison purposes we have used three different methods: (1) the asymptotic fitting method based on radiative transfer computations by using the doubling method with $N = 59$; (2) Eq. (70) with the redistribution matrices $\mathbf{h}^{\pm}$, and hence all other matrices appearing in this expression, modified following the delta-M truncation method[33]; and (3) Eq. (70) with the truncated multiple-plus-single-scattering (TMS) method,[23] which is an improvement of the delta-M method that is particularly significant for improving the accuracy of intensity calculations and uses a small number of Gaussian quadrature points. For most applications, we have found that the TMS method with $N = 10$ is sufficiently accurate, whereas the familiar delta-M method requires a larger quadraturization N. This observation implies that the TMS method improves the accuracy of intensity calculations even for optically thick layers, a feature not demonstrated in the original paper.[23]

Accurate computations of the reflection and transmission functions of optically thick layers are generally more difficult to obtain than they appear at first glance, since the effects of many different error sources tend to be amplified while the calculations are performed. For example, a small round-off error in the doubling method increases rapidly as the optical thickness of the layer increases.[34,20] Since we were

Table 7. Reflection Function of a Semi-Infinite Atmosphere with Normal Incidence ($\mu_0 = 1$) Derived by Several Different Methods[a]

ω_0	$\mu = 1.0$	$\mu = 0.5$	$\mu = 0.1$	N	Method
1.0	1.12835	0.94462	0.58135	59	Asymptotic fitting
	1.12838	0.94462	0.58136	20	TMS
	1.12957	0.94453	0.58073	10	TMS
	1.13520	0.94450	0.58540	5	TMS
	1.14132	0.94768	0.60031	3	TMS
	1.14444	0.92479	0.58336	2	TMS
	1.11212	0.94534	0.58278	10	Delta-M
0.999	0.86185	0.75785	0.47984	59	Asymptotic fitting
	0.86188	0.75785	0.47985	20	TMS
	0.86307	0.75777	0.47928	10	TMS
	0.86866	0.75769	0.48362	5	TMS
	0.87502	0.76099	0.49707	3	TMS
	0.87983	0.73874	0.47829	2	TMS
	0.85469	0.75858	0.48132	10	Delta-M
0.9	0.08513	0.11573	0.10576	59	Asymptotic fitting
	0.08515	0.11573	0.10576	20	TMS
	0.08612	0.11566	0.10548	10	TMS
	0.09036	0.11543	0.10767	5	TMS
	0.09424	0.11751	0.11266	3	TMS
	0.09433	0.10527	0.09876	2	TMS
	0.07860	0.11638	0.10732	10	Delta-M
0.6	0.01020	0.01910	0.02877	59	Asymptotic fitting
	0.01021	0.01910	0.02877	20	TMS
	0.01063	0.01907	0.02868	10	TMS
	0.01225	0.01896	0.02925	5	TMS
	0.01320	0.01952	0.03027	3	TMS
	0.01249	0.01697	0.02718	2	TMS
	0.00567	0.01955	0.02989	10	Delta-M

[a]Henyey–Greenstein phase function with $g = 0.85$.

aware of this problem, we checked the results of the Nakajima and Tanaka[20] algorithm and have found that it yields the same answer as the results shown in Table 7 up to $\tau_c \simeq 10^6$ for double precision calculations on an IBM 3081 computer. Therefore, another possibility for computing the asymptotic functions is simply to use the ordinary TMS method with the DOM algorithm of Nakajima and Tanaka[23] or Stamnes *et al.*[22] and let $\tau_c = 10^6$.

6. Concluding Remarks

We have derived the asymptotic limit of the radiative transfer equation in optically thick and vertically homogeneous plane-parallel layers from first principles by using the discrete ordinates method (DOM). Our derivation differs substantially from the heuristic thought experiment derivation of van de Hulst[3,6] and the mathematical derivation based on the formal solution of the radiative transfer equation presented by Sobolev.[4,5] Furthermore, we have shown how to calculate the various functions and constants arising in asymptotic theory by using the matrices and eigenvalues routinely computed in discrete ordinates algorithms. The asymptotic expressions for the scaled reflection and transmission matrices, for example, are given in Eqs. (35) and (36), for which the scaled escape function vector is given in Eq. (38b), the scaled diffusion pattern vectors by Eqs. (47), and the scaled reflection matrix of a semi-infinite layer by Eq. (37). Equations (57) further show how to convert these scaled matrices and vectors into physical functions of μ and μ_0 as in Eqs. (1)–(3).

By using these and other formulas presented in the previous sections, we have found several different methods for computing the escape function, diffusion pattern, and reflection function of a semi-infinite layer. These methods include (a) the asymptotic limit of the DOM, namely the direct use of Eqs. (38b), (47), and (37), (b) a hybrid method based on Eqs. (62), (63), and (70), (c) the direct method, namely the use of the ordinary algorithm of the TMS method[23] with sufficiently large τ_c ($\simeq 10^6$), and (d) the asymptotic fitting method[15,6] in which doubling computations at three optical thicknesses ($\tau_c = 8$, 16, and 32) are matched to the asymptotic formulas to obtain the required functions and constants.

The aforementioned methods have several advantages and disadvantages. Method (a) is simple in its formulation but relatively inefficient in performing numerical computations. Method (b) is the most computationally efficient, but requires a more complicated formulation. In this method, a discrete quadrature of order $N = 10$ is sufficiently accurate for most applications. Method (c) is not limited to cases of large τ_c but requires a large τ_c to obtain the asymptotic fields. This necessarily adds the possibility of numerical instability to the solution. The asymptotic fitting method is highly stable but requires the use of the doubling method to compute the reflection and transmission functions to an optical thickness of at least $\tau_c = 32$. This method has been utilized extensively in our earlier work.[8–11,18,26]

Comparison of the reflection function of a semi-infinite layer obtained by several different methods (see Table 7) has shown that the TMS method of Nakajima and Tanaka[23] improves the efficiency and accuracy of intensity calculations even for optically thick atmospheres.

Finally, we have demonstrated the accuracy of a number of numerically efficient methods for calculating the radiative intensity field in any plane-parallel optically thick atmosphere. Since DOM computer codes subdivide a vertically inhomogeneous layer into several homogeneous sublayers, it is possible to implement the asymptotic formulations given here into DOM codes for rapid treatment of any sublayer of sufficient optical thickness ($\tau_c \geq 8$).

Appendix A

To derive the expression for the internal scattered radiation field deep within an optically thick multiple scattering media, we begin by substituting Eqs. (47) into Eq. (44), leading to

$$\mathbf{v}(\tau) = \frac{1}{k}\left[(1 - c_1 + c_3)\hat{\mathbf{P}}^{+} - c_2\hat{\mathbf{P}}^{-}\right]. \quad \text{(A1)}$$

Substituting Eq. (A1) back into Eqs. (45) leads to

$$c_1 = \frac{\gamma_a}{k}\hat{\mathbf{K}}^T\hat{\mathbf{R}}_\infty[(1 - c_1 + c_3)\hat{\mathbf{P}}^+ - c_2\hat{\mathbf{P}}^-], \tag{A2a}$$

$$c_2 = \frac{\gamma_b}{k}\hat{\mathbf{K}}^T[(1 - c_1 + c_3)\hat{\mathbf{P}}^+ - c_2\hat{\mathbf{P}}^-], \tag{A2b}$$

$$c_3 = \frac{\gamma_a\gamma_b}{k}\hat{\mathbf{K}}^T\hat{\mathbf{K}} * \hat{\mathbf{K}}^T[(1 - c_1 + c_3)\hat{\mathbf{P}}^+ - c_2\hat{\mathbf{P}}^-]. \tag{A2c}$$

By making further use of Eqs. (47) and (49) it follows that

$$\begin{aligned}\hat{\mathbf{K}}^T\hat{\mathbf{R}}_\infty\hat{\mathbf{P}}^+ &= \hat{\mathbf{K}}^T\hat{\mathbf{P}}^- \\ &= l, \qquad\qquad\qquad\qquad (\text{A3a})\\ \hat{\mathbf{K}}^T\hat{\mathbf{R}}_\infty\hat{\mathbf{P}}^- &= \hat{\mathbf{K}}^T\hat{\mathbf{R}}_\infty{}^2\hat{\mathbf{P}}^+, \\ &= \hat{\mathbf{K}}^T[\hat{\mathbf{P}}^+ - (\mathbf{I} - \hat{\mathbf{R}}_\infty{}^2)\hat{\mathbf{P}}^+] \\ &= 1 - k\hat{\mathbf{K}}^T\hat{\mathbf{K}}. \qquad\qquad (\text{A3b})\end{aligned}$$

Substituting Eqs. (A3) back into Eqs. (A2) results in the following system of linear equations:

$$c_1 = \frac{\gamma_a}{k}[(1 - c_1 + c_3)l - c_2(1 - k\hat{\mathbf{K}}^T\hat{\mathbf{K}})], \tag{A4a}$$

$$c_2 = \frac{\gamma_b}{k}(1 - c_1 + c_3 - c_2 l), \tag{A4b}$$

$$c_3 = \gamma_a c_2\hat{\mathbf{K}}^T\hat{\mathbf{K}}, \tag{A4c}$$

which can be solved simultaneously for each of the constants c_1, c_2, and c_3. The solution of this system of linear equations can be obtained in the form

$$c_2 = (1 - c_1 + c_3)l\exp[-2k(\tau_c - \tau)], \tag{A5a}$$

$$1 - c_1 + c_3 = \frac{1 - l^2\exp(-2k\tau)}{1 - l^2\exp(-2k\tau_c)}. \tag{A5b}$$

Substituting Eqs. (A5) back into Eq. (A1) yields the following solution for column vector $\mathbf{v}(\tau)$:

$$\mathbf{v}(\tau) = \frac{1 - l^2\exp(-2k\tau)}{k - kl^2\exp(-2k\tau_c)} \times \{\hat{\mathbf{P}}^+ - l\exp[-2k(\tau_c - \tau)]\hat{\mathbf{P}}^-\}. \tag{A6}$$

This is just the solution given by Eq. (50).

When Eq. (A6) is substituted back into Eq. (43), it readily follows that

$$\hat{\mathbf{u}}^+(\tau) = \frac{\exp(-k\tau)}{1 - l^2\exp(-2k\tau_c)} \times \{\hat{\mathbf{P}}^+ - l\exp[-2k(\tau_c - \tau)]\hat{\mathbf{P}}^-\} * \hat{\mathbf{K}}^T\hat{\mathbf{u}}^+(0). \tag{A7}$$

The derivation of the corresponding formula for $\hat{\mathbf{u}}^-(\tau)$ is considerably more difficult, requiring the evaluation of

$$\begin{aligned}\hat{\mathbf{R}}_b\{\hat{\mathbf{P}}^+ &- l\exp[-2k(\tau_c - \tau)]\hat{\mathbf{P}}^-\} \\ &= (\hat{\mathbf{R}}_\infty - \gamma_b\hat{\mathbf{K}} * \hat{\mathbf{K}}^T) \\ &\quad\times\{\hat{\mathbf{P}}^+ - l\exp[-2k(\tau_c - \tau)]\hat{\mathbf{P}}^-\}, \\ &= \hat{\mathbf{P}}^- - l\exp[-2k(\tau_c - \tau)](\hat{\mathbf{P}}^+ - k\hat{\mathbf{K}}) \\ &\quad - \gamma_b\hat{\mathbf{K}} * \hat{\mathbf{K}}^T\{\hat{\mathbf{P}}^+ - l\exp[-2k(\tau_c - \tau)]\hat{\mathbf{P}}^-\}, \qquad (\text{A8})\end{aligned}$$

in which we have made use of Eqs. (35), (46b), and (47). The last term in this expression can be further reduced as follows:

$$\gamma_b\hat{\mathbf{K}} * \hat{\mathbf{K}}^T\{\hat{\mathbf{P}}^+ - l\exp[-2k(\tau_c - \tau)]\hat{\mathbf{P}}^-\} = kl\exp[-2k(\tau_c - \tau)]\hat{\mathbf{K}}, \tag{A9}$$

in which we have made use of the normalization conditions in Eq. (49). Finally, combining Eqs. (A8) and (A9) and substituting the resulting expression back into Eq. (A7) yields

$$\begin{aligned}\hat{\mathbf{u}}^-(\tau) &= \hat{\mathbf{R}}_b\hat{\mathbf{u}}^+(\tau), \\ &= \frac{\exp(-k\tau)}{1 - l^2\exp(-2k\tau_c)} \\ &\quad\times\{\hat{\mathbf{P}}^- - l\exp[-2k(\tau_c - \tau)]\hat{\mathbf{P}}^+\} * \hat{\mathbf{K}}^T\hat{\mathbf{u}}^+(0).\end{aligned} \tag{A10}$$

Equations (A7) and (A10) represent, respectively, the asymptotic solutions for the downward and upward propagating intensities deep within an optically thick medium. When combined these equations can be written as Eq. (51).

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