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# THE EFFECT OF THE HOT SPOT ON THE TRANSPORT EQUATION IN PLANT CANOPIES

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**Abstract**—The governing equation for photon transport in plant canopies is discussed. The specular component of leaf reflectance comprising the presence of the leaf-wax layer is introduced. The scattering phase function is derived for this case. By separating the first-order scattering into an independent problem, the leaf size is taken into account in obtaining the hot-spot effect of the canopy. The correlation function characterizing the probability that a point inside the canopy or on the soil is seen from two directions is considered. Numerical results for the transport equation obtained by the method of discrete ordinates are compared with those obtained by using the Monte Carlo method and experimental data.

## 1. INTRODUCTION

By the beginning of the 1960s, scientists had come very close to a mathematical description of the radiative regime in the plant canopy. The future development of this subject demands the construction of an adequate model of photon transfer in the canopy. The theory of radiative transfer in a turbid medium has been sufficiently well developed to solve problems arising in astrophysics, nuclear physics and atmospheric physics.[1–5]

Development of a model for radiative transfer in the canopy has been discussed by Ross and his colleagues.[6] These authors have described the main mathematical characteristics of radiative transfer, e.g., the mean projection of a unit foliage area on the propagation direction and the scattering phase function of the canopy. They have also derived the transport equation for the plate medium and determined the possible ways of its solutions. Because of the presence of the oriented plates, there are some essential differences between the transport equation in the plate and in turbid media. The main differences are the following:[6] (i) the phase function is not rotationally invariant; (ii) the extinction coefficient is not necessarily independent of the direction of photon travel. In some cases the first difference makes the solution of the governing equation much more complicated.

Recently, papers by Myneni[7,8] and his colleagues have provided new impetus to the development of the theory of radiative transfer in vegetation canopies. In Refs. 7 and 8, they have described the transport equation more accurately and have adapted the method of discrete ordinates in solving the equation. Various numerical results have shown good agreement between solutions of the transport equation and measured results.

However, it is, in principle, impossible to consider such important canopy parameters as leaf and stem dimensions, the effective distances between neighbouring leaves, the non-random distribution of leaves, the azimuthal angle between successive leaves on the genetic spiral, etc. in the transport equation in a turbid medium. The Monte Carlo method proposed by Ross and Marshak[9,10] allows us to take the influences of these parameters into account in the canopy bidirectional reflectance distribution function (see also the three-dimensional model of Kimes and Kirchner[11]). As a matter of fact, the Monte Carlo models are rather complicated and have serious disadvantages in inversion.

The approximation model of Nilson and Kuusk[12,13] also involves consideration of such an important parameter as leaf size by means of simple analytical formulae. It allows us to obtain and study the effect of leaf size on the hot spot. Their model is also easily reversible[13] but the contribution of multiple scattering is calculated approximately according to the Schwarzschild approximation.[6]

The aim of the present paper is to introduce consideration of the canopy hot spot into the model of the radiative regime described by the transport equation. By separating the calculation of first-order scattering in the original problem, we change the extinction function by taking into account the sizes of leaves (plates) and by using the correlation function introduced by Nilson and Kuusk.[12] We then add the specular component of leaf reflection that allows us to describe reflectance from the leaf surface more adequately.[10,13-15]

In Sec. 2, we consider the transport equation in vegetation canopies and, in Sec. 3, we discuss the boundary conditions using results obtained by Gerstl and Zardecki.[16] Section 4 deals with separation of first-order scattering and consideration of the size of plates needed to obtain the hot-spot effect. For uniform azimuthal orientation of leaves in the canopy, the transport equation is considerably simplified and we are able to decrease the dimensions of the problem (Sec. 5). In Sec. 6, we discuss the numerical results for a variety of agricultural canopies and compare these with the results of calculations using the Monte Carlo method[10] and with measured results. In the Appendix, we present mathematical results showing the conditions under which the problem has a unique solution.

## 2. TRANSPORT EQUATION IN THE PLANT CANOPY

We consider the radiative transfer equation in a flat, horizontal medium for a plane-parallel slab of depth $T$, the upper surface of which is illuminated by diffuse and direct radiation. We follow the notation of Ref. 7. For simplicity, we assume that the distribution of plate orientations does not depend on the height. We then write the transport equation in terms of the radiance distribution function $I$ as follows:[6,7]

$$-\mu[\partial I(\tau,\Omega)/\partial\tau] + G(\Omega)I(\tau,\Omega) = (1/\pi)\int_{4\pi} I(\tau,\Omega')\Gamma(\Omega'\to\Omega)\,d\Omega', \tag{1}$$

where the unit vector $\Omega = \Omega(\mu, \phi)$ has an azimuthal angle $\phi$ and a polar angle $\theta = \cos^{-1}(\mu)$ with respect to the outward normal directed opposite to the $z$-axis. Here, $0 \leqslant \tau \leqslant H$ denotes an analogue to the optical depth, viz.

$$\tau(z) = \int_0^z u_L(z')\,dz', \quad H = \tau(T) = \int_0^T u_L(z')\,dz',$$

and $u_L(z)$ is the vertical distribution of the total one-sided leaf area per unit volume of the canopy at the depth $z$. The function $G(\Omega)$ is the mean projection of a unit foliage area in the direction $\Omega$, i.e.

$$G(\Omega) = (1/2\pi)\int_{2\pi+} g_L(\Omega_L)|\Omega_L\cdot\Omega|\,d\Omega_L \tag{2}$$

and $g_L(\Omega_L)$ is the probability density $[(1/2\pi)\int_{2\pi+} g_L(\Omega_L)\,d\Omega_L = 1]$ of the distribution of the leaf normals with respect to the upper hemisphere.[6] Here, $2\pi+$ and $2\pi-$ are the upper and lower hemispheres, respectively. The area scattering phase function $\Gamma$ is given by

$$\Gamma(\Omega'\to\Omega) = (1/2)\int_{2\pi+} g_L(\Omega_L)|\Omega'\cdot\Omega_L| f(\Omega'\to\Omega,\Omega_L)\,d\Omega, \tag{3}$$

where $f$ is the leaf-scattering distribution-function. The following simple optical model of the leaf is proposed: the transmission is isotropic and characterized by a diffuse spectral function $t_L(\alpha')$. The reflection consists of two components: a diffuse spectral function $r_L(\alpha')$ and a specular component. Here, $\alpha' = \Omega'\cdot\Omega_L$ is the cosine of the angle between the incident ray $\Omega'$ and the leaf normal $\Omega_L$. Thus,

$$f(\Omega'\to\Omega,\Omega_L) = f_D(\Omega'\to\Omega,\Omega_L) + f_{sp}(\Omega'\to\Omega,\Omega_L). \tag{4}$$

We hypothesize that diffuse scattering from the leaves follows the bi-Lambertian scattering model, viz.

$$f_D(\Omega' \to \Omega, \Omega_L) = \begin{cases} r_L(\alpha')|\alpha|/\pi, & \alpha\alpha' < 0, \\ t_L(\alpha')|\alpha|/\pi, & \alpha\alpha' > 0, \end{cases}$$

with $\alpha = \Omega \cdot \Omega_L$. In other words, the functions $r_L$ and $t_L$ define the radii of hemispheres that depend on the angle between $\Omega'$ and $\Omega_L$. We can show that

$$\int_{4\pi} f_D(\Omega' \to \Omega, \Omega_L)\, d\Omega = r_L(\alpha') + t_L(\alpha'). \tag{5}$$

The specular component of reflection is determined by the presence of the wax layer on the leaf surface and it depends on the following three factors:[14] $\alpha'$, wax refractive index $n$, and the smoothness of the leaf surface. Hence, we can define $f_{sp}$ as

$$f_{sp}(\Omega' \to \Omega, \Omega_L) = K(k, \alpha')F(n, \alpha')\delta_2(\Omega \cdot \Omega^*). \tag{6}$$

We will now examine the last equality. The function $F$ is the intensity of the refracted ray defined by Fresnel's law, i.e.

$$F(n, \alpha') = \frac{1}{2}\left[\frac{\sin^2(j-i)}{\sin^2(j+i)} + \frac{\tan^2(j-i)}{\tan^2(j+i)}\right],$$

where $j = \cos^{-1}(|\alpha'|)$, $i = \sin^{-1}(\sqrt{1-(\alpha')^2}/n)$. The function $K$ defines the roughness of the leaf surface[14] ($0 \leqslant K \leqslant 1$) and the argument $k \geqslant 0$ characterizes the surface. The function $\delta_2$ is a surface delta-function:[5]

$$\delta_2(\Omega \cdot \Omega') = 0, \quad \Omega \neq \Omega', \quad \int_{4\pi} q(\Omega)\delta_2(\Omega \cdot \Omega')\, d\Omega = q(\Omega').$$

The vector $\Omega^* = \Omega^*(\Omega', \Omega_L)$ defines the direction of specular reflection. Integration of Eq. (6) leads to

$$\int_{4\pi} f_{sp}(\Omega' \to \Omega, \Omega_L)\, d\Omega = K(k, \alpha')F(n, \alpha'). \tag{7}$$

Combination of Eqs. (5) and (6) yields

$$\int_{4\pi} f(\Omega' \to \Omega, \Omega_L)\, d\Omega = r_L(\alpha') + t_L(\alpha') + K(k, \alpha')F(n, \alpha'). \tag{8}$$

We now return to the scattering phase function $\Gamma$. Taking into account Eq. (4), it follows from Eq. (3) that

$$\Gamma(\Omega' \to \Omega) = \Gamma_D(\Omega' \to \Omega) + \Gamma_{sp}(\Omega' \to \Omega), \tag{9}$$

where

$$\Gamma_D(\Omega' \to \Omega) = \frac{1}{2\pi}\int_{\Omega^+} g_L(\Omega_L) t_L(\alpha')\alpha'\alpha\, d\Omega_L - \frac{1}{2\pi}\int_{\Omega^-} g_L(\Omega_L) r_L(\alpha')\alpha'\alpha\, d\Omega_L.$$

Here, $\Omega^{\pm}$ is a part of the hemisphere for which $\pm\alpha'\alpha > 0$, $\alpha' = \Omega'\cdot\Omega_L$, $\alpha = \Omega\cdot\Omega_L$. To obtain $\Gamma_{sp}$, we note that[14] $d\Omega_L = d\Omega^*/4|\Omega'\cdot\Omega_L|$. Then

$$\Gamma_{sp}(\Omega' \to \Omega) = \frac{1}{2}\int_{2\pi+} g_L(\Omega_L)|\Omega'\cdot\Omega_L| K(k, \Omega'\cdot\Omega_L)F(n, \Omega'\cdot\Omega_L)\delta_2[\Omega\cdot\Omega^*(\Omega', \Omega_L)]\, d\Omega_L$$

$$= \frac{1}{8}\int_{4\pi} g_L(\Omega_L)K(k, \Omega'\cdot\Omega_L)F(n, \Omega'\cdot\Omega_L)\delta_2(\Omega\cdot\Omega^*)\, d\Omega^*$$

$$= \frac{1}{8} g_L(\Omega_L^*)K(k, \Omega'\cdot\Omega_L^*)F(n, \Omega'\cdot\Omega_L^*),$$

where $\Omega_L^* = \Omega_L^*(\Omega', \Omega)$ defines the direction of the appropriate leaf normal for specular scattering between the incident and the reflected rays. It is not difficult to show[15] that $\Omega_L^* \sim (\mu_L^*, \phi_L^*)$, where

$$\mu_L^* = \frac{|\mu' - \mu|}{\sqrt{2(1 - \Omega\cdot\Omega')}}, \quad \tan\phi_L^* = \frac{\sqrt{[1-(\mu')^2]}\sin\phi' - \sqrt{(1-\mu^2)}\sin\phi}{\sqrt{[1-(\mu')^2]}\cos\phi' - \sqrt{(1-\mu^2)}\cos\phi}.$$

It is evident that $\Gamma_{sp}$ is symmetrical. However, for symmetry of $\Gamma_D$, it is sufficient to require constancy of the reflection and transmission functions, i.e.

$$r_L(\alpha') \equiv r_L, \quad t_L(\alpha') \equiv t_L. \tag{10}$$

Then

$$\Gamma(\Omega' \to \Omega) = \Gamma(\Omega \to \Omega'). \tag{11}$$

The theorem of optical reciprocity is valid if[4]

$$\Gamma(\Omega' \to \Omega) = \Gamma(-\Omega' \to -\Omega)$$

holds. However, this condition will hold if the functions $K$, $r_L$ and $t_L$ are even, i.e., $K(k, \alpha) = K(k, -\alpha)$, $r_L(\alpha) = r_L(-\alpha)$, $t_L(\alpha) = t_L(-\alpha)$.

Using the analogy of Ref. 7, we determine a normalized scattering phase function. Taking into account Eqs. (3) and (8), we obtain $(1/4\pi)\int_{4\pi} P(\Omega' \to \Omega)\, d\Omega = 1$, where

$$P(\Omega' \to \Omega) = 4\Gamma(\Omega' \to \Omega)/[G_1(\Omega') + G_2(\Omega')]$$

and

$$G_1(\Omega') = \frac{1}{2\pi}\int_{2\pi+} g_L(\Omega_L)|\alpha'|[r_L(\alpha') + t_L(\alpha')]\, d\Omega_L, \tag{12}$$

$$G_2(\Omega') = \frac{1}{2\pi}\int_{2\pi+} g_L(\Omega_L)|\alpha'| K(k, \alpha')F(n, \alpha')\, d\Omega_L. \tag{13}$$

## 3. THE BOUNDARY CONDITIONS

Equation (1) alone does not provide an adequate description of the transfer process. It is also necessary to specify the incident radiation at the canopy boundaries, i.e., to set up the applicable boundary conditions. Depending on the aim of investigations, different boundary conditions apply.[16,17]

*3.1. Problem A: transfer of radiation in the atmosphere adjoining the canopy*

Let $H_0$ be the optical depth of the atmosphere. In this case, we obtain the following boundary conditions:

$$I(-H_0, \Omega) = \pi I_0 \delta(\Omega - \Omega_0), \qquad \mu < 0,$$

$$I(0, \Omega) = (1/\pi) \int_{2\pi -} |\mu'| R(\Omega', \Omega) I(0, \Omega') \, d\Omega', \quad \mu > 0.$$

Here, $\Omega_0 \sim (\mu_0, \phi_0)$ is the direction of the monodirectional solar radiation, $I_0$ is its intensity and $R(\Omega', \Omega)$ is a canopy reflectance function.

*3.2. Problem B: the canopy is considered to be an independent layer*

There is now no interaction between the canopy and the atmosphere. Then

$$I(0, \Omega) = F(\Omega), \quad \mu < 0, \tag{14}$$

$$I(H, \Omega) = (1/\pi) \int_{2\pi -} R_s(\Omega', \Omega) |\mu'| I(H, \Omega') \, d\Omega', \quad \mu > 0. \tag{15}$$

The last condition describes reflection from the soil.[17] For Lambertian reflectance from the soil, $R_s(\Omega', \Omega) = R_s = \text{constant}$ and

$$I(H, \Omega) = (R_s/\pi) \int_{2\pi -} |\mu'| I(H, \Omega') \, d\Omega', \quad \mu > 0.$$

The equality (14) leads to two practically useful problems.

*3.2.1. Problem B1: the function $R(\Omega', \Omega)$ for canopy reflection for Problem A.* In this case,

$$F(\Omega) = \pi \delta(\Omega - \Omega'), \quad \mu < 0.$$

Solving now the transport equation with the boundary conditions (14), (15) for each $\Omega'$, we obtain the distribution of the intensity for the plant canopy-function $I(\tau, \Omega)$. Hence,[17]

$$R(\Omega', \Omega) = \pi I(0, \Omega)/|\mu'|, \quad \mu' < 0, \quad \mu > 0.$$

*3.2.2. Problem B2: the distribution $I(\tau, \Omega)$ of the radiative intensity in the plant canopy if the incident radiation is weakened and scattered by the atmosphere.* For this condition, we can define the function $F$ by the expression

$$F(\Omega) = I_D(\Omega) + \pi I_0 \exp(-H_0/|\mu_0|) \delta(\Omega - \Omega_0), \quad \mu < 0, \tag{16}$$

where $I_D$ is the diffuse solar radiance.

*3.3. Problem C: a two-layer problem with the atmosphere reflecting radiation*

A photon that has escaped from the canopy can return as the result of interaction with the atmosphere. In this case, only the first boundary condition (14) is changed:

$$F(\Omega) = (1/\pi) \int_{2\pi +} R_a(\Omega', \Omega) |\mu'| I(0, \Omega') \, d\Omega' + I_D(\Omega) + \pi I_0 \exp(-H_0/|\mu_0|) \delta(\Omega - \Omega_0), \quad \mu > 0,$$

where $R_a(\Omega', \Omega)$ is an atmosphere reflectance function.[17]

*3.4. Problem D: the standard problem*

The transport equation will now be solved with the boundary conditions $I(0, \Omega) = f_0(\Omega)$, $\mu < 0$, $I(H, \Omega) = f_H(\Omega)$, $\mu > 0$, where $f_0$ and $f_H$ are known functions.

These problems are generally solved by iterative methods.[18] At each step of the iteration, Problem D is solved without the boundary conditions imposed in Problems A–C, since, in place of the unknown function, there is now a known function that has been determined in the previous step.

The mathematical foundations for this approach have been presented in Ref. 19. We note that Problem D can be reduced to zero boundary conditions, i.e.

$$I(0,\Omega)=0,\quad \mu<0,\quad I(H,\Omega)=0,\quad \mu>0. \tag{17}$$

Mathematical questions concerning the existence of solutions, their uniqueness and continuous dependence on the initial data have been studied in detail by many authors (see, for example, Refs. 4 and 5). Some questions connected with the existence and uniqueness of the solution of the transport equation in a plant canopy are considered in the Appendix.

## 4. SEPARATION OF FIRST-ORDER SCATTERING

### *4.1. Reasons for separation*

First-order scattering in some spectral regions is so significant that it is natural to separate this problem from the total analysis. Furthermore, our model of radiative transfer in the plant canopy, as described by the transport equation, does not allow for the sizes of the leaves (plates) with respect to the depth of the canopy (layer). However, the parameter $\kappa = l_L/T$, where $l_L$ is the length of the mean chord of the leaf and $T$ the height of the canopy, plays a significamt role in the generation of the canopy hot spot. The canopy reflectance function in the region of the hot spot is very sensitive to variations of $\kappa$.[9,12,20,21] It is impossible to consider this effect with the use of the transport equation (1). The mean value of the photon free path in Eq. (1) is direction-dependent and is defined only by the leaf orientation. In real vegetation canopies, the photon mean free path also depends on the direction of the photon before interaction. Actually, the stream of photons is not weakened by interactions in the backward direction and in the close-to-backward directions it is weakened less than by $\exp[-G(\Omega)\tau/\mu]$.

This effect was allowed for in some papers[12,20,21] by using a correction factor. We shall take it into account in first-order scattering by changing the extinction function. Why do we consider this phenomenon only for first-order scattering? The answer is that the transfer equation in a plate medium, where the size of the plate has a fixed meaning, is unknown. Secondly, the main contribution to the hot-spot effect is made by first-order scattering when a unidirectional stream of photons escapes from the source. For diffuse radiation or multiple scattering, the effect is averaged over all directions and does not play an essential role.

### *4.2. Separation of the radiation stream that has not undergone any interactions*

We now consider Problem B2. We assume for simplicity that the solar radiation coming through the atmosphere has unit intensity. Combining Eqs. (1), (14), (15), and (16), we get

$$\begin{cases} -\mu[\partial I(\tau,\Omega)/\partial\tau]+G(\Omega)I(\tau,\Omega)=(1/\pi)\displaystyle\int_{4\pi}\Gamma(\Omega'\to\Omega)I(\tau,\Omega')\,d\Omega', & \\ I(0,\Omega)=I_D(\Omega)+\delta(\Omega-\Omega_0), & \mu<0, \\ I(H,\Omega)=\displaystyle\int_{2\pi-}q(\Omega',\Omega)I(H,\Omega')\,d\Omega', & \mu>0, \end{cases} \tag{18}$$

where $q(\Omega',\Omega)=R_s(\Omega',\Omega)|\mu'|/\pi$. We represent the solution of the boundary-value problem (18) by the sum of three components, viz.

$$I=I_1^{un}+I_2^{un}+I^c, \tag{19}$$

where $I_1^{un}$ and $I_2^{un}$ are, respectively, the incident diffuse and direct radiation streams that have not undergone any interactions in the canopy, and $I^c$ is the intensity of the photons which have been scattered one or more times in the canopy. It then follows from the boundary conditions that

$$\begin{cases} -\mu[\partial I_1^{un}(\tau,\Omega)/\partial\tau] + G(\Omega)I_1^{un}(\tau,\Omega) = 0, \\ I_1^{un}(0,\Omega) = I_D(\Omega), \quad \mu < 0, \quad I_1^{un}(H,\Omega) = \int_{2\pi-} q(\Omega',\Omega)I_1^{un}(H,\Omega')\,d\Omega', \quad \mu > 0, \end{cases} \tag{20}$$

and

$$I_1^{un}(\tau,\Omega) = \begin{cases} I_D(\Omega)\exp[-G(\Omega)\tau/|\mu|], \quad \mu < 0, \\ \int_{2\pi-} q(\Omega',\Omega)I_1^{un}(H,\Omega')\,d\Omega' \exp[-G(\Omega)(H-\tau)/\mu], \quad \mu > 0. \end{cases} \tag{21}$$

For $I_2^{un}$, we obtain the problem

$$\begin{cases} -\mu[\partial I_2^{un}(\tau,\Omega)/\partial\tau] + G(\Omega)I_2^{un}(\tau,\Omega) = 0, \\ I_2^{un}(0,\Omega) = \delta(\Omega-\Omega_0), \quad \mu < 0, \quad I_2^{un}(H,\Omega) = \int_{2\pi-} q(\Omega',\Omega)I_2^{un}(H,\Omega')\,d\Omega', \quad \mu > 0, \end{cases} \tag{22}$$

which we will divide into two problems. We take

$$I_2^{un}(\tau,\Omega) = \begin{cases} I_{down}^{un}(\tau,\Omega), & \mu < 0, \\ I_{up}^{un}(\tau,\Omega), & \mu > 0, \end{cases} \tag{23}$$

where

$$I_{down}^{un}(\tau,\Omega) = \exp[-G(\Omega)\tau/|\mu|]\delta(\Omega-\Omega_0), \quad \mu < 0. \tag{24}$$

The function $I_{up}^{un}$ satisfies the initial-value problem

$$\begin{cases} -\mu[\partial I_{up}^{un}(\tau,\Omega)/\partial\tau] + G(\Omega)I_{up}^{un}(\tau,\Omega) = 0, & \mu > 0, \\ I_{up}^{un}(H,\Omega) = q(\Omega_0,\Omega)\exp[-G(\Omega_0)H/|\mu_0|], & \mu > 0, \end{cases} \tag{25}$$

the solution of which will be considered below.

It only remains for us to define the boundary-value problem for the radiation that has interacted, $I^c$. Taking into account Eqs. (20) and (22), it follows from Eqs. (18) and (19) that

$$-\mu[\partial I^c(\tau,\Omega)/\partial\tau] + G(\Omega)I^c(\tau,\Omega) = (1/\pi)\int_{4\pi} \Gamma(\Omega'\to\Omega)I^c(\tau,\Omega')\,d\Omega' + F(\tau,\Omega),$$

$$I^c(0,\Omega) = 0, \quad \mu < 0, \quad I^c(H,\Omega) = \int_{2\pi-} q(\Omega',\Omega)I^c(H,\Omega')\,d\Omega', \quad \mu > 0.$$

Here,

$$F(\tau,\Omega) = F_1(\tau,\Omega) + F_{up}(\tau,\Omega) + F_{down}(\tau,\Omega),$$

where

$$F_1(\tau,\Omega) = (1/\pi)\int_{4\pi} \Gamma(\Omega'\to\Omega)I_1^{un}(\tau,\Omega')\,d\Omega',$$

$$F_{up}(\tau,\Omega) = (1/\pi)\int_{2\pi+} \Gamma(\Omega'\to\Omega)I_2^{un}(\tau,\Omega')\,d\Omega',$$

$$F_{down}(\tau,\Omega) = (1/\pi)\Gamma(\Omega_0\to\Omega)\exp[-G(\Omega_0)\tau/|\mu_0|].$$

*4.3. Separation of first-order scattering*

We represent the solution of the specified problem by the sum of two components, i.e.

$$I^c = I_1^c + I_M^c,$$

where $I_1^c$ is the intensity of the photons which have been scattered only once in the canopy. Here, $I_1^c$ satisfies the boundary-value problem

$$\begin{cases} -\mu[\partial I_1^c(\tau,\Omega)/\partial\tau] + G(\Omega)I_1^c(\tau,\Omega) = F_{\text{down}}(\tau,\Omega), \\ I_1^c(0,\Omega) = 0, \quad \mu < 0, \quad I_1^c(H,\Omega) = 0, \quad \mu > 0. \end{cases} \tag{26}$$

The function $I_M^c$ is the intensity of multiply-scattered photons that satisfy the problem

$$\begin{cases} -\mu[\partial I_M^c(\tau,\Omega)/\partial\tau] + G(\Omega)I_M^c(\tau,\Omega) = (1/\pi)\int_{4\pi} \Gamma(\Omega'\to\Omega)I_M^c(\tau,\Omega')\,d\Omega' + Q(\tau,\Omega), \\ I_M^c(0,\Omega) = 0, \quad \mu < 0, \quad I_M^c(H,\Omega) = \int_{2\pi-} q(\Omega',\Omega)I_M^c(H,\Omega')\,d\Omega' + F_S(\Omega), \quad \mu > 0. \end{cases} \tag{27}$$

Here,

$$Q(\tau,\Omega) = F_1(\tau,\Omega) + F_{\text{up}}(\tau,\Omega) + F_L(\tau,\Omega),$$

$$F_L(\tau,\Omega) = \frac{1}{\pi}\int_{4\pi} \Gamma(\Omega'\to\Omega)I_1^c(\tau,\Omega')\,d\Omega',$$

$$F_S(\tau,\Omega) = \int_{2\pi-} q(\Omega',\Omega)I_1^c(H,\Omega')\,d\Omega'.$$

By analogy with Eq. (22), we split Eq. (26) into two initial-value problems as follows:

$$\begin{cases} -\mu[\partial I_1^c(\tau,\Omega)/\partial\tau] + G(\Omega)I_1^c(\tau,\Omega) = F_{\text{down}}(\tau,\Omega), & \mu < 0, \\ I_1^c(0,\Omega) = 0, & \mu < 0. \end{cases} \tag{28}$$

and

$$\begin{cases} -\mu[\partial I_1^c(\tau,\Omega)/\partial\tau] + G(\Omega)I_1^c(\tau,\Omega) = F_{\text{down}}(\tau,\Omega), & \mu > 0 \\ I_1^c(H,\Omega) = 0, & \mu > 0. \end{cases} \tag{29}$$

We can easily obtain the solution of the initial-value problem (28) for $\mu < 0$. It is

$$I_1^c(\tau,\Omega) = \begin{cases} \dfrac{|\mu_0|\Gamma(\Omega_0\to\Omega)}{\pi[G(\Omega)|\mu_0| - G(\Omega_0)|\mu|]}[\exp(-G(\Omega_0)\tau/|\mu_0|) - \exp(-G(\Omega)\tau/|\mu|)], & \Omega \neq \Omega_0, \\ \dfrac{\tau\Gamma(\Omega_0\to\Omega)}{\pi|\mu_0|}\exp[-G(\Omega_0)\tau/|\mu_0|], & \Omega = \Omega_0. \end{cases} \tag{30}$$

### *4.4. Consideration of the canopy hot-spot effect*

We begin with the two initial-value problems defined in Eqs. (25) and (29) that describe the movement of photons reflected from the soil and leaves in the upward directions. In Eq. (25), the unidirectional radiation has descreased from the upper to the lower boundary (soil) by $\exp[-G(\Omega_0)H/|\mu_0|]$. The portion $q(\Omega_0,\Omega)$ is reflected from the soil and escapes into a unit solid angle about the direction $\Omega$. The function $G(\Omega)$ is defined by Eq. (2) and is placed before the second term in the transport equation; it characterizes the extinction of radiation in the direction $\Omega$. For simplicity, we assume that $u_L(z) \equiv u_L =$ constant, i.e., the plates are uniformly distributed with respect to depth. Therefore, extinction applies only for directions that are far from $\Omega_0$. In real plant canopies, the leaves have a fixed size and the value $\exp[-G(\Omega_0)H/|\mu_0|]$ denotes the probability that there is a gap in the direction $\Omega_0$ through which the soil can be seen. It is evident that if a photon has come through the gap, it does directly back unweakened to the upper boundary with unit probability. Since the leaves have a finite size, this probability decreases with the distance from $\Omega_0$. Thus, the directions in which the photon flies into and out of the canopy (after reflection from the soil) are not independent. Nilson and Kuusk[12,13] have considered the correlation function $r_{\Omega_0,\Omega}(H)$ characterizing the probability that a point on the soil illuminated in the direction $\Omega_0$ would be seen in the direction $\Omega$. They have found that $r_{\Omega_0,\Omega}$ depends on $\kappa$, i.e., on the relation between the length of the average chord and the height of the canopy. Approximating this function by the exponent and considering the results of Refs. 12 and 13, we obtain the new coefficient of extinction

$$\gamma(\tau, \Omega_0, \Omega) = G(\Omega)\{1 - \sqrt{A_0/A}\ \exp[-C(\Omega_0, \Omega)\tau/\kappa H]\}, \quad \mu > 0, \tag{31}$$

where $A_0 = G(\Omega_0)/|\mu_0|$, $A = G(\Omega)/|\mu|$ and $C(\Omega_0, \Omega) = (\mu_0^{-2} + \mu^{-2} + 2(\Omega_0 \cdot \Omega)/|\mu_0 \mu|)^{1/2}$. For $\Omega = -\Omega_0$, back reflection occurs and $\Omega \cdot \Omega_0 = -1$. Hence, it follows that $C(\Omega_0, \Omega) = 0$ and $\gamma(\tau, \Omega_0, \Omega) = 0$, which indicates the absence of extinction.

Taking into account Eq. (31), the initial-value problem (25) is transformed into

$$\begin{cases} -\mu[\partial I_{\mathrm{up}}^{\mathrm{un}}(\tau, \Omega)/\partial\tau] + \gamma(\tau, \Omega_0, \Omega) I_{\mathrm{up}}^{\mathrm{un}}(\tau, \Omega) = 0, & \mu > 0, \\ I_{\mathrm{up}}^{\mathrm{un}}(H, \Omega) = q(\Omega_0, \Omega)\exp[-G(\Omega_0)H/|\mu_0|], & \mu > 0. \end{cases}$$

Similar reasoning reduces the initial-value problem (29) to

$$\begin{cases} -\mu[\partial I_1^{\mathrm{c}}(\tau, \Omega)/\partial\tau] + \gamma(\tau, \Omega_0, \Omega) I_1^{\mathrm{c}}(\tau, \Omega) = F_{\mathrm{down}}(\tau, \Omega), & \mu > 0, \\ I_1^{\mathrm{c}}(H, \Omega) = 0, & \mu > 0. \end{cases}$$

It is easy to obtain the solutions for the last two problems as follows:

$$I_{\mathrm{up}}^{\mathrm{un}}(\tau, \Omega) = I(H, \Omega)\exp\left[-(1/\mu)\int_\tau^H \gamma(\zeta, \Omega_0, \Omega)\,\mathrm{d}\zeta\right], \quad \mu > 0, \tag{32}$$

$$I_1^{\mathrm{c}}(\tau, \Omega) = (1/\mu)\int_\tau^H F_{\mathrm{down}}(\tau', \Omega)\exp\left[-(1/\mu)\int_\tau^{\tau'} \gamma(\zeta, \Omega_0, \Omega)\,\mathrm{d}\zeta\right]\mathrm{d}\tau', \quad \mu > 0, \tag{33}$$

where

$$(1/\mu)\int_\tau^{\tau'} \gamma(\zeta, \Omega_0, \Omega)\,\mathrm{d}\zeta = G(\Omega)(\tau' - \tau)/\mu$$
$$- [\sqrt{A_0 A}\,\kappa H/C(\Omega_0, \Omega)]\{\exp[-C(\Omega_0, \Omega)\tau/\kappa H] - \exp[-C(\Omega_0, \Omega)\tau'/\kappa H]\}.$$

The complete solution is the sum of the components, viz.

$$I = I_1^{\mathrm{un}} + I_2^{\mathrm{un}} + I_1^{\mathrm{c}} + I_{\mathrm{M}}^{\mathrm{c}},$$

where $I_1^{\mathrm{un}}$ is defined by the equality (21), $I_2^{\mathrm{un}}$ by Eqs. (23), (24) and (32), $I_1^{\mathrm{c}}$ by Eqs. (30) and (33), and $I_M^{\mathrm{c}}$ is the solution of the boundary-value problem (27). This problem may be solved by using the discrete ordinates method.[7,18]

## 5. UNIFORM AZIMUTHAL DISTRIBUTION FOR THE LEAF NORMALS

We assume that the polar and azimuthal angles of the distribution of the leaf normals are independent and the distribution in the azimuth is uniform. Then, $g_{\mathrm{L}}(\Omega_{\mathrm{L}}) \equiv g_{\mathrm{L}}(\mu_{\mathrm{L}})$ and[7,8]

$$G(\Omega) \equiv G(\mu) = \int_0^1 g_{\mathrm{L}}(\mu_{\mathrm{L}})\left[(1/2\pi)\int_0^{2\pi} |\Omega \cdot \Omega_{\mathrm{L}}|\,\mathrm{d}\phi_{\mathrm{L}}\right]\mathrm{d}\mu_{\mathrm{L}}.$$

The second integral has been evaluated in Refs. 6 and 7. In addition to experimental data, there exist some theoretical models for the function $g_{\mathrm{L}}$ for various forms of the canopy. Examples are a trigonometric representation[22]

$$g_{\mathrm{L}}(\theta_{\mathrm{L}}) = (a + b\cos 2\theta_{\mathrm{L}} + c\cos 4\theta_{\mathrm{L}})/\sin\theta_{\mathrm{L}}, \quad \theta_{\mathrm{L}} = \cos^{-1}(\mu_{\mathrm{L}}), a, b, c = \text{constant},$$

and the statistical distributions: $g_{\mathrm{L}}(\theta_{\mathrm{L}}) \sim \beta(\mu, \nu)$, a beta distribution[23] and $g_{\mathrm{L}}(\theta_{\mathrm{L}}) \sim e(\nu, \epsilon)$, an elliptical distribution.[13]

In this case, if Eq. (10) holds, we can show that the phase function $\Gamma$ depends only on the difference between the azimuthal angles of the incident and reflected rays; however, it does not depend on each ray separately, i.e.[24]

$$\Gamma(\Omega' \to \Omega) = \Gamma(\mu, \mu', \phi - \phi'), \tag{34}$$

and

$$\Gamma(\mu, \mu', x) = \Gamma(\mu, \mu', -x) = \Gamma(\mu, \mu', x + 2\pi). \tag{35}$$

The indicated procedure allows us to reduce the dimensions of the problem significantly for numerical solution by using discrete ordinates. Furthermore, the symmetry properties, together with the conditions (34) and (35), allow us to use the "method of circulant matrix",[18] which, because of the dependence of the phase function on the differences of azimuthal angles and its periodicity with respect to this argument, transfers a problem of dimension $NM$ (where $N$ is the number of directions with respect to $\mu$ and $M$ is the number with respect to $\phi$) into $M$ problems of dimension of $N$ (see Ref. 18 for detail).

## 6. NUMERICAL RESULTS AND DISCUSSION

We will now illustrate the influence of various model parameters on the scattering phase function and canopy bidirectional reflectance.

From the definition (9), it follows that the phase function $\Gamma(\Omega' \to \Omega)$ represents the sum of the diffuse and specular components, each of which depends on the plate orientation. The diffuse component also depends on the reflectivity $r_L$ and on the transmissivity $t_L$ for each separate plate, whereas the specular component is described by the wax refractive index $n$ and the extinction parameter $k$.

In Fig. 1, we show three vertical cross-sections of the phase function ($\phi = \phi'$ and $\phi = \phi' + 180°$, $\phi = \phi' + 45°$ and $\phi = \phi' + 225°$, $\phi = \phi' + 90°$ and $\phi = \phi' + 270°$) to represent the planophile canopy (mainly horizontal leaves) and erectophile canopy (mainly vertical leaves) with $\mu' = -0.5$ ($\theta' = 120°$). In this case, the specular component is missing ($n = 1.0$) and the phase functions at $\mu < 0$ and $\mu > 0$ are similar. An increase in scattering in the near-nadir directions for planophile leaves and a decrease for the erectophile leaves can be explained by the Lambertian distribution of reflectance from their surfaces. Symmetry with respect to the nadir occurs only for the cross-sections $\phi = \phi' + 90°$ and $\phi = \phi' + 270°$.

Figure 2 illustrates the role of the parameters $n$ and $k$ in the phase function. The cross-sections of the phase function are presented in the principal plane $\phi = \phi'$, $\phi = \phi' + 180°$ at $\mu' = -\sqrt{3}/2$ ($\theta' = 150°$) for the erectophile canopy. Figure 2(a) shows the phase function for $\mu < 0$ (upward

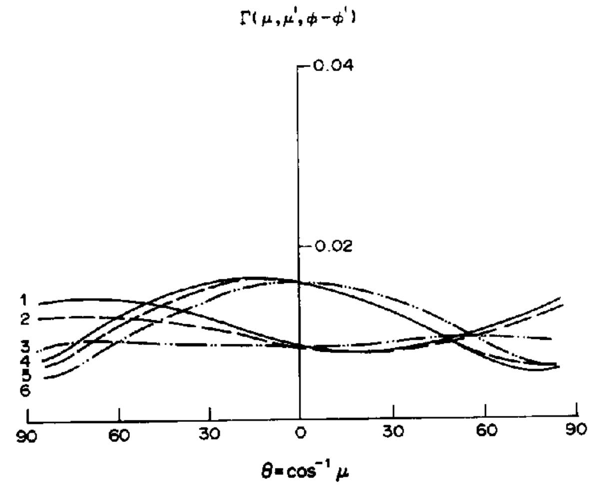


Fig. 1. The scattering phase function for various leaf-angle distributions (1–3—erectophile; 4–6—planophile). Curves 1 and 4 correspond to the cross-sections $\phi = \phi'$ and $\phi = \phi' + 180°$; curves 2 and 5 refer to $\phi = \phi' + 45°$ and $\phi = \phi' + 225°$; curves 3 and 6 refer to $\phi = \phi' + 90°$ and $\phi = \phi' + 270°$. Here, $r_L = t_L = 0.04$, $n = 1.0$, $\mu' = -0.5$.

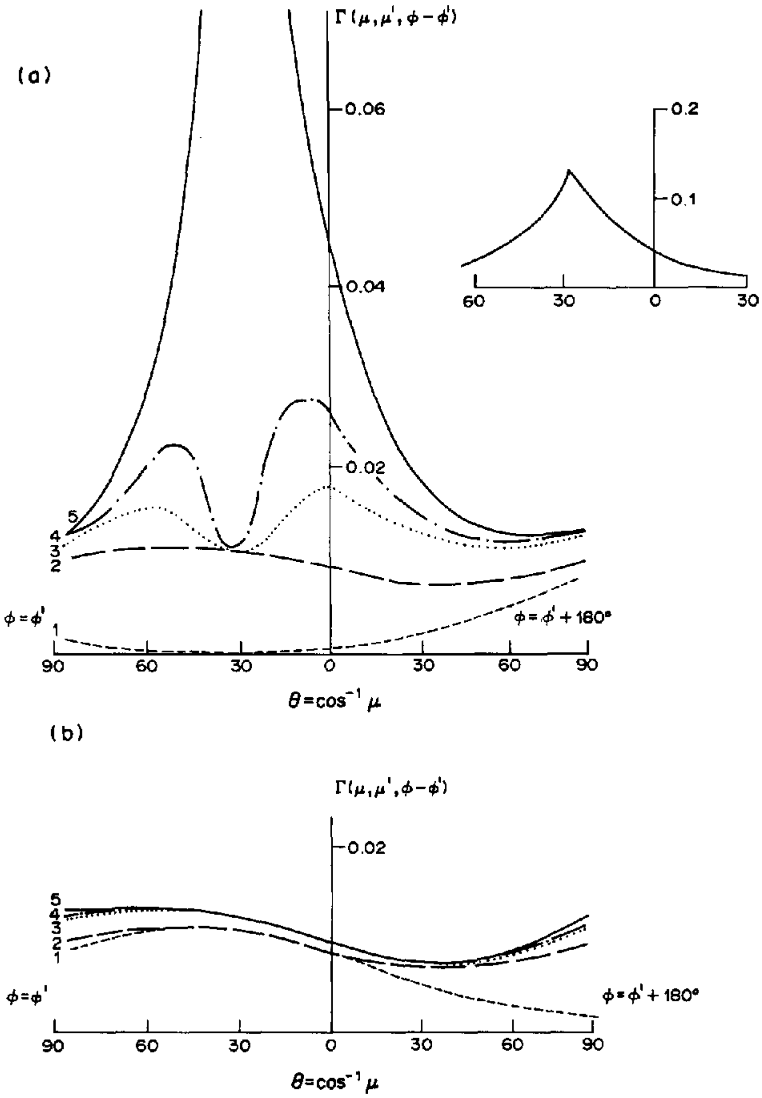


Fig. 2. The scattering phase function for the erectophile canopy at $\mu' = -\sqrt{3}/2$. The corresponding cross-section is $\phi = \phi'$ and $\phi = \phi' + 180°$; Fig. 2(a) refers to $\mu < 0$ and Fig. 2(b) to $\mu > 0$. Curve 1: $n = 1.0, r_L = 0.04, t_L = 0.0$; curve 2: $n = 1.0, r_L = t_L = 0.04$; curves 3–5: $n = 1.4, r_L = t_L = 0.04$; curve 3: $k = 0.6$; curve 4: $k = 0.3$; curve 5: $k = 0.0$.

scattering photons). Curve 1 represents only reflection ($r_L = 0.04$, $t_L = 0.0$) and curve 2 represents reflection and transmission ($r_L = t_L = 0.04$). It is obvious that their difference corresponds to the contribution of the transmittance that has a maximum at $\mu < 0$ and $\phi = \phi'$.

Curves 3, 4, and 5 denote the contributions of the specular component ($n = 1.4$) for different coefficients $k$, which characterize the presence ($k > 0$) and dimensions of the fibre on the leaf surface. The extinction function for the specular component is calculated from[13]

$$K(k, \alpha') = \exp[-2k \tan(\alpha')/\pi], \quad \alpha' = \cos^{-1}(\Omega' \cdot \Omega_L). \tag{36}$$

It is evident that a decrease in $k$ increases the contribution of the specular component. For an erectophile canopy at $\mu > 0$, this result is valid only for large reflection angles [Fig. 2(b)]. An increase in the phase function with decreasing $k$ is observed for scattering in the downward direction [Fig. 2(a)]. For $\theta = 30°$ and $\phi = \phi'$ (scattering along the leaf), a deep minimum is observed for $k > 0$. However, this minimum disappears with decreasing $k$. The presence of scattering maxima for both directions at $\theta = 30°$ is due to strong specular reflection (especially, in the case of the absence of fibre, curve 5) in the plane of the incident ray in the directions close to the leaf surface.

In the following figures, we represent reflectance functions of the plant canopy, which have been calculated from

$$R(\Omega_0, \Omega) = \pi I(0, \Omega)/F,$$

where

$$F = |\mu_0| + \int_{2\pi-} I_D(\Omega')|\mu'|\,d\Omega'.$$

Here, $I$ is the solution of boundary-value problem (18), $\Omega_0 \sim (\mu_0, \phi_0)$ the direction of the direct solar radiation, and $I_D$ the diffuse component. In the following numerical results, we assume that the sky was uniformly bright,[6] i.e. $I_D(\Omega) = \text{constant}$. The contribution of the direct radiation is expressed by $\beta = |\mu_0|/F$, which characterizes the part of the direct radiation in the total incident flux density.

The reflectance functions $R(\Omega_0, \Omega)$ for various leaf-angle distributions in the principal plane ($\phi = \phi_0, \phi = \phi_0 + 180°$), are presented in Fig. 3. By increasing the average leaf-inclination angle, the reflectance function increases for all view directions. Here, the effect of the canopy hot spot also increases and leads to asymmetry of the bidirectional reflectance relative to the nadir. There are three orientations: plagiophile (leaves mostly inclined at 45°), uniform [$g_L(\theta_L) = 2/\pi$] and extremophile (nearly horizontal and vertical leaves); these have the same average inclination angle ($E\theta_L = 45°$) but different dispersions (4.6, 11.8 and 18.9, respectively). There are significant differences in the reflectance functions for $\phi = \phi_0 + 180°$ within the region around $\theta = 90° - E\theta_L$. In Refs. 9 and 10, the same effect has been observed in calculations using the Monte Carlo method. On the other hand, for $\phi = \phi_0$ and large view directions, the reflection from the extremophile leaves is less than from the plagiophile. It is caused by the Lambertian law of reflectance from the leaf surface at $\theta_0 = 150°$.

Figure 4 illustrates the influence of the parameter characterizing the leaf dimensions on the reflectance function $R(\Omega_0, \Omega)$. For the planophile canopy at $\theta_0 = 120°$ ($\mu_0 = -0.5$) in the principal plane ($\phi = \phi_0, \phi = \phi_0 + 180°$), there are four reflectance functions, from $\kappa = 0.0$ (infinitesimally small leaves) up to $\kappa = 0.5$ (large leaves with a diameter exceeding 1 m). The relation between the

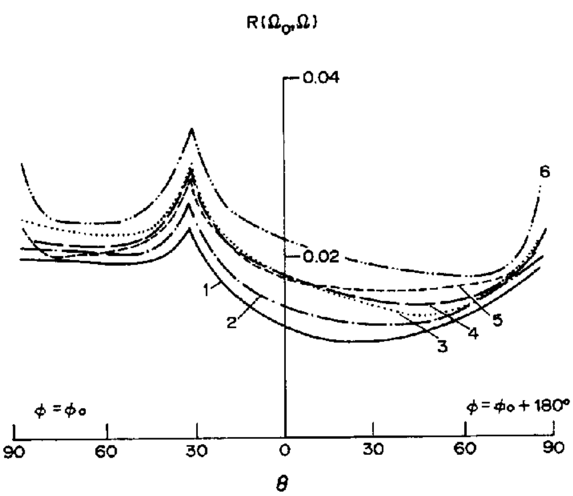


Fig. 3. The reflectance function of the plant canopy $R(\Omega_0, \Omega)$ for various leaf-angle distributions. Curve 1—erectophile; curve 2—spherical; curve 3—plagiophile; curve 4—uniform; curve 5—extremophile; curve 6—planophile. Here, $r_L = t_L = 0.04, n = 1.0, H = 3.0, \kappa = 0.08, R_s = 0.0, \beta = 1, \theta_0 = 150°, \phi = \phi_0$.

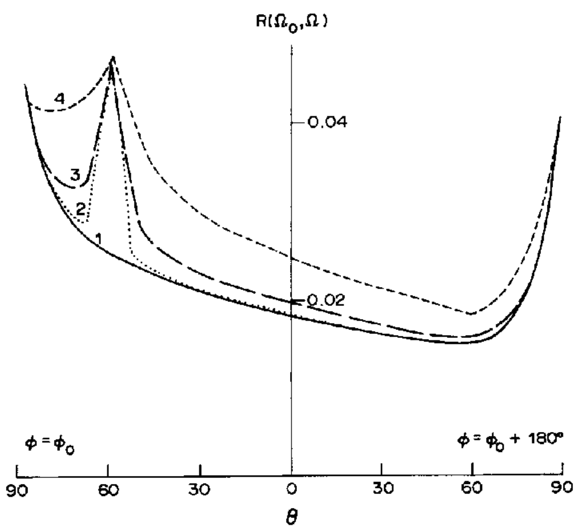


Fig. 4. The reflectance function of the plant canopy $R(\Omega_0, \Omega)$ for various leaf sizes. Curve 1—$\kappa = 0.00$; curve 2—$\kappa = 0.01$; curve 3—$\kappa = 0.08$; curve 4—$\kappa = 0.50$. Here, $r_L = t_L = 0.04$, $n = 1.0$, $H = 3.0$, $\kappa = 0.08$, $R_s = 0.0$, $\beta = 1$, $\theta_0 = 120°$, $\phi = \phi_0$.

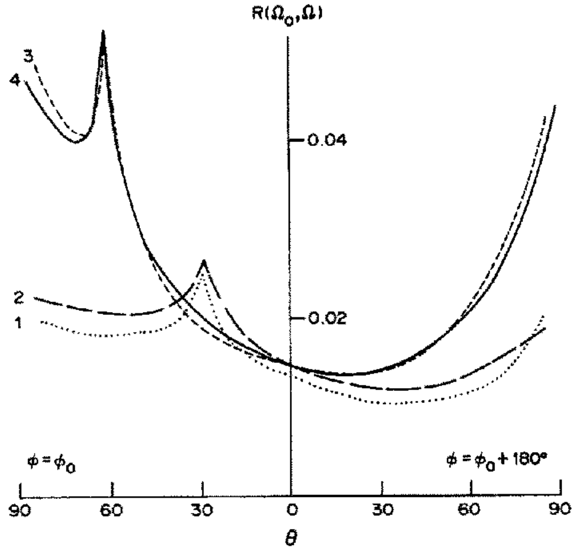


Fig. 5. Comparison of the reflectance functions calculated by the Monte Carlo method (curves 1 and 3) with values obtained by the discrete ordinates method, allowing for the canopy hot-spot effect (curves 2 and 4). Here $r_L = t_L = 0.04, \eta = 1.0, H = 3.0, \kappa = 0.08, R_s = 0.0, \beta = 1, \phi = \phi_0, \theta_0 = 150°$ (curves 1 and 2), $\theta_0 = 120°$ (curves 3 and 4). The leaf orientation is spherical.

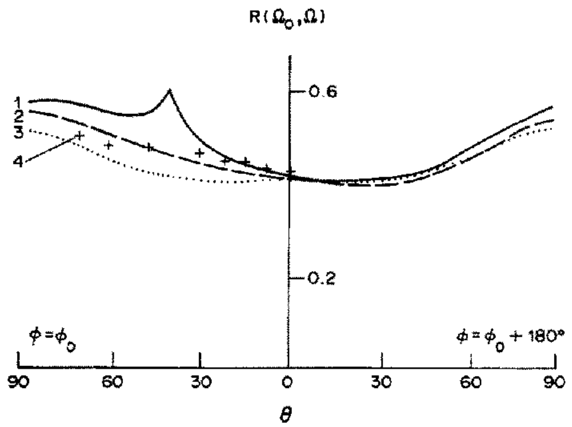


Fig. 6. Comparison of the calculated reflectance functions with measured data for the maize canopy in the near-i.r. spectral region (+). Here, $r_L = t_L = 0.46$, $n = 1.0$, $H = 4.0$, $\kappa = 0.08$, $R_s = 0.2$, $\beta = 0.8$, $\theta_0 = 140°$, $\phi_0 = 100°$. Curve 1—$\phi = \phi_0$ and $\phi = \phi_0 + 180°$; curve 2—$\phi = \phi_0 - 45°$ and $\phi = \phi_0 + 135°$; curve 3—$\phi = \phi_0 - 90°$ and $\phi = \phi_0 + 90°$; crosses 4—$\phi = 0$. The leaf orientation is spherical.

parameter $\kappa$, the average round leaf diameter $d_L$ and the canopy height has been found by Kuusk and Nilson[13] for horizontal leaves, namely, $\kappa = (\pi/4)(d_L/H)$. For spherically oriented leaves, Kuusk has obtained the equality $\kappa = (\pi/4)^2(d_L/H)$. With an increase in the relative leaf size, the bidirectional reflectance function in the region of the hot spot increases and the asymmetry relative to the nadir direction increases.

We now consider two models: the model of a plant canopy whose radiative regime is described by the transport equation (model 1) and the geometrical Ross–Marshak model, the radiative regime of which is calculated by the Monte Carlo method[9,10] (model 2). We propose for model 2 that stems are absent, the plants are planted in check-rows and the azimuthal angle between the successive leaves on the genetic spiral is 120°, with the average number of round leaves within a plant being 4.

In Fig. 5, we show four curves. Curves 1 and 2 refer to $\theta_0 = 150°$ and curves 3 and 4 illustrate $\theta_0 = 120°$. The curves calculated according to models 1 and 2 for $\theta_0 = 120°$ practically coincide. Under a high sun (at $\theta_0 = 150°$), the bidirectional reflectance for model 1 is greater than the bidirectional reflectance for model 2. This result may hold because of the following reasons. In the geometrical model, the phytoelements are distributed more regularly. In such a canopy, the probability to see the black soil ($R_s = 0$) at small view directions is greater than with randomly distributed leaves (model 1). By increasing the view angles, the difference between the regular and random canopies decreases as the mean photon free path grows longer and the gaps in the foliage become invisible. Therefore, curves 3 and 4 are similar to each other, whereas there are essential differences between curves 1 and 2.

We now compare our results with experimental data on the reflectance in the maize canopy obtained by Ranson and Biehl[25] (see also Refs. 7 and 8). There is good agreement between our data and experimental results (Fig. 6). The greatest difference can be seen in the principal plane, where the hot-spot effect is noticeable. The soil reflectance follows Lambert's law: $q(\Omega', \Omega) = R_s|\mu'|/\pi$.

Figure 7 is an illustration of the three-dimensional Cartesian plot of the reflectance function $R(\Omega_0, \Omega)$ in the infrared spectral region for the uniform leaf-angle distribution ($g_L(\theta_L) = 2/\pi$), where the specular component is also considered. The extinction of the specular component has been calculated by using Eq. (36).

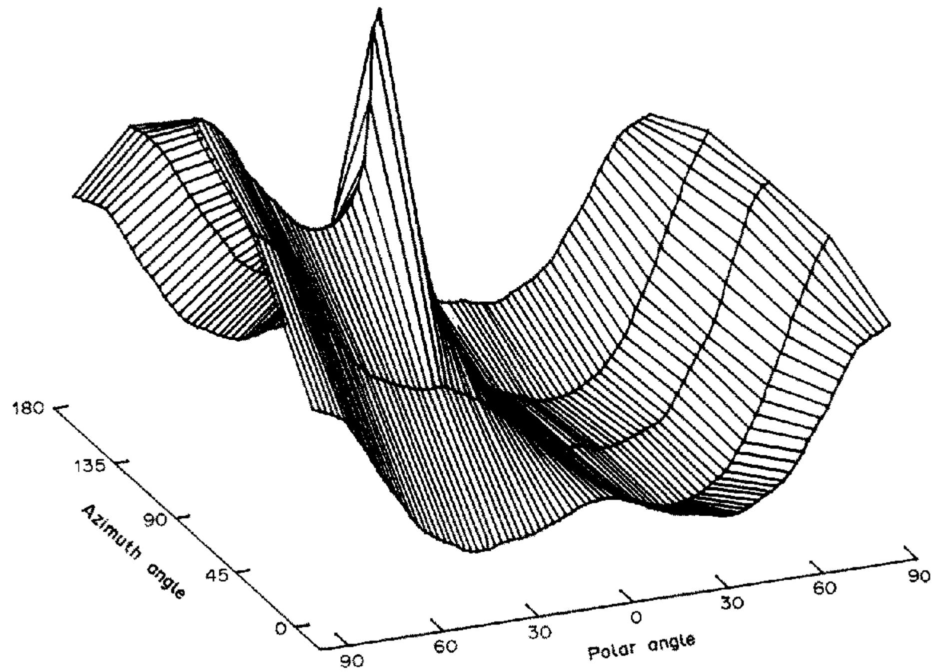


Fig. 7. The three-dimensional Cartesian reflectance function $R(\Omega_0, \Omega)$ is shown. Here, $r_L = 0.5$, $t_L = 0.35$, $n = 1.4$, $k = 0.3$, $H = 3.0$, $\kappa = 0.08$, $R_s = 0.0$, $\beta = 1$, $\theta_0 = 150°$, $\phi_0 = 90°$. The leaf orientation is uniform.

## 7. CONCLUSIONS

We supplement the transport equation in the vegetation canopy considered by Shultis and Myneni[7] by including the specular component in the scattering phase function.[10,13,14] The separation of first-order scattering allows us to consider the leaf size in the transfer equation for a plate medium. It has led to the formation of the canopy hot-spot effect. It is very sensitive to variations of the leaf sizes and orientations (Figs. 3 and 4). Therefore, it is the most informative region for remote sensing problems.

Comparison of the present model with much more complicated Monte Carlo model (Fig. 5) and with experimental data on real canopies (Fig. 6) shows good agreement.

Further development of the present model involves generalization to the three-dimensional case and inversion of geometrical and optical parameters by using data on canopy reflectance.

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## APPENDIX

### *Solvability of the Transport Equation for the Plant Canopy*

We now consider the transport equation with a non-zero absolute term [e.g., the equation for the boundary-value problem (27) with the standard boundary conditions (17)]. We use the operator notation

$$LI = SI + Q_1. \tag{A1}$$

Here, $L$ is the differential operator that satisfies the relation

$$(LI)(\tau, \Omega) = -[\mu/G(\Omega)][\partial I(\tau, \Omega)/\partial \tau] + I(\tau, \Omega).$$

An integral operator is denoted by $S$ such that the following relation is satisfied:

$$(SI)(\tau, \Omega) = [1/\pi G(\Omega)] \int_{4\pi} \Gamma(\Omega' \to \Omega) I(\tau, \Omega')\, d\Omega'.$$

Next, we define the function $Q_1 = Q/G$. We also introduce

$$W = \left\{ \Phi(\tau, \Omega) \colon \int_0^H \int_{4\pi} G(\Omega') |\Phi(\tau', \Omega)|\, d\Omega'\, d\tau' < \infty \right\}$$

and

$$D(L) = \{\Phi(\tau, \Omega) \colon \Phi(0, \Omega) = 0, \quad \mu < 0, \quad \Phi(H, \Omega) = 0, \quad \mu > 0, \quad L\Phi \in W\}.$$

It is known[4,19] that (i) $D(L) \subset W$ and for each function $f \in W$ there always exists a sequence from $D(L)$ that converges to $f$; the equation $LI = f$ has a unique solution $I \in D(L)$ for each $f \in W$, i.e., the inverse operator $L^{-1}$ exists such that $I = L^{-1} f$.

We denote the r.h.s. of Eq. (A1) by

$$f = SI + Q_1. \tag{A2}$$

After solving the boundary-value problem $LI = f$, we obtain

$$I = L^{-1} f. \tag{A3}$$

Substituting Eq. (A2) into Eq. (A3) we obtain the integral equation

$$I = L^{-1} SI + L^{-1} Q_1. \tag{A4}$$

Equation (A4) may be studied by using the Banach theorem for an operator equation of the second kind.[26] According to this theorem, both the existence and uniqueness of the solution from the inequality

$$\| L^{-1}S\Phi \| \leqslant q \| \Phi \|, \; q < 1, \quad \text{for any } \Phi \in W \tag{A5}$$

Here, $\| \Phi \|$ is given by

$$\| \Phi \| = \int_0^H \int_{4\pi} G(\Omega) | \Phi(\tau, \Omega) | \, d\Omega \, d\tau.$$

It may be proved that the inequality (A5) holds for $q < \lambda$, where

$$\lambda = \operatorname*{ess\,sup}_{-1 \leqslant \mu \leqslant 1, 0 \leqslant \phi \leqslant 2\pi} (1/\pi) \int_{4\pi} \Gamma(\Omega \to \Omega') \, d\Omega' / G(\Omega), \quad \Omega \sim (\mu, \phi).\dagger \tag{A6}$$

Hence, Eq. (A4) has a unique solution, which may be expressed as a Neiman series as follows:

$$I = L^{-1}Q_1 + L^{-1}SL^{-1}Q_1 + L^{-1}SL^{-1}SL^{-1}Q_1 + \cdots.$$

If $Q_1(\tau, \Omega) \geqslant 0$, then the $(L^{-1}S)^n L^{-1}Q_1$, $n = 0, 1, 2, \ldots$ are positive. Using Eqs. (12) and (13) in Eq. (A6), we obtain

$$\lambda = \operatorname*{ess\,sup}_{\Omega \in 4\pi} \left[ \sum_{i=1}^{2} G_i(\Omega)/G(\Omega) \right] = \operatorname*{ess\,sup}_{\Omega \in 4\pi} \frac{\int_{2\pi+} g_L(\tau, \Omega_L) | \Omega \cdot \Omega_L | b(\Omega, \Omega_L) \, d\Omega_L}{\int_{2\pi+} g_L(\tau, \Omega_L) | \Omega \cdot \Omega_L | \, d\Omega_L} \leqslant 1,$$

where

$$b(\Omega, \Omega_L) = (1/\pi) \int_{4\pi} | \Omega' \cdot \Omega_L | c(\Omega' \cdot \Omega_L) \, d\Omega' + K(k, \Omega \cdot \Omega_L) F(n, \Omega \cdot \Omega_L),$$

$$c(\alpha') = \begin{cases} r_L(\alpha'), & \alpha\alpha' < 0, \\ t_L(\alpha'), & \alpha\alpha' > 0, \end{cases}$$

$$\alpha = (\Omega \cdot \Omega_L), \quad \alpha' = (\Omega' \cdot \Omega_L).$$

The condition $\lambda \leqslant 1$ guarantees the existence and uniqueness of the solution of the transport equation in the plant canopy. For the special case $r_L(\alpha) \equiv r_L$, $t_L(\alpha) \equiv t_L$, we obtain

$$r_L + t_L + \max_{0 \leqslant \alpha \leqslant 1} K(k, \alpha) F(n, \alpha) \leqslant 1.$$

†Here, $\operatorname*{ess\,sup}_{t \in T} |x(t)| = \lim_{p \to \infty} \left[ \int_T |x(t)| \, dt \right]^{1/p}$.